

# Dynamics of dark energy

Edmund J. Copeland,<sup>1</sup> M. Sami,<sup>2,3</sup> and Shinji Tsujikawa<sup>4</sup>

<sup>1</sup>*School of Physics and Astronomy, University of Nottingham,  
University Park, Nottingham NG7 2RD, United Kingdom*

Email: ed.copeland@nottingham.ac.uk

<sup>2</sup>*Centre for Theoretical Physics, Jamia Millia Islamia, New Delhi, India*

<sup>3</sup>*Department of Physics, Jamia Millia Islamia, New Delhi, India*

Email: sami@iucaa.ernet.in; sami@jamia-physics.net

<sup>4</sup>*Department of Physics, Gunma National College of Technology, Gunma 371-8530, Japan*

Email: shinji@nat.gunma-ct.ac.jp

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In this paper we review in detail a number of approaches that have been adopted to try and explain the remarkable observation of our accelerating Universe. In particular we discuss the arguments for and recent progress made towards understanding the nature of dark energy. We review the observational evidence for the current accelerated expansion of the universe and present a number of dark energy models in addition to the conventional cosmological constant, paying particular attention to scalar field models such as quintessence, K-essence, tachyon, phantom and dilatonic models. The importance of cosmological scaling solutions is emphasized when studying the dynamical system of scalar fields including coupled dark energy. We study the evolution of cosmological perturbations allowing us to confront them with the observation of the Cosmic Microwave Background and Large Scale Structure and demonstrate how it is possible in principle to reconstruct the equation of state of dark energy by also using Supernovae Ia observational data. We also discuss in detail the nature of tracking solutions in cosmology, particle physics and braneworld models of dark energy, the nature of possible future singularities, the effect of higher order curvature terms to avoid a Big Rip singularity, and approaches to modifying gravity which leads to a late-time accelerated expansion without recourse to a new form of dark energy.

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## Contents

<b>I. Introduction</b>	3
<b>II. Elements of FRW cosmology</b>	5
A. Evolution equations	6
B. The evolution of the universe filled with a perfect fluid	7
<b>III. Observational evidence for dark energy</b>	7
A. Luminosity distance	7
B. Constraints from Supernovae Ia	9
C. The age of the universe and the cosmological constant	10
D. Constraints from the CMB and LSS	12
<b>IV. Cosmological constant</b>	13
A. Introduction of $\Lambda$	13
B. Fine tuning problem	14
C. $\Lambda$ from string theory	15
1. Four-form fluxes and quantization	15
2. The KKLT scenario	15
3. Relaxation of $\Lambda$ in string theory	17
4. $\Lambda$ from a self-tuning universe	17
5. $\Lambda$ through mixing of degenerate vacua	17
D. Causal sets and $\Lambda$	18
E. Anthropic selection of $\Lambda$	18

F. A Dynamical Approach to the Cosmological Constant	19
G. Observing dark energy in the laboratory ?	19
<b>V. Scalar-field models of dark energy</b>	20
A. Quintessence	20
B. K-essence	22
C. Tachyon field	23
D. Phantom (ghost) field	24
E. Dilatonic dark energy	25
F. Chaplygin gas	26
<b>VI. Cosmological dynamics of scalar fields in the presence of a barotropic perfect fluid</b>	26
A. Autonomous system of scalar-field dark energy models	27
1. Fixed or critical points	27
2. Stability around the fixed points	27
B. Quintessence	28
1. Constant $\lambda$	28
2. Dynamically changing $\lambda$	30
C. Phantom fields	30
D. Tachyon fields	30
1. Constant $\lambda$	31
2. Dynamically changing $\lambda$	31
E. Dilatonic ghost condensate	33
<b>VII. Scaling solutions in a general Cosmological background</b>	34
A. General Lagrangian for the existence of scaling solution	34
B. General properties of scaling solutions	35
C. Effective potential corresponding to scaling solutions	36
1. Ordinary scalar fields	36
2. Tachyon	36
3. Dilatonic ghost condensate	36
D. Autonomous system in Einstein gravity	37
<b>VIII. The details of quintessence</b>	37
A. Nucleosynthesis constraint	37
B. Exit from a scaling regime	38
C. Assisted quintessence	38
D. Particle physics models of Quintessence	39
1. Supergravity inspired models	39
2. Pseudo-Nambu-Goldstone models	42
E. Quintessential inflation	43
<b>IX. Coupled dark energy</b>	44
A. Critical points for coupled Quintessence	45
B. Stability of critical points	45
1. Ordinary field ( $\epsilon = +1$ )	46
2. Phantom field ( $\epsilon = -1$ )	47
C. General properties of fixed points	48
D. Can we have two scaling regimes ?	48
E. Varying mass neutrino scenario	50
F. Dark energy through brane-bulk energy exchange	50
<b>X. Dark energy and varying alpha</b>	51
A. Varying alpha from quintessence	51
B. Varying alpha from tachyon fields	52
<b>XI. Perturbations in a universe with dark energy</b>	54
A. Perturbation equations	54
B. Single-field system without a fluid	55

C. Evolution of matter perturbations	56
D. Perturbations in coupled dark energy	57
1. Analytic solutions in scalar-field matter dominant stage	57
2. Analytic solutions for scaling solutions	58
<b>XII. Reconstruction of dark energy models</b>	58
A. Application to specific cases	60
1. Case of $p = f(X) - V(\phi)$	60
2. Case of $p = f(X)V(\phi)$	60
3. Scaling solutions	60
B. Example of reconstruction	61
C. $w = -1$ crossing	61
<b>XIII. Observational constraints on the equation of state of dark energy</b>	62
A. Parametrization of $w_{\text{DE}}$	63
B. Observational constraints from SN Ia data	63
C. Observational constraints from CMB	65
D. Cross-correlation Tomography	68
E. Constraints from baryon oscillations	68
<b>XIV. The fate of a dark energy universe—future singularities</b>	69
A. Type I and III singularities	70
B. Type II singularity	70
C. Type IV singularity	70
<b>XV. Dark energy with higher-order curvature corrections</b>	71
A. Quantum effects from a conformal anomaly	71
B. String curvature corrections	72
<b>XVI. Cosmic acceleration from modified gravity and other alternatives to dark energy</b>	74
A. $f(R)$ gravities	75
B. DGP model	77
C. Dark energy arising from the Trans-Planckian Regime	78
D. Acceleration due to the backreaction of cosmological perturbations	79
<b>XVII. Conclusions</b>	80
<b>ACKNOWLEDGEMENTS</b>	81
<b>References</b>	82

## I. INTRODUCTION

Over the course of the past decade, evidence for the most striking result in modern cosmology has been steadily growing, namely the existence of a cosmological constant which is driving the current acceleration of the Universe as first observed in Refs. [1, 2]. Although it may not have come as such a surprise to a few theorists who were at that time considering the interplay between a number of different types of observations [3], for the majority it came as something of a bombshell. The Universe is not only expanding, it is accelerating. The results first published in Refs. [1, 2] have caused a sea change in the way we have started thinking about the universe.

Conventionally, the world of particle physics and cosmology has been seen as overlapping in the early universe, particle physics providing much needed sources of

energy density during that period, leading to processes like inflation, baryogenesis, phase transitions etc... Now though we need to understand the impact particle physics has on cosmology today, how else can we explain the nature of this apparent cosmological constant? Theorists never short of ideas, have come up with a number of particle physics related suggestions (as well as a number completely unrelated to particle physics) to help us understand the nature of the acceleration.

There is a key problem that we have to explain, and it is fair to say it has yet to be understood. The value of the energy density stored in the cosmological constant today, which rather paradoxically is called dark energy and has nothing to do with dark matter, this value has to be of order the critical density, namely  $\rho_{\Lambda} \sim 10^{-3} \text{ eV}^4$ . Unfortunately, no sensible explanation exists as to why a true cosmological constant should be at this scale, it

should naturally be much larger. Typically, since it is conventionally associated with the energy of the vacuum in quantum theory we expect it to have a size of order the typical scale of early Universe phase transitions. Even at the QCD scale it would imply a value  $\rho_\Lambda \sim 10^{-3} \text{ GeV}^4$ . The question then remains, why has  $\Lambda$  got the value it has today?

Rather than dealing directly with the cosmological constant a number of alternative routes have been proposed which skirt around this thorny issue [4, 5, 6, 7, 8]. They come in a number of flavors. An incomplete list includes: Quintessence models [9, 10] (see also Refs. [11, 12]) which invoke an evolving canonical scalar field with a potential (effectively providing an inflaton for today) and makes use of the scaling properties [13, 14] and tracker nature [15, 16] of such scalar fields evolving in the presence of other background matter fields; scalar field models where the small mass of the quintessence field is protected by an approximate global symmetry by making the field a pseudo-Nambu-Goldstone boson [17]; Chameleon fields in which the scalar field couples to the baryon energy density and is homogeneous being allowed to vary across space from solar system to cosmological scales [18, 19]; a scalar field with a non-canonical kinetic term, known as K-essence [20, 21, 22] based on earlier work of K-inflation [23]; modified gravity arising out of both string motivated [24] or more generally General Relativity modified [25, 26, 27] actions which both have the effect of introducing large length scale corrections and modifying the late time evolution of the Universe; the feedback of non-linearities into the evolution equations which can significantly change the background evolution and lead to acceleration at late times without introducing any new matter [28]; Chaplygin gases which attempt to unify dark energy and dark matter under one umbrella by allowing for a fluid with an equation of state which evolves between the two [29, 30, 31]; tachyons [32, 33] arising in string theory [34]; the same scalar field responsible for both inflation in the early Universe and again today, known as Quintessential inflation [35]; the possibility of a network of frustrated topological defects forcing the universe into a period of accelerated expansion today [36]; Phantom Dark Energy [37] and Ghost Condensates [38, 39]; de-Sitter vacua with the flux compactifications in string theory [40]; the String Landscape arising from the multiple numbers of vacua that exist when the string moduli are made stable as non-abelian fluxes are turned on [41]; the Cyclic Universe [42]; causal sets in the context of Quantum Gravity [43]; direct anthropic arguments [44, 45, 46, 47], all of these are more or less exotic solutions to the dark energy question.

These possibilities and more, have been discussed in the literature and many of them will be discussed in detail in this review. Given the strength of the data which are all effectively indicating the presence of a cosmological constant type term today, then any dynamically evolving contribution must resemble a cosmological constant today. If we are to see evidence of dynamics in

the dark energy equation of state, we have to probe back in time. A number of routes in that direction have been suggested and plans are underway to extend this even further. For example by looking at the detailed patterns of the anisotropies in the cosmic microwave background (CMB), we are seeing when and under what conditions the photons left the surface of last scattering. As they propagated towards us today, they will have traveled through gravitational potentials determined by the nature of the dark matter and dark energy, and so different forms of dark energy could in principle have led to different contributions to quantities such as the separation of CMB Peaks [48, 49, 50], the integrated Sachs Wolfe effect [51], the nature of galaxy formation [52], the clustering of large scale structure (LSS) as measured through quantities such as  $\sigma_8$  [53, 54], the propagation of light through weak and strong gravitational lenses [55, 56], and simply through the evolution of the Hubble expansion rate itself which is a function of the energy contributions to the Friedmann equation [57].

On the other hand, what if the data is misleading us and we do not require an effective cosmological constant [58]? A minority of cosmologists have argued forcefully that the majority of the data as it presently stands can be interpreted without recourse to a cosmological constant, rather we can explain it through other physical processes, for example by relaxing the hypothesis that the fluctuation spectrum can be described by a single power law [58]. On the other hand perhaps we do not yet fully understand how Type Ia supernova evolve and we may have to eventually think of alternative explanations. Although this might well be the case, there is a growing body of evidence for the presence of a cosmological constant which does not rely on the supernova data to support it (in relation to this and the comment above see Ref. [59]).

In the same vein Plaga recently discussed observations of a cluster of galaxies “Abell 194” and has argued that the distribution of galaxy redshifts is fitted better with an Einstein-Straus vacuole region of space time as opposed to the cosmological concordance model with a  $\Lambda$  [60]. Of course, this is based on limited data, but we should remember the need to always be prepared to test the standard model against observation.

However, the more accepted interpretation of the data is that it is becoming clear that consistency between the anisotropies in the CMB [61, 62] and LSS [63] observations imply we live in a Universe where the energy density is dominated by a cosmological constant type contribution. An impressive aspect of this consistency check is the fact that the physics associated with each epoch is completely different and of course it occurs on different time scales. It appears that consistency is obtained for a spatially flat universe with the fractional energy density in matter contributing today with  $\Omega_m^{(0)} \sim 0.3$  whereas for the cosmological constant we have  $\Omega_\Lambda^{(0)} \sim 0.7$  [64].

In this review we assume that the dark energy is really there in some form, either dominating the energy density

or through some form of modified gravity, in both cases driving our Universe into a second period of accelerated expansion around a redshift of  $z = \mathcal{O}(1)$ . Most of the observational results are based on the years of analysing the first year WMAP data [62], and has not yet reached the stage of analysing the beautiful new data published around the same time as this review was completed [61]. We have attempted to include the new results where possible and where appropriate. Fortunately for us, many of the key results of WMAP1 have stood the test of time and statistics and appear to be holding true in the three year data as well (with some notable exceptions of course).

Our goal is to introduce the reader to some of the theoretical model building that has gone into understanding the nature of dark energy. We will include string inspired models, un-inspired models, phenomenological models, modified gravity models, etc. We will look into the observational implications associated with dynamical dark energy, and investigate the ways we may determine whether or not there may be a  $\Lambda$  term out there governing our Universe today.

Now a word of caution. The reader is about to spend a great deal of time learning (we hope!) about models of dark energy. The fact remains that although many of us believe some sort of dynamics is responsible for the dark energy, such is the sensitivity of current observations, there is no evidence of an evolving dark energy component, everything remains perfectly consistent with the simplest model (not from the particle physics point of view) of a time independent cosmological constant [51]. Indeed if we include the number of required extra parameters needed to allow for dynamical dark energy as a part of the selection criteria and apply Bayesian information criteria to carry out cosmological model selection, then there is no need at present to allow anything other than the cosmological constant [65, 66]. Nevertheless this may change in the future as observations improve even more, and it remains important to pursue alternative models of dark energy to distinguish them from the cosmological constant observationally.

Before we set off, it is worth mentioning here the approach we are adopting with regard the way we are classifying models, because to some, having a list of apparently unrelated possibilities may not seem the best way forward. We are treating all of these possibilities separately, whereas in principle a number of them can be related to each other as variants of theories carrying the same sort of signature – see for example Refs. [67, 68, 69]. Our reason for doing this is that we believe the models themselves have now become accepted in their own right and have had so much work done on them that they are better being treated separately without trying in this review to discuss the conformal transformations which link them - although we take on board the fact that some of them can be related.

This paper is organized as follows. In Sec. II we introduce Einstein's equations in a homogeneous and isotropic background and provide the basic tools to study the dy-

namics of dark energy. In Sec. III we discuss the observational evidence for dark energy coming from supernova constraints. Sec. IV is devoted to the discussion of the cosmological constant, whereas in Sec. V we introduce a number of scalar-field dark energy models which can act as alternatives to the cosmological constant. This is followed in Sec. VI where the cosmological dynamics of scalar-field dark energy models in the presence of a barotropic fluid is presented. In Sec. VII we derive the condition for the existence of scaling solutions for more general scalar-field Lagrangians. In Sec. VIII we turn to discuss a number of aspects of quintessence scenarios, paying particular attention to particle physics models of Quintessence. In Sec. IX we present coupled dark energy scenarios showing how accelerated expansion can be realized for a class of scaling solutions. Sec. X is devoted to a discussion of varying fine structure constant ( $\alpha$ ) models which although somewhat controversial opens up an important avenue, allowing us in principle to distinguish between quintessence and a cosmological constant observationally. In Sec. XI we study the evolution of cosmological perturbations in a dark energy universe and show several situations in which analytic solutions for perturbations can be obtained. This is followed in Sec. XII where we provide reconstruction equations for a general scalar-field Lagrangian including a coupling to dark matter. Sec. XIII is devoted to a number of approaches to reconstructing the equation of state of the dark energy by parameterizing it in terms of the redshift  $z$ . In Sec. XIV we investigate a possibility that there may be future singularities in a dark energy scenario, and classify these into five classes. In Sec. XV we study the effect of higher-order curvature terms to the cosmological evolution around the singularities discussed in Sec. XIV and in Sec. XVI we discuss modified gravity theories in which an accelerated expansion can be realized without recourse to dark energy. We conclude in the final section.

Throughout the review we adopt natural units  $c = \hbar = 1$  and have a metric signature  $(-, +, +, +)$ . We denote the Planck mass as  $m_{\text{pl}} = G^{-1/2} = 1.22 \times 10^{19}$  GeV and the reduced Planck mass as  $M_{\text{pl}} = (8\pi G)^{-1/2} = 2.44 \times 10^{18}$  GeV. Here  $G$  is Newton's gravitational constant. We define  $\kappa^2 = 8\pi G = 8\pi m_{\text{pl}}^{-2} = M_{\text{pl}}^{-2}$  and will use the unit  $\kappa^2 = 1$  in some sections (but will make it clear when we are doing so).

Finally we would like to provide guide lines for approaching this review. Some of the sections/subsections are of specific interest and may be skipped over in the first reading. For many, it may be preferable first time round to skip over the details of the KKLt scenario described in Sec. IV. Similarly a brief look at sections XIV and XV may be sufficient for a first reading of the review.

## II. ELEMENTS OF FRW COSMOLOGY

The dynamics of the universe is described by the Einstein equations which are in general complicated non-

linear equations. However they exhibit simple analytical solutions in the presence of generic symmetries. The Friedmann-Robertson-Walker (FRW) metric is based upon the assumption of homogeneity and isotropy of the universe which is approximately true on large scales. The small deviation from homogeneity at early epochs played a very important role in the dynamical history of our universe. Small initial density perturbations grew via gravitational instability into the structure we see today in the universe. The temperature anisotropies observed in the Cosmic Microwave Background (CMB) are believed to have originated from quantum fluctuations generated during an inflationary stage in the early universe. See Refs. [70, 71, 72, 73, 74, 75, 76] for details on density perturbations predicted by inflationary cosmology. In this section we shall review the main features of the homogeneous and isotropic cosmology necessary for the subsequent sections.

The FRW metric is given by [70, 77, 78, 79]

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (1)$$

where  $a(t)$  is scale factor with cosmic time  $t$ . The coordinates  $r$ ,  $\theta$  and  $\phi$  are known as *comoving* coordinates. A freely moving particle comes to rest in these coordinates.

Equation (1) is a purely kinematic statement. In this problem the dynamics is associated with the scale factor  $a(t)$ . Einstein equations allow us to determine the scale factor provided the matter content of the universe is specified. The constant  $K$  in the metric (1) describes the geometry of the spatial section of space time, with closed, flat and open universes corresponding to  $K = +1, 0, -1$ , respectively.

It may be convenient to write the metric (1) in the following form:

$$ds^2 = -dt^2 + a^2(t) [d\chi^2 + f_K^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)], \quad (2)$$

where

$$f_K(\chi) = \begin{cases} \sin\chi, & K = +1, \\ \chi, & K = 0, \\ \sinh\chi, & K = -1. \end{cases} \quad (3)$$

### A. Evolution equations

The differential equations for the scale factor and the matter density follow from Einstein's equations [77]

$$G_\nu^\mu \equiv R_\nu^\mu - \frac{1}{2}\delta_\nu^\mu R = 8\pi G T_\nu^\mu, \quad (4)$$

where  $G_\nu^\mu$  is the Einstein tensor, and  $R_\nu^\mu$  is the Ricci tensor which depends on the metric and its derivatives,  $R$  is the Ricci scalar and  $T_\nu^\mu$  is the energy momentum

tensor. In the FRW background (1) the curvature terms are given by [78]

$$R_0^0 = \frac{3\ddot{a}}{a}, \quad (5)$$

$$R_j^i = \left( \frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2K}{a^2} \right) \delta_j^i, \quad (6)$$

$$R = 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2} \right), \quad (7)$$

where a dot denotes a derivative with respect to  $t$ .

Let us consider an ideal perfect fluid as the source of the energy momentum tensor  $T_\nu^\mu$ . In this case we have

$$T_\nu^\mu = \text{Diag}(-\rho, p, p, p), \quad (8)$$

where  $\rho$  and  $p$  are the energy density and the pressure density of the fluid, respectively. Then Eq. (4) gives the two independent equations

$$H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G\rho}{3} - \frac{K}{a^2}, \quad (9)$$

$$\dot{H} = -4\pi G(p + \rho) + \frac{K}{a^2}, \quad (10)$$

where  $H$  is the Hubble parameter,  $\rho$  and  $p$  denote the total energy density and pressure of all the species present in the universe at a given epoch.

The energy momentum tensor is conserved by virtue of the Bianchi identities, leading to the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (11)$$

Equation (11) can be derived from Eqs. (9) and (10), which means that two of Eqs. (9), (10) and (11) are independent. Eliminating the  $K/a^2$  term from Eqs. (9) and (10), we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \quad (12)$$

Hence the accelerated expansion occurs for  $\rho + 3p < 0$ .

One can rewrite Eq. (9) in the form:

$$\Omega(t) - 1 = \frac{K}{(aH)^2}, \quad (13)$$

where  $\Omega(t) \equiv \rho(t)/\rho_c(t)$  is the dimensionless density parameter and  $\rho_c(t) = 3H^2(t)/8\pi G$  is the critical density. The matter distribution clearly determines the spatial geometry of our universe, i.e.,

$$\Omega > 1 \quad \text{or} \quad \rho > \rho_c \quad \rightarrow \quad K = +1, \quad (14)$$

$$\Omega = 1 \quad \text{or} \quad \rho = \rho_c \quad \rightarrow \quad K = 0, \quad (15)$$

$$\Omega < 1 \quad \text{or} \quad \rho < \rho_c \quad \rightarrow \quad K = -1. \quad (16)$$

Observations have shown that the current universe is very close to a spatially flat geometry ( $\Omega \simeq 1$ ) [61]. This is actually a natural result from inflation in the early universe [70]. Hence we will therefore consider a flat universe ( $K = 0$ ) in the rest of this section.

## B. The evolution of the universe filled with a perfect fluid

Let us consider the evolution of the universe filled with a barotropic perfect fluid with an equation of state

$$w = p/\rho, \quad (17)$$

where  $w$  is assumed to be constant. Then by solving the Einstein equations given in Eqs. (9) and (10) with  $K = 0$ , we obtain

$$H = \frac{2}{3(1+w)(t-t_0)}, \quad (18)$$

$$a(t) \propto (t-t_0)^{\frac{2}{3(1+w)}}, \quad (19)$$

$$\rho \propto a^{-3(1+w)}, \quad (20)$$

where  $t_0$  is constant. We note that the above solution is valid for  $w \neq -1$ . The radiation dominated universe corresponds to  $w = 1/3$ , whereas the dust dominated universe to  $w = 0$ . In these cases we have

$$\text{Radiation: } a(t) \propto (t-t_0)^{1/2}, \quad \rho \propto a^{-4}, \quad (21)$$

$$\text{Dust: } a(t) \propto (t-t_0)^{2/3}, \quad \rho \propto a^{-3}. \quad (22)$$

Both cases correspond to a decelerated expansion of the universe.

From Eq. (12) an accelerated expansion ( $\ddot{a}(t) > 0$ ) occurs for the equation of state given by

$$w < -1/3. \quad (23)$$

In order to explain the current acceleration of the universe, we require an exotic energy dubbed ‘‘dark energy’’ with equation of state satisfying Eq. (23). We note that Newton gravity can not account for the accelerated expansion. Let us consider a homogeneous sphere whose radius and energy density are  $a$  and  $\rho$ , respectively. The Newton’s equation of motion for a point particle with mass  $m$  on this sphere is give by

$$\begin{aligned} m\ddot{a} &= -\frac{Gm}{a^2} \left( \frac{4\pi a^3 \rho}{3} \right), \\ \rightarrow \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3} \rho. \end{aligned} \quad (24)$$

The difference compared to the Einstein equation (12) is the absence of the pressure term,  $p$ . This appears in Einstein equations by virtue of relativistic effects. The condition (23) means that we essentially require a large negative pressure in order to give rise to an accelerated expansion. We stress here that Newton gravity only leads to a decelerated expansion of the universe.

From Eq. (11) the energy density  $\rho$  is constant for  $w = -1$ . In this case the Hubble rate is also constant from Eq. (9), giving the evolution of the scale factor:

$$a \propto e^{Ht}, \quad (25)$$

which is the de-Sitter universe. As we will see in the Sec. IV, this exponential expansion also arises by including a cosmological constant,  $\Lambda$ , in the Einstein equations.

So far we have restricted our attention to the equation of state:  $w \geq -1$ . Recent observations suggest that the equation of state which is less than  $-1$  can be also allowed [80]. This specific equation of state corresponds to a *phantom* (ghost) dark energy [37] component and requires a separate consideration (see also Ref. [81]). We first note that Eq. (19) describes a contracting universe for  $w < -1$ . There is another expanding solution given by

$$a(t) = (t_s - t)^{\frac{2}{3(1+w)}}, \quad (26)$$

where  $t_s$  is constant. This corresponds to a super-inflationary solution where the Hubble rate and the scalar curvature grow:

$$H = \frac{n}{t_s - t}, \quad n = -\frac{2}{3(1+w)} > 0, \quad (27)$$

$$R = 6 \left( 2H^2 + \dot{H} \right) = \frac{6n(2n+1)}{(t_s - t)^2}. \quad (28)$$

The Hubble rate diverges as  $t \rightarrow t_s$ , which corresponds to an infinitely large energy density at a finite time in the future. The curvature also grows to infinity as  $t \rightarrow t_s$ . Such a situation is referred to as a Big Rip singularity [82]. This cataclysmic conclusion is not inevitable in these models, and can be avoided in specific models of phantom fields with a top-hat potential [83, 84]. It should also be emphasized that we expect quantum effects to become important in a situation when the curvature of the universe becomes large. In that case we should take into account higher-order curvature corrections to the Einstein Hilbert action which crucially modifies the structure of the singularity, as we will see in Sec. XIV.

## III. OBSERVATIONAL EVIDENCE FOR DARK ENERGY

In this section we briefly review the observational evidence for dark energy, concentrating on the types of observation that have been introduced. Later, in Sec. XIII we will return to discuss in more detail the observational constraints on the dark energy equation of state.

### A. Luminosity distance

In 1998 the accelerated expansion of the universe was pointed out by two groups from the observations of Type Ia Supernova (SN Ia) [1, 2]. We often use a redshift to describe the evolution of the universe. This is related to the fact that light emitted by a stellar object becomes red-shifted due to the expansion of the universe. The

wavelength  $\lambda$  increases proportionally to the scale factor  $a$ , whose effect can be quantified by the redshift  $z$ , as

$$1 + z = \frac{\lambda_0}{\lambda} = \frac{a_0}{a}, \quad (29)$$

where the subscript zero denotes the quantities given at the present epoch.

Another important concept related to observational tools in an expanding background is associated to the definition of a distance. In fact there are several ways of measuring distances in the expanding universe. For instance one often deals with the comoving distance which remains unchanged during the evolution and the physical distance which scales proportionally to the scale factor. An alternative way of defining a distance is through the luminosity of a stellar object. The distance  $d_L$  known as the luminosity distance, plays a very important role in astronomy including the Supernova observations.

In Minkowski space time the absolute luminosity  $L_s$  of the source and the energy flux  $\mathcal{F}$  at a distance  $d$  is related through  $\mathcal{F} = L_s/(4\pi d^2)$ . By generalizing this to an expanding universe, the luminosity distance,  $d_L$ , is defined as

$$d_L^2 \equiv \frac{L_s}{4\pi\mathcal{F}}. \quad (30)$$

Let us consider an object with absolute luminosity  $L_s$  located at a coordinate distance  $\chi_s$  from an observer at  $\chi = 0$  [see the metric (2)]. The energy of light emitted from the object with time interval  $\Delta t_1$  is denoted as  $\Delta E_1$ , whereas the energy which reaches at the sphere with radius  $\chi_s$  is written as  $\Delta E_0$ . We note that  $\Delta E_1$  and  $\Delta E_0$  are proportional to the frequencies of light at  $\chi = \chi_s$  and  $\chi = 0$ , respectively, i.e.,  $\Delta E_1 \propto \nu_1$  and  $\Delta E_0 \propto \nu_0$ . The luminosities  $L_s$  and  $L_0$  are given by

$$L_s = \frac{\Delta E_1}{\Delta t_1}, \quad L_0 = \frac{\Delta E_0}{\Delta t_0}. \quad (31)$$

The speed of light is given by  $c = \nu_1 \lambda_1 = \nu_0 \lambda_0$ , where  $\lambda_1$  and  $\lambda_0$  are the wavelengths at  $\chi = \chi_s$  and  $\chi = 0$ . Then from Eq. (29) we find

$$\frac{\lambda_0}{\lambda_1} = \frac{\nu_1}{\nu_0} = \frac{\Delta t_0}{\Delta t_1} = \frac{\Delta E_1}{\Delta E_0} = 1 + z, \quad (32)$$

where we have also used  $\nu_0 \Delta t_0 = \nu_1 \Delta t_1$ . Combining Eq. (31) with Eq. (32), we obtain

$$L_s = L_0(1 + z)^2. \quad (33)$$

The light traveling along the  $\chi$  direction satisfies the geodesic equation  $ds^2 = -dt^2 + a^2(t)d\chi^2 = 0$ . We then obtain

$$\chi_s = \int_0^{\chi_s} d\chi = \int_{t_1}^{t_0} \frac{dt}{a(t)} = \frac{1}{a_0 H_0} \int_0^z \frac{dz'}{h(z')}, \quad (34)$$

where  $h(z) = H(z)/H_0$ . Note that we have used the relation  $\dot{z} = -H(1+z)$  coming from Eq. (29). From the

metric (2) we find that the area of the sphere at  $t = t_0$  is given by  $S = 4\pi(a_0 f_K(\chi_s))^2$ . Hence the observed energy flux is

$$\mathcal{F} = \frac{L_0}{4\pi(a_0 f_K(\chi_s))^2}. \quad (35)$$

Substituting Eqs. (34) and (35) for Eq. (30), we obtain the luminosity distance in an expanding universe:

$$d_L = a_0 f_K(\chi_s)(1 + z). \quad (36)$$

In the flat FRW background with  $f_K(\chi) = \chi$  we find

$$d_L = \frac{1 + z}{H_0} \int_0^z \frac{dz'}{h(z')}, \quad (37)$$

where we have used Eq. (34). Then the Hubble rate  $H(z)$  can be expressed in terms of  $d_L(z)$ :

$$H(z) = \left\{ \frac{d}{dz} \left( \frac{d_L(z)}{1 + z} \right) \right\}^{-1}. \quad (38)$$

If we measure the luminosity distance observationally, we can determine the expansion rate of the universe.

The energy density  $\rho$  on the right hand side of Eq. (9) includes all components present in the universe, namely, non-relativistic particles, relativistic particles, cosmological constant, etc:

$$\rho = \sum_i \rho_i^{(0)} (a/a_0)^{-3(1+w_i)} = \sum_i \rho_i^{(0)} (1 + z)^{3(1+w_i)}, \quad (39)$$

where we have used Eq. (29). Here  $w_i$  and  $\rho_i^{(0)}$  correspond to the equation of state and the present energy density of each component, respectively.

Then from Eq. (9) the Hubble parameter takes the convenient form

$$H^2 = H_0^2 \sum_i \Omega_i^{(0)} (1 + z)^{3(1+w_i)}, \quad (40)$$

where  $\Omega_i^{(0)} \equiv 8\pi G \rho_i^{(0)} / (3H_0^2) = \rho_i^{(0)} / \rho_c^{(0)}$  is the density parameter for an individual component at the present epoch. Hence the luminosity distance in a flat geometry is given by

$$d_L = \frac{(1 + z)}{H_0} \int_0^z \frac{dz'}{\sqrt{\sum_i \Omega_i^{(0)} (1 + z')^{3(1+w_i)}}}. \quad (41)$$

In Fig. 1 we plot the luminosity distance (41) for a two component flat universe (non-relativistic fluid with  $w_m = 0$  and cosmological constant with  $w_\Lambda = -1$ ) satisfying  $\Omega_m^{(0)} + \Omega_\Lambda^{(0)} = 1$ . Notice that  $d_L \simeq z/H_0$  for small values of  $z$ . The luminosity distance becomes larger when the cosmological constant is present.



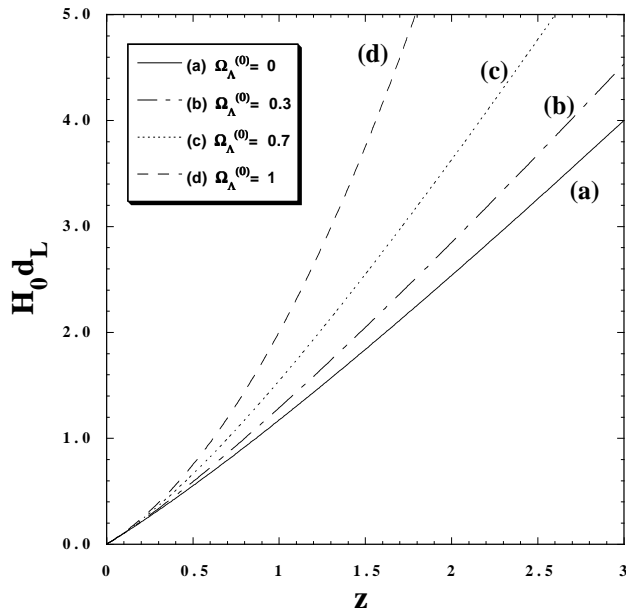


FIG. 1: Luminosity distance  $d_L$  in the units of  $H_0^{-1}$  for a two component flat universe with a non-relativistic fluid ( $w_m = 0$ ) and a cosmological constant ( $w_\Lambda = -1$ ). We plot  $H_0 d_L$  for various values of  $\Omega_\Lambda^{(0)}$ .

## B. Constraints from Supernovae Ia

The direct evidence for the current acceleration of the universe is related to the observation of luminosity distances of high redshift supernovae [1, 2]. The apparent magnitude  $m$  of the source with an absolute magnitude  $M$  is related to the luminosity distance  $d_L$  via the relation [4, 6]

$$m - M = 5 \log_{10} \left( \frac{d_L}{\text{Mpc}} \right) + 25. \quad (42)$$

This comes from taking the logarithm of Eq. (30) by noting that  $m$  and  $M$  are related to the logarithms of  $\mathcal{F}$  and  $L_s$ , respectively. The numerical factors arise because of conventional definitions of  $m$  and  $M$  in astronomy.

The Type Ia supernova (SN Ia) can be observed when white dwarf stars exceed the mass of the Chandrasekhar limit and explode. The belief is that SN Ia are formed in the same way irrespective of where they are in the universe, which means that they have a common absolute magnitude  $M$  independent of the redshift  $z$ . Thus they can be treated as an ideal standard candle. We can measure the apparent magnitude  $m$  and the redshift  $z$  observationally, which of course depends upon the objects we observe.

In order to get a feeling of the phenomenon let us consider two supernovae 1992P at low-redshift  $z = 0.026$  with  $m = 16.08$  and 1997ap at high-redshift redshift  $z = 0.83$  with  $m = 24.32$  [1]. As we have already mentioned, the luminosity distance is approximately given

by  $d_L(z) \simeq z/H_0$  for  $z \ll 1$ . Using the apparent magnitude  $m = 16.08$  of 1992P at  $z = 0.026$ , we find that the absolute magnitude is estimated by  $M = -19.09$  from Eq. (42). Here we adopted the value  $H_0^{-1} = 2998h^{-1} \text{Mpc}$  with  $h = 0.72$ . Then the luminosity distance of 1997ap is obtained by substituting  $m = 24.32$  and  $M = -19.09$  for Eq. (42):

$$H_0 d_L \simeq 1.16, \quad \text{for } z = 0.83. \quad (43)$$

From Eq. (41) the theoretical estimate for the luminosity distance in a two component flat universe is

$$H_0 d_L \simeq 0.95, \quad \Omega_m^{(0)} \simeq 1, \quad (44)$$

$$H_0 d_L \simeq 1.23, \quad \Omega_m^{(0)} \simeq 0.3, \quad \Omega_\Lambda^{(0)} \simeq 0.7. \quad (45)$$

This estimation is clearly consistent with that required for a dark energy dominated universe as can be seen also in Fig. 1.

Of course, from a statistical point of view, one can not strongly claim that that our universe is really accelerating by just picking up a single data set. Up to 1998 Perlmutter *et al.* [supernova cosmology project (SCP)] had discovered 42 SN Ia in the redshift range  $z = 0.18-0.83$  [1], whereas Riess *et al.* [high- $z$  supernova team (HSST)] had found 14 SN Ia in the range  $z = 0.16-0.62$  and 34 nearby SN Ia [2]. Assuming a flat universe ( $\Omega_m^{(0)} + \Omega_\Lambda^{(0)} = 1$ ), Perlmutter *et al.* found  $\Omega_m^{(0)} = 0.28^{+0.09}_{-0.08}$  ( $1\sigma$  statistical)  $^{+0.05}_{-0.04}$  (identified systematics), thus showing that about 70 % of the energy density of the present universe consists of dark energy.

In 2004 Riess *et al.* [85] reported the measurement of 16 high-redshift SN Ia with redshift  $z > 1.25$  with the Hubble Space Telescope (HST). By including 170 previously known SN Ia data points, they showed that the universe exhibited a transition from deceleration to acceleration at  $> 99$  % confidence level. A best-fit value of  $\Omega_m^{(0)}$  was found to be  $\Omega_m^{(0)} = 0.29^{+0.05}_{-0.03}$  (the error bar is  $1\sigma$ ). In Ref. [86] a likelihood analysis was performed by including the SN data set by Tonry *et al.* [87] together with the one by Riess *et al.* [85]. Figure 2 illustrates the observational values of the luminosity distance  $d_L$  versus redshift  $z$  together with the theoretical curves derived from Eq. (41). This shows that a matter dominated universe without a cosmological constant ( $\Omega_m^{(0)} = 1$ ) does not fit to the data. A best-fit value of  $\Omega_m^{(0)}$  obtained in a joint analysis of Ref. [86] is  $\Omega_m^{(0)} = 0.31^{+0.08}_{-0.08}$ , which is consistent with the result by Riess *et al.* [85]. See also Refs. [88] for recent papers about the SN Ia data analysis.

In Ref. [89], a comparison is made of the constraints on models of dark energy from supernova and CMB observations. The authors argue that models preferred by these observations lie in distinct parts of the parameter space but there is no overlap of regions allowed at the 68% confidence level. They go on to suggest that this may indicate unresolved systematic errors in one of the observations, with supernova observations being more likely to suffer from this problem due to the very heterogeneous

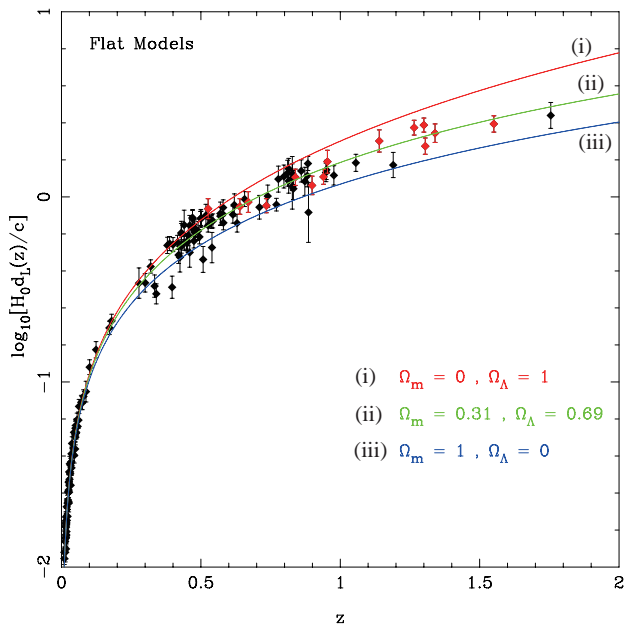


FIG. 2: The luminosity distance  $H_0 d_L$  (log plot) versus the redshift  $z$  for a flat cosmological model. The black points come from the “Gold” data sets by Riess *et al.* [85], whereas the red points show the recent data from HST. Three curves show the theoretical values of  $H_0 d_L$  for (i)  $\Omega_m^{(0)} = 0$ ,  $\Omega_\Lambda^{(0)} = 1$ , (ii)  $\Omega_m^{(0)} = 0.31$ ,  $\Omega_\Lambda^{(0)} = 0.69$  and (iii)  $\Omega_m^{(0)} = 1$ ,  $\Omega_\Lambda^{(0)} = 0$ . From Ref. [86].

nature of the data sets available at the time. Recently observations of high redshift supernovae from the Supernova Legacy Survey have been released [91]. The survey has aimed to reduce systematic errors by using only high quality observations based on using a single instrument to observe the fields. The claim is that through a rolling search technique the sources are not lost and data is of superior quality. Jassal *et al.* claim that the data set is in better agreement with WMAP [92]. In other words the high redshift supernova data from the SNLS (SuperNova Legacy Survey) project is in excellent agreement with CMB observations. It leaves open the current state of supernova observations and their analysis, as compared to that of the CMB. The former is still in a state of flux and any conclusions reached using them need to be understood giving due regard to underlying assumptions.

It should be emphasized that the accelerated expansion is by cosmological standards really a late-time phenomenon, starting at a redshift  $z \sim 1$ . From Eq. (40) the deceleration parameter,  $q \equiv -a\ddot{a}/\dot{a}^2$ , is given by

$$q(z) = \frac{3 \sum_i \Omega_i^{(0)} (1 + w_i) (1 + z)^{3(1+w_i)}}{2 \sum_i \Omega_i^{(0)} (1 + z)^{3(1+w_i)}} - 1. \quad (46)$$

For the two component flat cosmology, the universe enters an accelerating phase ( $q < 0$ ) for

$$z < z_c \equiv \left( \frac{2\Omega_\Lambda^{(0)}}{\Omega_m^{(0)}} \right)^{1/3} - 1. \quad (47)$$

When  $\Omega_m^{(0)} = 0.3$  and  $\Omega_\Lambda^{(0)} = 0.7$ , we have  $z_c = 0.67$ . The problem of why an accelerated expansion should occur now in the long history of the universe is called the “coincidence problem”.

We have concentrated in this section on the use of SN Ia as standard candles. There are other possible candles that have been proposed and are actively being investigated. One such approach has been to use FRIB radio galaxies [93, 94]. From the corresponding redshift-angular size data it is possible to constrain cosmological parameters in a dark energy scalar field model. The derived constraints are found to be consistent with but generally weaker than those determined using Type Ia supernova redshift-magnitude data.

However, in Ref. [95], the authors have gone further and developed a model-independent approach (i.e. independent of assumptions about the form of the dark energy) using a set of 20 radio galaxies out to a redshift  $z \sim 1.8$ , which is further than the SN Ia data can reach. They conclude that the current observations indicate the universe transits from acceleration to deceleration at a redshift greater than 0.3, with a best fit estimate of about 0.45, and have best fit values for the matter and dark energy contributions to  $\Omega$  in broad agreement with the SN Ia estimates.

Another suggested standard candle is that of Gamma Ray Bursts (GRB), which may enable the expansion rate of our Universe to be measured out to very high redshifts ( $z > 5$ ). Hooper and Dodelson [96] have explored this possibility and found that GRB have the potential to detect dark energy at high statistical significance, but in the short term are unlikely to be competitive with future supernovae missions, such as SNAP, in measuring the properties of the dark energy. If however, it turns out there is appreciable dark energy at early times, GRB’s will provide an excellent probe of that regime, and will be a real complement for the SN Ia data. This is a rapidly evolving field and there has recently been announced tentative evidence for a dynamical equation of state for dark energy, based on GRB data out to redshifts of order 5 [97]. It is far too early to say whether this is the correct interpretation, or whether GRB are good standard candles, but the very fact they can be seen out to such large redshifts, means that if they do turn out to be standard candles, they will be very significant complements to the SN Ia data sets, and potentially more significant.

### C. The age of the universe and the cosmological constant

Another interesting piece of evidence for the existence of a cosmological constant emerges when we compare the age of the universe ( $t_0$ ) to the age of the oldest stellar populations ( $t_s$ ). For consistency we of course require  $t_0 > t_s$ , but it is difficult to satisfy this condition for a flat cosmological model with a normal form of matter as we will see below. Remarkably, the presence of cosmological

constant can resolve this age problem.

First we briefly mention the age of the oldest stellar objects have been constrained by a number of groups. For example, Jimenez *et al.* [98] determined the age of Globular clusters in the Milky Way as  $t_1 = 13.5 \pm 2$  Gyr by using a distance-independent method. Using the white dwarfs cooling sequence method, Richer *et al.* [99] and Hansen *et al.* [100] constrained the age of the globular cluster M4 to be  $t_1 = 12.7 \pm 0.7$  Gyr. Then the age of the universe needs to satisfy the lower bound:  $t_0 > 11$ -12 Gyr. Assuming a  $\Lambda$ CDM model, the most recent WMAP3 data produces a best fit value of  $t_0 = 13.73^{+0.13}_{-0.17}$  Gyrs for the age of the universe [61].

Let us calculate the age of the universe from the Friedmann equation (9) with  $\rho$  given by (39). We shall consider three contributions: radiation ( $w_r = 1/3$ ), pressureless dust ( $w_m = 0$ ) and cosmological constant ( $w_\Lambda = -1$ ). Then Eq. (9) is written as

$$H^2 = H_0^2 [\Omega_r^{(0)} (a/a_0)^{-4} + \Omega_m^{(0)} (a/a_0)^{-3} + \Omega_\Lambda^{(0)} - \Omega_K^{(0)} (a/a_0)^{-2}], \quad (48)$$

where  $\Omega_K^{(0)} \equiv K/(a_0^2 H_0^2)$ . Then by using Eq. (29) one can express  $\dot{H}$  in terms of  $z$ . The age of the universe is given by

$$\begin{aligned} t_0 &= \int_0^{t_0} dt = \int_0^\infty \frac{dz}{H(1+z)} \\ &= \int_0^\infty \frac{dz}{H_0 x [\Omega_r^{(0)} x^4 + \Omega_m^{(0)} x^3 + \Omega_\Lambda^{(0)} - \Omega_K^{(0)} x^2]^{1/2}}, \end{aligned} \quad (49)$$

where  $x(z) \equiv 1+z$ . It is a good approximation to neglect the contribution of the radiation term in Eq. (49) since the radiation dominated period is much shorter than the total age of the universe. In other words the integral coming from the region  $z \gtrsim 1000$  hardly affects the total integral (49). Hence we set  $\Omega_r^{(0)} = 0$  when we evaluate  $t_0$ .

We shall first study the case in which the cosmological constant is absent ( $\Omega_\Lambda^{(0)} = 0$ ). Since  $\Omega_K^{(0)} = \Omega_m^{(0)} - 1$  from Eq. (48), the age of the universe is given by

$$t_0 = \frac{1}{H_0} \int_0^\infty \frac{dz}{(1+z)^2 \sqrt{1 + \Omega_m^{(0)} z}}. \quad (50)$$

For a flat universe ( $\Omega_K^{(0)} = 0$  and  $\Omega_m^{(0)} = 1$ ), we obtain

$$t_0 = \frac{2}{3H_0}. \quad (51)$$

From the observations of the Hubble Space Telescope Key project [101] the present Hubble parameter is constrained to be

$$H_0^{-1} = 9.776h^{-1} \text{ Gyr}, \quad 0.64 < h < 0.80. \quad (52)$$

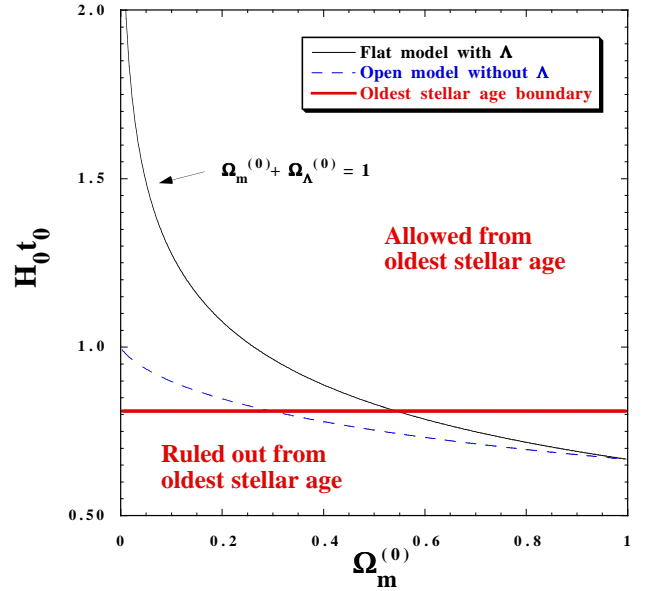


FIG. 3: The age of the universe (in units of  $H_0^{-1}$ ) is plotted against  $\Omega_m^{(0)}$  for (i) a flat model with  $\Omega_m^{(0)} + \Omega_\Lambda^{(0)} = 1$  (solid curve) and (ii) a open model (dashed curve). We also show the border  $t_0 = 11$  Gyr coming from the bound of the oldest stellar ages. The region above this border is allowed for consistency. This constraint strongly supports the evidence of dark energy.

This is consistent with the conclusions arising from observations of the CMB [61] and large scale structure [63, 64]. Then Eq. (51) gives  $t_0 = 8$ -10 Gyr, which does not satisfy the stellar age bound:  $t_0 > 11$ -12 Gyr. Hence a flat universe without a cosmological constant suffers from a serious age problem.

In an open universe model ( $\Omega_m^{(0)} < 1$ ), Eq. (50) shows that the age of the universe is larger than the flat model explained above. This is understandable, as the amount of matter decreases, it would take longer for gravitational interactions to slow down the expansion rate to its present value. In this case Eq. (50) is integrated to give

$$H_0 t_0 = \frac{1}{1 - \Omega_m^{(0)}} - \frac{\Omega_m^{(0)}}{2(1 - \Omega_m^{(0)})^{3/2}} \ln \left( \frac{1 - \sqrt{1 - \Omega_m^{(0)}}}{1 + \sqrt{1 - \Omega_m^{(0)}}} \right), \quad (53)$$

from which we have  $H_0 t_0 \rightarrow 1$  for  $\Omega_m^{(0)} \rightarrow 0$  and  $H_0 t_0 \rightarrow 2/3$  for  $\Omega_m^{(0)} \rightarrow 1$ . As illustrated in Fig. 3,  $t_0$  monotonically increases toward  $t_0 = H_0^{-1}$  with the decrease of  $\Omega_m^{(0)}$ . The observations of the CMB [61] constrain the curvature of the universe to be very close to flat, i.e.,  $|\Omega_K^{(0)}| = |\Omega_m^{(0)} - 1| \ll 1$ . However, since  $\Omega_m^{(0)} \simeq 1$  in this case, the age of the universe does not become larger than the oldest stellar age (see Fig. 3).

The age problem can easily be solved in a flat universe ( $K_0 = 0$ ) with a cosmological constant ( $\Omega_\Lambda^{(0)} \neq 0$ ). In

this case Eq. (49) gives

$$\begin{aligned}
 H_0 t_0 &= \int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_m^{(0)}(1+z)^3 + \Omega_\Lambda^{(0)}}} \\
 &= \frac{2}{3\sqrt{\Omega_\Lambda^{(0)}}} \ln \left( \frac{1 + \sqrt{\Omega_\Lambda^{(0)}}}{\sqrt{\Omega_m^{(0)}}} \right), \quad (54)
 \end{aligned}$$

where  $\Omega_m^{(0)} + \Omega_\Lambda^{(0)} = 1$ . The asymptotic values are  $H_0 t_0 \rightarrow \infty$  for  $\Omega_m^{(0)} \rightarrow 0$  and  $H_0 t_0 \rightarrow 2/3$  for  $\Omega_m^{(0)} \rightarrow 1$ . In Fig. 3 we plot the age  $t_0$  versus  $\Omega_m^{(0)}$ . The age of the universe increases as  $\Omega_m^{(0)}$  decreases. When  $\Omega_m^{(0)} = 0.3$  and  $\Omega_\Lambda^{(0)} = 0.7$  one has  $t_0 = 0.964 H_0^{-1}$ , which corresponds to  $t_0 = 13.1$  Gyr for  $h = 0.72$ . Hence this easily satisfies the constraint  $t_0 > 11$ -12 Gyr coming from the oldest stellar populations. Thus the presence of  $\Lambda$  elegantly solves the age-crisis problem. In [103], the authors manage to go further and find the solution for the scale factor in a flat Universe driven by dust plus a component characterized by a constant parameter of state which dominates in the asymptotic future.

#### D. Constraints from the CMB and LSS

The observations related to the CMB [61] and large-scale structure (LSS) [63, 64] independently support the ideas of a dark energy dominated universe. The CMB anisotropies observed by COBE in 1992 and by WMAP in 2003 exhibited a nearly scale-invariant spectra of primordial perturbations, which agree very well with the prediction of inflationary cosmology. However, note that the best fit power-law flat  $\Lambda$ CDM model obtained from using only the WMAP data now gives a scalar spectral tilt of  $n_s = 0.951_{-0.019}^{+0.015}$ , significantly less than scale invariant! [61]. The position of the first acoustic peak around  $l = 200$  constrains the curvature of the universe to be  $|1 - \Omega_{\text{total}}| = 0.030_{-0.025}^{+0.026} \ll 1$  [102] as predicted by the inflationary paradigm. It is worth pointing out that Weinberg in Ref. [104] provides an analytic expression for the position of the first peak showing how it depends on the background distribution of energy densities between matter and a cosmological constant.

Using the most recent WMAP data [61] with an assumption of constant equation of state  $w_{\text{DE}} = -1$  for dark energy, then combining WMAP and the Supernova legacy Survey implies  $\Omega_K^{(0)} = -0.015_{-0.016}^{+0.02}$ , consistent with a flat universe. Combining with the HST key project constraint on  $H_0$  provides a tighter constraint,  $\Omega_K^{(0)} = -0.010_{-0.009}^{+0.016}$  and  $\Omega_\Lambda^{(0)} = 0.72 \pm 0.04$  (to be compared with earlier pre WMAP3 results  $\Omega_\Lambda^{(0)} = 0.69_{-0.06}^{+0.03}$ , which assumed a flat universe with a prior for the Hubble constant  $h = 0.71 \pm 0.076$  [105]).

In Fig. 4 we plot the confidence regions coming from SN Ia, CMB(WMAP1) and large-scale galaxy clustering [106] (see Ref. [107] for an earlier work introducing

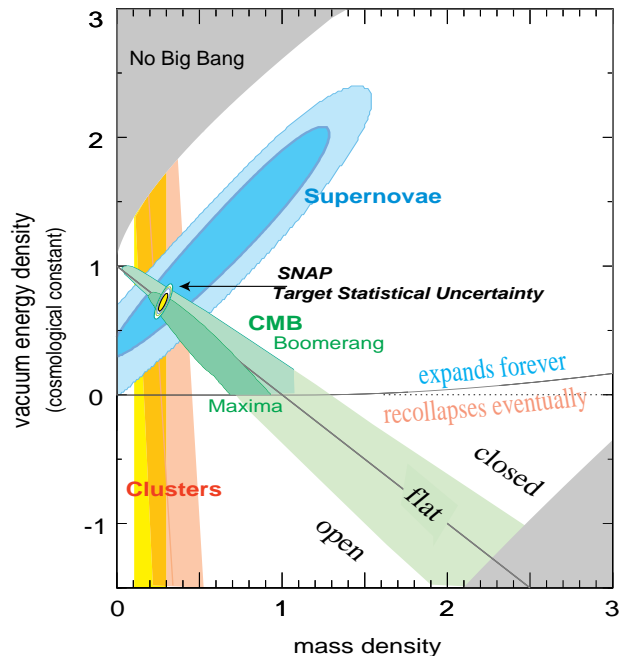


FIG. 4: The  $\Omega_m^{(0)}$ - $\Omega_\Lambda^{(0)}$  confidence regions constrained from the observations of SN Ia, CMB and galaxy clustering. We also show the expected confidence region from a SNAP satellite for a flat universe with  $\Omega_m^{(0)} = 0.28$ . From Ref. [106].

the “cosmic triangle”). Clearly the flat universe without a cosmological constant is ruled out. The compilation of three different cosmological data sets strongly reinforces the need for a dark energy dominated universe with  $\Omega_\Lambda^{(0)} \simeq 0.7$  and  $\Omega_m^{(0)} \simeq 0.3$ . Amongst the matter content of the universe, baryonic matter amounts to only 4%. The rest of the matter (27%) is believed to be in the form of a non-luminous component of non-baryonic nature with a dust like equation of state ( $w = 0$ ) known as Cold Dark Matter (CDM). Dark energy is distinguished from dark matter in the sense that its equation of state is different ( $w < -1/3$ ), allowing it to give rise to an accelerated expansion.

The discussion in this section has been based on the assumption that the equation of state of dark energy is constant ( $w_\Lambda = -1$ ). This scenario, the so called  $\Lambda$ CDM model, has become the standard model for modern cosmology. However, it may be that this is not the true origin of dark energy. If scalar fields turn out to be responsible for it, then the equation of state of dark energy can be dynamical. In order to understand the origin of dark energy it is important to distinguish between the cosmological constant and dynamical dark energy models. The observations of SN Ia alone are still not sufficient to establish evidence of a dynamically changing equation of state, but this situation could well improve through future observations. In a dark energy dominated universe the gravitational potential varies unlike the case of matter dominated universe, which leads to an imprint on

the CMB power spectrum [108]. This phenomenon, the so called Integrated Sachs-Wolfe (ISW) effect [109], could also be important in helping to distinguish the cosmological constant and dynamical dark energy models, since the evolution of the gravitational potential strongly depends upon the dynamical property of the equation of state of dark energy.

At present the observations of WMAP are perfectly consistent with a non varying dark energy contributed by a cosmological constant. Tensions which appeared to exist between the WMAP and the Gold SN data set [89] appear to have disappeared in the more recent SNLS data [91, 92], although it is still early days in the search for the true nature of the dark energy. However given the consistency of a true cosmological constant, we shall first discuss the problem and highlight recent progress that has been made in determining the existence of a pure cosmological constant, before proceeding to discuss dynamical dark energy models in subsequent sections.

#### IV. COSMOLOGICAL CONSTANT

As mentioned earlier, the cosmological constant  $\Lambda$ , was originally introduced by Einstein in 1917 to achieve a static universe. After Hubble's discovery of the expansion of the universe in 1929, it was dropped by Einstein as it was no longer required. From the point of view of particle physics, however, the cosmological constant naturally arises as an energy density of the vacuum. Moreover, the energy scale of  $\Lambda$  should be much larger than that of the present Hubble constant  $H_0$ , if it originates from the vacuum energy density. This is the "cosmological constant problem" [45] and was well known to exist long before the discovery of the accelerated expansion of the universe in 1998.

There have been a number of attempts to solve this problem. An incomplete list includes: adjustment mechanisms [110, 111], anthropic considerations [44, 47, 112, 113, 114, 115, 116], changing gravity [117], quantum gravity [118], degenerate vacua [119], higher-dimensional gravity [120, 121], supergravity [122, 123], string theory [40, 124, 125, 127, 128], space-time foam approach [129] and vacuum fluctuations of the energy density [130] (see also [131]). In this section we shall first address the fine-tuning problem associated with the cosmological constant  $\Lambda$ . We will then discuss recent progress to construct de-Sitter vacua in the context of string theory [40] and proceed to discuss several attempts to explain the origin of  $\Lambda$ .

##### A. Introduction of $\Lambda$

The Einstein tensor  $G^{\mu\nu}$  and the energy momentum tensor  $T^{\mu\nu}$  satisfy the Bianchi identities  $\nabla_\nu G^{\mu\nu} = 0$  and energy conservation  $\nabla_\nu T^{\mu\nu} = 0$ . Since the metric  $g^{\mu\nu}$  is constant with respect to covariant derivatives

( $\nabla_\alpha g^{\mu\nu} = 0$ ), there is a freedom to add a term  $\Lambda g_{\mu\nu}$  in the Einstein equations (see Refs. [132] for a nice discussion on the related theme). Then the modified Einstein equations are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (55)$$

By taking a trace of this equation, we find that  $-R + 4\Lambda = 8\pi GT$ . Combining this relation with Eq. (55), we obtain

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} \right). \quad (56)$$

Let us consider Newtonian gravity with metric  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ , where  $h_{\mu\nu}$  is the perturbation around the Minkowski metric  $\eta_{\mu\nu}$ . If we neglect the time-variation and rotational effect of the metric,  $R_{00}$  can be written by a gravitational potential  $\Phi$ , as  $R_{00} \simeq -(1/2)\Delta h_{00} = \Delta\Phi$ . Note that  $g_{00}$  is given by  $g_{00} = -1 - 2\Phi$ . In the relativistic limit with  $|p| \ll \rho$ , we have  $T_{00} \simeq -T \simeq \rho$ . Then the 00 component of Eq. (56) gives

$$\Delta\Phi = 4\pi G\rho - \Lambda. \quad (57)$$

In order to reproduce the Poisson equation in Newtonian gravity, we require that  $\Lambda = 0$  or  $\Lambda$  is sufficiently small relative to the  $4\pi G\rho$  term in Eq. (57). Since  $\Lambda$  has dimensions of  $[\text{Length}]^{-2}$ , the scale corresponding to the cosmological constant needs to be much larger than the scale of stellar objects on which Newtonian gravity works well. In other words the cosmological constant becomes important on very large scales.

In the FRW background given by (1) the modified Einstein equations (55) give

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \quad (58)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \quad (59)$$

This clearly demonstrates that the cosmological constant contributes negatively to the pressure term and hence exhibits a repulsive effect.

Let us consider a static universe ( $a = \text{const}$ ) in the absence of  $\Lambda$ . Setting  $H = 0$  and  $\ddot{a}/a = 0$  in Eqs. (9) and (12), we find

$$\rho = -3p = \frac{3K}{8\pi Ga^2}. \quad (60)$$

Equation (60) shows that either  $\rho$  or  $p$  needs to be negative. When Einstein first tried to construct a static universe, he considered that the above solution is not physical<sup>1</sup> and so added the cosmological constant to the original field equations (4).

<sup>1</sup> We note however that the negative pressure can be realized by scalar fields.

Using the modified field equations (58) and (59) in a dust-dominated universe ( $p = 0$ ), we find that the static universe obtained by Einstein corresponds to

$$\rho = \frac{\Lambda}{4\pi G}, \quad \frac{K}{a^2} = \Lambda. \quad (61)$$

Since  $\rho > 0$  we require that  $\Lambda$  is positive. This means that the static universe is a closed one ( $K = +1$ ) with a radius  $a = 1/\sqrt{\Lambda}$ . Equation (61) shows that the energy density  $\rho$  is determined by  $\Lambda$ .

The requirement of a cosmological constant to achieve a static universe can be understood by having a look at the Newton's equation of motion (24). Since gravity pulls the point particle toward the center of the sphere, we need a repulsive force to realize a situation in which  $a$  is constant. This corresponds to adding a cosmological constant term  $\Lambda/3$  on the right hand side of Eq. (24).

The above description of the static universe was abandoned with the discovery of the redshift of distant stars, but it is intriguing that such a cosmological constant should return in the 1990's to explain the observed acceleration of the universe.

Introducing the modified energy density and pressure

$$\tilde{\rho} = \rho + \frac{\Lambda}{8\pi G}, \quad \tilde{p} = p - \frac{\Lambda}{8\pi G}, \quad (62)$$

we find that Eqs. (58) and (59) reduce to Eqs. (9) and (12). In the subsequent sections we shall use the field equations (9) and (12) when we study the dynamics of dark energy.

## B. Fine tuning problem

If the cosmological constant originates from a vacuum energy density, then this suffers from a severe fine-tuning problem. Observationally we know that  $\Lambda$  is of order the present value of the Hubble parameter  $H_0$ , that is

$$\Lambda \approx H_0^2 = (2.13h \times 10^{-42} \text{ GeV})^2. \quad (63)$$

This corresponds to a critical density  $\rho_\Lambda$ ,

$$\rho_\Lambda = \frac{\Lambda m_{\text{pl}}^2}{8\pi} \approx 10^{-47} \text{ GeV}^4. \quad (64)$$

Meanwhile the vacuum energy density evaluated by the sum of zero-point energies of quantum fields with mass  $m$  is given by

$$\begin{aligned} \rho_{\text{vac}} &= \frac{1}{2} \int_0^\infty \frac{d^3\mathbf{k}}{(2\pi)^3} \sqrt{k^2 + m^2} \\ &= \frac{1}{4\pi^2} \int_0^\infty dk k^2 \sqrt{k^2 + m^2}. \end{aligned} \quad (65)$$

This exhibits an ultraviolet divergence:  $\rho_{\text{vac}} \propto k^4$ . However we expect that quantum field theory is valid up to

some cut-off scale  $k_{\text{max}}$  in which case the integral (65) is finite:

$$\rho_{\text{vac}} \approx \frac{k_{\text{max}}^4}{16\pi^2}. \quad (66)$$

For the extreme case of General Relativity we expect it to be valid to just below the Planck scale:  $m_{\text{pl}} = 1.22 \times 10^{19} \text{ GeV}$ . Hence if we pick up  $k_{\text{max}} = m_{\text{pl}}$ , we find that the vacuum energy density in this case is estimated as

$$\rho_{\text{vac}} \approx 10^{74} \text{ GeV}^4, \quad (67)$$

which is about  $10^{121}$  orders of magnitude larger than the observed value given by Eq. (64). Even if we take an energy scale of QCD for  $k_{\text{max}}$ , we obtain  $\rho_{\text{vac}} \approx 10^{-3} \text{ GeV}^4$  which is still much larger than  $\rho_\Lambda$ .

We note that this contribution is related to the ordering ambiguity of fields and disappears when normal ordering is adopted. Since this procedure of throwing away the vacuum energy is ad hoc, one may try to cancel it by introducing counter terms. However this requires a fine-tuning to adjust  $\rho_\Lambda$  to the present energy density of the universe. Whether or not the zero point energy in field theory is realistic is still a debatable question.

A nice resolution of the zero point energy is provided by supersymmetry. In supersymmetric theories every bosonic degree of freedom has its Fermi counter part which contributes to the zero point energy with an opposite sign compared to the bosonic degree of freedom thereby canceling the vacuum energy. Indeed, for a field with spin  $j > 0$ , the expression (65) for the vacuum energy generalizes to

$$\begin{aligned} \rho_{\text{vac}} &= \frac{1}{2} (-1)^{2j} (2j+1) \int_0^\infty \frac{d^3\mathbf{k}}{(2\pi)^3} \sqrt{k^2 + m^2} \\ &= \frac{(-1)^{2j} (2j+1)}{4\pi^2} \int_0^\infty dk k^2 \sqrt{k^2 + m^2}. \end{aligned} \quad (68)$$

Exact supersymmetry implies an equal number of fermionic and bosonic degrees of freedom for a given value of the mass  $m$  such that the net contribution to the vacuum energy vanishes. It is in this sense that supersymmetric theories do not admit a non-zero cosmological constant. However, we know that we do not live in a supersymmetric vacuum state and hence it should be broken today. For a viable supersymmetric scenario, for instance if it is to be relevant to the hierarchy problem, the supersymmetry breaking scale should be around  $M_{\text{SUSY}} \sim 10^3 \text{ GeV}$ . Indeed, the presence of a scalar field (Higgs field) in the standard model of particle physics (SM) is necessary to ensure the possibility of a spontaneous breakdown of the gauge symmetry.

However, the same scalar field creates what has come to be known as the ‘‘hierarchy problem’’. The origin of this problem lies in the quadratic nature of the divergence of the scalar self-energy arising out of scalar loops. A way out of this is supersymmetry (SUSY) which as

we have mentioned demands a fermionic partner for every boson and vice versa with the two having the same mass [133, 134]. Since fermionic loops come with an overall negative sign, the divergence in the scalar self energy due to the scalar loop and its SUSY partner cancel out. However, particles in nature do not come with degenerate partners as demanded by SUSY and hence SUSY must be broken. With a broken SUSY, one of course wants to ensure that no new scales are introduced between the electroweak scale of about 246 GeV and the Planck scale. The superpartners of the Standard Model particles thus are expected to have masses of the order of TeV. Masses much lower than this are ruled out from null experimental results in present day accelerators and specific bounds for the masses for the various superpartners of SM particles are available from analysis of experimental data. Theoretically, a consistent scheme of spontaneous breakdown of SUSY is technically far more complicated than in the non SUSY version. Nevertheless several approaches are available where this can be achieved.

With supersymmetry breaking around  $10^3$  GeV, we are still far away from the observed value of  $\Lambda$  by many orders of magnitudes. At present we do not know how the Planck scale or SUSY breaking scales are really related to the observed vacuum scale.

The above cosmological constant problem has led many many authors to try a different approach to the dark energy issue. Instead of assuming we have a small cosmological constant, we ignore it, presume it is zero due to some as yet unknown mechanism, and investigate the possibility that the dark energy is caused by the dynamics of a light scalar field. It does not solve the cosmological constant problem, but it does open up another avenue of attack as we will shortly see.

### C. $\Lambda$ from string theory

Recently there has been much progress in constructing de-Sitter vacua in string theory or supergravity. According to the no-go theorem in Refs. [135, 136] it is not possible to find de-Sitter solutions only in the presence of the lowest order terms in the 10 or 11 dimensional supergravity action. However this situation is improved when  $\alpha'$  or quantum corrections to the tree-level action are taken into account or extended objects like D-branes are present. In fact Kachru, Kallosh, Linde and Trivedi (KKLT) [40] constructed de-Sitter vacua by incorporating nonperturbative corrections to a superpotential in the context of type IIB string theory compactified on a Calabi-Yau manifold in the presence of flux. The importance of flux to insolving the cosmological constant problem was originally realized in Ref. [124]. In what follows we shall briefly discuss the effect of a four-form gauge flux to construct de-Sitter vacua [124] and then proceed to the review of the KKLT scenario using flux compactification.

#### 1. Four-form fluxes and quantization

Let us consider a four-form flux field  $F_4^2 = F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma}$  which appears in M theory. The starting point in Ref. [124] is a four dimensional gravity action in the presence of a negative bare cosmological constant  $-\Lambda_b$  and the four-form flux field:

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + \Lambda_b - \frac{1}{2 \cdot 4!} F_4^2 \right), \quad (69)$$

which arises as an effective action arising, e.g., from a  $M^4 \times S^7$  compactification. The bare cosmological constant is should be negative at the perturbative regime of string theory if it exists.

The four-form equation of motion,  $\nabla_\mu (\sqrt{-g} F^{\mu\nu\rho\sigma}) = 0$ , gives the solution  $F^{\mu\nu\rho\sigma} = c \epsilon^{\mu\nu\rho\sigma}$ , where  $F^{\mu\nu\rho\sigma}$  is an antisymmetric tensor with  $c$  being constant. Since  $F_4^2 = -24c^2$ , we find that the effective cosmological constant is given by

$$\Lambda = -\Lambda_b - \frac{1}{48} F_4^2 = -\Lambda_b + \frac{c^2}{2}. \quad (70)$$

This shows that it is possible to explain a small value of  $\Lambda$  provided that the bare cosmological constant is nearly canceled by the term coming from the four-form flux. However as long as the contribution of the flux is continuous, one can not naturally obtain the observed value of  $\Lambda$ .

Bousso and Polchinski tackled this problem by quantizing the value of  $c$  [124]. This implies that  $c$  is discontinuous as  $c = nq$ , where  $n$  is an integer. Although a single flux is not sufficient to explain the small values of  $\Lambda$  because of the large steps involved, this situation is improved by considering  $J$  multiple fluxes. In this case an effective cosmological constant is given by

$$\Lambda = -\Lambda_b + \frac{1}{2} \sum_{i=1}^J n_i^2 q_i^2. \quad (71)$$

It was shown in Ref. [124] that one can explain the observed value of  $\Lambda$  with  $J$  of order 100, which is not unrealistic. The work of Bousso and Polchinski did not address the problem of the stabilization of the modulus fields, but it opened up a new possibility for constructing large numbers of de-Sitter vacua using fluxes—and this has been called the “string landscape” [41].

#### 2. The KKLT scenario

KKLT [40] provided a mechanism to construct de-Sitter vacua of type IIB string theory based on flux compactifications on a Calabi-Yau manifold. They first of all fixed all the moduli associated with the compactification in an anti de-Sitter vacua by preserving supersymmetry. Then they incorporated nonperturbative corrections to the superpotential to obtain de-Sitter vacua.



The low energy effective action of string/M-theory in four dimensions is described by  $N = 1$  supergravity [134]

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{pl}}^2}{2} R + g^{\mu\nu} K_{\alpha\bar{\beta}} \partial_\mu \varphi^\alpha \partial_\nu \bar{\varphi}^\beta - e^{K/M_{\text{pl}}^2} \left( K^{\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} - \frac{3}{M_{\text{pl}}^2} |W|^2 \right) \right], \quad (72)$$

where  $\alpha, \beta$  run over all moduli fields  $\varphi$ . Here  $W(\varphi^\alpha)$  and  $K(\varphi^\alpha, \bar{\varphi}^\beta)$  are the superpotential and the Kähler potential, respectively, and

$$K_{\alpha\bar{\beta}} \equiv \frac{\partial^2 K}{\partial \varphi^\alpha \partial \bar{\varphi}^\beta}, \quad D_\alpha W \equiv \frac{\partial W}{\partial \varphi^\alpha} + \frac{W}{M_{\text{pl}}^2} \frac{\partial K}{\partial \varphi^\alpha}. \quad (73)$$

The supersymmetry is unbroken only for the vacua in which  $D_\alpha W = 0$  for all  $\alpha$ , which means that the effective cosmological constant is not positive from the action (72). We use the units  $M_{\text{pl}}^2 = 1$  for the rest of this section.

The authors in Ref. [137] adopted the following tree level functions for  $K$  and  $W$  in the flux compactification of Type IIB string theory [137, 138]:

$$K = -3 \ln[-i(\rho - \bar{\rho})] - \ln[-i(\tau - \bar{\tau})] - \ln[-i \int_{\mathcal{M}} \Omega \wedge \bar{\Omega}], \quad (74)$$

$$W = \int_{\mathcal{M}} G_3 \wedge \Omega, \quad (75)$$

where  $\rho$  is the volume modulus which includes the volume of the Calabi-Yau space and an axion coming from the R-R 4-form  $C_{(4)}$ , and  $\tau = C_{(0)} + i e^{-\Phi}$  is the axion-dilaton modulus.  $\Omega$  is the holomorphic three-form on the Calabi-Yau space and  $G_3$  is defined by  $G_3 = F_3 - \tau H_3$  where  $F_3$  and  $H_3$  are the R-R flux and the NS-NS flux, respectively, on the 3-cycles of the internal Calabi-Yau manifold  $\mathcal{M}$ .

Since  $W$  is not a function of  $\rho$ , we obtain  $K^{\rho\bar{\rho}} D_\rho W D_{\bar{\rho}} \bar{W} = 3|W|^2$ . Then Eq. (72) gives the supergravity potential

$$V = e^K \left( K^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} \right), \quad (76)$$

where  $i, j$  run over all moduli fields except for  $\rho$ . The condition  $D_i W = 0$  fixes all complex moduli except for  $\rho$  [137], which gives a zero effective cosmological constant. On the other hand, the supersymmetric vacua satisfying  $D_\rho W = 0$  gives  $W = 0$ , whereas, the non-supersymmetric vacua yield  $W = W_0 \neq 0$ .

To fix the volume modulus  $\rho$  as well, KKLT [40] added a non-perturbative correction [139] to the superpotential, which is given by

$$W = W_0 + A e^{ia\rho}, \quad (77)$$

where  $A$  and  $a$  are constants and  $W_0 \equiv \int G_3 \wedge \Omega$  is the tree level contribution. This correction is actually related to the effect of brane instantons. Note that the conditions

$D_i W = 0$  are automatically satisfied. For simplicity we set the axion-dilaton modulus to be zero and take  $\rho = i\sigma$ . Taking real values of  $A, a$  and  $W_0$ , we find that the supersymmetric condition  $D_\rho W = 0$  gives

$$W_0 = -A e^{-a\sigma_c} \left( 1 + \frac{2}{3} a\sigma_c \right), \quad (78)$$

which fixes the volume modulus  $\rho$  in terms of  $W_0$ . This produces the anti de-Sitter vacua, that is

$$V_{\text{AdS}} = -3e^K |W|^2 = -\frac{a^2 A^2 e^{-2a\sigma_c}}{6\sigma_c}. \quad (79)$$

Hence all the moduli are stabilized while preserving supersymmetry with a negative cosmological constant.

In order to obtain a de-Sitter vacuum, KKLT introduced an anti-D3 brane in a warped background. Since fluxes  $F_3$  and  $H_3$  are also the sources for a warp factor [137, 138], models with fluxes generically correspond to a warped compactification, whose metric is given by

$$ds_{10}^2 = e^{2B(y)} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{-2B(y)} \tilde{g}_{mn}(y) dy^m dy^n, \quad (80)$$

where the factor,  $e^B$ , can be computed in the regions closed to a conifold singularity of the Calabi-Yau manifold. This warp factor is exponentially suppressed at the tip of the throat, depending on the fluxes as

$$e^{B_{\text{min}}} \sim \exp\left(-\frac{2\pi N}{3g_s M}\right), \quad (81)$$

where  $g_s$  is the string coupling, integers  $M$  and  $N$  are the R-R and NS-NS three-form flux, respectively. While the warp factor is of order one at generic points in the  $y$ -space, its minimum value can be extremely small for a suitable choice of fluxes.

The background fluxes generate a potential for the world-volume scalars of the anti-D3 brane, which means that they do not introduce additional moduli [140]. The anti-D3 brane, however, provides an additional energy to the supergravity potential [40, 140]:

$$\delta V = \frac{2b_0^4 T_3}{g_s^4} \frac{1}{(\text{Im}\rho)^3}, \quad (82)$$

where  $T_3$  is the brane tension and  $b_0$  is the warp factor at the location of the anti-D3 brane. The anti-D3 brane energetically prefers to sit at the tip of the throat, giving  $b_0 = e^{B_{\text{min}}}$ . The total potential is the sum of Eqs. (79) and (82), that is

$$V = \frac{2b_0^4 T_3}{g_s^4} \frac{1}{(\text{Im}\rho)^3} - \frac{a^2 A^2 e^{-2a\sigma_c}}{6\sigma_c}. \quad (83)$$

Then one can obtain positive cosmological constant by tuning the flux integers  $M$  and  $N$ .

The life time of the vacua was found to be larger than the age of the universe and hence these solutions can be



considered as stable for practical purposes [40]. Although a fine-tuning problem of  $\Lambda$  still remains in this scenario, it is interesting that string theory in principle gives rise to a stable de-Sitter vacua with all moduli fixed. A remarkable and somewhat controversial argument about the nature of the cosmological constant problem has developed recently out of this realisation that there are many possible de-Sitter vacua. The fact that there are a vast number of different choices of fluxes leads in principle to a complicated string landscape with more than  $10^{100}$  vacuum [41]. Surely, the argument goes, it should be possible to find a vacuum which is identical to the one we live in! In an interesting paper, Liddle and Urena-Lopez have examined the conditions needed to unify the description of dark matter, dark energy and inflation within the context of the string landscape [126]. They claim that incomplete decay of the inflaton field offers the possibility that a single field might be responsible for all of inflation, dark matter and dark energy, whereas, unifying dark matter and dark energy into a single field which is separate from the inflaton appears very difficult.

### 3. Relaxation of $\Lambda$ in string theory

In Ref. [127], the authors developed an earlier approach in Refs. [111] to relax the effective cosmological constant through the nucleation of branes coupled to a three-index gauge potential. The influence of string theory in the new approach is important, the brane depends on the compactification of the extra dimensions which in turn can provide the required very small quantized unit for jumps in the effective cosmological term. As well as this feature, when considering multiple coincident branes, in Ref. [127], the authors show that the internal degrees of freedom for such a configuration can dramatically enhance tunneling rates by exponentially large density of states factors.

For consistency, the dynamics of the system must be such that the cosmological constant relaxes quickly enough from high energy scales, but today remains stable on a time scale of the universe, a constraint which leads to a non-trivial relation between the scale of supersymmetry breaking and the value of the cosmological constant. In particular the constraint becomes

$$M_{\text{SUSY}}^2 \leq (10^{-3}\text{eV})(M_{\text{Planck}}), \quad (84)$$

which rules out large supersymmetry breaking scales for these relaxation models, with the largest possible scale still viable in nature. Time will tell whether the relaxation mechanism is sufficiently versatile to uniquely pick out the actual vacuum we live in, but it is certainly a novel approach to determining it.

### 4. $\Lambda$ from a self-tuning universe

In [120], both sets of authors develop an approach to the cosmological constant problem which relies on the presence of an extra dimension. Rather than making the vacuum energy small, this approach proceeds by removing the gravitational effect of vacuum energy on the expansion of the universe. Considering Poincare invariant domain wall (“3-brane”) solutions to some 5-dimensional effective theories which can arise naturally in string theory, the basic idea behind the models is that the Standard Model vacuum energy “warps” the higher-dimensional spacetime while preserving 4D flatness. In the strong curvature region the size of the extra dimension is effectively cut off (under certain assumptions about the nature of the singularity in the strong curvature regime), giving rise to macroscopic 4D gravity without a cosmological constant. Although the higher-dimensional gravity dynamics is treated classically, the Standard Model is fully quantum field-theoretic, leading the authors to argue that 4D flatness of their solutions is stable against Standard Model quantum loops and changes to Standard Model couplings.

In [141], the authors point out how such a self tuning scenario requires changing of the Friedmann equation of conventional cosmology, and investigate in the context of specific toy models of self tuning the difficulties that arise in obtaining cosmological evolution compatible with observation in this context. It remains to be seen whether this mechanism will eventually work, but the idea that by making the metric insensitive to the value of the cosmological constant as opposed to trying to make the vacuum energy small itself is intriguing.

### 5. $\Lambda$ through mixing of degenerate vacua

In Refs. [142], the authors suggest a mechanism in string theory, where the large number  $N$  of connected degenerate vacua that could exist, can lead to a ground state with much lower energy than that of any individual vacuum. This is because of the effect of level repulsion in quantum theory for the wavefunction describing the Universe. To make it more quantitative, they consider a scenario where initial quantum fluctuations give an energy density  $\sim m_{\text{SUSY}}^2 m_{\text{pl}}^2$ , but the universe quickly cascades to an energy density  $\sim m_{\text{SUSY}}^2 m_{\text{pl}}^2 / N$ . The argument then proceeds, as the universe expands and undergoes a series of phase transitions there are large contributions to the energy density and consequent rearrangement of levels, each time followed by a rapid cascade to the ground state or near it. The ground state which eventually describes our world is then a superposition of a large number of connected string vacua, with shared superselection sets of properties such as three families etc.. The observed value of the cosmological constant is given in terms of the Planck mass, the scale of supersymmetry breaking and the number of connected string vacua, and

they argue can quite easily be very small.

#### D. Causal sets and $\Lambda$

String theory is not the only candidate for a quantum theory of gravity. There are a number of others, and one in particular is worthy of mention in that it makes a prediction for the order of magnitude expected of the cosmological constant. In the context of Causal sets (for a review of Causal sets see [143]), Sorkin [43], back in the early 1990's, predicted that a fluctuating cosmological term  $\Lambda(x)$  would arise under the specific modification of General Relativity motivated by causal sets. The predicted fluctuations arise as a residual (and non-local) quantum effect from the underlying space-time discreteness.

Roughly speaking, the space-time discreteness leads to a finite number  $N$  of elements, and the space-time volume  $\mathcal{V}$  is a direct reflection of  $N$ . Now  $\Lambda$  is conjugate to  $\mathcal{V}$ , and fluctuations in  $\mathcal{V}$  arise from the Poisson fluctuations in  $N$  (which have a typical scale  $\sqrt{N}$ ), implying there will be ever decreasing fluctuations in  $\Lambda$  given by [43]

$$\Delta\Lambda \sim \frac{1}{\Delta\mathcal{V}} \sim \frac{1}{\sqrt{\mathcal{V}}}. \quad (85)$$

This could be used to explain why  $\Lambda$  is not exactly zero today, but why is it so near to zero? Sorkin addresses this issue by pointing out that the space-time volume  $\mathcal{V}$  is roughly equal to the fourth power of the Hubble radius  $H^{-1}$ . It follows that at all times we expect the energy density in the cosmological constant to be of order the critical density  $\rho_c$ , i.e.,

$$\rho_\Lambda \sim \mathcal{V}^{-1/2} \sim H^2 \sim \rho_{\text{crit}}. \quad (86)$$

Therefore, the prediction for today's  $\Lambda$  has the right order of magnitude that agrees with current observations of the dark energy, and it fluctuates about zero due to the non-discrete nature of space-time. Interestingly another prediction is that this agreement is true for all times, implying a kind of scaling (or tracking) behaviour arising out of causal sets. In [144], this basic paradigm is put to the test against observations, and appears to have survived the first set of tests showing evidence of "tracking" behaviour with no need for fine tuning and consistency with nucleosynthesis and structure formation constraints. This is a fascinating idea and certainly deserves further attention to compare it with more detailed observations, although of course the actual mechanism to generate the cosmological constant based on causal sets remains to be solved.

#### E. Anthropic selection of $\Lambda$

The use of the anthropic principle has generated much debate in the cosmology community. It has been used

in physics on many occasions to explain some of the observed features of our Universe, without necessarily explaining the features from an underlying theory. For many, it is the solution you introduce when you have given up on finding any physical route to a solution. For others, it is a perfectly plausible weapon in the physicists armoury and can be brought out and used when the need arises.

In a cosmological context, it could be argued that discussions related to the use of the anthropic principle were meaningless without an underlying cosmological model, to place it in context with. The Inflationary Universe provided such a paradigm and Linde discussed the anthropic principle in this context in the famous Proceedings of the Nuffield Symposium in 1982 [145]. In [44], he proposed a possible anthropic solution to the cosmological constant problem. Assuming unsuppressed quantum creation of the universe at the Planck energy density, he noted that vacuum energy density could be written as a sum of contributions from the effective potential of the scalar field  $V(\phi)$  and that of fluxes  $V(F)$ . The condition for the universe to form was that the sum of these two terms matched the Planck energy density, i.e.,  $V(\phi) + V(F) = 1$  in suitable units. However as the universe inflates, the field slowly rolls to its minimum at some different value  $\phi_0$ , leaving a different vacuum energy density  $\Lambda = V(\phi_0) + V(F)$ . Since  $V(\phi)$  can take any value subject to the initial constraint  $V(\phi) + V(F) = 1$ , it leads to a flat probability distribution for the final value of the cosmological constant  $\Lambda = V(\phi_0) + V(F)$  a condition which is required for the anthropic solution of the cosmological constant problem.

If ever a problem required an anthropic argument to explain it, then it could well be that the cosmological constant is that problem. There has been considerable work in this area over the past twenty or so years [45, 112].

In [115], the authors extended an idea first explored in [146], the possibility that the dark energy is due to a potential of a scalar field and that the magnitude and the slope of this potential in our part of the universe are largely determined by anthropic selection effects. A class of models are consistent with observations in that the most probable values of the slope are very small, implying that the dark energy density stays constant to very high accuracy throughout cosmological evolution. However, in other models, the most probable values of the slope make it hard to have sufficient slow-roll condition, leading to a re-collapse of the local universe on a time-scale comparable to the lifetime of the sun. Such a situation leads to a rapidly varying effective equation of state with the redshift, leading to a number of testable predictions (see also [147] for a related model).

According to the anthropic principle, only specific values of the fundamental constants of nature can have lead to intelligent life in our universe. Weinberg [45] was the first to point out that once the cosmological constant comes to dominate the dynamics of the universe, then

structure formation stops because density perturbations cease to grow. Thus structure formation should be completed before the domination of vacuum energy, otherwise there could be no observers now. This leads to the following bound arising out of an anthropic argument [45]

$$\rho_\Lambda < 500\rho_m^{(0)}, \quad (87)$$

which is two orders of magnitude away from the observed value of the vacuum energy density.

The situation can change if the vacuum energy differs in different regions of the universe. In this case one should define a conditional probability density to observe a given value of  $\rho_\Lambda$  [46, 47]

$$d\mathcal{P}(\rho_\Lambda) = \mathcal{P}_*(\rho_\Lambda)n_G(\rho_\Lambda)d\rho_\Lambda, \quad (88)$$

where  $n_G(\rho_\Lambda)$  is the average number of galaxies that can form per unit volume for a given value of the vacuum energy density and  $\mathcal{P}_*(\rho_\Lambda)$  is the *a priori* probability density distribution. For a flat distribution of  $\mathcal{P}_*(\rho_\Lambda)$ , it was shown in Ref. [116] that  $\mathcal{P}(\rho_\Lambda)$  peaks around  $\rho_{\text{vac}} \sim 8\rho_m^{(0)}$ .

There are two important aspects of the anthropic selection, one is related to the prediction of the a priori probability and the other to the possibility of  $\Lambda$  assuming different values in different regions of the universe. The existence of a vast landscape of de-Sitter vacua in string theory makes the anthropic approach especially interesting. On the other hand, the prediction of a priori probability arising out of fundamental theory is of course non-trivial (perhaps impossible!) and this could perhaps be more difficult than the derivation of an observed value of  $\Lambda$  itself. The anthropic arguments can not tell us how the present observed scale of  $\Lambda$  is related to the scales arising in particle physics, e.g., SUSY breaking scale, but many believe it is important to carry on the investigation of whether or not the anthropic principle has real predictive power in the context of the cosmological constant.

### F. A Dynamical Approach to the Cosmological Constant

In [148] a dynamical approach to the cosmological constant is investigated. The novel feature is that a scalar field exists which has non-standard kinetic terms whose coefficient diverges at zero curvature. Moreover, as well as having the standard kinetic term, the field has a potential whose minimum occurs at a generic, but negative, value for the vacuum energy. The divergent coefficient of the kinetic term means that the lowest energy state is never achieved. Instead, the cosmological constant automatically stalls at or near zero. The authors argue that the model is stable under radiative corrections, leads to stable dynamics, despite the singular kinetic term, and can reduce the required fine-tuning by at least 60 orders of magnitude. They also point out that the model could

provide a new mechanism for sampling possible cosmological constants and implementing the anthropic principle.

### G. Observing dark energy in the laboratory ?

At present we do not really know how quantum field theory could *naturally* lead to the present observed scale of cosmological constant. Assuming that we have solved this problem let us ask whether it is possible to observe the cosmological constant directly through laboratory experiments? It is a question that is fascinating and has generated quite a bit of debate. There is no consensus yet as to the answer.

We remind the reader that so far all the evidence for the presence of dark energy has been astrophysical in nature. In fact there is little doubt that vacuum fluctuations are found in nature. One example of their role, is that they are responsible for the quantum noise found in dissipative systems, noise which has been detected experimentally. Quantum noise should emerge in a dissipative system due to uncertainty principal for a simple reason. Classically, the stable state of a dissipative system corresponds to a zero momentum state which is not permissible quantum mechanically. Thus quantum noise should be present in the system which would keep it going [149].

For simplicity let us assume that the vacuum fluctuations are electromagnetic in nature [150]. These fluctuations are then represented by an ideal gas of harmonic oscillators. Quantum statistical mechanics tells us that the spectral energy density of the fluctuations with a frequency  $\nu$  and a temperature  $T$  is given by

$$\rho(\nu, T) = \rho_0(\nu) + \rho_{\text{rad}}(\nu, T), \quad (89)$$

where

$$\rho_0(\nu) = \frac{4\pi h\nu^3}{c^3} \quad (90)$$

corresponds to the zero-point fluctuation, and

$$\rho_{\text{rad}}(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp(h\nu/k_B T) - 1} \quad (91)$$

describes the thermal fluctuations of a Planck spectrum. Note that we have explicitly written Planck's constant  $h$ , the speed of light  $c$  and the Boltzmann constant  $k_B$  following standard convention.

The energy density  $\rho_0(\nu)$  is formally infinite, so, as before, we introduce a cut-off  $\nu = \nu_\Lambda$  to handle it [150]

$$\rho_v \equiv \int_0^{\nu_\Lambda} \rho_0(\nu)d\nu = \frac{\pi h}{c^3} \nu_\Lambda^4. \quad (92)$$

Identifying the vacuum energy density with the observed value of the dark energy we obtain an estimate for the cut-off frequency

$$\nu = \nu_\Lambda \simeq 1.7 \times 10^{12} \text{ Hz}. \quad (93)$$

If the vacuum fluctuations are responsible for dark energy, we should observe a cut-off (93) in the spectrum of fluctuations.

Let us now briefly describe an experimental set up to investigate the nature of vacuum fluctuations. Over two decades ago, Koch *et al.* carried out experiments with devices based upon Josephson junctions [151, 152]. They were interested in obtaining the spectrum of quantum noise present in their particular experiment that could remove the thermal part of the noise because it ran at low temperatures. The results of this experiment are in agreement with Eq. (89) up to the maximum frequency of  $\nu_{\max} = 6 \times 10^{11}$  Hz they could reach in their experiment.

The results of Koch *et al.* demonstrate the existence of vacuum fluctuations in the spectrum through the linear part of the spectrum. However, on the basis of these findings, we can say nothing about the inter-relation of vacuum fluctuations to dark energy. We still need to investigate the spectrum up to frequencies three times larger than  $\nu_{\max}$  to beat the threshold. And if a cut-off is observed in the spectrum around  $\nu_{\Lambda}$ , it will be suggestive that vacuum fluctuations could be responsible for dark energy. In the next few years it would be possible to cross the threshold frequency as suggested in Ref. [153] (see also [154]). The outcome of such an experiment may be dramatic not only for cosmology but also for string theory [155]. However, we should remind the reader that there is some debate as to whether this technique can actually produce evidence of a  $\Lambda$  in the laboratory. In [156], Jetzer and Straumann claim that Dark Energy contributions can not be determined from noise measurements of Josephson junctions as assumed in [153]. This claim is then rebutted by Beck and Mackey in [157], with Jetzer and Straumann arguing against that conclusion in [158] (see also Ref.[159] on the related theme). Time will tell who (if either) are correct.

From now on we assume we have solved the underlying  $\Lambda$  problem. It is zero for some reason and dark energy is to be explained by some other mechanism. Readers only interested in a constant  $\Lambda$ , may want to skip to Sec. XIII on the observational features of dark energy as a way of testing for  $\Lambda$ .

## V. SCALAR-FIELD MODELS OF DARK ENERGY

The cosmological constant corresponds to a fluid with a constant equation of state  $w = -1$ . Now, the observations which constrain the value of  $w$  today to be close to that of the cosmological constant, these observations actually say relatively little about the time evolution of  $w$ , and so we can broaden our horizons and consider a situation in which the equation of state of dark energy changes with time, such as in inflationary cosmology. Scalar fields naturally arise in particle physics including string theory and these can act as candidates for dark energy. So far a wide variety of scalar-field dark energy models have

been proposed. These include quintessence, phantoms, K-essence, tachyon, ghost condensates and dilatonic dark energy amongst many. We shall briefly describe these models in this section. We will also mention the Chaplygin gas model, although it is different from scalar-field models of dark energy. We have to keep in mind that the contribution of the dark matter component needs to be taken into account for a complete analysis. Their dynamics will be dealt with in detail in Sec. VI. In the rest of the paper we shall study a flat FRW universe ( $K = 0$ ) unless otherwise specified.

### A. Quintessence

Quintessence is described by an ordinary scalar field  $\phi$  minimally coupled to gravity, but as we will see with particular potentials that lead to late time inflation. The action for Quintessence is given by

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2}(\nabla\phi)^2 - V(\phi) \right], \quad (94)$$

where  $(\nabla\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$  and  $V(\phi)$  is the potential of the field. In a flat FRW spacetime the variation of the action (94) with respect to  $\phi$  gives

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \quad (95)$$

The energy momentum tensor of the field is derived by varying the action (94) in terms of  $g^{\mu\nu}$ :

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}. \quad (96)$$

Taking note that  $\delta\sqrt{-g} = -(1/2)\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ , we find

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu} \left[ \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi) \right]. \quad (97)$$

In the flat Friedmann background we obtain the energy density and pressure density of the scalar field:

$$\rho = -T_0^0 = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p = T_i^i = \frac{1}{2}\dot{\phi}^2 - V(\phi). \quad (98)$$

Then Eqs. (9) and (12) yield

$$H^2 = \frac{8\pi G}{3} \left[ \frac{1}{2}\dot{\phi}^2 + V(\phi) \right], \quad (99)$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3} \left[ \dot{\phi}^2 - V(\phi) \right]. \quad (100)$$

We recall that the continuity equation (11) is derived by combining these equations.

From Eq. (100) we find that the universe accelerates for  $\dot{\phi}^2 < V(\phi)$ . This means that one requires a flat potential

to give rise to an accelerated expansion. In the context of inflation the slow-roll parameters

$$\epsilon = \frac{m_{\text{pl}}^2}{16\pi} \left( \frac{1}{V} \frac{dV}{d\phi} \right)^2, \quad \eta = \frac{m_{\text{pl}}^2}{8\pi} \frac{1}{V} \frac{d^2V}{d\phi^2}, \quad (101)$$

are often used to check the existence of an inflationary solution for the model (94) [70]. Inflation occurs if the slow-roll conditions,  $\epsilon \ll 1$  and  $|\eta| \ll 1$ , are satisfied. In the context of dark energy these slow-roll conditions are not completely trustworthy, since there exists dark matter as well as dark energy. However they still provide a good measure to check the existence of a solution with an accelerated expansion. If we define slow-roll parameters in terms of the time-derivatives of  $H$  such as  $\epsilon = -\dot{H}/H^2$ , this is a good measure to check the existence of an accelerated expansion since they implement the contributions of both dark energy and dark matter.

The equation of state for the field  $\phi$  is given by

$$w_\phi = \frac{p}{\rho} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}. \quad (102)$$

In this case the continuity equation (11) can be written in an integrated form:

$$\rho = \rho_0 \exp \left[ - \int 3(1 + w_\phi) \frac{da}{a} \right], \quad (103)$$

where  $\rho_0$  is an integration constant. We note that the equation of state for the field  $\phi$  ranges in the region  $-1 \leq w_\phi \leq 1$ . The slow-roll limit,  $\dot{\phi}^2 \ll V(\phi)$ , corresponds to  $w_\phi = -1$ , thus giving  $\rho = \text{const}$  from Eq. (103). In the case of a stiff matter characterized by  $\dot{\phi}^2 \gg V(\phi)$  we have  $w_\phi = 1$ , in which case the energy density evolves as  $\rho \propto a^{-6}$  from Eq. (103). In other cases the energy density behaves as

$$\rho \propto a^{-m}, \quad 0 < m < 6. \quad (104)$$

Since  $w_\phi = -1/3$  is the border of acceleration and deceleration, the universe exhibits an accelerated expansion for  $0 \leq m < 2$  [see Eq. (20)].

It is of interest to derive a scalar-field potential that gives rise to a power-law expansion:

$$a(t) \propto t^p. \quad (105)$$

The accelerated expansion occurs for  $p > 1$ . From Eq. (10) we obtain the relation  $\dot{H} = -4\pi G \dot{\phi}^2$ . Then we find that  $V(\phi)$  and  $\dot{\phi}$  can be expressed in terms of  $H$  and  $\dot{H}$ :

$$V = \frac{3H^2}{8\pi G} \left( 1 + \frac{\dot{H}}{3H^2} \right), \quad (106)$$

$$\dot{\phi} = \int dt \left[ -\frac{\dot{H}}{4\pi G} \right]^{1/2}. \quad (107)$$

Here we chose the positive sign of  $\dot{\phi}$ . Hence the potential giving the power-law expansion (105) corresponds to

$$V(\phi) = V_0 \exp \left( -\sqrt{\frac{16\pi}{p}} \frac{\phi}{m_{\text{pl}}} \right), \quad (108)$$

where  $V_0$  is a constant. The field evolves as  $\phi \propto \ln t$ . The above result shows that the exponential potential may be used for dark energy provided that  $p > 1$ .

In addition to the fact that exponential potentials can give rise to an accelerated expansion, they possess cosmological *scaling solutions* [14, 160] in which the field energy density ( $\rho_\phi$ ) is proportional to the fluid energy density ( $\rho_m$ ). Exponential potentials were used in one of the earliest models which could accommodate a period of acceleration today within it, the loitering universe [161] (and see [162] for an example of a loitering universe in the braneworld context).

In Sec. VI we shall carry out a detailed analysis of the cosmological dynamics of an exponential potential in the presence of a barotropic fluid.

The above discussion shows that scalar-field potentials which are not steep compared to exponential potentials can lead to an accelerated expansion. In fact the original quintessence models [10, 15] are described by the power-law type potential

$$V(\phi) = \frac{M^{4+\alpha}}{\phi^\alpha}, \quad (109)$$

where  $\alpha$  is a positive number (it could actually also be negative [163]) and  $M$  is constant. Where does the fine tuning arise in these models? Recall that we need to match the energy density in the quintessence field to the current critical energy density, that is

$$\rho_\phi^{(0)} \approx m_{\text{pl}}^2 H_0^2 \approx 10^{-47} \text{ GeV}^4. \quad (110)$$

The mass squared of the field  $\phi$  is given by  $m_\phi^2 = \frac{d^2V}{d\phi^2} \approx \rho_\phi/\phi^2$ , whereas the Hubble expansion rate is given by  $H^2 \approx \rho_\phi/m_{\text{pl}}^2$ . The universe enters a tracking regime in which the energy density of the field  $\phi$  catches up that of the background fluid when  $m_\phi^2$  decreases to of order  $H^2$  [10, 15]. This shows that the field value at present is of order the Planck mass ( $\phi_0 \sim m_{\text{pl}}$ ), which is typical of most of the quintessence models. Since  $\rho_\phi^{(0)} \approx V(\phi_0)$ , we obtain the mass scale

$$M = \left( \rho_\phi^{(0)} m_{\text{pl}}^\alpha \right)^{\frac{1}{4+\alpha}}. \quad (111)$$

This then constrains the allowed combination of  $\alpha$  and  $M$ . For example the constraint implies  $M = 1 \text{ GeV}$  for  $\alpha = 2$  [16]. This energy scale can be compatible with the one in particle physics, which means that the severe fine-tuning problem of the cosmological constant is alleviated. Nevertheless a general problem we always have to tackle is finding such quintessence potentials in particle

physics. One of the problems is highlighted in Ref. [12]. The Quintessence field must couple to ordinary matter, which even if suppressed by the Planck scale, will lead to long range forces and time dependence of the constants of nature. There are tight constraints on such forces and variations and any successful model must satisfy them. In Sec. VIII we shall present a number of quintessence models motivated by particle physics.

### B. K-essence

Quintessence relies on the potential energy of scalar fields to lead to the late time acceleration of the universe. It is possible to have a situation where the accelerated expansion arises out of modifications to the kinetic energy of the scalar fields. Originally kinetic energy driven inflation, called K-inflation, was proposed by Armendariz-Picon *et al.* [23] to explain early universe inflation at high energies. This scenario was first applied to dark energy by Chiba *et al.* [20]. The analysis was extended to a more general Lagrangian by Armendariz-Picon *et al.* [21, 22] and this scenario was called ‘‘K-essence’’.

K-essence is characterized by a scalar field with a non-canonical kinetic energy. The most general scalar-field action which is a function of  $\phi$  and  $X \equiv -(1/2)(\nabla\phi)^2$  is given by

$$S = \int d^4x \sqrt{-g} p(\phi, X), \quad (112)$$

where the Lagrangian density  $p(\phi, X)$  corresponds to a pressure density. We note that the action (112) includes quintessence models. Usually K-essence models are restricted to the Lagrangian density of the form [20, 21, 22]:

$$p(\phi, X) = f(\phi)\hat{p}(X). \quad (113)$$

One of the motivations to consider this type of Lagrangian originates from string theory [23]. The low-energy effective string theory generates higher-order derivative terms coming from  $\alpha'$  and loop corrections (here  $\alpha'$  is related to the string length scale  $\lambda_s$  via the relation  $\alpha' = \lambda_s/2\pi$ ). The four-dimensional effective string action is generally given by

$$S = \int d^4x \sqrt{-\tilde{g}} \{ B_g(\phi) \tilde{R} + B_\phi^{(0)}(\phi) (\tilde{\nabla}\phi)^2 - \alpha' [c_1^{(1)} B_\phi^{(1)}(\phi) (\tilde{\nabla}\phi)^4 + \dots] + \mathcal{O}(\alpha'^2) \}, \quad (114)$$

where  $\phi$  is the dilaton field that controls the strength of the string coupling  $g_s^2$  via the relation  $g_s^2 = e^\phi$  [164]. Here we set  $\kappa^2 = 8\pi G = 1$ . In the weak coupling regime ( $e^\phi \ll 1$ ) the coupling functions have the dependence  $B_g \simeq B_\phi^{(0)} \simeq B_\phi^{(1)} \simeq e^{-\phi}$ . As the string coupling becomes of order unity, the form of the couplings should take more complicated forms. If we make a conformal transformation  $g_{\mu\nu} = B_g(\phi) \tilde{g}_{\mu\nu}$ , the string-frame action (114) is

transformed to the Einstein-frame action [23, 164, 165]:

$$S_E = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R + K(\phi) X + L(\phi) X^2 + \dots \right], \quad (115)$$

where

$$K(\phi) = \frac{3}{2} \left( \frac{1}{B_g} \frac{dB_g}{d\phi} \right)^2 - \frac{B_\phi^{(0)}}{B_g}, \quad (116)$$

$$L(\phi) = 2c_1^{(1)} \alpha' B_\phi^{(1)}(\phi). \quad (117)$$

Hence this induces a Lagrangian with noncanonical kinetic terms:

$$p(\phi, X) = K(\phi) X + L(\phi) X^2. \quad (118)$$

If we make the field redefinition

$$\phi_{\text{new}} = \int^{\phi_{\text{old}}} d\phi \sqrt{\frac{L}{|K|}}, \quad (119)$$

the Lagrangian (118) transforms into [20]

$$p(\phi, X) = f(\phi)(-X + X^2), \quad (120)$$

where  $\phi \equiv \phi_{\text{new}}$ ,  $X \equiv X_{\text{new}} = (L/|K|)X_{\text{old}}$  and  $f(\phi) = K^2(\phi_{\text{old}})/L(\phi_{\text{old}})$ . This shows that the model given by (118) falls into the category of K-essence (113) with a choice  $\hat{p}(X) = -X + X^2$  after an appropriate field definition.

For the pressure density (120) we find that the energy density of the field  $\phi$  is given by

$$\rho = 2X \frac{\partial p}{\partial X} - p = f(\phi)(-X + 3X^2). \quad (121)$$

Then the equation of state of the field is given by

$$w_\phi = \frac{p}{\rho} = \frac{1 - X}{1 - 3X}. \quad (122)$$

This shows that  $w_\phi$  does not vary for constant  $X$ . For example we obtain the equation of state of a cosmological constant ( $w_\phi = -1$ ) for  $X = 1/2$ . The equation of state giving rise to an accelerated expansion is  $w_\phi < -1/3$ , which translates into the condition  $X < 2/3$ .

We recall that the energy density  $\rho$  satisfies the continuity equation (11). During the radiation or matter dominant era in which the equation of state of the background fluid is  $w_m$ , the evolution of the Hubble rate is given by  $H = 2/[3(1 + w_m)(t - t_0)]$  from Eq. (18). Then the energy density  $\rho$  of the field  $\phi$  satisfies

$$\dot{\rho} = -\frac{2(1 + w_\phi)}{(1 + w_m)(t - t_0)} \rho. \quad (123)$$

For constant  $X$  (i.e., constant  $w_\phi$ ) the form of  $f(\phi)$  is constrained to be

$$f(\phi) \propto (\phi - \phi_0)^{-\alpha}, \quad \alpha = \frac{2(1 + w_\phi)}{1 + w_m}, \quad (124)$$

where we used Eqs. (121) and (123).

When  $w_\phi = w_m$  the function  $f(\phi) \propto (\phi - \phi_0)^{-2}$  in the radiation or matter dominant era. This corresponds to the scaling solutions, as we will see in Sec. VII. In the case of  $w_\phi = -1$  we find that  $f(\phi) = \text{const}$  with  $X = 1/2$ . This corresponds to the ghost condensate scenario proposed in Ref. [38]. In order to apply this to dark energy we need to fine-tune  $f(\phi)$  to be of order the present energy density of the universe. We caution that the above function  $f(\phi)$  is obtained by assuming that the energy density of the field is much smaller than that of the background fluid ( $\rho \ll \rho_m$ ). Hence this is no longer applicable for a dark energy dominated universe. For example even for  $f(\phi) \propto (\phi - \phi_0)^{-2}$  there exists another solution giving an accelerated expansion other than the scaling solutions at late times. In fact this case marks the border between acceleration and deceleration. We will clarify these issues in Sec. VI.

Equation (122) shows that the kinetic term  $X$  plays a crucial role in determining the equation of state of  $\phi$ . As long as  $X$  belongs in the range  $1/2 < X < 2/3$ , the field  $\phi$  behaves as dark energy for  $0 \leq \alpha \leq 2$ . The model (120) describes one of the examples of K-essence. In fact Armendariz-Picon *et al.* [21, 22] extended the analysis to more general forms of  $\hat{p}(X)$  in Eq. (113) to solve the coincident problem of dark energy. See Refs. [166] for various aspects of K-essence.

### C. Tachyon field

Recently it has been suggested that rolling tachyon condensates, in a class of string theories, may have interesting cosmological consequences. Sen [167] showed that the decay of D-branes produces a pressureless gas with finite energy density that resembles classical dust (see also Refs. [168]). A rolling tachyon has an interesting equation of state whose parameter smoothly interpolates between  $-1$  and  $0$  [169]. This has led to a flurry of attempts being made to construct viable cosmological models using the tachyon as a suitable candidate for the inflaton at high energy [170]. However tachyon inflation in open string models is typically plagued by several difficulties [171] associated with density perturbations and reheating<sup>2</sup>. Meanwhile the tachyon can also act as a source of dark energy depending upon the form of the tachyon potential [174, 175, 176, 177, 178, 179]. In what follows we shall consider the tachyon as a field from which it is possible to obtain viable models of dark energy.

The effective Lagrangian for the tachyon on a non-BPS

D3-brane is described by

$$S = - \int d^4x V(\phi) \sqrt{-\det(g_{ab} + \partial_a \phi \partial_b \phi)}, \quad (125)$$

where  $V(\phi)$  is the tachyon potential. The effective potential obtained in open string theory has the form [181]

$$V(\phi) = \frac{V_0}{\cosh(\phi/\phi_0)}, \quad (126)$$

where  $\phi_0 = \sqrt{2}$  for the non-BPS D-brane in the superstring and  $\phi_0 = 2$  for the bosonic string. Note that the tachyon field has a ground state at  $\phi \rightarrow \infty$ . There exists another type of tachyon potential which appears as the excitation of massive scalar fields on the anti-D branes [172]. In this case the potential is given by  $V(\phi) = V_0 e^{\frac{1}{2} m^2 \phi^2}$  and it has a minimum at  $\phi = 0$ . In this review we keep the tachyon potential as general as possible and will carry out a detailed analysis of the associated dynamics in Sec. VI.

The energy momentum tensor which follows from the action (125) has the form

$$T_{\mu\nu} = \frac{V(\phi) \partial_\mu \phi \partial_\nu \phi}{\sqrt{1 + g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi}} - g_{\mu\nu} V(\phi) \sqrt{1 + g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi}. \quad (127)$$

In a flat FRW background the energy density  $\rho$  and the pressure density  $p$  are given by

$$\rho = -T_0^0 = \frac{V(\phi)}{\sqrt{1 - \dot{\phi}^2}}, \quad (128)$$

$$p = T_i^i = -V(\phi) \sqrt{1 - \dot{\phi}^2}. \quad (129)$$

From Eqs. (9) and (11) we obtain the following equations of motion:

$$H^2 = \frac{8\pi G V(\phi)}{3\sqrt{1 - \dot{\phi}^2}}, \quad (130)$$

$$\frac{\ddot{\phi}}{1 - \dot{\phi}^2} + 3H\dot{\phi} + \frac{1}{V} \frac{dV}{d\phi} = 0. \quad (131)$$

Combining these equations gives

$$\frac{\ddot{a}}{a} = \frac{8\pi G V(\phi)}{3\sqrt{1 - \dot{\phi}^2}} \left( 1 - \frac{3}{2} \dot{\phi}^2 \right). \quad (132)$$

Hence an accelerated expansion occurs for  $\dot{\phi}^2 < 2/3$ .

The equation of state of the tachyon is given by

$$w_\phi = \frac{p}{\rho} = \dot{\phi}^2 - 1. \quad (133)$$

Now the tachyon dynamics is very different from the standard field case. Irrespective of the steepness of the tachyon potential, the equation of state varies between  $0$

<sup>2</sup> We note that these problems are alleviated in D-branes in a warped metric [172] or in the case of the geometrical tachyon [173].

and  $-1$ , in which case the tachyon energy density behaves as  $\rho \propto a^{-m}$  with  $0 < m < 3$  from Eq. (103).

One can express  $V(\phi)$  and  $\phi$  in terms of  $H$  and  $\dot{H}$ , as we did in the case of Quintessence<sup>3</sup>. From Eqs. (130) and (132) we find  $\dot{H}/H^2 = -(3/2)\dot{\phi}^2$ . Then together with Eq. (130) we obtain [174]

$$V = \frac{3H^2}{8\pi G} \left( 1 + \frac{2\dot{H}}{3H^2} \right)^{1/2}, \quad (134)$$

$$\phi = \int dt \left( -\frac{2\dot{H}}{3H^2} \right)^{1/2}. \quad (135)$$

Then the tachyon potential giving the power-law expansion,  $a \propto t^p$ , is

$$V(\phi) = \frac{2p}{4\pi G} \left( 1 - \frac{2}{3p} \right)^{1/2} \phi^{-2}. \quad (136)$$

In this case the evolution of the tachyon is given by  $\phi = \sqrt{2/3pt}$  (where we set an integration constant to zero). The above inverse square power-law potential corresponds to the one in the case of scaling solutions [177, 179], as we will see later. Tachyon potentials which are not steep compared to  $V(\phi) \propto \phi^{-2}$  lead to an accelerated expansion. In Sec. VI we will consider the cosmological evolution for a more general inverse power-law potential given by  $V(\phi) \propto \phi^{-n}$ . There have been a number of papers written concerning the cosmology of tachyons. A fairly comprehensive listing can be seen in Ref. [183].

#### D. Phantom (ghost) field

Recent observational data indicates that the equation of state parameter  $w$  lies in a narrow strip around  $w = -1$  and is quite consistent with being below this value [51, 80]. The scalar field models discussed in the previous subsections correspond to an equation of state  $w \geq -1$ . The region where the equation of state is less than  $-1$  is typically referred to as a being due to some form of phantom (ghost) dark energy. Specific models in braneworlds or Brans-Dicke scalar-tensor gravity can lead to phantom energy [184, 185]. Meanwhile the simplest explanation for the phantom dark energy is provided by a scalar field with a negative kinetic energy [37]. Such a field may be motivated from  $S$ -brane constructions in string theory [186].

Historically, phantom fields were first introduced in Hoyle's version of the steady state theory. In adherence to the perfect cosmological principle, a creation field (C-field) was introduced by Hoyle to reconcile the model

with the homogeneous density of the universe by the creation of new matter in the voids caused by the expansion of the universe [187]. It was further refined and reformulated in the Hoyle and Narlikar theory of gravitation [188] (see also Ref. [189] on a similar theme). The action of the phantom field minimally coupled to gravity is given by

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} (\nabla\phi)^2 - V(\phi) \right], \quad (137)$$

where the sign of the kinetic term is opposite compared to the action (94) for an ordinary scalar field. Since the energy density and pressure density are given by  $\rho = -\dot{\phi}^2/2 + V(\phi)$  and  $p = -\dot{\phi}^2/2 - V(\phi)$  respectively, the equation of state of the field is

$$w_\phi = \frac{p}{\rho} = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)}. \quad (138)$$

Then we obtain  $w_\phi < -1$  for  $\dot{\phi}^2/2 < V(\phi)$ .

As discussed in Sec. II the curvature of the universe grows toward infinity within a finite time in the universe dominated by a phantom fluid. In the case of a phantom scalar field this Big Rip singularity may be avoided if the potential has a maximum, e.g.,

$$V(\phi) = V_0 \left[ \cosh \left( \frac{\alpha\phi}{m_{\text{pl}}} \right) \right]^{-1}, \quad (139)$$

where  $\alpha$  is constant [84]. Due to its peculiar properties, the phantom field evolves towards the top of the potential and crosses over to the other side. It turns back to execute a period of damped oscillations about the maximum of the potential at  $\phi = 0$ . After a certain period of time the motion ceases and the field settles at the top of the potential to mimic the de-Sitter like behavior ( $w_\phi = -1$ ). This behavior is generic if the potential has a maximum, see e.g., Ref. [83]. In the case of exponential potentials the system approaches a constant equation of state with  $w_\phi < -1$  [190], as we will see in Sec. VI.

Although the above behavior of the phantom field is intriguing as a ‘‘classical cosmological’’ field, unfortunately phantom fields are generally plagued by severe Ultra-Violet (UV) quantum instabilities. Since the energy density of a phantom field is unbounded from below, the vacuum becomes unstable against the production of ghosts and normal (positive energy) fields [83]. Even when ghosts are decoupled from matter fields, they couple to gravitons which mediate vacuum decay processes of the type: vacuum  $\rightarrow 2$  ghosts +  $2\gamma$ . It was shown by Cline *et al.* [191] that we require an unnatural Lorenz invariance breaking term with cut off of order  $\sim$  MeV to prevent an overproduction of cosmic gamma rays. Hence the fundamental origin of the phantom field still poses an interesting challenge for theoreticians. See Refs. [192] for a selection of papers covering various cosmological aspects of phantom fields.

<sup>3</sup> Note that a ‘‘first-order formalism’’ which relates the potential to the Hubble parameter is given in Ref. [180]



### E. Dilatonic dark energy

We have already mentioned in the previous subsection that the phantom field with a negative kinetic term has a problem with quantum instabilities. Let us consider the stability of perturbations by decomposing the field  $\phi$  into a homogeneous part  $\phi_0$  and a fluctuation  $\delta\phi$ , as

$$\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x}). \quad (140)$$

Since we are concerned with the UV instability of the vacuum, it is not too restrictive to choose a Minkowski background metric when studying quantum fluctuations, because we are interested in high energy, short distance effects.

Let us start with a general Lagrangian density  $p(\phi, X)$ . Expanding  $p(X, \phi)$  to second order in  $\delta\phi$  it is straightforward to find the Lagrangian together with the Hamiltonian for the fluctuations. The Hamiltonian is given by [39]

$$\begin{aligned} \mathcal{H} = & (p_{,X} + 2Xp_{,XX}) \frac{(\delta\dot{\phi})^2}{2} \\ & + p_{,X} \frac{(\nabla\delta\phi)^2}{2} - p_{,\phi\phi} \frac{(\delta\phi)^2}{2}, \end{aligned} \quad (141)$$

where  $p_{,X} \equiv \partial p / \partial X$ . It is positive as long as the following conditions hold

$$\xi_1 \equiv p_{,X} + 2Xp_{,XX} \geq 0, \quad \xi_2 \equiv p_{,X} \geq 0, \quad (142)$$

$$\xi_3 \equiv -p_{,\phi\phi} \geq 0. \quad (143)$$

The speed of sound is given by

$$c_s^2 \equiv \frac{p_{,X}}{\rho_{,X}} = \frac{\xi_2}{\xi_1}, \quad (144)$$

which is often used when we discuss the stability of classical perturbations, since it appears as a coefficient of the  $k^2/a^2$  term ( $k$  is a comoving wavenumber). Although the classical fluctuations may be regarded to be stable when  $c_s^2 > 0$ , the stability of quantum fluctuations requires both  $\xi_1 > 0$  and  $\xi_2 \geq 0$ . We note that the instability prevented by the condition (143) is essentially an Infra-Red (IR) instability which is less dramatic compared to the instability associated with the violation of the condition (142). In fact this IR instability appears in the context of density perturbations generated in inflationary cosmology. Hence we shall adopt (142) but not (143) as the fundamental criteria for the consistency of the theory. These two conditions prevent an instability related to the presence of negative energy ghost states which render the vacuum unstable under a catastrophic production of ghosts and photons pairs [191]. This is essentially an Ultra-Violet instability with which the rate of production from the vacuum is simply proportional to the phase space integral on all possible final states.

In the case of a phantom scalar field  $\phi$  with a potential  $V(\phi)$ , i.e.,  $p = -X - V(\phi)$ , we find that  $\xi_1 = \xi_2 =$

$-1$ . Hence the system is quantum mechanically unstable even though the speed of sound is positive ( $c_s^2 > 0$ ). It was shown in Ref. [38] that a scalar field with a negative kinetic term does not necessarily lead to inconsistencies, provided that a suitable structure of higher-order kinetic terms are present in the effective theory. The simplest model that realizes this stability is  $p = -X + X^2$  [38]. In this case one has  $\xi_1 = -1 + 6X$  and  $\xi_2 = -1 + 2X$ . When  $\xi_1 > 0$  and  $\xi_2 \geq 0$ , corresponding to  $X \geq 1/2$ , the system is completely stable at the quantum level. In the region of  $0 \leq X < 1/6$  one has  $\xi_1 < 0$  and  $\xi_2 < 0$  so that the perturbations are classically stable due to the positive sign of  $c_s^2$ . This vacuum state is, however, generally quantum mechanically unstable.

It is difficult to apply the model  $p = -X + X^2$  for dark energy as it is. This is because the small energy density of the scalar field relative to the Planck density gives the condition  $|X| \gg X^2$ , in which case one can not ensure the stability of quantum fluctuations. Instead one may consider the following *dilatonic ghost condensate* model:

$$p = -X + ce^{\lambda\phi} X^2, \quad (145)$$

where  $c$  is a positive constant. This is motivated by dilatonic higher-order corrections to the tree-level action in low energy effective string theory [39]. We assume that the dilaton is effectively decoupled from gravity in the limit  $\phi \rightarrow \infty$ . This is the so-called the runaway dilaton scenario [193] in which the coupling functions in Eq. (114) are given by

$$B_g(\phi) = C_g + D_g e^{-\phi} + \mathcal{O}(e^{-2\phi}), \quad (146)$$

$$B_\phi^{(0)}(\phi) = C_\phi^{(0)} + D_\phi^{(0)} e^{-\phi} + \mathcal{O}(e^{-2\phi}). \quad (147)$$

In this case  $B_g(\phi)$  and  $B_\phi^{(0)}(\phi)$  approach constant values as  $\phi \rightarrow \infty$ . Hence the dilaton gradually decouples from gravity as the field evolves toward the region  $\phi \gg 1$  from the weakly coupled regime.

In the Einstein frame the function  $K(\phi)$  given by Eq. (116) also approaches a constant value, whose sign depends upon the coefficients of  $B_g(\phi)$  and  $B_\phi^{(0)}(\phi)$ . The dilatonic ghost condensate model corresponds to negative  $K(\phi)$ . From Eq. (117) we find that the coefficient in front of the  $(\nabla\phi)^4$  term has a dependence  $B_\phi^{(1)} \propto e^{\lambda\phi}$  in the dilatonic ghost condensate. Since the  $e^{\lambda\phi}$  term in Eq. (145) can be large for  $\phi \rightarrow \infty$ , the second term in Eq. (145) can stabilize the vacuum even if  $X$  is much smaller than the Planck scale. The condition for quantum stability is characterized by the condition  $ce^{\lambda\phi} X \geq 1/2$  from Eq. (142).

It is worth mentioning that the Lagrangian density (145) is transformed to Eq. (120) with  $f(\phi) \propto (\phi - \phi_0)^{-2}$  by a field redefinition. In subsection B we showed that this case has a scaling solution in the radiation or matter dominating era. This means that dilatonic ghost condensate model has scaling solutions. In Sec. VII we will show this in a more rigorous way and carry out a detailed

analysis in Sec. VI about the cosmological evolution for the Lagrangian density (145). The above discussion explicitly tells us that (dilaton) ghost condensate models fall into the category of K-essence.

Gasperini *et al.* proposed a runaway dilatonic quintessence scenario [193] in which  $K(\phi)$  approaches a positive constant as  $\phi \rightarrow \infty$ . They assumed the presence of an exponential potential  $V(\phi) = V_0 e^{-\lambda\phi}$  which vanishes for  $\phi \rightarrow \infty$ . The higher-order kinetic term  $X^2$  is neglected in their analysis. They took into account the coupling between the field  $\phi$  and dark matter, since the dilaton is naturally coupled to matter fields. This model is also an interesting attempt to explain the origin of dark energy using string theory.

## F. Chaplygin gas

So far we have discussed a number of scalar-field models of dark energy. There exist another interesting class of dark energy models involving a fluid known as a Chaplygin gas [29]. This fluid also leads to the acceleration of the universe at late times, and in its simplest form has the following specific equation of state:

$$p = -\frac{A}{\rho}, \quad (148)$$

where  $A$  is a positive constant. We recall that  $p = -V^2(\phi)/\rho$  for the tachyon from Eqs. (128) and (129). Hence the Chaplygin gas can be regarded as a special case of a tachyon with a constant potential.

The equation of state for the Chaplygin gas can be derived from the Nambu-Goto action for a D-brane moving in the  $D + 1$  dimensional bulk [194, 195]. For the case of the moving brane (via the Born-Infeld Lagrangian), the derivation of the Chaplygin gas equation of state was first discussed in the context of braneworld cosmologies in [196].

With the equation of state (148) the continuity equation (11) can be integrated to give

$$\rho = \sqrt{A + \frac{B}{a^6}}, \quad (149)$$

where  $B$  is a constant. Then we find the following asymptotic behavior:

$$\rho \sim \frac{\sqrt{B}}{a^3}, \quad a \ll (B/A)^{1/6}, \quad (150)$$

$$\rho \sim -p \sim \sqrt{A} \quad a \gg (B/A)^{1/6}. \quad (151)$$

This is the intriguing result for the Chaplygin gas. At early times when  $a$  is small, the gas behaves as a pressureless dust. Meanwhile it behaves as a cosmological constant at late times, thus leading to an accelerated expansion.

One can obtain a corresponding potential for the Chaplygin gas by treating it as an ordinary scalar field  $\phi$ . Using Eqs. (148) and (149) together with  $\rho = \dot{\phi}^2/2 + V(\phi)$

and  $p = \dot{\phi}^2/2 - V(\phi)$ , we find

$$\dot{\phi}^2 = \frac{B}{a^6 \sqrt{A + B/a^6}}, \quad (152)$$

$$V = \frac{1}{2} \left[ \sqrt{A + B/a^6} + \frac{A}{\sqrt{A + B/a^6}} \right]. \quad (153)$$

We note that this procedure is analogous to the reconstruction methods we adopted for the quintessence and tachyon potentials. Since the Hubble expansion rate is given by  $H = (8\pi\rho/3m_{\text{pl}}^2)^{1/2}$ , we can rewrite Eq. (152) in terms of the derivative of  $a$ :

$$\frac{\kappa}{\sqrt{3}} \frac{d\phi}{da} = \frac{\sqrt{B}}{a\sqrt{Aa^6 + B}}. \quad (154)$$

This is easily integrated to give

$$a^6 = \frac{4Be^{2\sqrt{3}\kappa\phi}}{A(1 - e^{2\sqrt{3}\kappa\phi})^2}. \quad (155)$$

Substituting this for Eq. (153) we obtain the following potential:

$$V(\phi) = \frac{\sqrt{A}}{2} \left( \cosh \sqrt{3}\kappa\phi + \frac{1}{\cosh \sqrt{3}\kappa\phi} \right). \quad (156)$$

Hence, a minimally coupled field with this potential is equivalent to the Chaplygin gas model.

Chaplygin gas provides an interesting possibility for the unification of dark energy and dark matter. However it was shown in Ref. [197] that the Chaplygin gas models are under strong observational pressure from CMB anisotropies (see also Ref. [30, 198]). This comes from the fact that the Jeans instability of perturbations in Chaplygin gas models behaves similarly to cold dark matter fluctuations in the dust-dominant stage given by (150) but disappears in the acceleration stage given by (151). The combined effect of the suppression of perturbations and the presence of a non-zero Jeans length gives rise to a strong integrated Sachs-Wolfe (ISW) effect, thereby leading to the loss of power in CMB anisotropies. This situation can be alleviated in the generalized Chaplygin gas model introduced in Ref. [31] with  $p = -A/\rho^\alpha$ ,  $0 < \alpha < 1$ . However, even in this case the parameter  $\alpha$  is rather severely constrained, i.e.,  $0 \leq \alpha < 0.2$  at the 95% confidence level [197]. For further details of the cosmology associated with generalized Chaplygin gas models, see Refs. [199].

## VI. COSMOLOGICAL DYNAMICS OF SCALAR FIELDS IN THE PRESENCE OF A BAROTROPIC PERFECT FLUID

In order to obtain viable dark energy models, we require that the energy density of the scalar field remains subdominant during the radiation and matter dominating eras, emerging only at late times to give rise to the

current observed acceleration of the universe. In this section we shall carry out cosmological dynamics of a scalar field  $\phi$  in the presence of a barotropic fluid whose equation of state is given by  $w_m = p_m/\rho_m$ . We denote pressure and energy densities of the scalar field as  $p_\phi$  and  $\rho_\phi$  with an equation of state  $w_\phi = p_\phi/\rho_\phi$ . Equations (9) and (10) give

$$H^2 = \frac{8\pi G}{3}(\rho_\phi + \rho_m), \quad (157)$$

$$\dot{H} = -4\pi G(\rho_\phi + p_\phi + \rho_m + p_m). \quad (158)$$

Here the energy densities  $\rho_\phi$  and  $\rho_m$  satisfy

$$\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = 0, \quad (159)$$

$$\dot{\rho}_m + 3H(1 + w_m)\rho_m = 0. \quad (160)$$

In what follows we shall assume that  $w_m$  is constant, which means that the fluid energy is given by  $\rho_m = \rho_0 a^{-3(1+w_m)}$ . Meanwhile  $w_\phi$  dynamically changes in general.

Of particular importance in the investigation of cosmological scenarios are those solutions in which the energy density of the scalar field mimics the background fluid energy density. Cosmological solutions which satisfy this condition are called “*scaling solutions*” [14] (see also Refs. [163, 200, 201, 202, 203, 204, 205, 206]). Namely scaling solutions are characterized by the relation

$$\rho_\phi/\rho_m = C, \quad (161)$$

where  $C$  is a *nonzero* constant. As we have already mentioned in the previous section, exponential potentials give rise to scaling solutions and so can play an important role in quintessence scenarios, allowing the field energy density to mimic the background being sub-dominant during radiation and matter dominating eras. In this case, as long as the scaling solution is the attractor, then for any generic initial conditions, the field would sooner or later enter the scaling regime, thereby opening up a new line of attack on the fine tuning problem of dark energy.

We note that the system needs to exit from the scaling regime characterized by Eq. (161) in order to give rise to an accelerated expansion. This is realized if the slope of the field potential becomes shallow at late times compared to the one corresponding to the scaling solution [160, 207]. We shall study these models in more details in Sec. VIII. It is worth mentioning that scaling solutions live on the border between acceleration and deceleration. Hence the energy density of the field catches up to that of the fluid provided that the potential is shallow relative to the one corresponding to the scaling solutions. In what follows we shall study the dynamics of scalar fields in great detail for a variety of dark energy models. First, we explain the property of an autonomous system before entering the detailed analysis.

## A. Autonomous system of scalar-field dark energy models

A dynamical system which plays an important role in cosmology belongs to the class of so called autonomous systems [14, 209]. We first briefly present some basic definitions related to dynamical systems (see also [210, 211] for a related approach). For simplicity we shall study the system of two first-order differential equations, but the analysis can be extended to a system of any number of equations. Let us consider the following coupled differential equations for two variables  $x(t)$  and  $y(t)$ :

$$\dot{x} = f(x, y, t), \quad \dot{y} = g(x, y, t), \quad (162)$$

where  $f$  and  $g$  are the functions in terms of  $x, y$  and  $t$ . The system (162) is said to be autonomous if  $f$  and  $g$  do not contain explicit time-dependent terms. The dynamics of the autonomous systems can be analyzed in the following way.

### 1. Fixed or critical points

A point  $(x_c, y_c)$  is said to be a *fixed point* or a *critical point* of the autonomous system if

$$(f, g)|_{(x_c, y_c)} = 0. \quad (163)$$

A critical point  $(x_c, y_c)$  is called an *attractor* when it satisfies the condition

$$(x(t), y(t)) \rightarrow (x_c, y_c) \text{ for } t \rightarrow \infty. \quad (164)$$

### 2. Stability around the fixed points

We can find whether the system approaches one of the critical points or not by studying the stability around the fixed points. Let us consider small perturbations  $\delta x$  and  $\delta y$  around the critical point  $(x_c, y_c)$ , i.e.,

$$x = x_c + \delta x, \quad y = y_c + \delta y. \quad (165)$$

Then substituting into Eqs. (162) leads to the first-order differential equations:

$$\frac{d}{dN} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}, \quad (166)$$

where  $N = \ln(a)$  is the number of  $e$ -foldings which is convenient to use for the dynamics of dark energy. The matrix  $\mathcal{M}$  depends upon  $x_c$  and  $y_c$ , and is given by

$$\mathcal{M} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x=x_c, y=y_c)}. \quad (167)$$

This possesses two eigenvalues  $\mu_1$  and  $\mu_2$ . The general solution for the evolution of linear perturbations can be

written as

$$\delta x = C_1 e^{\mu_1 N} + C_2 e^{\mu_2 N}, \quad (168)$$

$$\delta y = C_3 e^{\mu_1 N} + C_4 e^{\mu_2 N}, \quad (169)$$

where  $C_1, C_2, C_3, C_4$  are integration constants. Thus the stability around the fixed points depends upon the nature of the eigenvalues. One generally uses the following classification [14, 212]:

- (i) Stable node:  $\mu_1 < 0$  and  $\mu_2 < 0$ .
- (ii) Unstable node:  $\mu_1 > 0$  and  $\mu_2 > 0$ .
- (iii) Saddle point:  $\mu_1 < 0$  and  $\mu_2 > 0$  (or  $\mu_1 > 0$  and  $\mu_2 < 0$ ).
- (iv) Stable spiral: The determinant of the matrix  $\mathcal{M}$  is negative and the real parts of  $\mu_1$  and  $\mu_2$  are negative.

A fixed point is an attractor in the cases (i) and (iv), but it is not so in the cases (ii) and (iii).

## B. Quintessence

Let us consider a minimally coupled scalar field  $\phi$  with a potential  $V(\phi)$  whose Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi), \quad (170)$$

where  $\epsilon = +1$  for an ordinary scalar field. Here we also allow for the possibility of a phantom ( $\epsilon = -1$ ) as we see in the next subsection. For the above Lagrangian density (170), Eqs. (157), (158) and (159) read

$$H^2 = \frac{\kappa^2}{3} \left[ \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi) + \rho_m \right], \quad (171)$$

$$\dot{H} = -\frac{\kappa^2}{2} \left[ \epsilon\dot{\phi}^2 + (1 + w_m)\rho_m \right], \quad (172)$$

$$\epsilon\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \quad (173)$$

Let us introduce the following dimensionless quantities

$$\begin{aligned} x &\equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H}, & y &\equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \\ \lambda &\equiv -\frac{V_{,\phi}}{\kappa V}, & \Gamma &\equiv \frac{VV_{,\phi\phi}}{V_{,\phi}^2}, \end{aligned} \quad (174)$$

where  $V_{,\phi} \equiv dV/d\phi$ . Then the above equations can be

written in the following autonomous form [14, 203]:

$$\begin{aligned} \frac{dx}{dN} &= -3x + \frac{\sqrt{6}}{2}\epsilon\lambda y^2 \\ &+ \frac{3}{2}x [(1 - w_m)\epsilon x^2 + (1 + w_m)(1 - y^2)], \end{aligned} \quad (175)$$

$$\begin{aligned} \frac{dy}{dN} &= -\frac{\sqrt{6}}{2}\lambda xy \\ &+ \frac{3}{2}y [(1 - w_m)\epsilon x^2 + (1 + w_m)(1 - y^2)], \end{aligned} \quad (176)$$

$$\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2(\Gamma - 1)x, \quad (177)$$

together with a constraint equation

$$\epsilon x^2 + y^2 + \frac{\kappa^2 \rho_m}{3H^2} = 1. \quad (178)$$

The equation of state  $w_\phi$  and the fraction of the energy density  $\Omega_\phi$  for the field  $\phi$  is

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\epsilon x^2 - y^2}{\epsilon x^2 + y^2}, \quad (179)$$

$$\Omega_\phi \equiv \frac{\kappa^2 \rho_\phi}{3H^2} = \epsilon x^2 + y^2. \quad (180)$$

We also define the total effective equation of state:

$$\begin{aligned} w_{\text{eff}} &\equiv \frac{p_\phi + p_m}{\rho_\phi + \rho_m} \\ &= w_m + (1 - w_m)\epsilon x^2 - (1 + w_m)y^2. \end{aligned} \quad (181)$$

An accelerated expansion occurs for  $w_{\text{eff}} < -1/3$ . In this subsection we shall consider the case of quintessence ( $\epsilon = +1$ ). We define new variables  $\gamma_\phi$  and  $\gamma$  as  $\gamma_\phi \equiv 1 + w_\phi$  and  $\gamma \equiv 1 + w_m$ .

### 1. Constant $\lambda$

From Eq. (174) we find that the case of constant  $\lambda$  corresponds to an exponential potential [14, 203]:

$$V(\phi) = V_0 e^{-\kappa\lambda\phi}. \quad (182)$$

In this case Eq. (177) is trivially satisfied because  $\Gamma = 1$ . One can obtain the fixed points by setting  $dx/dN = 0$  and  $dy/dN = 0$  in Eqs. (175) and (176). We summarize the fixed points and their stabilities for quintessence ( $\epsilon = +1$ ) in TABLE I.

The eigenvalues of the matrix  $\mathcal{M}$  given in Eq. (166) are as follows.

- Point (a):

$$\mu_1 = -\frac{3}{2}(2 - \gamma), \quad \mu_2 = \frac{3}{2}\gamma. \quad (183)$$

- Point (b1):

$$\mu_1 = 3 - \frac{\sqrt{6}}{2}\lambda, \quad \mu_2 = 3(2 - \gamma). \quad (184)$$

Name	$x$	$y$	Existence	Stability	$\Omega_\phi$	$\gamma_\phi$
(a)	0	0	All $\lambda$ and $\gamma$	Saddle point for $0 < \gamma < 2$	0	–
(b1)	1	0	All $\lambda$ and $\gamma$	Unstable node for $\lambda < \sqrt{6}$ Saddle point for $\lambda > \sqrt{6}$	1	2
(b2)	-1	0	All $\lambda$ and $\gamma$	Unstable node for $\lambda > -\sqrt{6}$ Saddle point for $\lambda < -\sqrt{6}$	1	2
(c)	$\lambda/\sqrt{6}$	$[1 - \lambda^2/6]^{1/2}$	$\lambda^2 < 6$	Stable node for $\lambda^2 < 3\gamma$ Saddle point for $3\gamma < \lambda^2 < 6$	1	$\lambda^2/3$
(d)	$(3/2)^{1/2} \gamma/\lambda$	$[3(2 - \gamma)\gamma/2\lambda^2]^{1/2}$	$\lambda^2 > 3\gamma$	Stable node for $3\gamma < \lambda^2 < 24\gamma^2/(9\gamma - 2)$ Stable spiral for $\lambda^2 > 24\gamma^2/(9\gamma - 2)$	$3\gamma/\lambda^2$	$\gamma$

TABLE I: The properties of the critical points for the quintessence model (170) with  $\epsilon = +1$  for the exponential potential given by Eq. (182).

- Point (b2):

$$\mu_1 = 3 + \frac{\sqrt{6}}{2}\lambda, \quad \mu_2 = 3(2 - \gamma). \quad (185)$$

- Point (c):

$$\mu_1 = \frac{1}{2}(\lambda^2 - 6), \quad \mu_2 = \lambda^2 - 3\gamma. \quad (186)$$

- Point (d):

$$\mu_{1,2} = -\frac{3(2 - \gamma)}{4} \left[ 1 \pm \sqrt{1 - \frac{8\gamma(\lambda^2 - 3\gamma)}{\lambda^2(2 - \gamma)}} \right]. \quad (187)$$

In what follows we clarify the properties of the five fixed points given in TABLE I. Basically we are interested in a fluid with  $0 < \gamma < 2$ . The point (a) corresponds to a fluid dominated solution and is a saddle point since  $\mu_1 < 0$  and  $\mu_2 > 0$ . The points (b1) and (b2) are either an unstable node or a saddle point depending upon the value of  $\lambda$ . The point (c) is a stable node for  $\lambda^2 < 3\gamma$ , whereas it is a saddle point for  $3\gamma < \lambda^2 < 6$ . Since the effective equation of state is  $w_{\text{eff}} = w_\phi = -1 + \lambda^2/3$  from Eqs. (179) and (181), the universe accelerates for  $\lambda^2 < 2$  in this case. The point (d) corresponds to a scaling solution in which the energy density of the field  $\phi$  decreases proportionally to that of the barotropic fluid ( $\gamma_\phi = \gamma$ ). Since both  $\mu_1$  and  $\mu_2$  are negative for  $\lambda^2 > 3\gamma$  from Eq. (187), the point (d) is stable in this case. Meanwhile it is a saddle point for  $\lambda^2 < 3\gamma$ , but this case is not realistic because the condition,  $\Omega_\phi \leq 1$ , is not satisfied. We note that the point (d) becomes a stable spiral for  $\lambda^2 > 24\gamma^2/(9\gamma - 2)$ .

In Fig. 5 we show the phase plane plot for  $\lambda = 2$  and  $\gamma = 1$ . We note that the trajectories are confined inside the circle given by  $x^2 + y^2 = 1$  with  $y \geq 0$ . In this case the point (c) is a saddle point, whereas the point (d) is a stable spiral. Hence the late-time attractor is the scaling solution (d) with  $x = y = \sqrt{3}/8$ . This behavior is clearly seen in Fig. 5.

The above analysis of the critical points shows that one can obtain an accelerated expansion provided that

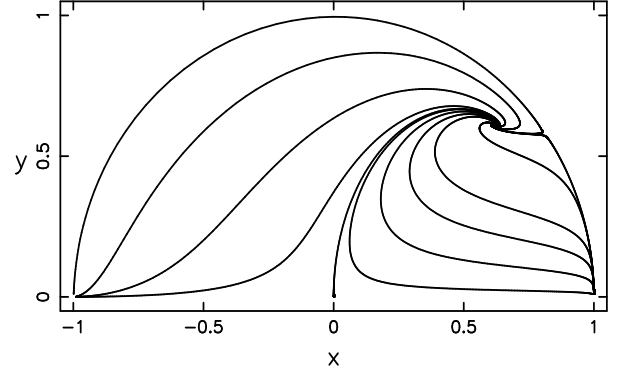


FIG. 5: The phase plane for  $\lambda = 2$  and  $\gamma = 1$ . The scalar field dominated solution (c) is a saddle point at  $x = (2/3)^{1/2}$  and  $y = (1/3)^{1/2}$ . Since the point (d) is a stable spiral in this case, the late-time attractor is the scaling solution with  $x = y = (3/8)^{1/2}$ . From Ref. [14].

the solutions approach the fixed point (c) with  $\lambda^2 < 2$ , in which case the final state of the universe is the scalar-field dominated one ( $\Omega_\phi = 1$ ). The scaling solution (d) is not viable to explain a late-time acceleration. However this can be used to provide the cosmological evolution in which the energy density of the scalar field decreases proportionally to that of the background fluid in either a radiation or matter dominated era. If the slope of the exponential potential becomes shallow enough to satisfy  $\lambda^2 < 2$  near to the present, the universe exits from the scaling regime and approaches the fixed point (c) giving rise to an accelerated expansion [160, 207]. This of course requires an effective  $\lambda$  which changes with time, and we turn to that case in the next subsection. However before we do that, we mention that in [208], the authors discuss the possibility that the field has not yet reached the fixed point, and argue that (i) even for  $2 < \lambda^2 < 3$ , there is a non-trivial region of parameter space that can explain the observed values of the cosmological parameters, such as the equation of state, and (ii) the fine tuning for these models, is no worse than in other quintessential scenarios.

## 2. Dynamically changing $\lambda$

Exponential potentials correspond to constant  $\lambda$  and  $\Gamma = 1$ . Let us consider a potential  $V(\phi)$  along which the field rolls down toward plus infinity with  $\dot{\phi} > 0$ . This means that  $x > 0$  in Eq. (177). Then if the condition,

$$\Gamma > 1, \quad (188)$$

is satisfied,  $\lambda$  decreases toward 0. Hence the slope of the potential defined by Eq. (174) becomes flat, thereby giving rise to an accelerated expansion at late times. The condition (188) is regarded as the tracking condition under which the energy density of  $\phi$  eventually catches up that of the fluid [15]. In order to construct viable quintessence models, we require that the potential should satisfy the condition (188). For example, one has  $\Gamma = (n + 1)/n > 1$  for the inverse power-law potential  $V(\phi) = V_0\phi^{-n}$  with  $n > 0$ . This means that tracking behaviour occurs for this potential.

When  $\Gamma < 1$  the quantity  $\lambda$  increases towards infinity. Since the potential is steeper than the one corresponding to scaling solutions, the energy density of the scalar field becomes negligible compared to that of the fluid. Then we do not have an accelerated expansion at late times.

In order to obtain dynamical evolution of the system we need to solve Eq. (177) together with Eqs. (175) and (176). Although  $\lambda$  is a dynamically changing quantity, one can apply the discussion of constant  $\lambda$  to this case as well by considering ‘‘instantaneous’’ critical points [201, 203]. For example, the point (c) in TABLE I dynamically changes with time, i.e.,  $x(N) = \lambda(N)/\sqrt{6}$  and  $y(N) = [1 - \lambda^2(N)/6]^{1/2}$ . When  $\Gamma > 1$  this point eventually approaches  $x(N) \rightarrow 0$  and  $y(N) \rightarrow 1$  with an equation of state of a cosmological constant ( $\gamma_\phi \rightarrow 0$ ) as  $\lambda(N) \rightarrow 0$ . See Refs. [201, 203] for more details.

### C. Phantom fields

The phantom field corresponds to a negative kinetic sign, i.e.,  $\epsilon = -1$  in Eq. (170). Let us first consider the exponential potential given by Eq. (182). In this case Eq. (177) is dropped from the dynamical system. In Table II we show fixed points for the phantom field. The points  $(x, y) = (\pm 1, 0)$  which exist in the case of quintessence disappear for the phantom field. The point (a) corresponds to a saddle point, since the eigenvalues of the matrix  $\mathcal{M}$  are the same as in the quintessence case.

The point (b) is a scalar-field dominated solution whose equation of state is given by

$$w_\phi = -1 - \lambda^2/3, \quad (189)$$

which is less than  $-1$ . The eigenvalues of the matrix  $\mathcal{M}$  are  $\mu_1 = -(\lambda^2 + 6)/2$  and  $\mu_2 = -\lambda^2 - 3\gamma$ , which are both negative for  $\gamma > 0$ . Hence the fixed point (b) is a stable node. The scaling solution (c) exists only for the

phantom fluid ( $\gamma < 0$ ). The eigenvalues of the matrix  $\mathcal{M}$  are

$$\mu_{1,2} = -\frac{3(2-\gamma)}{4} \left[ 1 \pm \sqrt{1 - \frac{8\gamma(\lambda^2 + 3\gamma)}{\lambda^2(2-\gamma)}} \right]. \quad (190)$$

When  $\gamma < 0$  the point (c) is a saddle point for  $\lambda^2 > -3\gamma$ .

In the presence of a non-relativistic dark matter ( $\gamma = 1$ ) the system approaches the scalar-field dominated solution (b). Exponential potentials give rise to *constant* equation of state  $w_\phi$  smaller than  $-1$  [190]. Then the universe reaches a Big Rip singularity at which the Hubble rate and the energy density of the universe diverge. We recall that the phantom field rolls *up* the potential hill, which leads to the increase of the energy density.

When the potential of the phantom field is no longer a simple exponential, the quantity  $\lambda$  can evolve in time. In this case the point (b) can be regarded as an instantaneous critical point. For example, in the case of the bell-type potential introduced in Eq. (139),  $\lambda$  decreases to zero as the field settles on the top of the potential. Hence the equation of state finally approaches  $w_\phi = -1$ .

### D. Tachyon fields

The energy density and the pressure density of a tachyon field are given by Eqs. (128) and (129), with the tachyon satisfying the equation of motion (131). In the presence of a barotropic fluid whose equation of state is  $\gamma \equiv 1 + w_m = 1 + p_m/\rho_m$ , Equations (157) and (158) give

$$H^2 = \frac{\kappa^2}{3} \left[ \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}} + \rho_m \right], \quad (191)$$

$$\dot{H} = -\frac{\kappa^2}{2} \left[ \frac{\dot{\phi}^2 V(\phi)}{\sqrt{1-\dot{\phi}^2}} + \gamma \rho_m \right]. \quad (192)$$

Let us define the following dimensionless quantities:

$$x = \dot{\phi}, \quad y = \frac{\kappa\sqrt{V(\phi)}}{\sqrt{3}H}. \quad (193)$$

Then we obtain the following autonomous equations [177, 179]

$$\frac{dx}{dN} = -(1-x^2)(3x - \sqrt{3}\lambda y), \quad (194)$$

$$\frac{dy}{dN} = \frac{y}{2} \left[ -\sqrt{3}\lambda xy - \frac{3(\gamma - x^2)y^2}{\sqrt{1-x^2}} + 3\gamma \right], \quad (195)$$

$$\frac{d\lambda}{dN} = -\sqrt{3}\lambda^2 xy (\Gamma - 3/2), \quad (196)$$

together with a constraint equation

$$\frac{y^2}{\sqrt{1-x^2}} + \frac{\kappa^2 \rho_m}{3H^2} = 1. \quad (197)$$

Name	$x$	$y$	Existence	Stability	$\Omega_\phi$	$\gamma_\phi$
(a)	0	0	No for $0 \leq \Omega_\phi \leq 1$	Saddle point	0	-
(b)	$-\lambda/\sqrt{6}$	$[1 + \lambda^2/6]^{1/2}$	All values	Stable node	1	$-\lambda^2/3$
(c)	$(3/2)^{1/2} \gamma/\lambda$	$[-3(2-\gamma)\gamma/2\lambda^2]^{1/2}$	$\gamma < 0$	Saddle point for $\lambda^2 > -3\gamma$	$\frac{-3\gamma}{\lambda^2}$	$\gamma$

TABLE II: The properties of the critical points for a phantom scalar field ( $\epsilon = -1$ ).

Here  $\lambda$  and  $\Gamma$  are defined by

$$\lambda \equiv -\frac{V_{,\phi}}{\kappa V^{3/2}}, \quad \Gamma \equiv \frac{VV_{,\phi\phi}}{V_{,\phi}^2}. \quad (198)$$

The equation of state and the fraction of the energy density in the tachyon field are given by

$$\gamma_\phi = x^2, \quad \Omega_\phi = \frac{y^2}{\sqrt{1-x^2}}. \quad (199)$$

Then the allowed range of  $x$  and  $y$  in a phase plane is  $0 \leq x^2 + y^4 \leq 1$  from the requirement:  $0 \leq \Omega_\phi \leq 1$ .

### 1. Constant $\lambda$

From Eq. (196) we find that  $\lambda$  is constant for  $\Gamma = 3/2$ . This case corresponds to an inverse square potential

$$V(\phi) = M^2 \phi^{-2}. \quad (200)$$

As we showed in the previous section, this potential gives a power-law expansion,  $a \propto t^p$  [see Eq. (136)]. The fixed points for this potential have been obtained in Refs. [177, 179], and are summarized in Table III. One can study the stability of the critical points by evaluating the eigenvalues of the matrix  $\mathcal{M}$ . We do not present all the eigenvalues in this review, but note that they are given in Refs. [177, 179].

The fixed points (a), (b1) and (b2) are not stable, so they are not a late-time attractor. The point (c) is a scalar-field dominated solution ( $\Omega_\phi = 1$ ) with eigenvalues  $\mu_1 = -3 + \lambda^2(\sqrt{\lambda^4 + 36} - \lambda^2)/12$  and  $\mu_2 = -3\gamma + \lambda^2(\sqrt{\lambda^4 + 36} - \lambda^2)/6$ . Hence this point is stable for

$$\gamma \geq \gamma_s \equiv \frac{\lambda^2}{18}(\sqrt{\lambda^4 + 36} - \lambda^2). \quad (201)$$

In TABLE III the quantity  $y_s$  is given by

$$y_s = \left( \frac{\sqrt{\lambda^4 + 36} - \lambda^2}{6} \right)^{1/2}. \quad (202)$$

Since  $\gamma_\phi = \lambda^2 y_s^2/3$  for the point (c), an accelerated expansion occurs for  $\lambda^2 y_s^2 < 2$ . This translates into the condition  $\lambda^2 < 2\sqrt{3}$  [212].

The point (d) is a scaling solution which exists only for  $\gamma < 1$ , since  $\Omega_\phi$  is given by  $\Omega_\phi = 3\gamma/\lambda^2 \sqrt{1-\gamma}$ . From the condition  $\Omega_\phi \leq 1$  we obtain

$$\gamma \leq \gamma_s = \frac{\lambda^2}{18}(\sqrt{\lambda^4 + 36} - \lambda^2). \quad (203)$$

The eigenvalues of the matrix  $\mathcal{M}$  are

$$\mu_{1,2} = \frac{3}{4} \left[ \gamma - 2 \pm \sqrt{17\gamma^2 - 20\gamma + 4 + \frac{48}{\lambda^2} \gamma^2 \sqrt{1-\gamma}} \right]. \quad (204)$$

The real parts of  $\mu_1$  and  $\mu_2$  are both negative when the condition (203) is satisfied. When the square root in Eq. (204) is positive, the fixed point is a stable node. The fixed point is a stable spiral when the square root in Eq. (204) is negative. In any case the scaling solution is always stable for  $\Omega_\phi < 1$ , but this is not a realistic solution in applying to dark energy because of the condition  $\gamma < 1$ .

The above discussion shows that the only viable late-time attractor is the scalar-field dominated solution (c). When the solution approaches the fixed point (c), the accelerated expansion occurs for  $\lambda^2 < 2\sqrt{3}$ . Since  $\lambda$  is given by  $\lambda = 2M_{\text{pl}}/M$ , the condition for an accelerated expansion gives an energy scale which is close to a Planck mass, i.e.,  $M \gtrsim 1.1M_{\text{pl}} \simeq 2.6 \times 10^{18}$  GeV. The mass scale  $M$  becomes smaller for the inverse power-law potential  $V(\phi) = M^{4-n}\phi^{-n}$ , as we will see below.

### 2. Dynamically changing $\lambda$

When the potential is different from the inverse square potential given by Eq. (200),  $\lambda$  is a dynamically changing quantity. As we have seen in the case of quintessence, there are basically two cases: (i)  $\lambda$  evolves toward zero, or (ii)  $|\lambda|$  increases toward infinity. The case (i) is regarded as the tracking solution in which the energy density of the tachyon eventually dominates over that of the fluid. This situation is realized when the following condition is satisfied [213]

$$\Gamma > 3/2, \quad (205)$$

which is derived from Eq. (196). When  $\Gamma < 3/2$  the energy density of the scalar field becomes negligible compared to that of the fluid.

As an example let us consider the inverse power-law potential

$$V(\phi) = M^{4-n}\phi^{-n}, \quad n > 0. \quad (206)$$

Since  $\Gamma = (n+1)/n$  in this case, the scalar-field energy density dominates at late-times for  $n < 2$ . The system approaches the ‘‘instantaneous’’ critical point (c) for  $\gamma \geq$

Name	$x$	$y$	Existence	Stability	$\Omega_\phi$	$\gamma_\phi$
(a)	0	0	All $\lambda$ and $\gamma$	Unstable saddle for $\gamma > 0$ Stable node for $\gamma = 0$	0	0
(b1)	1	0	All $\lambda$ and $\gamma$	Unstable node	1	1
(b2)	-1	0	All $\lambda$ and $\gamma$	Unstable node	1	1
(c)	$\lambda y_s/\sqrt{3}$	$y_s$	All $\lambda$ and $\gamma$	Stable node for $\gamma \geq \gamma_s$ Saddle for $\gamma < \gamma_s$	1	$\lambda^2 y_s^2/3$
(d1)	$\sqrt{\gamma}$	$\sqrt{3\gamma}/\lambda$	$\lambda > 0$ and $\gamma < \gamma_s$	Stable for $\Omega_\phi < 1$	$\frac{3\gamma}{\lambda^2} \frac{1}{\sqrt{1-\gamma}}$	$\gamma$
(d2)	$-\sqrt{\gamma}$	$-\sqrt{3\gamma}/\lambda$	$\lambda < 0$ and $\gamma < \gamma_s$	Stable for $\Omega_\phi < 1$	$\frac{3\gamma}{\lambda^2} \frac{1}{\sqrt{1-\gamma}}$	$\gamma$

TABLE III: The critical points for the inverse square potential (200) in the case of tachyon.  $\gamma_s$  is defined in Eq. (201).

1. In the limit  $\lambda \rightarrow 0$  one has  $x \rightarrow 0$  and  $y \rightarrow 1$  for the point (c), which means that slow-roll approximations can be used at late-times. The slow-roll parameter for the tachyon is given by [170].

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \simeq \frac{M_{\text{pl}}^2}{2} \left(\frac{V_\phi}{V}\right)^2 \frac{1}{V} = \frac{n^2}{2} \left(\frac{M_{\text{pl}}}{M}\right)^2 \frac{1}{(\phi M)^{2-n}}. \quad (207)$$

We find that  $\epsilon$  decreases for  $n < 2$  as the field evolves toward large values. The condition for the accelerated expansion corresponds to  $\epsilon < 1$ , which gives

$$\phi M > \left(\frac{n}{\sqrt{2}} \frac{M_{\text{pl}}}{M}\right)^{2/(2-n)}. \quad (208)$$

The present potential energy is approximated as  $\rho_c^{(0)} \simeq V(\phi_0) = M^4/(\phi_0 M)^n \simeq 10^{-47} \text{ GeV}^4$ . Combining this relation with Eq. (208) we get

$$\frac{M}{M_{\text{pl}}} > \left[ \left(\frac{\rho_c^{(0)}}{M_{\text{pl}}^4}\right)^{1-n/2} \left(\frac{n}{\sqrt{2}}\right)^n \right]^{1/(4-n)}. \quad (209)$$

While  $M$  is close to the Planck scale for  $n = 2$ , this problem is alleviated for smaller  $n$ . For example one has  $M/M_{\text{pl}} \gtrsim 10^{-20}$  for  $n = 1$ . We note that the solutions approach instantaneous critical points:  $(x_c, y_c) = (\lambda(N)y_s(N)/\sqrt{3}, y_s(N))$  with  $y_s(N) = [(\sqrt{\lambda(N)^4 + 36} - \lambda(N)^2)/6]^{1/2}$ . This behavior is clearly seen in the numerical simulations in Fig. 6. Thus the discussion of constant  $\lambda$  can be applied to the case of varying  $\lambda$  after the system approaches the stable attractor solutions.

There exists another tachyon potential in which the quantity  $\lambda$  decreases toward zero with oscillations [179]. One example is provided by the potential

$$V(\phi) = V_0 e^{\frac{1}{2}M^2\phi^2}, \quad (210)$$

which, for example, appears as an excitation of the massive state on the anti D-brane [172]. In this case the scalar field approaches the potential minimum at  $\phi = 0$  with oscillations, after which the field stabilizes there. Since the potential energy  $V_0$  remains at  $\phi = 0$ , this works as a cosmological constant at late-times.

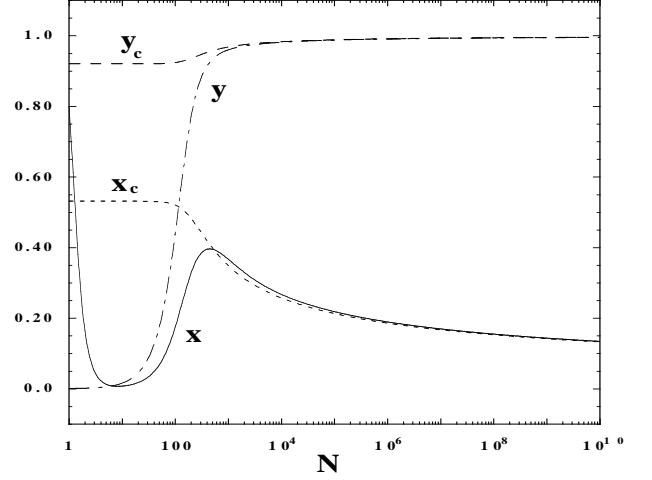


FIG. 6: Evolution of the parameters  $x$  and  $y$  together with the critical points  $x_c$  and  $y_c$  for tachyon with potential  $V(\phi) = M^3\phi^{-1}$  and a barotropic fluid with  $\gamma = 1$ . We choose initial conditions  $x_i = 0.8$ ,  $y_i = 5.0 \times 10^{-4}$  and  $\lambda_i = 1.0$ . The solution approaches instantaneous critical points whose asymptotic values are  $x_c = 0$  and  $y_c = 1$ . From Ref. [179].

There are a number of potentials which exhibit the behavior  $|\lambda| \rightarrow \infty$  asymptotically. For example  $V(\phi) = M^{4-n}\phi^{-n}$  with  $n > 2$  and  $V(\phi) = V_0 e^{-\mu\phi}$  with  $\mu > 0$ . In the latter case one has  $\Gamma = 1$  and  $d\lambda/dN = (\sqrt{3}/2)\lambda^2 xy$ , thereby leading to the growth of  $\lambda$  for  $x > 0$ . In the limit  $\lambda \rightarrow \infty$  the instantaneous critical point (c) approaches  $x_c(N) \rightarrow 1$  and  $y_c(N) \rightarrow 0$  with  $\gamma_\phi \rightarrow 1$ , which means the absence of an accelerated expansion. Although the accelerated expansion does not occur at late-times in this scenario, it is possible to have a temporal acceleration for  $\lambda \lesssim 1$  and have a deceleration for  $\lambda \gg 1$  [179]. If this temporal acceleration corresponds to the one at present, the universe will eventually enter the non-accelerating regime in which the tachyon field behaves as a pressureless dust.



### E. Dilatonic ghost condensate

Let us consider the dilatonic ghost condensate model given by Eq. (145). In this case the pressure density and the energy density of the field are given by  $p_\phi = p = -X + ce^{\lambda\phi}X^2$  and  $\rho_\phi = 2X\partial p_\phi/\partial X - p_\phi = -X + 3ce^{\lambda\phi}X^2$  with  $X = \dot{\phi}^2/2$ . Then Eqs. (157), (158) and (159) read

$$3H^2 = -\frac{1}{2}\dot{\phi}^2 + \frac{3}{4}ce^{\lambda\phi}\dot{\phi}^4 + \rho_m, \quad (211)$$

$$2\dot{H} = \dot{\phi}^2 - ce^{\lambda\phi}\dot{\phi}^4 - (1 + w_m)\rho_m, \quad (212)$$

$$\ddot{\phi}(3ce^{\lambda\phi}\dot{\phi}^2 - 1) + 3H\dot{\phi}(ce^{\lambda\phi}\dot{\phi}^2 - 1) + \frac{3}{4}c\lambda e^{\lambda\phi}\dot{\phi}^4 = 0, \quad (213)$$

where we set  $\kappa^2 = 1$ .

Introducing the following quantities

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}H}, \quad y \equiv \frac{e^{-\lambda\phi/2}}{\sqrt{3}H}, \quad (214)$$

the above equations can be written in an autonomous form

$$\frac{dx}{dN} = \frac{3}{2}x [1 + w_m + (1 - w_m)x^2(-1 + cY) - 2cw_mx^2Y] + \frac{1}{1 - 6cY} \left[ 3(-1 + 2cY)x + \frac{3\sqrt{6}}{2}\lambda cx^2Y \right], \quad (215)$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[1 + w_m + (1 - w_m)x^2(-1 + cY) - 2cw_mx^2Y], \quad (216)$$

where

$$Y \equiv \frac{x^2}{y^2} = Xe^{\lambda\phi}. \quad (217)$$

The equation of state and the fraction of the energy density for the field can now be written as

$$w_\phi = \frac{1 - cY}{1 - 3cY}, \quad (218)$$

$$\Omega_\phi = -x^2 + 3c\frac{x^4}{y^2}. \quad (219)$$

In Table IV we present the fixed points for the system of the dilatonic ghost condensate. The point (a) is not realistic, since we require a phantom fluid ( $w_m \leq -1$ ) to satisfy  $0 \leq \Omega_\phi \leq 1$ .

The points (b) and (c) correspond to the dark-energy dominated universe with  $\Omega_\phi = 1$ . The functions  $f_\pm(\lambda)$  are defined by

$$f_\pm(\lambda) \equiv 1 \pm \sqrt{1 + 16/(3\lambda^2)}. \quad (220)$$

The condition (142) for the stability of quantum fluctuations corresponds to  $cY \geq 1/2$ . From Eq. (218) one has  $w_\phi < -1$  for  $cY < 1/2$  and  $w_\phi > -1$  otherwise. The parameter range of  $Y$  for the point (b) is  $1/3 < cY < 1/2$ ,

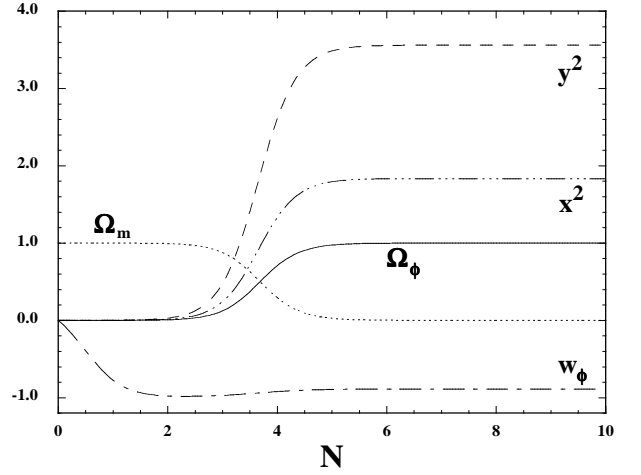


FIG. 7: Evolution of  $\Omega_\phi$ ,  $\Omega_m$ ,  $w_\phi$ ,  $x^2$  and  $y^2$  for  $c = 1$ ,  $w_m = 0$  and  $\lambda = 0.1$  with initial conditions  $x_i = 0.0085$  and  $y_i = 0.0085$ . The solution approaches the scalar-field dominated fixed point (c) with  $x^2 \simeq 1.834$ ,  $y^2 = 3.561$  and  $Y \simeq 0.515$ ,  $\Omega_\phi = 1$  and  $\Omega_m = 0$ . The equation of state in the attractor regime is given by  $w_\phi = -0.889$ . From Ref. [39].

which means that the field  $\phi$  behaves as a phantom. The point (c) belongs to the parameter range given by  $1/2 < cY < \infty$ , which means that the stability of quantum fluctuations is ensured. An accelerated expansion occurs for  $w_\phi < -1/3$ , i.e.,  $cY < 2/3$ . This corresponds to the condition  $\lambda^2 f_+(\lambda) < 8/3$ , i.e.,  $\lambda < \sqrt{6}/3$ . In the limit  $\lambda \rightarrow 0$  we have  $cY \rightarrow 1/2$ ,  $\Omega_\phi \rightarrow 1$  and  $w_\phi \rightarrow -1$  for both points (b) and (c). The  $\lambda = 0$  case is the original ghost condensate scenario proposed in Ref. [38], i.e.,  $p = -X + X^2$ . The point (d) corresponds to a scaling solution characterized by  $w_\phi = w_m$ , in which case we do not have an accelerated expansion unless  $w_m < -1/3$ .

We shall study the stability of the fixed points in the case of non-relativistic dark matter ( $w_m = 0$ ) with  $c = 1$ . Numerically the eigenvalues of the matrix  $\mathcal{M}$  were evaluated in Ref. [212] and it was shown that the determinant of the matrix  $\mathcal{M}$  for the point (b) is negative with negative real parts of  $\mu_1$  and  $\mu_2$ . Hence the phantom fixed point (b) is a stable spiral. The point (c) is a stable node for  $0 < \lambda < \sqrt{3}$ , whereas it is a saddle point for  $\lambda > \sqrt{3}$ . This critical value  $\lambda_* = \sqrt{3}$  is computed by setting the determinant of  $\mathcal{M}$  to be zero. The point (d) is physically meaningful for  $\lambda > \sqrt{3}$  because of the condition  $\Omega_\phi < 1$ , and it is a stable node [212]. Hence the point (d) is stable when the point (c) is unstable and vice versa. It was shown in Ref. [214] that this property holds for all scalar-field models which possess scaling solutions. We recall that the point (b) is not stable at the quantum level. The above discussion shows that the only viable attractor which satisfies the conditions of an accelerated expansion and the quantum stability is the point (c).

In Fig. 7 we plot the variation of  $\Omega_\phi$ ,  $\Omega_m$ ,  $w_\phi$ ,  $x^2$  and  $y^2$  for  $c = 1$ ,  $w_m = 0$  and  $\lambda = 0.1$ . We find that analytic values of the attractor points agree very well with our

Name	$x$	$cY$	$\Omega_\phi$	$w_\phi$
(a)	0	$\infty$	$\frac{3(w_m+1)}{3w_m-1}$	1/3
(b)	$-\frac{\sqrt{6\lambda}f_+(\lambda)}{4}$	$\frac{1}{2} + \frac{\lambda^2 f_-(\lambda)}{16}$	1	$\frac{-8+\lambda^2 f_-(\lambda)}{8+3\lambda^2 f_-(\lambda)}$
(c)	$-\frac{\sqrt{6\lambda}f_-(\lambda)}{4}$	$\frac{1}{2} + \frac{\lambda^2 f_+(\lambda)}{16}$	1	$\frac{-8+\lambda^2 f_+(\lambda)}{8+3\lambda^2 f_+(\lambda)}$
(d)	$\frac{\sqrt{6(1+w_m)}}{2\lambda}$	$\frac{1-w_m}{1-3w_m}$	$\frac{3(1+w_m)^2}{\lambda^2(1-3w_m)}$	$w_m$

TABLE IV: The critical points for the dilatonic ghost condensate model given by (145). Here  $Y$  and  $f_\pm(\lambda)$  are defined in Eqs. (217) and (220).

numerical results.

## VII. SCALING SOLUTIONS IN A GENERAL COSMOLOGICAL BACKGROUND

In the previous section we have seen that there exist scaling solutions in certain classes of dark energy models. It is convenient to know the existence of scaling solutions, since they give the border of acceleration and deceleration. This allows the field energy density to mimic the background whilst remaining sub-dominant during both the radiation and matter eras. Although one does not have an acceleration of the universe at late-times in this case, it is possible to obtain an accelerated expansion if a field  $\phi$  (dark energy) is coupled to a background fluid (dark matter) [215] (see also Ref. [216]). In this section we implement the coupling  $Q$  between the field and the barotropic fluid and derive a general form of the Lagrangian [39] for the existence of scaling solutions. We note that this includes uncoupled dark energy scenarios discussed in the previous section by taking the limit  $Q \rightarrow 0$ .

The existence of scaling solutions has been extensively studied in a number of cosmological scenarios—including standard General Relativity (GR), braneworlds [Randall-Sundrum (RS) and Gauss-Bonnet (GB)], tachyon and Cardassian scenarios [221]-[228, 229]. In what follows we present a unified framework to investigate scaling solutions in a general cosmological background characterized by  $H^2 \propto \rho_T^n$ , where  $\rho_T$  is the total energy density. The GR, RS, GB and Cardassian cases correspond to  $n = 1$ ,  $n = 2$ ,  $n = 2/3$  and  $n = 1/3$ , respectively. Our formalism provides a very generic method to study these solutions for all known scalar-field dark energy models [231].

### A. General Lagrangian for the existence of scaling solution

We start with the following general 4-dimensional action

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{pl}}^2}{2} R + p(X, \varphi) \right] + S_m(\varphi), \quad (221)$$

where  $X = -g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi / 2$  is a kinetic term of a scalar field  $\varphi$ .  $S_m$  is an action for a matter fluid which is generally dependent on  $\varphi$ . In what follows we set the reduced Planck mass  $M_{\text{pl}}$  to be unity.

Let us consider the following effective Friedmann equation in a flat FRW background:

$$H^2 = \beta_n^2 \rho_T^n, \quad (222)$$

where  $\beta_n$  and  $n$  are constants, and  $\rho_T$  is a total energy density of the universe. We note that a more general analysis can also be undertaken for the case where  $H^2 \propto L^2(\rho_T)$ , where  $L(\rho_T)$  is a general function of  $\rho_T$  [232]. We consider a cosmological scenario in which the universe is filled by the scalar field  $\varphi$  with equation of state  $w_\varphi = p_\varphi/\rho_\varphi$  and by one type of barotropic perfect fluid with equation of state  $w_m = p_m/\rho_m$ . Here the pressure density and the energy density of the field are given by  $p_\varphi = p$  and  $\rho_\varphi = 2X\partial p_\varphi/\partial X - p_\varphi$ .

We introduce a scalar charge  $\sigma$  corresponding to the coupling between the field  $\varphi$  and matter, which is defined by the relation  $\sigma = -(1/\sqrt{-g})\delta S_m/\delta\varphi$ . Then the continuity equation for the field  $\varphi$  is given by

$$\frac{d\rho_\varphi}{dN} + 3(1 + w_\varphi)\rho_\varphi = -Q\rho_m \frac{d\varphi}{dN}, \quad (223)$$

where  $N \equiv \ln a$  and  $Q \equiv \sigma/\rho_m$ . The energy density  $\rho_m$  of the fluid satisfies

$$\frac{d\rho_m}{dN} + 3(1 + w_m)\rho_m = Q\rho_m \frac{d\varphi}{dN}. \quad (224)$$

We define the fractional densities of  $\rho_\varphi$  and  $\rho_m$  as

$$\Omega_\varphi \equiv \frac{\rho_\varphi}{(H/\beta_n)^{2/n}}, \quad \Omega_m \equiv \frac{\rho_m}{(H/\beta_n)^{2/n}}, \quad (225)$$

which satisfy  $\Omega_\varphi + \Omega_m = 1$  from Eq. (222).

We are interested in asymptotic scaling solutions which satisfy the relation (161), in which case the fractional density  $\Omega_\varphi$  is constant. We also assume that  $w_\varphi$  and  $Q$  are constants in the scaling regime. Since Eq. (161) is equivalent to the condition  $d \log \rho_\varphi / dN = d \log \rho_m / dN$ , we obtain the following relation from Eqs. (223) and (224):

$$\frac{d\varphi}{dN} = \frac{3\Omega_\varphi}{Q}(w_m - w_\varphi) = \text{const.} \quad (226)$$

Then this gives the scaling behavior of  $\rho_\phi$  and  $\rho_m$ :

$$\frac{d\ln\rho_\phi}{dN} = \frac{d\ln\rho_m}{dN} = -3(1 + w_{\text{eff}}), \quad (227)$$

where the effective equation of state is

$$w_{\text{eff}} \equiv \frac{w_\phi\rho_\phi + w_m\rho_m}{\rho_\phi + \rho_m} = w_m + \Omega_\phi(w_\phi - w_m). \quad (228)$$

This expression of  $w_{\text{eff}}$  is valid irrespective of the fact that scaling solutions exist or not. The condition for an accelerated expansion corresponds to  $w_{\text{eff}} < -1/3$ .

From the definition of  $X$  we obtain

$$2X = H^2 \left( \frac{d\varphi}{dN} \right)^2 \propto H^2 \propto \rho_T^n. \quad (229)$$

This means that the scaling property of  $X$  is the same as  $\rho_\phi^n$  and  $\rho_m^n$ . Then we find

$$\frac{d\ln X}{dN} = -3n(1 + w_{\text{eff}}). \quad (230)$$

Since  $p_\phi = w_\phi\rho_\phi$  scales in the same way as  $\rho_\phi$ , one has  $d\ln p_\phi/dN = -3(1 + w_{\text{eff}})$ . Hence we obtain the following relation by using Eqs. (226) and (230):

$$n \frac{\partial \ln p_\phi}{\partial \ln X} - \frac{1}{\lambda} \frac{\partial \ln p_\phi}{\partial \varphi} = 1, \quad (231)$$

where

$$\lambda \equiv Q \frac{1 + w_m - \Omega_\phi(w_m - w_\phi)}{\Omega_\phi(w_m - w_\phi)}. \quad (232)$$

Equation (231) gives a constraint on the functional form of  $p(X, \varphi)$  for the existence of scaling solutions:

$$p(X, \varphi) = X^{1/n} g(Xe^{n\lambda\varphi}), \quad (233)$$

where  $g$  is any function in terms of  $Y \equiv Xe^{n\lambda\varphi}$ . This expression was first derived in the GR case ( $n = 1$ ) in Ref. [39] and was extended to the case of general  $n$  in Ref. [231]. One can easily show that  $Y$  is constant along the scaling solution:

$$Xe^{n\lambda\varphi} = Y_0 = \text{const}. \quad (234)$$

This property tells us that  $p$  is proportional to  $X^{1/n}$  by Eq. (233). This could be a defining property of scaling solutions which means that the Lagrangian or the pressure density depends upon the kinetic energy alone in the scaling regime. For an ordinary scalar field it leads to a constant ratio of the kinetic to potential energy which is often taken to be a definition of scaling solutions.

In deriving Eq. (233) we assumed that the coupling  $Q$  is a constant in the scaling regime. One can also obtain a scaling Lagrangian even when the coupling is a free function of the field  $\varphi$ , see Ref. [233]. It was also shown that we get the Lagrangian (233) by appropriate field redefinitions. This means that one can always work with the Lagrangian (233), no matter what kind of coupling one has in mind.

## B. General properties of scaling solutions

Combining Eq. (228) with Eq. (232) we find that the effective equation of state for scaling solutions is given by

$$w_{\text{eff}} = \frac{w_m\lambda - Q}{Q + \lambda}. \quad (235)$$

This property holds irrespective of the form of the function  $g(Y)$ . In the case of nonrelativistic dark matter ( $w_m = 0$ ) we have  $w_{\text{eff}} = 0$  for  $Q = 0$  and  $w_{\text{eff}} \rightarrow -1$  in the limit  $Q \gg \lambda > 0$ .

From the pressure density (233) we obtain the energy density  $\rho_\phi$  as  $\rho_\phi = X^{1/n}(2/n - 1 + 2Yg'/g)g$ , where a prime denotes a derivative in terms of  $Y$ . Then the equation of state  $w_\phi = p_\phi/\rho_\phi$  reads

$$w_\phi = \left( \frac{2}{n} - 1 + 2\alpha \right)^{-1}, \quad (236)$$

where

$$\alpha \equiv \frac{d \log g(Y)}{d \log Y} \Big|_{Y=Y_0}. \quad (237)$$

Using Eqs. (226), (229) and (232), we obtain the following relation for the scaling solutions:

$$3H^2 = \frac{2(Q + \lambda)^2}{3(1 + w_m)^2} X. \quad (238)$$

Then the fractional density (225) of the field  $\phi$  is given by

$$\Omega_\phi = \left[ \frac{9\beta_n^2(1 + w_m)^2}{2(Q + \lambda)^2} \right]^{1/n} \frac{g(Y_0)}{w_\phi}. \quad (239)$$

By combining Eq. (226) with Eq. (225) together with the relation  $w_\phi = p_\phi/\rho_\phi$ , we find that  $g$  in Eq. (233) can be written as

$$g(Y_0) = -Q \left( \frac{2}{9\beta_n^2} \right)^{1/n} \frac{w_\phi}{w_\phi - w_m} \left( \frac{1 + w_m}{Q + \lambda} \right)^{(n-2)/n}. \quad (240)$$

Then Eq. (239) yields

$$\Omega_\phi = \frac{Q}{Q + \lambda} \frac{1 + w_m}{w_m - w_\phi}. \quad (241)$$

Once the functional form of  $g(Y)$  is known, the equation of state  $w_\phi$  is determined by Eq. (236) with Eq. (237). We can then derive the fractional density  $\Omega_\phi$  from Eq. (241).

For scaling solutions we can define the acceleration parameter by

$$-q \equiv \frac{\ddot{a}a}{\dot{a}^2} = 1 - \frac{3n(1 + w_m)\lambda}{2(\lambda + Q)}. \quad (242)$$

When  $Q = 0$  the condition  $-q > 0$  corresponds to  $w_m < 2/(3n) - 1$ . For example  $w_m < -1/3$  for  $n = 1$ . In the case of non-relativistic dark matter ( $w_m = 0$ ), an accelerated expansion occurs only for  $n < 2/3$  (see for example [228], for the case of Cardassian cosmology, and [230] for a discussion of a class of Cardassian scenarios in terms of dynamical systems). If we account for the coupling  $Q$ , it is possible to get an acceleration even for  $n \geq 2/3$ . The condition for acceleration is then

$$\frac{Q}{\lambda} > \frac{3n(1 + w_m) - 2}{2}. \quad (243)$$

One has  $Q/\lambda > 1/2$  for  $w_m = 0$  and  $n = 1$ . We shall review coupled dark energy scenarios in detail in Sec. IX.

### C. Effective potential corresponding to scaling solutions

By using the results obtained in previous subsections we can obtain the effective potentials corresponding to scaling solutions.

#### 1. Ordinary scalar fields

We first study the case in which the Lagrangian density  $p$  is written in the form:

$$p(X, \varphi) = f(X) - V(\varphi). \quad (244)$$

By using Eq. (231) we find that the functions  $f(X)$  and  $V(\varphi)$  satisfy

$$nX \frac{df}{dX} - f(X) = -\frac{1}{\lambda} \frac{dV}{d\varphi} - V \equiv C, \quad (245)$$

where  $C$  is a constant. Hence we obtain  $f = c_1 X^{1/n} - C$  and  $V = c_2 e^{-\lambda\varphi} - C$  with  $c_1$  and  $c_2$  being constants. Then the Lagrangian density is given by

$$p = c_1 X^{1/n} - c_2 e^{-\lambda\varphi}. \quad (246)$$

This shows that when  $n = 1$  (GR) an exponential potential corresponds to the one for scaling solutions. In other cases ( $n \neq 1$ ) the Lagrangian density (246) does not have a standard kinetic term, but one can perform a transformation so that the kinetic term becomes a canonical one. By introducing a new variable  $\phi \equiv e^{\beta\lambda\varphi}$ , we find  $Y_0 = \tilde{X} \phi^{(n-2\beta)/\beta} / \beta^2 \lambda^2 = \text{const}$ , where  $\tilde{X} \equiv -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2$ . Hence the Lagrangian density (233) can be rewritten as

$$p = \frac{Y_0^{1/n}}{\phi^{1/\beta}} g(Y_0) = Y_0^{1/n} \left( \frac{\tilde{X}}{\beta^2 \lambda^2 Y_0} \right)^{1/(n-2\beta)} g(Y_0). \quad (247)$$

Since  $p$  is proportional to  $\tilde{X}^{1/(n-2\beta)}$ , the transformation that gives  $p \propto \tilde{X}$  corresponds to  $\beta = (n-1)/2$ ,

i.e.,  $\phi = e^{(n-1)\lambda\varphi/2}$ . Then we have  $p \propto \phi^{-2/(n-1)}$  from Eq. (247), which means that the potential of the field  $\phi$  corresponding to scaling solutions is

$$V(\phi) = V_0 \phi^{-2/(n-1)}, \quad (248)$$

where  $V_0$  is constant. In the case of the RS braneworld ( $n = 2$ ) one obtains an inverse square potential  $V(\phi) = V_0 \phi^{-2}$  [223]. The Gauss-Bonnet braneworld ( $n = 2/3$ ) gives the potential  $V(\phi) = V_0 \phi^6$ , as shown in Ref. [227]. The Cardassian cosmology ( $n = 1/3$ ) corresponds to the potential  $V(\phi) = V_0 \phi^3$ .

#### 2. Tachyon

At first glance the tachyon Lagrangian (129) does not seem to satisfy the condition for the existence of scaling solutions given in Eq. (233). However we can rewrite the Lagrangian (233) by introducing a new field  $\phi = e^{\beta\lambda\varphi}/(\beta\lambda)$ . Since the quantity  $Y$  is written as  $Y = \tilde{X} (\beta\lambda\phi)^{n/\beta-2}$  with  $\tilde{X} \equiv -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2$ , one has  $Y = \tilde{X}$  for  $\beta = n/2$ . Hence the Lagrangian density (233) yields

$$p = \left( \frac{n\lambda\phi}{2} \right)^{-2/n} \tilde{X}^{1/n} g(\tilde{X}), \quad (249)$$

which corresponds to a system  $p(\tilde{X}, \phi) = V(\phi) f(\tilde{X})$  with potential

$$V(\phi) = V_0 \phi^{-2/n}, \quad (250)$$

and  $f(\tilde{X}) = \tilde{X}^{1/n} g(\tilde{X})$ . We note that the tachyon Lagrangian density (129) is obtained by choosing

$$g(Y) = -cY^{-1/n} \sqrt{1 - 2Y}. \quad (251)$$

When  $n = 1$  (GR), Eq. (250) gives the inverse square potential  $V(\phi) = V_0 \phi^{-2}$ . We have earlier studied the dynamics of this system in Sec. VI. We also have  $V(\phi) = V_0 \phi^{-1}$  for  $n = 2$  (RS),  $V(\phi) = V_0 \phi^{-3}$  for  $n = 2/3$  (GB), and  $V(\phi) = V_0 \phi^{-6}$  for  $n = 1/3$  (Cardassian cosmology).

#### 3. Dilatonic ghost condensate

The dilatonic ghost condensate model (145) does not have a potential. Let us consider the GR case ( $n = 1$ ) in this model. The Lagrangian density (145) is derived by choosing  $g(Y) = -1 + cY$  in Eq. (233). Then by using the relations obtained in subsections A and B, we find

$$cY_0 = -\frac{2Q(Q + \lambda) - 3(1 - w_m^2)}{3(1 + w_m)(1 - 3w_m)}, \quad (252)$$

and

$$w_\varphi = \frac{-3(1 + w_m)w_m + Q(Q + \lambda)}{-3(1 + w_m) + 3Q(Q + \lambda)}, \quad (253)$$

$$\Omega_\varphi = \frac{3(1 + w_m)[1 + w_m - Q(Q + \lambda)]}{(Q + \lambda)^2(1 - 3w_m)}. \quad (254)$$

The condition for an accelerated expansion (243) gives  $Q/\lambda > (1 + 3w_m)/2$  for  $n = 1$ . The stability of quantum fluctuations discussed in Sec. VI requires  $cY_0 \geq 1/2$ , which translates into the condition  $Q(Q + \lambda) \leq 3(1 + w_m)^2/4$ . One can obtain viable scaling solutions if the coupling  $Q$  satisfies both conditions.

#### D. Autonomous system in Einstein gravity

In this subsection we shall derive autonomous equations for the Lagrangian density (233) with  $n = 1$  (GR), i.e.,  $p = Xg(Xe^{\lambda\varphi})$ . In this case the energy density of the field  $\varphi$  is given by  $\rho_\varphi = p(1 + 2Yg'/g)$ . We introduce two quantities  $x \equiv \dot{\varphi}/(\sqrt{6}H)$  and  $y \equiv e^{-\lambda\varphi/2}/(\sqrt{3}H)$ . Using Eqs. (222), (223) and (224), we obtain the following autonomous equations [212, 214, 234]

$$\frac{dx}{dN} = \frac{3x}{2} \left[ 1 + gx^2 - w_m(\Omega_\varphi - 1) - \frac{\sqrt{6}}{3}\lambda x \right] + \frac{\sqrt{6}A}{2} \left[ (Q + \lambda)\Omega_\varphi - Q - \sqrt{6}(g + Yg')x \right] \quad (255)$$

$$\frac{dy}{dN} = \frac{3y}{2} \left[ 1 + gx^2 - w_m(\Omega_\varphi - 1) - \frac{\sqrt{6}}{3}\lambda x \right], \quad (256)$$

where  $A \equiv (g + 5Yg' + 2Y^2g'')^{-1}$ . We note that  $\Omega_\varphi$  and  $w_\varphi$  are given by

$$\Omega_\varphi = x^2(g + 2Yg'), \quad w_\varphi = \frac{g}{g + 2Yg'}. \quad (257)$$

Since  $\partial p/\partial X = g + Yg'$ , we find

$$w_\varphi = -1 + \frac{2x^2}{\Omega_\varphi} \frac{\partial p}{\partial X}. \quad (258)$$

Eq. (258) shows that the field behaves as a phantom ( $w_\varphi < -1$ ) for  $\partial p/\partial X < 0$ .

Equation (238) means that there exists the following scaling solution for *any* form of the function  $g(Y)$ :

$$x = \frac{\sqrt{6}(1 + w_m)}{2(Q + \lambda)}. \quad (259)$$

In fact it is straightforward to show that this is one of the critical points for the autonomous system given by Eqs. (255) and (256). We recall that the effective equation of state  $w_{\text{eff}}$  is also independent of  $g(Y)$ , see Eq. (235). While  $x$  and  $w_{\text{eff}}$  do not depend on the form of  $g(Y)$ ,  $\Omega_\varphi$  and  $w_\varphi$  remain undetermined unless we specify the Lagrangian.

### VIII. THE DETAILS OF QUINTESSENCE

In this section we shall discuss various aspects of quintessence such as the nucleosynthesis constraint, tracking behavior, assisted quintessence, particle physics models and quintessential inflation.

#### A. Nucleosynthesis constraint

The tightest constraint on the energy density of dark energy during a radiation dominated era comes from primarily nucleosynthesis. The introduction of an extra degree of freedom (on top of those already present in the standard model of particle physics) like a light scalar field affects the abundance of light elements in the radiation dominated epoch. The presence of a quintessence scalar field changes the expansion rate of the universe at a given temperature. This effect becomes crucial at the nucleosynthesis epoch with temperature around 1 MeV when the weak interactions (which keep neutrons and protons in equilibrium) freeze-out.

The observationally allowed range of the expansion rate at this temperature leads to a bound on the energy density of the scalar field [13]

$$\Omega_\phi(T \sim 1\text{MeV}) < \frac{7\Delta N_{\text{eff}}/4}{10.75 + 7\Delta N_{\text{eff}}/4}, \quad (260)$$

where 10.75 is the effective number of standard model degrees of freedom and  $\Delta N_{\text{eff}}$  is the additional relativistic degrees of freedom. A conservative bound on the additional degrees of freedom used in the literature is  $\Delta N_{\text{eff}} \simeq 1.5$  [235], whereas a typical one is given by  $\Delta N_{\text{eff}} \simeq 0.9$  [236]. Taking a conservative one, we obtain the following bound

$$\Omega_\phi(T \sim 1\text{MeV}) < 0.2. \quad (261)$$

Any quintessence models need to satisfy this constraint at the epoch of nucleosynthesis. We note that Bean *et al.* [237] obtained a tighter constraint  $\Omega_\phi(T \sim 1\text{MeV}) < 0.045$  with the use of the observed abundances of primordial nuclides.

As we have already seen in Sec. VI, the exponential potential (182) possesses the following two attractor solutions in the presence of a background fluid:

(1)  $\lambda^2 > 3\gamma$ : the scalar field mimics the evolution of the barotropic fluid with  $\gamma_\phi = \gamma$ , and the relation  $\Omega_\phi = 3\gamma/\lambda^2$  holds.

(2)  $\lambda^2 < 3\gamma$ : the late time attractor is the scalar field dominated solution ( $\Omega_\phi = 1$ ) with  $\gamma_\phi = \lambda^2/3$ .

The case (1) corresponds to a scaling solution in which the field energy density mimics that of the background during radiation or matter dominated era, thus alleviating the problem of a cosmological constant. If this scaling solution exists by the epoch of nucleosynthesis ( $\gamma = 4/3$ ), the constraint (261) gives

$$\Omega_\phi = \frac{4}{\lambda^2} < 0.2 \quad \rightarrow \quad \lambda^2 > 20. \quad (262)$$

In this case, however, one can not have an accelerated expansion at late times, since the equation of state of the field is the same as that of the background. In order to lead to a late-time acceleration, the scaling solution (1) needs to exit to the scalar-field dominated solution (2) near to the present. In the next subsection we shall explain quintessence models which provide this transition.

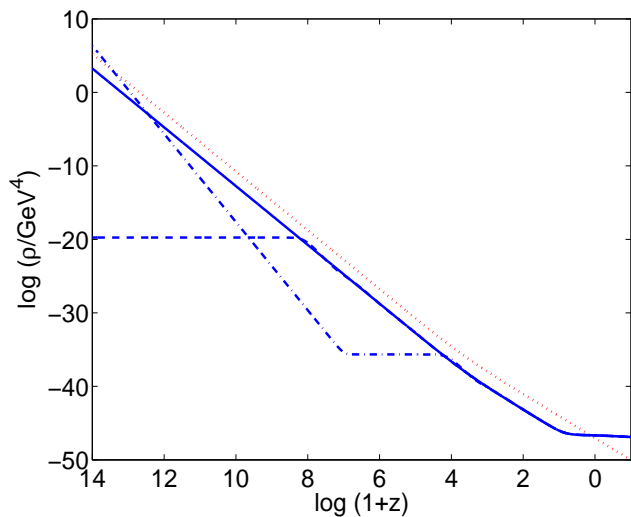


FIG. 8: Evolution of the energy density  $\rho_\phi$  for  $\lambda = 20$  and  $\mu = 0.5$ . The background energy density  $\rho_{\text{matter}} + \rho_{\text{radiation}}$  is plotted as a dotted line. Even when  $\rho_\phi$  is larger than  $\rho_{\text{matter}} + \rho_{\text{radiation}}$  at the initial stage, the solutions approach the scaling regime in which the scalar field energy density is subdominant and tracks the background fluid. We thank Nelson J. Nunes for providing us this figure.

### B. Exit from a scaling regime

In order to realize the exit from the scaling regime explained above, let us consider the following double exponential potential [160, 238]

$$V(\phi) = V_0 (e^{-\lambda\kappa\phi} + e^{-\mu\kappa\phi}), \quad (263)$$

where  $\lambda$  and  $\mu$  are positive. Such potentials are expected to arise as a result of compactifications in superstring models, hence are well motivated (although there remains an issue over how easy it is to obtain the required values of  $\mu$  and  $\lambda$ ). We require that  $\lambda$  satisfies the condition (262) under which the energy density of the field mimics the background energy density during radiation and matter dominated eras. When  $\mu^2 < 3$  the solution exits from the scaling regime and approaches the scalar-field dominated solution (2) with  $\Omega_\phi = 1$ . The accelerated expansion is realized at late times if  $\mu^2 < 2$ .

There is an important advantage to the above double exponential potential. For a wide range of initial conditions the solutions first enter the scaling regime, which is followed by an accelerated expansion of the universe once the potential becomes shallow. This behavior is clearly seen in Fig. 8. Interestingly it is acceptable to start with the energy density of the field  $\phi$  larger than that of radiation and then approach a subdominant scaling attractor.

Another model which is related to (263) was suggested by Sahni and Wang [207]:

$$V(\phi) = V_0 [\cosh(\kappa\lambda\phi) - 1]^n. \quad (264)$$

This potential has following asymptotic forms:

$$V(\phi) \simeq \begin{cases} \tilde{V}_0 e^{-n\kappa\lambda\phi} & (|\lambda\phi| \gg 1, \phi < 0), \\ \tilde{V}_0 (\kappa\lambda\phi)^{2n} & (|\lambda\phi| \ll 1), \end{cases} \quad (265)$$

where  $\tilde{V}_0 = V_0/2^n$ . Then the field energy density proportionally decreases to that of radiation and matter for  $|\lambda\phi| \gg 1$ , in which  $\Omega_\phi$  is given by  $\Omega_\phi = 3\gamma/n^2\lambda^2$ . As the field approaches the potential minimum at  $\phi = 0$ , the system exists from the scaling regime. During the oscillatory phase in which the potential is given by (265), the virial theorem gives the time-averaged relation  $\langle \dot{\phi}^2/2 \rangle = n\langle V(\phi) \rangle$ . Then the average equation of state for the field  $\phi$  is

$$\langle w_\phi \rangle = \frac{n-1}{n+1}. \quad (266)$$

When  $n < 1/2$  the field can satisfy the condition for an accelerated expansion ( $\langle w_\phi \rangle < -1/3$ ). In fact it was shown in Ref. [207] that tracking solutions which give the present-day values  $\Omega_\phi \simeq 0.7$  and  $\Omega_m \simeq 0.3$  can be obtained for a wide range of initial conditions. The field behaves as non-relativistic matter ( $\langle w_\phi \rangle = 0$ ) for  $n = 1$ . This scalar field can give rise to a tracking “scalar cold dark matter” if the mass of dark matter is  $m_{\text{CDM}} \sim 10^{-26}$  GeV [239]. An interesting attempt of unified description of dark matter and dark energy with a real scalar field is made in Ref.[240].

Albrecht and Skordis [241] have developed an interesting model which can be derived from string theory, in that they claim the parameters are all of order one in the underlying string theory. The potential has a local minimum which can be adjusted to have today’s critical energy density value (this is where the fine tuning is to be found as in all Quintessence models). The actual potential is a combination of exponential and power-law terms:

$$V(\phi) = V_0 e^{-\kappa\lambda\phi} [A + (\kappa\phi - B)^2]. \quad (267)$$

For early times the exponential term dominates the dynamics, with the energy density of  $\phi$  scaling as radiation and matter. For suitable choices of the parameters the field gets trapped in the local minimum because the kinetic energy during a scaling regime is small. The field then enters a regime of damped oscillations leading to  $w_\phi \rightarrow -1$  and an accelerating universe.

### C. Assisted quintessence

So far we have discussed the case of single-field quintessence. In early universe inflation it is known that multiple scalar fields with exponential potentials lead to the phenomenon of *assisted inflation* [242] whereby they collectively drive inflation even if each field has too steep a potential to do so on its own. This property also holds

in the context of quintessence with steep exponential potentials [160, 244, 245] (see also Ref. [246]). Here we shall briefly discuss the dynamics of assisted quintessence.

We consider two fields  $\phi_1$  and  $\phi_2$  each with a separate exponential potential

$$V(\phi_1, \phi_2) = Ae^{-\kappa\lambda_1\phi_1} + Be^{-\kappa\lambda_2\phi_2}, \quad (268)$$

where we do not implement interactions between fields. Note that such multi-field models may have a link to time-dependent compactifications of supergravity on symmetric (or twisted product) spaces, see e.g., Refs. [243]. The original assisted inflation scenario of Liddle *et al.* [242] corresponds to the case in which no matter is present, which gives an effective coupling

$$\frac{1}{\lambda_{\text{eff}}^2} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}. \quad (269)$$

Since the effective equation of state is given by  $\gamma_{\text{eff}} = \lambda_{\text{eff}}^2/3$ , the scale factor evolves as  $a \propto t^p$ , where  $p = 2/\lambda_{\text{eff}}^2$ . Hence an accelerated expansion occurs for  $\lambda_{\text{eff}} < \sqrt{2}$  even when both  $\lambda_1$  and  $\lambda_2$  are larger than  $\sqrt{2}$ .

Lets us take into account a barotropic fluid with an EOS given by  $\gamma = 1 + w_m$ . In the single field case the fixed points (c) and (d) in Table I are stable depending on the values of  $\lambda$  and  $\gamma$ . By replacing  $\lambda$  to  $\lambda_1$  and  $\lambda_2$ , we can obtain corresponding fixed points in the multi-field case [244, 245]. Once a second field is added, the new degrees of freedom always render those solutions unstable. The late-time attractors instead become either the assisted scalar-field dominated solution with  $\gamma_\phi = \lambda_{\text{eff}}^2/3$  and  $\Omega_\phi = 1$  (stable for  $\lambda_{\text{eff}}^2 < 3\gamma$ ) or the assisted scaling solution with  $\gamma_\phi = \gamma$  and  $\Omega_\phi = 3\gamma/\lambda_{\text{eff}}^2$  (stable for  $\lambda_{\text{eff}}^2 > 3\gamma$ ).

If there are a large number of exponential potentials with different initial conditions, more and more fields would join the assisted quintessence attractor, which reduces  $\lambda_{\text{eff}}$ . Eventually the attractor can switch from the scaling regime  $\lambda_{\text{eff}}^2 > 3\gamma$  into the regime of scalar field dominance  $\lambda_{\text{eff}}^2 < 3\gamma$  [244, 245]. This realizes an accelerated expansion at late-times, but we still have a fine-tuning problem to obtain a sufficiently negative value of EOS satisfying the current observational constraint ( $w_\phi \lesssim -0.8$ ).

In Ref. [214] a general analysis was given for scalar-field models which possess scaling solutions. Let us consider  $n$  scalar fields  $(\phi_1, \phi_2, \dots, \phi_n)$  with the Lagrangian density:

$$p = \sum_{i=1}^n X_i g(X_i e^{\lambda_i \phi_i}), \quad (270)$$

where  $X_i = -g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i / 2$  and  $g$  is an arbitrary function. It was shown that the scalar fominant fixed point exists for this system with an equation of state:

$$w_\phi = -1 + \frac{\lambda_{\text{eff}}^2}{3p_{,X}}, \quad (271)$$

where  $\lambda_{\text{eff}}^2$  is given by [214]

$$\frac{1}{\lambda_{\text{eff}}^2} = \sum_{i=1}^n \frac{1}{\lambda_i^2}. \quad (272)$$

Here  $p_{,X} = g(Y) + Yg'(Y)$  where  $Y \equiv Y_1 = Y_2 = \dots = Y_i = \dots = Y_n$  with  $Y_i \equiv X_i e^{\lambda_i \phi_i}$ .

The presence of multiple scalar fields leads to the decrease of the effective  $\lambda_{\text{eff}}^2$  relative to the single-field case. Since the quantity  $p_{,X}$  is not affected by introducing more scalar fields [214], the presence of many scalar fields works to shift the equation of state toward  $w_\phi = -1$ . Thus for a non-phantom scalar field ( $p_X > 0$ ) assisted acceleration always occurs for all scalar-field models which have scaling solutions.

## D. Particle physics models of Quintessence

### 1. Supergravity inspired models

We turn our attention to the thorny issue of finding examples of Quintessence in particle physics. Recall, one of the constraints we need to satisfy is that the Quintessence potential remains flat enough so that we can have slow roll inflation today, or some mechanism to trap the scalar field today. One of the most interesting approaches is to be found in Refs. [247, 248] (see also [249, 250]). In Ref. [247], Townsend considered the possibility of Quintessence arising in M-theory. He demonstrated that there is a version of  $N = 8, D = 4$  supergravity that has a positive exponential potential, obtainable from a “non-compactification” of M-theory, and this potential can lead to an accelerating cosmological solution that realizes “Eternal Quintessence”.

There is a problem that such models need to be able to address. To derive a Quintessence model from string/M-theory, we would expect that any  $D = 4$  dimensional solution should be a solution of  $D = 11$  supergravity or IIB  $D = 10$  supergravity. Unfortunately this is not so straightforward. There exists a no-go theorem due to Gibbons [135] (more recently extended by Maldacena and Nuñez [136]), which states that when the six or seven dimensional “internal” space is a time-independent non-singular compact manifold without boundary there can not be a scalar field with a positive potential, hence ruling out the possibility of late-time acceleration in any effective  $D = 4$  supergravity model based on an embedding in  $D = 11$  or  $D = 10$  supergravity.

The basic problem concerns the strong energy condition in going from  $D$  spacetime dimensions to  $d < D$  spacetime dimensions under a general warped compactification on a compact non-singular manifold of dimension  $n = D - d$ . If the non-singular  $D$ -dimensional metric can be written in the form

$$ds_D^2 = f(y) ds_d^2(x) + ds_n^2(y), \quad (273)$$

where  $y$  is the compact dimension, then the positivity of  $R_{00}$  (the Ricci tensor in  $D$  dimensions) implies positivity of  $r_{00}$  (the Ricci tensor in  $d$  dimensions). Hence for such compactifications, the strong energy condition in  $D$  spacetime dimensions implies that the strong energy condition holds in spacetime dimension  $d < D$ . From Einstein equations this then implies  $|g_{00}|V(\phi) \leq (d-2)\dot{\phi}^2$ , and hence the scalar field potential must satisfy  $V(\phi) \leq 0$  if initial conditions can be chosen such that  $\dot{\phi} = 0$ . This fact that the  $d$ -dimensional strong energy condition forbids an accelerating  $d$ -dimensional universe was emphasised in Refs. [251, 252], in showing how difficult it is to embed accelerating cosmologies into string/M-theory, where the strong energy condition is satisfied by both  $D = 11$  supergravity and IIB  $D = 10$  supergravity.

There exist a number of ways of avoiding the no-go theorem and these have been exploited to come up with Quintessence scenarios within string/M-theory. One route is to have a ‘‘compactifying’’ space that is actually non-compact [253]. In Ref. [247], Townsend adopted this approach and showed that a particular non-compact gauged  $N = 8, D = 4$  supergravity, obtainable from a warped ‘‘non-compactification’’ of M-theory, has a positive exponential potential leading to an accelerating universe, with an equation of state  $p = -(7/9)\rho$ . Of course, it leaves open the question of how realistic are these classes of non-compactified theories, a question we will not address here.

Another way round the problem is to allow the compact dimension to be time-dependent. A number of authors have adopted this approach, but in Ref. [248] the authors pointed out that in order to have a transient period of acceleration in the Einstein frame in  $D = 4$  dimensions, what is required is a hyperbolic compact internal space evolving in time (because the analogous solution of the vacuum Einstein equations for an internal manifold of positive curvature does not allow acceleration). The advantages of such compactifications have been discussed in Refs. [254, 255]. Of course, as with all Quintessence models to date, all is not rosy for this class of time dependent Hyperbolic space solutions. In [256] the authors developed a four-dimensional interpretation of the solutions with a transient accelerating phase obtained from compactification in hyperbolic manifolds. The solutions correspond to bouncing the radion field off its exponential potential, with acceleration occurring at the turning point, when the radion stops and the potential energy momentarily dominates. There is a degree of fine tuning involved in establishing the radion field is close enough to the turning point for sufficient inflation to occur. Moreover, in this interpretation in terms of the four dimensional effective theory, the precursor of the inflationary phase is a period of kinetic domination, whereas we believe the Universe was matter dominated before it became dominated by the Quintessence field. Another problem has been highlighted in [257] where the authors studied the time evolution of the corresponding effective 4d cosmological model this time including

cold dark matter. They concluded that even though it is marginally possible to describe the observational data for the late-time cosmic acceleration in this model, during the compactification of  $11d \rightarrow 4d$  the Compton wavelengths of the Kaluza Klein modes in this model are of the same order as the size of the observable part of the universe. This problem has yet to be resolved. Even so, assuming there is a resolution it is encouraging that it is possible to obtain late time inflationary solutions in M-theory. Another serious problem associated with the approach under consideration is related to the fact that masses of KK-modes in this class of models are of the order of the present value of Hubble parameter [257].

An interesting and possibly one of the most promising approaches to addressing the origin of dark energy in particle physics is due to Burgess and Quevedo along with their collaborators [123, 258, 259, 260, 261, 262, 263, 264, 265, 266]. The key idea is that the presence of two large extra dimensions (they consider 6-dimensional supersymmetric models) provides a natural mechanism to generate the small size of the observed dark energy today. Of particular note in the approach is the fact that the authors not only ask the question why has the dark energy the value it has today, but also why is that value stable to integrating over higher energy contributions? This question of *technical naturalness* is a vital one in particle physics, and is at the heart of our understanding the hierarchy problem. Therefore it makes sense to adopt it in determining the nature and value of the dark energy today, which after all is a very small amount compared to the natural scale we would expect it to take. Apart from the case of dark energy arising out of axion models (which we will come to shortly), the majority of quintessence scenarios do not possess such a protection mechanism for the mass of the field [267].

To be a bit more specific, following the nice review in [268], consider the case where a parameter  $\lambda$  is small when measured in an experiment performed at an energy scale  $\mu$ . We wish to understand this in terms of a microscopic theory of physics which is defined at energies  $\Lambda \gg \mu$ . The prediction for  $\lambda$  is given by

$$\lambda(\mu) = \lambda(\Lambda) + \delta\lambda(\mu, \Lambda), \quad (274)$$

where  $\lambda(\Lambda)$  represents the direct contribution to  $\lambda$  due to the parameters in the microscopic theory, and  $\delta\lambda$  represents the contributions to  $\lambda$  which are obtained as we integrate out all of the physics in the energy range  $\mu < E < \Lambda$ . For  $\lambda$  to be small, barring some miraculous cancellation we require that both  $\lambda(\Lambda)$  and  $\delta\lambda(\mu, \Lambda)$  are both equally small, for any chosen value of  $\Lambda$ . Most models of dark energy to date can generate  $\lambda(\mu)$  as being small, but can not guarantee the smallness of  $\delta\lambda(\mu, \Lambda)$  [45]. As we have seen the vacuum’s energy density today is  $\rho \sim (10^{-3} \text{ eV})^4$ , but for a particle of mass  $m$  it typically contributes an amount  $\delta\rho \sim m^4$  when it is integrated out. It follows that such a small value for  $\rho$  can only be understood in a technically natural way if  $\Lambda \sim 10^{-3} \text{ eV}$  or less. However, the majority of the elementary particles



(including the electron, for which  $m_e \sim 5 \times 10^5$  eV) have  $m \gg 10^{-3}$  eV, which violates the requirement on  $\Lambda$ .

In order to overcome this four dimensional problem, Burgess argues that we need to modify the response of gravity to physics at scales  $E > \mu \sim 10^{-3}$  eV, whilst maintaining that the modification does not ruin the excellent agreement with the non-gravitational experiments which have been performed covering the energy range  $\mu < E < \Lambda$ , with  $\Lambda \sim 10^{11}$  eV. The approach adopted to achieve this goal makes use of the framework of Large Extra Dimensions [269]. The observed particles (except the graviton) are constrained to live on a (3+1) dimensional surface, within an extra dimensional space, with only gravity probing the extra dimensions. (For the case of a non-supersymmetric string model that can realize the extra dimensions see [270, 271]).

The present upper limit for the size of the extra dimensions is  $r < 100 \mu\text{m}$ , or  $1/r > 10^{-3}$  eV which is very close to the scale  $\mu$  just described and where a natural understanding of the vacuum energy breaks down. Also, it turns out that there must be two of these extra dimensions if they are to be this large, and if the fundamental scale,  $M_g$ , of the extra-dimensional physics is around 10 TeV, due to the relation  $M_{\text{pl}} = M_g^2 r$  which relates  $M_g$  and  $r$  to the observed Planck mass:  $M_{\text{pl}} = (8\pi G)^{-1/2} \sim 10^{27}$  eV [268]. With this observation in mind, the idea of Supersymmetric Large Extra Dimensions (SLED) was introduced [123], which has at its heart the existence of two large (i.e.,  $r \sim 10 \mu\text{m}$ ) extra dimensions, within a supersymmetric theory, the supersymmetry allowing for the cancellation between bosons and fermions which appear in the vacuum energy.

Gravitational physics is effectively 6-dimensional for any energies above the scale,  $1/r \sim 10^{-2}$  eV, and so the cosmological constant problem has to be discussed in 6 dimensions. This means integrating out the degrees of freedom between the scales  $M_g \sim 10$  TeV and  $1/r \sim 10^{-2}$  eV. Once this is done, the cosmological constant within the effective 4D theory is obtained, describing gravitational physics on scales much larger than  $r$ . The basic procedure undertaken in integrating over modes having energies  $1/r < E < M_g$  is as follows [268]. First integrate out (exactly) all of the degrees of freedom on the branes, giving the low-energy brane dependence on the massless 4D graviton mode. The effect of this is to obtain a large effective brane tension,  $T \sim M_g^4$  for each of the 3-branes which might be present, which includes the vacuum energies of all of the presently-observed elementary particles. Following this, there is a classical part of the integration over the bulk degrees of freedom. This is achieved by solving the classical supergravity equations to determine how the extra dimensions curve in response to the brane sources which are scattered throughout the extra dimensions. The key result from this part of the calculation is that this classical response cancels the potentially large contributions from the branes obtained in the first step. Finally, the quantum part of the integration over the bulk degrees of freedom is performed. It

is this contribution which is responsible for the fact that the present-day dark energy density is nonzero, in other words it is a quantum feature!

Moreover for specific cases they find the small size of the 4D vacuum energy is attributed to the very small size with which supersymmetry breaks in the bulk relative to the scale with which it breaks on the branes. We will not go into details of the calculations here, the interested reader is referred to [268] and references therein. There are a few points worth highlighting though. In the third part, where the quantum part of the integration is performed, in the SLED model for a class of 6D supergravities, these quantum corrections lift the flat directions of the classical approximation, and those loops involving bulk fields do so by an amount leading to a potential [263]

$$V(r) \sim \frac{1}{r^4} (a + b \log r), \quad (275)$$

where  $a$  and  $b$  are calculable constants and the logarithmic corrections generically arise due to the renormalization of UV divergences in even dimensions.

Such a potential is similar to the Quintessence form introduced by Albrecht and Skordis [241], and it predicts that scalar-potential domination occurs when  $\log(M_{\text{pl}} r)$  is of order  $a/b$ , which can be obtained given a modest hierarchy amongst the coefficients,  $a/b \sim 70$ . The SLED proposal requiring the world to become six-dimensional at sub-eV energies is falsifiable (a useful attribute for a model!). This is because it has a number of knock on consequences for phenomenology that implies particle physics may soon rule it out. There is a deviation from the inverse square law for gravity around  $r/2\pi \sim 1 \mu\text{m}$  [265]; distinctive missing-energy signals in collider experiments at the LHC due to the emission of particles into the extra dimensions [272] and potential astrophysical signals arising from loss of energy into the extra dimensions by stars and supernovae. It is an interesting proposal which takes seriously the issue of technical naturalness and has a possible resolution of it in the context of Quintessence arising in six dimensional supersymmetric models. Before finishing this section on large extra dimensions we should mention the recent work of Sorokin [273], in which he argues that the true quantum gravity scale cannot be much larger than the Planck length. If it were then the quantum gravity-induced fluctuations in  $\Lambda$  would be insufficient to produce the observed dark energy.

In Sec. V we discussed dilatonic dark energy models based upon the low-energy effective string action. Another approach to supergravity inspired models of quintessence makes use of the inverse power-law potentials which arise in supersymmetric gauge theories due to non-perturbative effects, see Refs. [274] for details. In a toy model, taking into account a supergravity correction to globally supersymmetric theories, Brax and Martin [275] constructed a quintessence potential which possesses a minimum. The F-term in the scalar potential in

general is given by

$$V = e^{\kappa^2 K} [(W_i + \kappa^2 W K_i) K^{j*} (W_j + \kappa^2 W K_j)^* - 3\kappa^2 |W|^2], \quad (276)$$

where  $W$  and  $K$  are the superpotential and the Kähler potential, respectively. The subscript  $i$  represents the derivative with respect to the  $i$ -th field.

With the choice of a superpotential  $W = \Lambda^{3+\alpha} \phi^{-\alpha}$  and a flat Kähler potential,  $K = \phi \phi^*$ , we obtain the following potential

$$V(\phi) = e^{\frac{\kappa^2}{2} \phi^2} \frac{\Lambda^{4+\beta}}{\phi^\beta} \left[ \frac{(\beta-2)^2}{4} - (\beta+1) \frac{\kappa^2}{2} \phi^2 + \frac{\kappa^4}{4} \phi^4 \right], \quad (277)$$

where  $\beta = 2\alpha + 2$ . This means that the potential can be negative in the presence of supergravity corrections for  $\phi \sim m_{\text{pl}}$ . In order to avoid this problem, Brax and Martin imposed the condition that the expectation value of the superpotential vanishes, i.e.,  $\langle W \rangle = 0$ . In this case the potential (277) takes the form

$$V(\phi) = \frac{\Lambda^{4+\alpha}}{\phi^\alpha} e^{\frac{\kappa^2}{2} \phi^2}. \quad (278)$$

Although setting  $\langle W \rangle = 0$  is restrictive, Brax and Martin argued that this can be realized in the presence of matter fields in addition to the quintessence field [275].

The potential (278) has a minimum at  $\phi = \phi_* \equiv \sqrt{\alpha}/\kappa$ . If  $V(\phi_*)$  is of order the present critical density  $\rho_c^{(0)} \sim 10^{-47} \text{ GeV}^4$ , it is possible to explain the current acceleration of the universe. From Eq. (278) the mass squared at the potential minimum is given by  $m^2 \equiv \frac{d^2 V}{d\phi^2} = 2\kappa^2 V(\phi_*)$ . Since  $3H_0^2 \simeq \kappa^2 V(\phi_*)$ , we find

$$m^2 \simeq 6H_0^2. \quad (279)$$

This is a very small mass scale of order  $m \sim H_0 \sim 10^{-33} \text{ eV}$ . Such a tiny mass is very difficult to reconcile with fifth force experiments, unless there is a mechanism to prevent  $\phi$  from having interactions with the other matter fields. As mentioned this is a problem facing many quintessence models.

One can choose more general Kähler potentials when studying supergravity corrections. Let us consider a theory with superpotential  $W = \Lambda^{3+\alpha} \varphi^{-\alpha}$  and a Kähler  $K = -\ln(\kappa\varphi + \kappa\varphi^*)/\kappa^2$ , which appears at tree-level in string theory [276]. Then the potential for a canonically normalized field,  $\phi = (\ln \kappa\varphi)/\sqrt{2}\kappa$ , is

$$V(\phi) = M^4 e^{-\sqrt{2}\kappa\beta\phi}, \quad (280)$$

where  $M^4 = \Lambda^{5+\beta} \kappa^{1+\beta} (\beta^2 - 3)/2$  and  $\beta = 2\alpha + 1$ . We note that  $\beta$  needs to be larger than  $\sqrt{3}$  to allow for positivity of the potential. Thus we can obtain an exponential potential giving rise to scaling solutions in the context of supergravity.

Kolda and Lyth [267] argued that supergravity inspired models suffer from the fact that loop corrections always couple the quintessence field to other sources of matter so as to lift the potential thereby breaking the flatness criteria required for quintessence today. We now go on to discuss a class of models where this problem can be avoided. In the context of  $N \geq 2$  extended supergravity models the mass squared of any ultra-light scalar fields is quantized in unit of the Hubble constant  $H_0$ , i.e.,  $m^2 = nH_0^2$ , where  $n$  are of order unity [277, 278].

To be concrete let us consider the potential  $V(\phi) = \Lambda + (1/2)m^2\phi^2$  around the extremum at  $\phi = 0$ . In extended supergravity theories the mass  $m$  is related to  $\Lambda$  ( $> 0$ ) via the relation  $m^2 = n\Lambda/3M_{\text{pl}}^2$ , where  $n$  are integers. Since  $H_0^2 = \Lambda/3M_{\text{pl}}^2$  in de Sitter space, this gives  $m^2 = nH_0^2$ . In the context of  $N = 2$  gauged supergravity we have  $m^2 = 6H_0^2$  for a stable de Sitter vacuum [278], which gives

$$V(\phi) = 3H_0^2 M_{\text{pl}}^2 \left[ 1 + (\phi/M_{\text{pl}})^2 \right]. \quad (281)$$

The  $N = 8$  supergravity theories give the negative mass squared,  $m^2 = -6H_0^2$  [277], in which case we have

$$V(\phi) = 3H_0^2 M_{\text{pl}}^2 \left[ 1 - (\phi/M_{\text{pl}})^2 \right]. \quad (282)$$

We note that the constant  $\Lambda$  determines the energy scale of supersymmetry breaking. In order to explain the present acceleration we require  $\Lambda \sim m^2 M_{\text{pl}}^2 \sim H_0^2 M_{\text{pl}}^2 \sim 10^{-47} \text{ GeV}^4$ . This energy scale is so small that quantum corrections to  $\Lambda$  and  $m$  are suppressed. Hence we naturally obtain ultra-light scalars which are stable against quantum corrections.

## 2. Pseudo-Nambu-Goldstone models

Another approach to dark energy which avoids the serious problem posed by Kolda and Lyth [267] is to consider models in which the light mass of the Quintessence field can be protected by an underlying symmetry. Such a situation arises in cases where we have a pseudo Nambu Goldstone boson acting as the Quintessence field. This idea was first introduced by Frieman *et al.* [279] (see also [280]), in response to the first tentative suggestions that the universe may actually be dominated by a cosmological constant. These axion dark energy models based on  $N = 1$  supergravity have similar properties to the extended supergravity models discussed above.

The axion potential is

$$V(\phi) = \Lambda [C + \cos(\phi/f)], \quad (283)$$

where the model given by Frieman *et al.* [279] corresponds to  $C = 1$ . The model with  $C = 0$  can be obtained by using the superpotential and Kähler potential motivated from M/string theory [280]. The mass of the field  $\phi$  at the potential maximum is  $m^2 = -\Lambda/f^2$ . If

this energy at potential maximum is responsible for the current accelerated expansion, we have  $3H_0^2 \simeq \Lambda/M_{\text{pl}}^2$ . Then when  $f$  is of order  $M_{\text{pl}}$ , we get

$$m^2 = -\Lambda/M_{\text{pl}}^2 \simeq -3H_0^2. \quad (284)$$

The field is frozen at the potential maximum when  $|m^2|$  is smaller than  $H^2$ , but begins to roll down around present ( $|m^2| \sim H_0^2$ ). Since the energy at the potential minimum ( $\phi = \pi f$ ) is negative for  $C = 0$ , the universe collapses in the future within the next 10-20 billion years [277].

The possibility of there being an approximate global symmetry being present to suppress the natural couplings of the Quintessence field to matter, which generally result in long range forces, was investigated by Carroll in Ref. [12]. He also showed how such a symmetry could allow a coupling of  $\phi$  to the pseudoscalar  $F_{\mu\nu}\tilde{F}^{\mu\nu}$  of electromagnetism, the effect being to rotate the polarisation state of radiation from distant source. Such an effect, although well constrained today, could conceivably be used as a way of detecting a cosmological scalar field.

More recently the possibility of the axion providing the dark energy has been further developed by Kim and Nilles [281], as well as Hall and collaborators [282, 283, 284] and Hung [285]. In Ref. [281], the authors consider the model independent axion present in string theory, which has a decay constant of order the Planck scale. They propose the ‘‘quintaxion’’ as the dark energy candidate field, the field being made of a linear combination of two axions through the hidden sector supergravity breaking. The light cold dark matter axion solves the strong CP problem with decay constant determined through a hidden sector squark condensation ( $F_a \sim 10^{12}$  GeV), and the quintaxion with a decay constant as expected for model independent axion of string theory ( $F_q \sim 10^{18}$  GeV). For suitable ranges of couplings, they argue that the potential for the quintaxion is responsible for the observed vacuum energy of  $(0.003 \text{ eV})^4$ , which remains very flat, because of the smallness of the hidden sector quark masses. Hence it is ideal for Quintessence with the Quintessence mass protected through the existence of the global symmetry associated with the pseudo Nambu-Goldstone boson.

In Ref. [283], the authors consider an axion model which leads to a time dependent equation of state parameter  $w(z)$  for the Quintessence field. As before, the small mass scale is protected against radiative corrections. The novel feature they introduce is the seesaw mechanism, which allows for two natural scales to play a vital role in determining all the other fundamental scales. These are the weak scale,  $v$ , and the Planck scale,  $M_{\text{pl}}$ . For example, the dark energy density  $\rho_{\text{DE}}^{1/4} \propto v^2/M_{\text{pl}}$ , and the radiatively stable mass  $m_\phi \propto v^4/M_{\text{pl}}^3$ . Adopting a cosine quintessence potential they construct an explicit hidden axion model, and find a distinctive form for the equation of state  $w(z)$ . The dark energy resides in the potential of the axion field which is generated by a new QCD-like

force that gets strong at the scale  $\Lambda \approx v^2/M_{\text{pl}} \approx \rho_{\text{DE}}^{1/4}$ . The evolution rate is given by a second seesaw that leads to the axion mass,  $m_\phi \approx \Lambda^2/f$ , with  $f \approx M_{\text{pl}}$ .

Many particle physicists believe that the best route to find quintessence will be through the axion, and so we can expect much more progress in this area over the next few years.

## E. Quintessential inflation

We now turn our attention to the case of quintessential inflation, first developed by Peebles and Vilenkin [35] (see also Ref. [286] for an early example which includes some of the features). One of the major drawbacks often used to attack models of quintessence is that it introduces yet another weakly interacting scalar field. Why can't we use one of those scalars already ‘‘existing’’ in cosmology, to act as the quintessence field?

This is precisely what Peebles and Vilenkin set about doing (see also Ref. [287]). They introduced a potential for the field  $\phi$  which allowed it to play the role of the inflaton in the early Universe and later to play the role of the quintessence field. To do this it is important that the potential does not have a minimum in which the inflaton field would completely decay at the end of the initial period of inflation. They proposed the following potential

$$\begin{aligned} V(\phi) &= \lambda(\phi^4 + M^4) \quad \text{for } \phi < 0, \\ &= \frac{\lambda M^4}{1 + (\phi/M)^\alpha} \quad \text{for } \phi \geq 0. \end{aligned} \quad (285)$$

For  $\phi < 0$  we have ordinary chaotic inflation. Much later on, for  $\phi > 0$  the universe once again begins to inflate but this time at the lower energy scale associated with quintessence. Needless to say quintessential inflation also requires a degree of fine tuning, in fact perhaps even more than before as there are no tracker solutions we can rely on for the initial conditions. The initial period of inflation must produce the observed density perturbations, which constrains the coupling to be of order  $\lambda \sim 10^{-13}$  [70]. Demanding that  $\Omega_\phi^{(0)} \sim 0.7$ , we can constrain the parameter space of  $(\alpha, M)$ . For example, for  $\alpha = 4$ , we have  $M \sim 10^5$  GeV [35]. Reheating after inflation should have proceeded via gravitational particle production (see [286] for an early example of its effect in ending inflation) because of the absence of the potential minimum, but this mechanism is very inefficient. However this problem may be alleviated [288] in the instant preheating scenario [289] in the presence of an interaction  $(1/2)g^2\phi^2\chi^2$  between the inflaton  $\phi$  and another field  $\chi$ . Of note in the quintessential inflation model is that one gets a kinetic phase (driven by the kinetic energy of the field) before entering the radiation phase. This has the effect of changing the density of primordial gravitational waves [35, 291].

An interesting proposal making use of the protected axion as the quintessence field has been made in

Ref. [292]. One of the problems facing the models just described is that the potentials are simply constructed to solve the problem at hand, namely to give two periods of inflation at early and late times. As such, they are generally non-renormalisable. The authors of Ref. [292] introduce a renormalizable complex scalar field potential as the Quintessential inflation field. They suggest using a complex scalar field with a global  $U(1)_{PQ}$  symmetry which is spontaneously broken at a high energy scale. This then generates a flat potential for the imaginary part of the field (“axion”), which is then lifted (explicitly broken) by small instanton effects at a much lower energy. In this sense it combines the original idea of Natural Inflation [293] and the more recent idea of using a pseudo-Nambu Goldstone boson for the Quintessence field [279, 294]. The result is that the model can give both early universe inflation (real part of scalar field) and late time inflation (imaginary part of the scalar field). We also note that there is an interesting quintessential inflation model by Dimopoulos [290] that allows inflation to occur at lower energy scales than GUT in the context of the curvaton mechanism.

Complex scalar fields have also been introduced in the context of quintessence models called “spintessence” [295, 296]. In these models the usefulness of the complex nature manifests itself in that the model allows for a unified description of both dark matter and dynamical dark energy. The field  $\phi$  is spinning in a  $U(1)$ -symmetric potential  $V(\phi) = V(|\phi|)$ , such that as the Universe expands, the field spirals slowly toward the origin. It has internal angular momentum which helps drive the cosmic evolution and fluctuations of the field. Depending on the nature of the spin, and the form of the potential, the net equation of state for the system can model either that of an evolving dark energy component or self-interacting, fuzzy cold dark matter [296] (see also Ref. [297]).

One of the main difficulties for the realistic construction of quintessential inflation is that we need a flat potential during inflation but also require a steep potential during the radiation and matter dominated periods. The above mentioned axionic models provide one way to guarantee that can happen. The possibility that a pseudo-Nambu-Goldstone boson could arise in the bulk in a higher dimensional theory was investigated in Ref. [298]. Another route is through quintessential inflation [299] in braneworld scenarios [300]. Because of the modification of the Friedmann equation in braneworlds [301, 302] ( $H^2 \propto \rho^2$ ), it is possible to obtain inflationary solutions even in the case of a steep exponential potential. Although the ratio of tensor perturbations to scalar perturbations is large and the exponential potential is outside the  $2\sigma$  observational bound [303], the model can be allowed if a Gauss-Bonnet term is present in the five dimensional bulk [304]. We finish this section with the observation that in Ref. [305], the authors proposed a mechanism to generate the baryon asymmetry of the universe in a class of quintessential inflation models.

## IX. COUPLED DARK ENERGY

The possibility of a scalar field  $\phi$  coupled to a matter and its cosmological consequences were discussed in Refs. [306]. Amendola later proposed a quintessence scenario coupled with dark matter [215] as an extension of nonminimally coupled theories [216]. A related approach in which the dark matter and dark energy interact with each other exchanging energy has been proposed by Szydlowski in [217] and a method of testing for it has been developed in [218]. He is able to show that the cubic correction to the Hubble law, as measured by distant supernovae type Ia, can probe this interaction, and by considering flat decaying  $\Lambda(t)$  FRW cosmologies, he argues for the possibility of measuring the energy transfer through determination of the cubic and higher corrections to Hubble’s law.

An interesting aspect of the coupled dark energy scenario [215] is that the system can approach scaling solutions (characterized by  $\Omega_\phi \simeq 0.7$ ) with an associated accelerated expansion.

Earlier in Sec. VII we presented a coupling  $Q$  between dark energy and a barotropic fluid. This is actually the same coupling studied in Refs. [215, 216], and in order to show an example of this, let us consider the following 4-dimensional Lagrangian density with a scalar field  $\varphi$  and a barotropic perfect fluid:

$$\tilde{\mathcal{L}} = \frac{1}{2}F(\varphi)\tilde{R} - \frac{1}{2}\zeta(\varphi)(\tilde{\nabla}\varphi)^2 - U(\varphi) - \tilde{\mathcal{L}}_m, \quad (286)$$

where  $F(\varphi)$ ,  $\zeta(\varphi)$  and  $U(\varphi)$  are the functions of  $\varphi$ . This includes a wide variety of gravity models—such as Brans-Dicke theories, non-minimally coupled scalar fields and dilaton gravity. In fact a number of authors have studied quintessence scenarios with a nonminimally coupled scalar field [205, 307]. This is related to coupled quintessence scenario as we will see below. We set  $\kappa^2 = 1$  in this section.

After a conformal transformation  $g_{\mu\nu} = F(\varphi)\tilde{g}_{\mu\nu}$ , the above action reduces to that of the Einstein frame:

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi) - \mathcal{L}_m(\phi), \quad (287)$$

where

$$\phi \equiv \int G(\varphi)d\varphi, \quad G(\varphi) \equiv \sqrt{\frac{3}{2} \left( \frac{F_{,\varphi}}{F} \right)^2 + \frac{\zeta}{F}}, \quad (288)$$

and  $F_{,\varphi} \equiv dF/d\varphi$ . We note that several quantities in the Einstein frame are related to those in the Jordan frame via  $a = \sqrt{F}\tilde{a}$ ,  $dt = \sqrt{F}d\tilde{t}$ ,  $\rho_m = \tilde{\rho}_m/F^2$ ,  $p_m = \tilde{p}_m/F^2$  and  $V = U/F^2$ .

In the Jordan frame the energy density  $\tilde{\rho}_m$  obeys the continuity equation  $d\tilde{\rho}_m/d\tilde{t} + 3\tilde{H}(\tilde{\rho}_m + \tilde{p}_m) = 0$ . By rewriting this equation in terms of the quantities in the Einstein frame, we find

$$\dot{\rho}_m + 3H(\rho_m + p_m) = -\frac{F_{,\varphi}}{2FG}(\rho_m - 3p_m)\dot{\phi}. \quad (289)$$

In the case of cold dark matter ( $p_m = 0$ ) this corresponds to Eq. (224) with a coupling

$$Q(\varphi) = -\frac{F_{,\varphi}}{2FG} = -\frac{F_{,\varphi}}{2F} \left[ \frac{3}{2} \left( \frac{F_{,\varphi}}{F} \right)^2 + \frac{\zeta}{F} \right]^{-1/2}. \quad (290)$$

For example a nonminimally coupled scalar field with a coupling  $\xi$  corresponds to  $F(\varphi) = 1 - \xi\varphi^2$  and  $\zeta(\varphi) = 1$ . This gives

$$Q(\varphi) = \frac{\xi\varphi}{[1 - \xi\varphi^2(1 - 6\xi)]^{1/2}}. \quad (291)$$

In the limit  $|\xi| \rightarrow \infty$  the coupling approaches a constant value  $Q(\varphi) \rightarrow \pm 1/\sqrt{6}$ .

Thus a nonminimally coupled scalar field naturally leads to the coupling between dark energy and a barotropic fluid. In what follows we will derive critical points and study their stabilities in a coupled Quintessence scenario based on the coupling which appears on the RHS of Eqs. (223) and (224).

Before we investigate in detail the nature of coupled dark energy, it is worth mentioning a couple of important points that have been emphasised in [219] and [220]. In [219] it is pointed out that if there is an interaction between dark matter and dark energy then this will generically result in an effective dark energy equation of state of  $w < -1$ , arising because the interaction alters the redshift-dependence of the matter density. Therefore an observer who fits the data treating the dark matter as non-interacting will infer an effective dark energy fluid with  $w < -1$ . The authors go on to argue that the coupled dark energy model is consistent with all current observations, the tightest constraint coming from estimates of the matter density at different redshifts.

In [220] it is shown that cluster number counts can be used to test dark energy models where the dark energy candidates are coupled to dark matter. Increasing the coupling reduces significantly the cluster number counts, whereas dark energy inhomogeneities increase cluster abundances. Of possible significance is the fact that wiggles in cluster number counts are shown to be a specific signature of coupled dark energy models. Such oscillations could possibly be detected in future experiments, allowing us to discriminate among the different dark energy models.

### A. Critical points for coupled Quintessence

We now consider a coupled Quintessence scenario in Einstein gravity with an exponential potential i.e.,

$$p(X, \phi) = \epsilon X - ce^{-\lambda\phi}. \quad (292)$$

Here we allow the possibility of a phantom field ( $\epsilon < 0$ ). As we have already shown, exponential potentials possess scaling solutions. In fact the above Lagrangian density

corresponds to the choice  $g(Y) = \epsilon - c/Y$  and  $n = 1$  in Eq. (233).

The autonomous equations for a general function of  $g(Y)$  are given by Eqs. (255) and (256). Substituting  $g(Y) = \epsilon - c/Y$  with  $Y = x^2/y^2$  for Eqs. (255) and (256), we obtain the following differential equations for  $x = \dot{\phi}/(\sqrt{6}H)$  and  $y = e^{-\lambda\phi}/(\sqrt{3}H)$ :

$$\begin{aligned} \frac{dx}{dN} = & -3x + \frac{\sqrt{6}}{2}\epsilon\lambda cy^2 + \frac{3}{2}x[(1 - w_m)\epsilon x^2 \\ & + (1 + w_m)(1 - cy^2)] - \frac{\sqrt{6}Q}{2}\epsilon(1 - \epsilon x^2 - cy^2), \end{aligned} \quad (293)$$

$$\begin{aligned} \frac{dy}{dN} = & -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1 - w_m)\epsilon x^2 \\ & + (1 + w_m)(1 - cy^2)]. \end{aligned} \quad (294)$$

When  $Q = 0$  and  $c = 1$  these equations coincide with Eqs. (175) and (176). We note that  $w_\phi$ ,  $\Omega_\phi$  and  $w_{\text{eff}}$  are derived by changing  $y^2$  to  $cy^2$  in Eqs. (179), (180) and (181).

The fixed points for the above system can be obtained by setting  $dx/dN = 0$  and  $dy/dN = 0$ . We present the fixed points in Table V.

- (i) Ordinary field ( $\epsilon = +1$ )

The point (a) gives some fraction of the field energy density for  $Q \neq 0$ . However this does not provide an accelerated expansion, since the effective equation of state  $w_{\text{eff}}$  is positive for  $0 \leq w_m < 1$ . The points (b1) and (b2) are kinetically driven solutions with  $\Omega_\phi = 1$  and do not satisfy the condition  $w_{\text{eff}} < -1/3$ . The point (c) is a scalar-field dominating solution ( $\Omega_\phi = 1$ ), which gives an acceleration of the universe for  $\lambda^2 < 2$ . The point (d) corresponds to the cosmological scaling solution, which satisfies  $w_\phi = w_m$  for  $Q = 0$ . When  $Q \neq 0$  the accelerated expansion occurs for  $Q > \lambda(1 + 3w_m)/2$ . The points (b1), (b2) and (c) exist irrespective of the presence of the coupling  $Q$ .

- (ii) Phantom field ( $\epsilon = -1$ )

The point (a) corresponds to an unrealistic situation because of the condition  $\Omega_\phi < 0$  for  $0 \leq w_m < 1$ . The critical points (b1) and (b2) do not exist for the phantom field. Since  $w_{\text{eff}} = -1 - \lambda^2/3 < -1$  for the point (c), the universe accelerates independent of the values of  $\lambda$  and  $Q$ . The point (d) gives an accelerated expansion for  $Q > \lambda(1 + 3w_m)/2$ , and is similar to the case of a normal field.

### B. Stability of critical points

We shall study the stability around the fixed points. The eigenvalues of the matrix  $\mathcal{M}$  for the perturbations  $\delta x$  and  $\delta y$  in Eq. (167) are [212]

Name	$x$	$y$	$\Omega_\phi$	$w_\phi$	$w_{\text{eff}}$
(a)	$-\frac{\sqrt{6}Q}{3\epsilon(1-w_m)}$	0	$\frac{2Q^2}{3\epsilon(1-w_m)}$	1	$w_m + \frac{2Q^2}{3\epsilon(1-w_m)}$
(b1)	$\frac{1}{\sqrt{\epsilon}}$	0	1	1	1
(b2)	$-\frac{1}{\sqrt{\epsilon}}$	0	1	1	1
(c)	$\frac{\epsilon\lambda}{\sqrt{6}}$	$[\frac{1}{c}(1 - \frac{\epsilon\lambda^2}{6})]^{1/2}$	1	$-1 + \frac{\epsilon\lambda^2}{3}$	$-1 + \frac{\epsilon\lambda^2}{3}$
(d)	$\frac{\sqrt{6}(1+w_m)}{2(\lambda+Q)}$	$[\frac{2Q(\lambda+Q)+3\epsilon(1-w_m^2)}{2c(\lambda+Q)^2}]^{1/2}$	$\frac{Q(\lambda+Q)+3\epsilon(1+w_m)}{(\lambda+Q)^2}$	$\frac{-Q(\lambda+Q)+3\epsilon w_m(1+w_m)}{Q(\lambda+Q)+3\epsilon(1+w_m)}$	$\frac{w_m\lambda-Q}{\lambda+Q}$

TABLE V: The critical points for the ordinary (phantom) scalar field with an exponential potential in the presence of the coupling  $Q$ . The points (b1) and (b2) do not exist for the phantom field.

- Point (a):

$$\mu_1 = -\frac{3}{2}(1-w_m) + \frac{Q^2}{\epsilon(1-w_m)},$$

$$\mu_2 = \frac{1}{\epsilon(1-w_m)} \left[ Q(\lambda+Q) + \frac{3\epsilon}{2}(1-w_m^2) \right]. \quad (295)$$

- Point (b1):

$$\mu_1 = 3 - \frac{\sqrt{6}}{2}\lambda, \quad \mu_2 = 3(1-w_m) + \sqrt{6}Q. \quad (296)$$

- Point (b2):

$$\mu_1 = 3 + \frac{\sqrt{6}}{2}\lambda, \quad \mu_2 = 3(1-w_m) - \sqrt{6}Q. \quad (297)$$

- Point (c):

$$\mu_1 = \frac{1}{2}(\epsilon\lambda^2 - 6), \quad \mu_2 = \epsilon\lambda(\lambda+Q) - 3(1+w_m). \quad (298)$$

- Point (d):

$$\mu_{1,2} = -\frac{3\{\lambda(1-w_m) + 2Q\}}{4(\lambda+Q)} [1 \pm \sqrt{1 + f(\lambda, Q)}], \quad (299)$$

where

$$f = \frac{8[3(1+w_m) - \epsilon\lambda(\lambda+Q)][3\epsilon(1-w_m^2) + 2Q(\lambda+Q)]}{3\{\lambda(1-w_m) + 2Q\}^2}. \quad (300)$$

### 1. Ordinary field ( $\epsilon = +1$ )

We first study the dynamics of an ordinary scalar field in the presence of a fluid with an equation of state:  $0 \leq w_m < 1$ . We shall consider the case of  $Q > 0$  and  $\lambda > 0$  for simplicity, but it is easy to extend the analysis to other cases.

- Point (a):

For the point (a)  $\mu_1$  is negative if  $Q < \sqrt{3/2}(1-w_m)$  and positive otherwise. Meanwhile  $\mu_2$  is positive for any value of  $Q$  and  $\lambda$ . Therefore this is a

saddle point for  $Q < \sqrt{3/2}(1-w_m)$  and an unstable node for  $Q > \sqrt{3/2}(1-w_m)$ . We obtain the condition  $Q < \sqrt{(3/2)(1-w_m)}$  from the requirement  $\Omega_\phi < 1$ . Hence the point (a) is a saddle point for  $w_m = 0$  under this condition.

- Point (b1):

While  $\mu_2$  is always positive,  $\mu_1$  is negative if  $\lambda > \sqrt{6}$  and positive otherwise. Then (b1) is a saddle point for  $\lambda > \sqrt{6}$  and an unstable node for  $\lambda < \sqrt{6}$ .

- Point (b2):

Since  $\mu_1$  is always positive and  $\mu_2$  is negative for  $Q > (3/2)^{1/2}(1-w_m)$  and positive otherwise, the point (c) is either a saddle point or an unstable node.

- Point (c):

The requirement of the existence of the point (c) gives  $\lambda < \sqrt{6}$ , which means that  $\mu_1$  is always negative. The eigenvalue  $\mu_2$  is negative for  $\lambda < (\sqrt{Q^2 + 12(1+w_m)} - Q)/2$  and positive otherwise. Hence the point (c) presents a stable node for  $\lambda < (\sqrt{Q^2 + 12(1+w_m)} - Q)/2$ , whereas it is a saddle point for  $(\sqrt{Q^2 + 12(1+w_m)} - Q)/2 < \lambda < \sqrt{6}$ .

- Point (d):

We first find that  $-3\{\lambda(1-w_m) + 2Q\}/4(\lambda+Q) < 0$  in the expression of  $\mu_1$  and  $\mu_2$ . Secondly we obtain  $\lambda(\lambda+Q) > 3(1+w_m)$  from the condition,  $\Omega_\phi < 1$ . Then the point (d) corresponds to a stable node for  $3(1+w_m)/\lambda - \lambda < Q < Q_*$  and is a stable spiral for  $Q > Q_*$ , where  $Q_*$  satisfies the following relation

$$8[\lambda(\lambda+Q_*) - 3(1+w_m)][2Q_*(\lambda+Q_*) + 3(1-w_m^2)] = 3[\lambda(1-w_m) + 2Q_*]^2. \quad (301)$$

For example  $Q_* = 0.868$  for  $\lambda = 1.5$  and  $w_m = 0$ .

The stability around the fixed points and the condition for an acceleration are summarized in Table VI. The scaling solution (d) is always stable provided that  $\Omega_\phi < 1$ , whereas the stability of the point (c) is dependent on the values of  $\lambda$  and  $Q$ . It is important to note that the

Name	Stability	Acceleration	Existence
(a)	Saddle point for $Q < (3/2)^{1/2}(1 - w_m)$ Unstable node for $Q > (3/2)^{1/2}(1 - w_m)$	No	$Q < (3/2)^{1/2}(1 - w_m)^{1/2}$
(b1)	Saddle point for $\lambda > \sqrt{6}$ Unstable node for $\lambda < \sqrt{6}$	No	All values
(b2)	Saddle point for $Q > (3/2)^{1/2}(1 - w_m)$ Unstable node for $Q < (3/2)^{1/2}(1 - w_m)$	No	All values
(c)	Saddle point for $([Q^2 + 12(1 + w_m)]^{1/2} - Q)/2 < \lambda < \sqrt{6}$ Stable node for $\lambda < ([Q^2 + 12(1 + w_m)]^{1/2} - Q)/2$	$\lambda < \sqrt{2}$	$\lambda < \sqrt{6}$
(d)	Stable node for $3(1 + w_m)/\lambda - \lambda < Q < Q_*$ Stable spiral for $Q > Q_*$	$Q > \lambda(1 + 3w_m)/2$	$Q > 3(1 + w_m)/\lambda - \lambda$

TABLE VI: The conditions for stability, acceleration and existence for an ordinary scalar field ( $\epsilon = +1$ ). We consider the situation with positive values of  $Q$  and  $\lambda$ . Here  $Q_*$  is the solution of Eq. (301).

eigenvalue  $\mu_2$  for the point (c) is positive when the condition for the existence of the point (d) is satisfied, i.e.,  $\lambda(\lambda + Q) > 3(1 + w_m)$ . Therefore the point (c) is unstable for the parameter range of  $Q$  and  $\lambda$  in which the scaling solution (d) exists [214].

Amendola [215] implemented radiation together with cold dark matter and the scalar field  $\phi$ . Unsurprisingly there exist more critical points in this case, but we can use the analysis we have just presented to describe the dynamics of the system once radiation becomes dynamically unimportant. In Fig. 9 we show the evolution of the fractional energy densities:  $\Omega_R$  (radiation:  $w_m = 1/3$ ),  $\Omega_M$  (CDM:  $w_m = 0$ ),  $\Omega_\phi$  (dark energy) for (i)  $\lambda = 0.1$ ,  $Q = 0$  and (ii)  $\lambda = 0.1$ ,  $Q = 0.245$ . In the absence of the coupling  $Q$ ,  $\Omega_\phi$  is negligibly small compared to  $\Omega_M$  during the matter dominated era. Meanwhile when the coupling  $Q$  is present there are some portions of the energy density of  $\phi$  in the matter dominated era. This corresponds to the critical point (a) characterized by  $\Omega_\phi = 2Q^2/3$  and  $w_{\text{eff}} = 2Q^2/3$ . The presence of this phase (“ $\phi$ MDE” [215]) can provide a distinguishable feature for matter density perturbations, as we will see in the next section. Since the critical point (a) is not stable, the final attractor is either the scalar-field dominated solution (c) or the scaling solution (d). Figure 9 corresponds to the case in which the system approaches the fixed point (c).

The system approaches the scaling solution (d) with constant  $\Omega_\phi$  provided that the coupling satisfies the condition  $Q > 3/\lambda - \lambda$ . In addition we have an accelerated expansion for  $Q > \lambda/2$ . Then one can consider an interesting situation in which the present universe is a global attractor with  $\Omega_\phi \simeq 0.7$ . However it was pointed out in Ref. [215] the universe soon enters the attractor phase after the radiation dominated era for the coupling  $Q$  satisfying the condition for an accelerated expansion ( $Q/\lambda > (1 + 3w_m)/2$ ). This means the absence of a matter dominated era, which is problematic for structure formation. It comes from the fact that the coupling  $Q$  required for acceleration is too large to keep  $\Omega_\phi = 2Q^2/3$  small during the matter dominant era.

This problem is overcome if we consider a non-linear coupling that changes between a small  $Q_1$  to a large  $Q_2$

[308, 309]. The authors in Ref. [308] adopted the following coupling:

$$Q(\phi) = \frac{1}{2} \left[ (Q_2 - Q_1) \tanh \left( \frac{\phi_1 - \phi}{\Delta} \right) + Q_2 + Q_1 \right]. \quad (302)$$

In order to keep  $\Omega_\phi$  small during the matter dominated era but to get  $\Omega_\phi \simeq 0.7$  with an accelerated expansion, we need to impose the condition  $Q_1 \ll \lambda \ll Q_2$ . In Fig. 10 we plot the evolution of  $\Omega_R$ ,  $\Omega_M$  and  $\Omega_\phi$  together with an effective equation of state  $\gamma_{\text{eff}} = w_{\text{eff}} + 1$  for  $\lambda = 30$ ,  $Q_1 = 0$  and  $Q_2 = 57.15$ . We find that there exists a matter dominated era with a small value of  $\Omega_\phi$ , which allows large-scale structure to grow. The solution eventually approaches a stationary global attractor characterized by  $\Omega_\phi \simeq 0.7$  with an accelerated expansion.

## 2. Phantom field ( $\epsilon = -1$ )

The fixed points (b) and (c) do not exist for the phantom field.

- Point (a):

In this case  $\mu_1$  is always negative, whereas  $\mu_2$  can be either positive or negative depending on the values of  $Q$  and  $\lambda$ . Then this point is a saddle for  $Q(Q + \lambda) < (3/2)(1 - w_m^2)$  and a stable node for  $Q(Q + \lambda) > (3/2)(1 - w_m^2)$ . However, since  $\Omega_\phi = -2Q^2/3(1 - w_m) < 0$  for  $0 \leq w_m < 1$ , the fixed point (a) is not realistic.

- Point (c):

Since both  $\mu_1$  and  $\mu_2$  are negative independent of the values of  $\lambda$  and  $Q$ , the point (c) is a stable node.

- Point (d):

From the condition  $y^2 > 0$ , we require that  $2Q(Q + \lambda) > 3(1 - w_m^2)$  for the existence of the critical point (d). Under this condition we find that  $\mu_1 < 0$  and  $\mu_2 > 0$ . Therefore the point (d) corresponds to a saddle point.

Name	Stability	Acceleration	Existence
(a)	Saddle point for $Q(Q + \lambda) < (3/2)(1 - w_m^2)$ Stable node for $Q(Q + \lambda) > (3/2)(1 - w_m^2)$	$Q^2 > (1 - w_m)(1 + 3w_m)/2$	No if the condition $0 \leq \Omega_\phi \leq 1$ is imposed
(c)	Stable node	All values	All values
(d)	Saddle	Acceleration for $Q > \lambda(1 + 3w_m)/2$	$Q(Q + \lambda) > (3/2)(1 - w_m^2)$

TABLE VII: The conditions for stability & acceleration & existence for a phantom scalar field ( $\epsilon = -1$ ). We consider the situation with positive values of  $Q$  and  $\lambda$ .

The properties of critical points are summarized in Table VII. The scaling solution always becomes unstable for phantom fields. Therefore one can not construct a coupled dark energy scenario in which the present value of  $\Omega_\phi$  ( $\simeq 0.7$ ) is a late-time attractor. This property is different from the case of an ordinary field in which scaling solutions can be stable fixed points. The only viable stable attractor for phantom fields is the fixed point (c), giving the dark energy dominated universe ( $\Omega_\phi = 1$ ) with an equation of state  $w_\phi = -1 - \lambda^2/3 < -1$ .

### C. General properties of fixed points

In the previous subsections we have considered the case of a minimally coupled scalar field with an exponential potential. This coupled quintessence scenario can be applied to other scalar-field dark energy models such as tachyon and dilatonic ghost condensate. For the dark energy models that possess scaling solutions, the procedure to derive fixed points is very simple. The functional form  $g(Y)$  is determined by specifying the model. Then plugging this into Eqs. (255) and (256), we obtain the fixed points for the system. The stability of fixed points is known by evaluating the eigenvalues of the matrix  $\mathcal{M}$ .

In fact we can study the stability of fixed points relevant to dark energy for the scalar-field models which possess scaling solutions without specifying any form of  $g(Y)$ . From Eqs. (255) and (256) the fixed points we are interested in ( $y \neq 0$ ) satisfy the following equations:

$$\sqrt{6}\lambda x = 3 [1 + gx^2 - w_m(\Omega_\phi - 1)] , \quad (303)$$

$$\sqrt{6}(g + Yg')x = (Q + \lambda)\Omega_\phi - Q . \quad (304)$$

Since  $g + Yg' = \Omega_\phi(1 + w_\phi)/2x^2$  from Eq. (257), Eqs. (303) and (304) can be written in the form:

$$x = \frac{\sqrt{6}[1 + (w_\phi - w_m)\Omega_\phi + w_m]}{2\lambda} \quad (305)$$

$$= \frac{\sqrt{6}(1 + w_\phi)\Omega_\phi}{2[(Q + \lambda)\Omega_\phi - Q]} , \quad (306)$$

This leads to

$$(\Omega_\phi - 1) [(w_\phi - w_m)(Q + \lambda)\Omega_\phi + Q(1 + w_m)] = 0. \quad (307)$$

Hence we obtain

- (i) A scalar-field dominant solution with

$$\Omega_\phi = 1 . \quad (308)$$

- (ii) A scaling solution with

$$\Omega_\phi = \frac{(1 + w_m)Q}{(w_m - w_\phi)(Q + \lambda)} . \quad (309)$$

In the case (i) Eqs. (257) and (303) give

$$w_\phi = -1 + \frac{\sqrt{6}\lambda x}{3} = -1 + \frac{\lambda^2}{3p_{,X}} . \quad (310)$$

In the last equality we used the relation  $x = \lambda/\sqrt{6}p_{,X}$  which is derived from Eq. (304). Since the scalar-field dominates the dynamics, the effective equation is given by  $w_{\text{eff}} = w_\phi$ .

Substituting Eq. (309) for Eq. (305), we obtain the value of  $x$  given by Eq. (259). Hence the fixed point (ii) is actually the scaling solution.

In Ref. [214] the stability of the fixed points relevant to dark energy was studied without specifying the form of  $g(Y)$ . In the presence of non-relativistic dark matter with a non-phantom scalar field, the final attractor is either a scaling solution with constant  $\Omega_\phi$  satisfying  $0 < \Omega_\phi < 1$  or a scalar-field dominant solution with  $\Omega_\phi = 1$ . Meanwhile a phantom scalar-field dominant fixed point ( $\Omega_\phi = 1$  and  $p_{,X} < 0$ ) is always classically stable. Then the universe is eventually dominated by the energy density of a scalar field if phantom is responsible for dark energy. See Ref. [214] for details about the stability of fixed points.

### D. Can we have two scaling regimes ?

In the case of Quintessence with an exponential potential there exists a “ $\phi$ MDE” fixed point (a) presented in Table V. Since  $\Omega_\phi = w_{\text{eff}} = 2Q^2/3$  for  $w_m = 0$ , this also corresponds to a scaling solution if  $Q$  is a constant. As we mentioned in subsection B, it is not possible to have a sequence of the “ $\phi$ MDE” (a) and the accelerated scaling attractor (d) with  $\Omega_\phi \simeq 0.7$ . Then a question arises. Can we have two scaling regimes for a general class of coupled scalar field Lagrangians?

Let us consider the scaling Lagrangian (233) in Einstein gravity ( $n = 1$ ). The  $\phi$ MDE is a kinetic solution



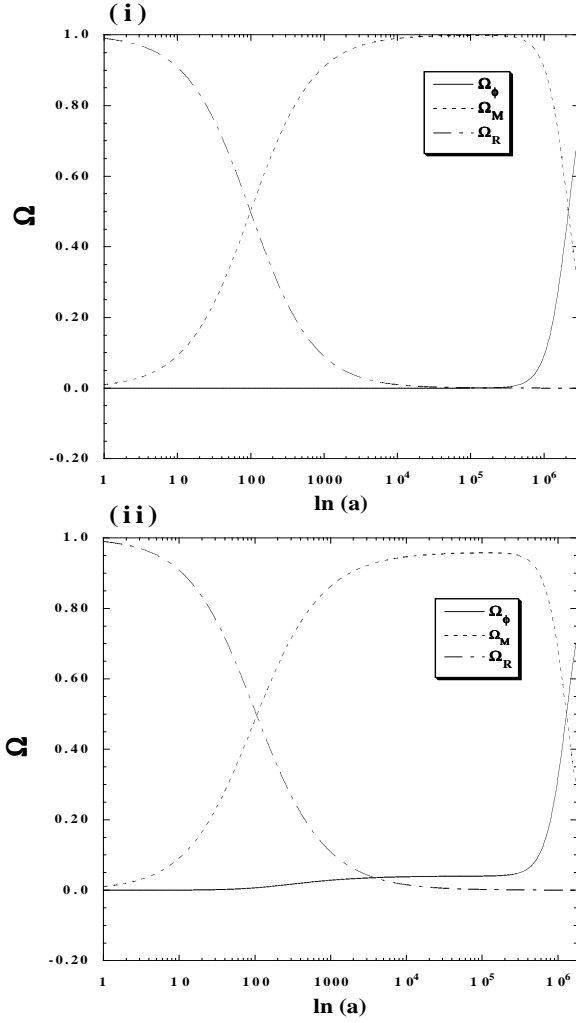


FIG. 9: Evolution of  $\Omega_R$ ,  $\Omega_M$  and  $\Omega_\phi$  in a coupled quintessence scenario for (i)  $\lambda = 0.1$ ,  $Q = 0$  and (ii)  $\lambda = 0.1$ ,  $Q = 0.245$ . In these cases the late-time attractor is the scalar-field dominated fixed point (d) in Table VI. In the case (ii) there exists a transient fixed point (a) characterized by  $\Omega_\phi = 2Q^2/3 \simeq 0.04$  during the matter dominated era, whereas this behavior is absent in the case (i).

which corresponds to  $y = 0$  in Eq. (294). This point exists only if  $g = g(x^2/y^2)$  is non-singular, i.e., only if one can expand  $g$  in positive powers of  $y^2/x^2$ ,

$$g = c_0 + \sum_{n>0} c_n \left(\frac{y^2}{x^2}\right)^n. \quad (311)$$

In this case Eq. (293) is given by

$$\frac{dx}{dN} = \frac{1}{2} \left(3c_0x + \sqrt{6}Q\right) \left(x^2 - \frac{1}{c_0}\right) = 0. \quad (312)$$

For  $c_0 = 0$  this equation gives no real solutions. For  $c_0 \neq 0$  we get the  $\phi$ MDE point  $x = -\sqrt{6}Q/3c_0$  together

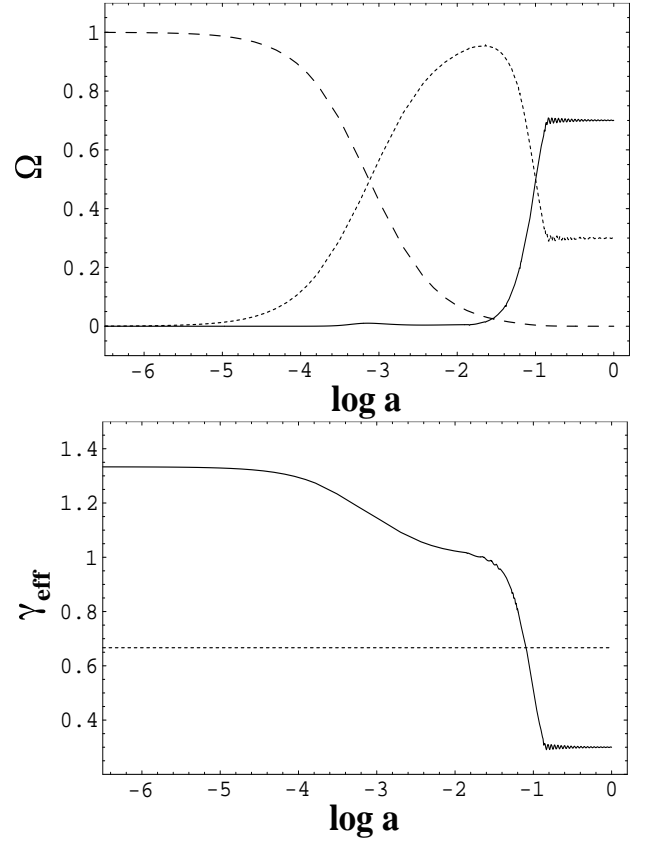


FIG. 10: Top panel shows the evolution of  $\Omega_R$  (dashed),  $\Omega_M$  (dotted) and  $\Omega_\phi$  (solid) with  $\lambda = 30$  in the case where the coupling  $Q$  changes rapidly from  $Q_1 = 0$  to  $Q_2 = 57.15$ . Bottom panel shows the evolution of the effective equation of state  $\gamma_{\text{eff}} = w_{\text{eff}} + 1$ . It first equals  $1/3$ , then goes down to  $1$ , and approaches  $-0.7$  with an accelerated expansion. From Ref. [308].

with pure kinematic solutions  $x = \pm 1/\sqrt{c_0}$  (which exists for  $c_0 > 0$ ).

In Ref. [233] it was shown that a sequence of the  $\phi$ MDE and the scaling attractor is not realized for the model (311) if  $n$  are *integers*. The main reason is that there exist two singularities at  $x = 0$  and  $A^{-1} = 0$ , where  $A = g + 5Yg' + 2Y^2g'' = \rho_{,X}^{-1}$  is related to a sound speed via  $c_s^2 = Ap_{,X}$ . For the fractional Lagrangian

$$g(Y) = c_0 - cY^{-u}, \quad (313)$$

we have

$$\left|\frac{dy/dN}{dx/dN}\right|_{x \rightarrow 0} \rightarrow \infty, \quad (u \neq 1). \quad (314)$$

Thus the solutions can not pass the line  $x = 0$  except for  $u = 1$  (the case of Quintessence with an exponential potential already excluded). When  $c_0 > 0$  this singularity is inevitable to be hit when the solutions move from the  $\phi$ MDE to the accelerated scaling solution [233]. When  $c_0 < 0$  one needs to cross either the singularity

at  $A^{-1} = 0$  or at  $A = 0$  ( $x = 0$ ), but both cases are forbidden.

There is an interesting case in which a sequence of the nearly matter dominated era and the accelerated scaling solution can be realized. This is the model (313) with  $0 < u < 1$ . Although the phase space is separated into positive and negative abscissa subspaces because of the condition (314), one can have a matter-dominated era in the region  $x > 0$  followed by the scaling attractor with  $x > 0$  (when  $Q$  is positive). It would be of interest to investigate whether this special case satisfies observational constraints.

### E. Varying mass neutrino scenario

There is an interesting model called mass-varying neutrinos (MaVaNs) in which neutrinos are coupled to dark energy [310, 311]. This makes use of the fact that the scale of neutrino mass-squared differences  $(0.01 \text{ eV})^2$  is similar to the scale of dark energy  $(10^{-3} \text{ eV})^4$ . According to this scenario the neutrino mass depends upon a scalar field called *acceleron*,  $\mathcal{A}$ , which has an instantaneous minimum that varies slowly as a function of the density of neutrinos. The mass of the acceleron can be heavy relative to the Hubble rate unlike the case of a slowly rolling light scalar field (Quintessence).

The energy density of non-relativistic neutrinos is given by  $\rho_\nu = n_\nu m_\nu$ , where  $n_\nu$  and  $m_\nu$  are the number density and the mass of neutrinos, respectively. When the acceleron field has a potential  $V_0(\mathcal{A})$ , the total effective potential for MaVaNs is

$$V = n_\nu m_\nu(\mathcal{A}) + V_0(\mathcal{A}). \quad (315)$$

Even if the potential  $V_0(\mathcal{A})$  does not have a minimum, the presence of the  $n_\nu m_\nu(\mathcal{A})$  term induces an instantaneous minimum that varies with time. Since  $\partial V / \partial \mathcal{A} = 0$  at the potential minimum, we obtain

$$n_\nu = -\frac{\partial V_0}{\partial m_\nu}, \quad (316)$$

if  $\partial m_\nu / \partial \mathcal{A} \neq 0$ .

Neglecting the contribution of the kinetic energy of the acceleron field, the energy density and the pressure density of the system is given by  $\rho \simeq n_\nu m_\nu + V_0$  and  $p \simeq -V_0$ . Hence the equation of state for the neutrino/acceleron system is

$$w = \frac{p}{\rho} = -1 + \frac{n_\nu m_\nu}{V}. \quad (317)$$

Then we have  $w \simeq -1$  provided that the energy density of neutrinos is negligible relative to that of the acceleron field ( $n_\nu m_\nu \ll V_0$ ).

The cosmological consequences of this scenario have been studied by a host of authors, see, e.g., Refs. [312, 313]. Since neutrinos are coupled to dark energy, it is expected that we may find similar cosmological evolution

to the one obtained in previous subsections of coupled dark energy. One difference from the discussions in previous subsections is that neutrinos should be described by a distribution function  $f(x^i, p^i, t)$  in phase space instead of being treated by a fluid [313]. When neutrinos are collisionless,  $f$  is not dependent on time  $t$ . Solving a Boltzman equation we obtain the energy density of neutrinos, as

$$\rho_\nu = \frac{1}{a^4} \int q^2 dq d\Omega \epsilon f_0(q), \quad (318)$$

together with the pressure density

$$p_\nu = \frac{1}{3a^4} \int q^2 dq d\Omega \epsilon f_0(q) \frac{q^2}{\epsilon}, \quad (319)$$

where  $f_0$  is a background neutrino Fermi-Dirac distribution function. Here  $\epsilon$  is defined by  $\epsilon^2 = q^2 + m_\nu^2 a^2$ , where  $q$  is the comoving momentum.

If neutrinos decouple while they are still relativistic, the phase-space density is the function of  $q$  only. When the mass of neutrinos depends on a field  $\phi$ , Eqs. (318) and (319) give [313]

$$\dot{\rho}_\nu + 3H(\rho_\nu + p_\nu) = \frac{\partial \ln m_\nu}{\partial \phi} (\rho_\nu - 3p_\nu) \dot{\phi}. \quad (320)$$

For non-relativistic neutrinos ( $p_\nu = 0$ ), comparison of Eq. (320) with Eq. (224) shows that the coupling between neutrinos and dark energy is given by  $Q(\phi) = \partial \ln m_\nu / \partial \phi$ . Hence we can apply the results of cosmological evolution in previous subsections to the neutrinos coupled to dark energy. In Refs. [313] the effect of mass-varying neutrinos on CMB and LSS was studied as well as cosmological background evolution in the case where a light scalar field (Quintessence) is coupled to massive neutrinos. This is somewhat different from the original MaVaNs scenario in which the mass of the acceleron field is much larger than the Hubble rate. See Refs. [314] for other models of coupled dark energy.

### F. Dark energy through brane-bulk energy exchange

In [315] the authors investigate the brane cosmological evolution involving a different method of energy exchange, this one being between the brane and the bulk, in the context of a non-factorizable background geometry with vanishing effective cosmological constant on the brane. A number of brane cosmologies are obtained, depending on the mechanism underlying the energy transfer, the equation of state of brane-matter and the spatial topology. Of particular note in their analysis is that accelerating eras are generic features of their solutions. The driving force behind the observed cosmic acceleration is due to the flow of matter from the bulk to the brane.

The observational constraints on these type of models in which the bulk is not empty has been explored in

[316]. Allowing for the fact that the effect of this energy exchange is to modify the evolution of matter fields for an observer on the brane the authors determine the constraints from various cosmological observations on the flow of matter from the bulk into the brane. Intriguingly they claim that a  $\Lambda = 0$  cosmology to an observer in the brane is allowed which satisfies standard cosmological constraints including the CMB temperature fluctuations, Type Ia supernovae at high redshift, and the matter power spectrum. Moreover it can account for the observed suppression of the CMB power spectrum at low multipoles. The cosmology associated with these solutions predicts that the present dark-matter content of the universe may be significantly larger than that of a  $\Lambda$ CDM cosmology, its influence, being counterbalanced by the dark-radiation term.

This is an interesting approach to dark energy, is well motivated by brane dynamics and has generated quite a bit of interest over the past few years [317, 318] and for a review see [319].

## X. DARK ENERGY AND VARYING ALPHA

In this section we investigate a possible way to distinguish dynamical dark energy models from a cosmological constant— through temporal variation of the effective fine structure constant  $\alpha$ . This is just one aspect of the more general approach of allowing for the variation of constants in general (such as for example the dilaton, Newton’s constant and possibly the speed of light). For a detailed overview of the Fundamental constants and their variation see the excellent review of Uzan [320]<sup>4</sup>.

In 2001, Webb *et al.* [322] reported observational evidence for the change of  $\alpha$  over a cosmological time between  $z \simeq 0.5$  and  $z = 0$ . Now, there are a number of existing constraints on the allowed variation of  $\alpha$ . For example, the Oklo natural fission reactor [323] found the variation  $\Delta\alpha/\alpha \equiv (\alpha - \alpha_0)/\alpha_0$  is constrained by  $-0.9 \times 10^{-7} < \Delta\alpha/\alpha < 1.2 \times 10^{-7}$  at a redshift  $z \sim 0.16$  [323] (here  $\alpha_0$  is the present value of the fine structure constant). The absorption line spectra of distance quasars [324, 325, 326] provides another route. In Ref. [327] it is claimed that  $\Delta\alpha/\alpha = (-0.574 \pm 0.102) \times 10^{-5}$  for  $0.2 < z < 3.7$ . The recent detailed analysis of high quality quasar spectra [328] gives the lower variation  $\Delta\alpha/\alpha = (-0.06 \pm 0.06) \times 10^{-5}$  over the redshift range  $0.4 < z < 2.3$ . Although there still remains the possibility of systematic errors [329], should the variation of  $\alpha$  hold up to closer scrutiny, it will have important implications for cosmology.

One powerful conclusion that follows from a varying  $\alpha$  is the existence of massless or nearly massless fields coupled to gauge fields. Quite independently there is a

need for a light scalar field to explain the origin of dark energy. Thus it is natural to consider that quintessence or another type of scalar field model could be responsible for the time variation of  $\alpha$ . In fact many authors have studied the change of  $\alpha$  based on quintessence by assuming specific forms for the interaction between a field  $\phi$  and an electromagnetic field  $F_{\mu\nu}$  [330, 331, 332, 333]. Now in general the inclusion of a non-renormalizable interaction of the form  $B_F(\phi)F_{\mu\nu}F^{\mu\nu}$  at the quantum level requires the existence of a momentum cut-off  $\Lambda_{UV}$ . Unfortunately, any particle physics motivated choice of  $\Lambda_{UV}$  destabilizes the potential of quintessence, i.e., it could induce a mass term much larger than the required one of order  $H_0$ . However, because the nature of this fine-tuning is similar to the one required for the smallness of the cosmological constant, we are open to proceed hoping that both problems could be resolved simultaneously in future.

Originally Bekenstein [334] introduced the exponential form for the coupling of the scalar field to the electromagnetic field which in practice can always be taken in the linear form  $B_F(\phi) = 1 - \zeta\kappa\phi$ . From the tests of the equivalence principle the coupling is constrained to be  $|\zeta| < 10^{-3}$  [331]. Although the existence of the coupling  $\zeta$  alone is sufficient to lead to the variation of  $\alpha$ , the resulting change of  $\alpha$  was found to be of order  $10^{-10}$ - $10^{-9}$  [334], which is too small to be compatible with observations. This situation is improved by including a potential for the field  $\phi$  or by introducing a coupling of order 1 between the field and dark matter [331, 332].

In the next subsection we shall study the time variation of  $\alpha$  for a minimal Bekenstein-like coupling in the presence of a Quintessence potential. We then discuss the case of a Dirac-Born-Infeld dark energy model in which the tachyon is naturally coupled to electromagnetic fields [335]. In this case we do not need an ad-hoc assumption for the form of the coupling.

### A. Varying alpha from quintessence

Let us consider an interaction between a Quintessence field  $\phi$  and an electromagnetic field  $F_{\mu\nu}$ , whose Lagrangian density is given by

$$\mathcal{L}_F(\phi) = -\frac{1}{4}B_F(\phi)F_{\mu\nu}F^{\mu\nu}. \quad (321)$$

We shall assume a linear dependence of the coupling  $B_F(\phi)$ :

$$B_F(\phi) = 1 - \zeta\kappa(\phi - \phi_0), \quad (322)$$

where the subscript 0 represents the present value of the quantity. We note that this is just one example for the form of the coupling. The fine structure “constant”  $\alpha$  is inversely proportional to  $B_F(\phi)$ , which can be expressed by  $\alpha = \alpha_0/B_F(\phi)$ . Then the variation of  $\alpha$  is given by

$$\frac{\Delta\alpha}{\alpha} \equiv \frac{\alpha - \alpha_0}{\alpha_0} = \zeta\kappa(\phi - \phi_0). \quad (323)$$

<sup>4</sup> See also Ref.[321] on the related theme.

If one uses the information of quasar absorption lines,  $\Delta\alpha/\alpha \simeq -10^{-5}$ , around  $z = 3$  [322], the value of the coupling  $\zeta$  can be evaluated as

$$\zeta \simeq -\frac{10^{-5}}{\kappa\phi(z=3) - \kappa\phi(z=0)}. \quad (324)$$

The bound of atomic clocks is given by  $|\dot{\alpha}/\alpha| < 4.2 \times 10^{-15} \text{ yr}^{-1}$  at  $z = 0$  [336]. In our model the ratio of the variation of  $\alpha$  around the present can be evaluated as

$$\frac{\dot{\alpha}}{\alpha} \simeq \zeta \kappa \dot{\phi} \simeq -\zeta \frac{d(\kappa\phi)}{d(1+z)} H_0. \quad (325)$$

As an example let us consider an exponential potential  $V(\phi) = V_0 e^{-\kappa\lambda\phi}$  in the presence of the coupling  $Q$  between dark energy and dark matter. The universe can reach the scaling attractor (d) in Table V with an accelerated expansion. Since  $x = \sqrt{6}(1+w_m)/2(Q+\lambda)$  in the scaling regime, we find

$$\kappa(\phi - \phi_0) = -\frac{3\gamma}{\lambda + Q} \ln(1+z), \quad (326)$$

where  $\gamma = 1 + w_m$  and  $\phi_0$  is the present value of the field. Substituting this for Eq. (323), one obtains

$$\frac{\Delta\alpha}{\alpha} = -\frac{3\zeta\gamma}{\lambda + Q} \ln(1+z). \quad (327)$$

From Eq. (324) the coupling  $\zeta$  consistent with quasar absorption lines is

$$\zeta = \frac{10^{-5}}{\ln(4)} \frac{\lambda + Q}{3\gamma}. \quad (328)$$

Substituting Eqs. (326) and (328) for Eq. (325), we obtain

$$\frac{\dot{\alpha}}{\alpha} = \frac{10^{-5}}{\ln 4} H_0 \simeq 4.8 \times 10^{-16} \text{ yr}^{-1}. \quad (329)$$

This satisfies the constraint of atomic clocks.

The constraint coming from the test of the equivalence principle corresponds to  $|\zeta| < 10^{-3}$ . When  $Q = 0$  this gives the upper bound for  $\lambda$  from Eq. (328). If we also use the constraint (262) coming from nucleosynthesis, we can restrict the value of  $\lambda$  to be  $4.5 < \lambda < 415$ . Of course the uncoupled case is not viable to explain the accelerated expansion at late times. If there exists another source for dark energy, this induces errors in the estimation of the evolution of  $\alpha$  and the coupling  $\zeta$ .

The acceleration of the universe is realized in the presence of the coupling  $Q$ . In Sec. IX we showed that a matter dominated era does not last sufficiently long for large-scale structure to grow if the field  $\phi$  is coupled to all dark matter and drives an accelerated expansion with a scaling attractor  $\Omega_\phi \simeq 0.7$ . In order to avoid this problem we shall assume the existence of two components of dark matter in which one is coupled to the scalar field

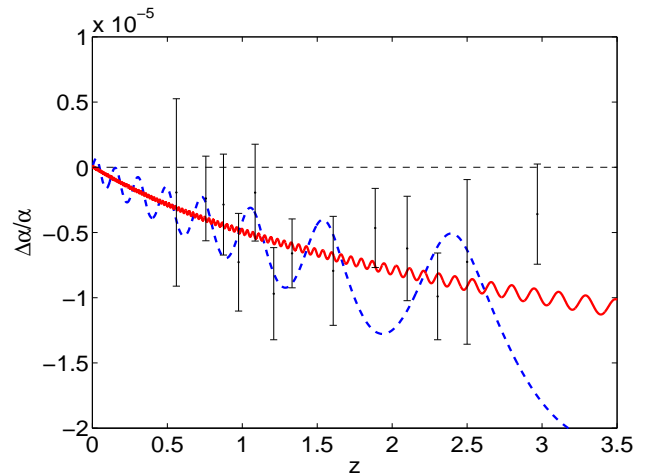


FIG. 11: Evolution of  $\Delta\alpha/\alpha$  for a coupled quintessence model with contributions  $\Omega_{m,c} = 0.05$  and  $\Omega_\phi = 0.7$  today. The solid and dashed curves correspond to  $\lambda = 100$  and  $\lambda = 10$ , respectively. We also show observational data with error bars. We thank Nelson J. Nunes for providing us this figure.

and another is not. This is an alternative approach to introducing a non-linear coupling given in Eq. (302).

In Fig. 11 we plot the evolution of  $\Delta\alpha/\alpha$  for two different values of  $\lambda$  when the coupled component of dark matter is  $\Omega_{m,c} = 0.05$  today. The coupling  $Q$  is determined by the condition that the scaling attractor corresponds to  $\Omega_{m,c} = 0.05$  and  $\Omega_\phi = 0.7$  [333]. The oscillation of  $\Delta\alpha/\alpha$  in Fig. 11 comes from the fact that the solution actually approaches a scaling attractor. For smaller values of  $\lambda$  we find that the attractor is reached at a later stage, which leads to the heavy oscillation of  $\Delta\alpha/\alpha$ .

From Eq. (328) we expect that the presence of the coupling  $Q$  gives larger values of  $\zeta$  compared to the uncoupled case. This is actually the case even when a part of dark matter is coupled to the scalar field. The bound of the equivalence principle  $|\zeta| < 10^{-3}$  can be satisfied provided that we choose smaller values of  $\lambda$  [333].

In Ref. [333] the evolution of  $\Delta\alpha/\alpha$  was obtained for a number of other quintessence potentials, in which they can in principle be consistent with observations if we fine-tune model parameters. We caution that there is a freedom to choose the coupling  $B_F(\phi)$  other than the one given in Eq. (322) and the evolution of  $\Delta\alpha/\alpha$  crucially depends upon the choice of this coupling. In Ref. [337] the possibility of reconstructing dark energy equation of state from varying  $\alpha$  was studied for the coupling given by Eq. (322); see also Ref. [338] on the similar theme.

## B. Varying alpha from tachyon fields

The change of  $\alpha$  may be explained in other types of scalar-field dark energy models such as those originating from tachyon fields. In fact a Dirac-Born-Infeld type effective 4-dimensional action given below naturally leads

to a coupling between a tachyon field  $\varphi$  and a Maxwell tensor  $F_{\mu\nu}$  [335]:

$$S = - \int d^4x \tilde{V}(\varphi) \sqrt{-\det(g_{\mu\nu} + \partial_\mu \varphi \partial_\nu \varphi + 2\pi\alpha' F_{\mu\nu})}, \quad (330)$$

where  $\tilde{V}(\varphi)$  is the potential of the field. Let us consider a situation in which a brane is located in a ten-dimensional spacetime with a warped metric [172]

$$ds_{10}^2 = \beta g_{\mu\nu}(x) dx^\mu dx^\nu + \beta^{-1} \tilde{g}_{mn}(y) dy^m dy^n, \quad (331)$$

where  $\beta$  is a warp factor.

For this metric the action (330) is written in the form

$$S = - \int d^4x V(\phi) \sqrt{-\det(g_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi + 2\pi\alpha' \beta^{-1} F_{\mu\nu})}, \quad (332)$$

where

$$\phi = \varphi/\sqrt{\beta}, \quad V(\phi) = \beta^2 \tilde{V}(\sqrt{\beta}\phi). \quad (333)$$

The warped metric (331) changes the mass scale on the brane from the string mass scale  $M_s = 1/\sqrt{\alpha'}$  to an effective mass which is  $m_{\text{eff}} = \sqrt{\beta} M_s$ . The expansion of the action (332) to second order in the gauge field, for an arbitrary metric, becomes

$$S \simeq \int d^4x \left[ -V(\phi) \sqrt{-\det(g_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi)} + \frac{(2\pi\alpha')^2 V(\phi)}{4\beta^2} \sqrt{-g} \text{Tr}(g^{-1} F g^{-1} F) \right]. \quad (334)$$

We have dropped other second order terms that involve the derivative of the field  $\phi$ . This should be justified provided that the kinetic energy of the field is relatively small compared to the potential energy of it (as it happens in the context of dark energy).

Comparing the above action with the standard Yang-Mills action, one finds that the effective fine-structure constant  $\alpha$  is given by

$$\alpha \equiv g_{\text{YM}}^2 = \frac{\beta^2 M_s^4}{2\pi V(\phi)}, \quad (335)$$

which depends on the field  $\phi$ . The variation of  $\alpha$  compared to the present value  $\alpha_0$  is given as

$$\frac{\Delta\alpha}{\alpha} = \frac{V(\phi_0)}{V(\phi)} - 1. \quad (336)$$

For the exponential potential  $V(\phi) = V_0 e^{-\mu\phi}$ , we get

$$\frac{\Delta\alpha}{\alpha} = e^{\mu(\phi - \phi_0)} - 1, \quad (337)$$

and for the massive rolling scalar potential  $V(\phi) = V_0 e^{\frac{1}{2} M^2 \phi^2}$  considered in Ref. [172], we have

$$\frac{\Delta\alpha}{\alpha} = e^{-\frac{1}{2} M^2 (\phi^2 - \phi_0^2)} - 1. \quad (338)$$

We recall that the present value of fine structure constant is  $\alpha_0 = 1/137$ . Since the potential energy of  $\phi$  at present is estimated as  $3H_0^2 \simeq 8\pi V(\phi_0)/m_{\text{pl}}^2$ , one finds the expression for the warp factor:

$$\beta^2 \simeq \frac{3}{548} \left( \frac{H_0}{M_s} \right)^2 \left( \frac{m_{\text{pl}}}{M_s} \right)^2. \quad (339)$$

When  $M_s \sim m_{\text{pl}}$  we have  $\beta \sim 10^{-62}$ .

The model parameters in the tachyon potentials are related to the string scale  $M_s$  and the brane tension  $T_3$  if they are motivated by string theory. The exponential potential  $V(\phi) = V_0 e^{-\mu\phi}$  introduced above appears in the context of the D3 and  $\bar{D}3$  branes [339]. The tachyon potential for the coincident D3- $\bar{D}3$  branes is twice the potential for the non-BPS D3-brane [340]. The latter is given by  $V(\phi) = 2\beta^2 T_3 / \cosh(\sqrt{\beta} M_s \phi)$  [341], where  $\beta$  is a warp factor at the position of the D3- $\bar{D}3$  in the internal compact space and  $T_3$  is the tension of the 3 branes. Then the potential behaves as  $V(\phi) = \beta^2 T_3 e^{-\sqrt{\beta} M_s \phi}$  for large  $\phi$ , which has a correspondence

$$V_0 = \beta^2 T_3, \quad \mu = \sqrt{\beta} M_s. \quad (340)$$

By using the equations (339) and (340), it was shown in Ref. [335] that the resulting value of  $\Delta\alpha/\alpha$  evaluated by Eq. (337) is  $|\Delta\alpha/\alpha| \gg 1$  for  $z = \mathcal{O}(1)$ , which contradicts the observational bounds, and hence implies that these particular string motivated models do not work as sources of dark energy.

Meanwhile for a rolling massive scalar potential  $V(\phi) = V_0 e^{\frac{1}{2} M^2 \phi^2}$  with parameters constrained by string theory ( $V_0 \sim \beta^2 T_3$  and  $M \sim \sqrt{\beta} M_s$ ), it is possible to explain the observed values of  $\Delta\alpha/\alpha$  at  $z = \mathcal{O}(1)$ . In this case the field oscillates around the potential minimum at  $\phi = 0$  and is given by  $\phi \simeq \Phi \cos(Mt)$  for  $\dot{\phi}^2 \ll 1$ , where  $\Phi$  is the amplitude of oscillation. The condition of an accelerated expansion for the tachyon case is  $\dot{\phi}^2 < 2/3$ . Taking the time average of  $\dot{\phi}^2$ , we find that  $M^2 \Phi^2 < 4/3$ . This then gives

$$\left| \frac{\Delta\alpha}{\alpha} \right| \simeq \frac{1}{2} M^2 |\phi^2 - \phi_0^2| \lesssim \frac{1}{2} M^2 \Phi^2. \quad (341)$$

It is possible to have  $|\Delta\alpha/\alpha| = 10^{-6}-10^{-5}$  if  $|M\Phi|$  is of order  $10^{-3}-10^{-2}$ . When  $M_s \sim m_{\text{pl}}$ , we have  $\beta \sim 10^{-62}$ , in which case the mass  $M = \sqrt{\beta} M_s$  is much larger  $H_0 \sim 10^{-42}$  GeV. Hence the field oscillates for many times while the universe evolves from  $z = \mathcal{O}(1)$  to present, which also leads to the oscillation of  $\Delta\alpha/\alpha$ .

In Ref. [335] it was found that inverse power-law potentials  $V(\phi) = M^{4-n} \phi^{-n}$  are not compatible with the observational data of  $\Delta\alpha/\alpha$  if one uses the mass scale obtained in the context of string theory. Thus a varying  $\alpha$  provides a powerful tool with which to constrain tachyon dark energy models.

## XI. PERTURBATIONS IN A UNIVERSE WITH DARK ENERGY

In order to confront models of dark energy with observations of say the Cosmic Microwave Background (CMB) and large-scale structure (LSS), it is important to study the evolution of density perturbations in a universe containing dark energy (see e.g., Refs. [342, 343, 344, 345]). Its presence can give rise to features such as the Integrated Sachs-Wolfe (ISW) effect, which alters the CMB power spectrum. In this section we provide the perturbation equations in a dark energy dominated universe with a barotropic fluid. The system we study covers most of scalar-field dark energy models including scalar-tensor theories. We shall also consider perturbations in coupled dark energy scenarios and derive analytic expressions for the solution of matter perturbations.

### A. Perturbation equations

A perturbed metric about a FRW background has the following form for scalar perturbations in an arbitrary gauge [346]:

$$ds^2 = -(1 + 2A)dt^2 + 2a\partial_i B dx^i dt + a^2 [(1 + 2\psi)\delta_{ij} + 2\partial_{ij} E] dx^i dx^j, \quad (342)$$

where  $\partial_i$  represents the spatial partial derivative  $\partial/\partial x^i$ . We will use lower case latin indices to run over the 3 spatial coordinates. We do not consider tensor and vector parts of perturbations.

The model we study is described by the following very general action

$$\begin{aligned} S &= \int d^4x \sqrt{-g} \left[ \frac{F(\phi)}{2} R + p(\phi, X) + \mathcal{L}_m \right] \\ &\equiv \int d^4x \sqrt{-g} \left[ \frac{1}{2} f(R, \phi, X) + \mathcal{L}_m \right], \end{aligned} \quad (343)$$

where  $F(\phi)$  is a function of a scalar field  $\phi$ ,  $p(\phi, X)$  is a function of  $\phi$  and  $X = -(1/2)(\nabla\phi)^2$ , and  $\mathcal{L}_m$  is the Lagrangian density for a barotropic perfect fluid. The action (343) includes a wide variety of gravity theories such as Einstein gravity, scalar-tensor theories and low-energy effective string theories.

The background equations for this system are given by

$$H^2 = \frac{1}{3F}(2Xp_{,X} - p - 3H\dot{F} + \rho_m), \quad (344)$$

$$\dot{H} = -\frac{1}{2F}(2Xp_{,X} + \ddot{F} - H\dot{F} + \rho_m + p_m), \quad (345)$$

$$\frac{1}{a^3}(a^3\dot{\phi}p_{,X})^\bullet - p_{,\phi} - \frac{1}{2}F_{,\phi}R = 0, \quad (346)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \quad (347)$$

We define the equation of state for the field  $\phi$ , as

$$w_\phi = \frac{p + \ddot{F} + 2H\dot{F}}{2Xp_{,X} - p - 3H\dot{F}}. \quad (348)$$

We have not implemented the coupling between the field and the fluid. The case of coupled dark energy will be discussed later. Note that there is another way of defining  $w_\phi$  when we confront it with observations, see e.g., Ref. [347].

We define several gauge-invariant variables of cosmological perturbations. Under a gauge transformation:  $t \rightarrow t + \delta t$  and  $x^i \rightarrow x^i + \delta^{ij}\partial_j \delta x$ , the scalar perturbations transform as [346]

$$\begin{aligned} A &\rightarrow A - \dot{\delta}t, & B &\rightarrow B - a^{-1}\delta t + a\dot{\delta}x, \\ \psi &\rightarrow \psi - H\delta t, & E &\rightarrow E - \delta x, \end{aligned} \quad (349)$$

together with the transformation of the field perturbation:

$$\delta\phi \rightarrow \delta\phi - \dot{\phi}\delta t. \quad (350)$$

The uniform-field gauge corresponds to a gauge transformation to a frame such that  $\delta\phi = 0$ , leaving the following gauge-invariant variable:

$$\mathcal{R} \equiv \psi - \frac{H}{\dot{\phi}}\delta\phi. \quad (351)$$

This is so-called comoving curvature perturbation first introduced by Lukash [348]. Meanwhile the longitudinal gauge corresponds to a transformation to a frame such that  $B = E = 0$ , giving the gauge-invariant variables:

$$\Phi \equiv A - \frac{d}{dt} \left[ a^2(\dot{E} + B/a) \right], \quad (352)$$

$$\Psi \equiv -\psi + a^2 H(\dot{E} + B/a). \quad (353)$$

The above two gauges are often used when we discuss cosmological perturbations. One can construct other gauge invariant variables, see e.g., [349].

The energy-momentum tensor can be decomposed as

$$\begin{aligned} T_0^0 &= -(\rho + \delta\rho), & T_\alpha^0 &= -(\rho + p)v_{,\alpha}, \\ T_\beta^\alpha &= (p + \delta p)\delta_\beta^\alpha + \Pi_\beta^\alpha, \end{aligned} \quad (354)$$

where  $\Pi_\beta^\alpha$  is a tracefree anisotropic stress. Note that  $\rho, \delta\rho$  e.t.c. can be written by the sum of the contribution of field and fluid, i.e.,  $\rho = \rho_\phi + \rho_m$  and  $\delta\rho = \delta\rho_\phi + \delta\rho_m$ .

We define the following new variables:

$$\chi \equiv a(B + a\dot{E}), \quad \xi \equiv 3(HA - \dot{\psi}) - \frac{\Delta}{a^2}\chi. \quad (355)$$

Considering perturbed Einstein equations at linear order

for the model (343), we obtain [349] (see also Ref. [350])

$$\frac{\Delta}{a^2}\psi + H\xi = -4\pi G\delta\rho, \quad (356)$$

$$HA - \dot{\psi} = 4\pi Ga(\rho + p)v, \quad (357)$$

$$\dot{\chi} + H\chi - A - \psi = 8\pi G\Pi, \quad (358)$$

$$\dot{\xi} + 2H\xi + \left(3\dot{H} + \frac{\Delta}{a^2}\right)A = 4\pi G(\delta\rho + 3\delta p), \quad (359)$$

$$\begin{aligned} & \delta\dot{\rho}_m + 3H(\delta\rho_m + \delta p_m) \\ &= (\rho_m + p_m) \left( \xi - 3HA + \frac{\Delta}{a}v_m \right), \end{aligned} \quad (360)$$

$$\frac{[a^4(\rho_m + p_m)v_m]^\bullet}{a^4(\rho_m + p_m)} = \frac{1}{a} \left( A + \frac{\delta p_m}{\rho_m + p_m} \right), \quad (361)$$

where

$$\begin{aligned} 8\pi G\delta\rho &= \frac{1}{F} \left[ -\frac{1}{2}(f_{,\phi}\delta\phi + f_{,X}\delta X) + \frac{1}{2}\dot{\phi}^2(f_{,X\phi}\delta\phi \right. \\ & \quad \left. + f_{,XX}\delta X) + f_{,X}\dot{\phi}\delta\dot{\phi} - 3H\delta\dot{F} \right. \\ & \quad \left. + \left(3\dot{H} + 3H^2 + \frac{\Delta}{a^2}\right)\delta F + \dot{F}\xi \right. \\ & \quad \left. + (3H\dot{F} - f_{,X}\dot{\phi}^2)A \right. \\ & \quad \left. + \delta\rho_m - \frac{\delta F}{F}\rho_m \right], \end{aligned} \quad (362)$$

$$\begin{aligned} 8\pi G\delta p &= \frac{1}{F} \left[ \frac{1}{2}(f_{,\phi}\delta\phi + f_{,X}\delta X) + \delta\ddot{F} + 2H\delta\dot{F} \right. \\ & \quad \left. - \left(\dot{H} + 3H^2 + \frac{2}{3}\frac{\Delta}{a^2}\right)\delta F - \frac{2}{3}\dot{F}\xi - \dot{F}\dot{A} \right. \\ & \quad \left. - 2(\ddot{F} + H\dot{F})A + \delta p_m - \frac{\delta F}{F}p_m \right], \end{aligned} \quad (363)$$

$$\begin{aligned} 8\pi G(\rho + p)v &= -\frac{1}{aF} \left[ -\frac{1}{2}f_{,X}\dot{\phi}\delta\phi - \delta\dot{F} \right. \\ & \quad \left. + H\delta F + \dot{F}A - \frac{a}{k}(\rho_m + p_m)v_m \right], \end{aligned} \quad (364)$$

$$8\pi G\Pi = \frac{1}{F}(\delta F - \dot{F}\chi). \quad (365)$$

Here we have  $X = \dot{\phi}^2/2$  and  $\delta X = \dot{\phi}\delta\phi - \dot{\phi}^2 A$ . Note that the definition of the sign of  $X$  is opposite compared to the one given in Ref. [349].

Equations (356)-(361) are written without fixing any gauge conditions (so called ‘‘gauge-ready’’ form [351]). This allows one to choose a temporal gauge condition depending upon a situation one is considering. Readers may be discouraged by rather complicated expressions (362)-(365), but in subsequent discussions we expect that readers will be impressed by beauty of cosmological perturbation theory!

## B. Single-field system without a fluid

Let us first discuss the case in which the barotropic perfect fluid is absent ( $\mathcal{L}_m = 0$ ). For the perturba-

tion system given above it is convenient to choose the uniform-field gauge ( $\delta\phi = 0$ ) and derive the equation for the curvature perturbation  $\mathcal{R}$ . Since  $\delta F = 0$  in this case, Eq. (357) gives

$$A = \frac{\dot{\mathcal{R}}}{H + \dot{F}/2F}. \quad (366)$$

From Eq. (356) together with the use of Eq. (366), we obtain

$$\begin{aligned} \xi &= -\frac{1}{H + \dot{F}/2F} \left[ \frac{\Delta}{a^2}\mathcal{R} \right. \\ & \quad \left. + \frac{3H\dot{F} - Xf_{,X} - 2X^2f_{,XX}}{2F(H + \dot{F}/2F)}\dot{\mathcal{R}} \right]. \end{aligned} \quad (367)$$

Substituting Eq. (345) for Eq. (359), we find

$$\begin{aligned} \dot{\xi} + \left(2H + \frac{\dot{F}}{F}\right)\xi + \frac{3\dot{F}}{2F}\dot{A} \\ + \left[ \frac{3\ddot{F} + 6H\dot{F} + Xf_{,X} + 2X^2f_{,XX}}{2F} + \frac{\Delta}{a^2} \right]A = 0. \end{aligned} \quad (368)$$

Plugging Eqs. (366) and (367) into Eq. (368), we finally get the following differential equation for each Fourier mode of  $\mathcal{R}$ :

$$\ddot{\mathcal{R}} + \frac{\dot{s}}{s}\dot{\mathcal{R}} + c_A^2\frac{k^2}{a^2}\mathcal{R} = 0, \quad (369)$$

where

$$s \equiv \frac{a^3(Xf_{,X} + 2X^2f_{,XX} + 3\dot{F}^2/2F)}{(H + \dot{F}/2F)^2}, \quad (370)$$

$$\begin{aligned} c_A^2 &\equiv \frac{Xf_{,X} + 3\dot{F}^2/2F}{Xf_{,X} + 2X^2f_{,XX} + 3\dot{F}^2/2F} \\ &= \frac{p_{,X} + 3\dot{F}^2/4FX}{\rho_{,X} + 3\dot{F}^2/4FX}. \end{aligned} \quad (371)$$

Here  $\rho_{,X} = p_{,X} + 2Xp_{,XX}$ .

In the large-scale limit ( $c_A^2 k^2 \rightarrow 0$ ) we have the following solution

$$\mathcal{R} = C_1 + C_2 \int \frac{1}{s} dt, \quad (372)$$

where  $C_1$  and  $C_2$  are integration constants. When the field  $\phi$  slowly evolves as in the contexts of dark energy and inflationary cosmology, the second-term can be identified as a decaying mode [73]. Then the curvature perturbation is conserved on super-Hubble scales.

On sub-Hubble scales the sign of  $c_A^2$  is crucially important to determine the stability of perturbations. When  $c_A^2$  is negative, this leads to a violent instability of perturbations. In Einstein gravity where  $F$  is constant,  $c_A^2$  coincides with Eq. (144). In this case  $c_A^2$  vanishes for

$p_{,X} = 0$ . Meanwhile  $w_\phi = -1$  for  $p_{,X} = 0$  from Eq. (348). Hence one has  $c_A^2 = 0$  at cosmological constant boundary ( $w_\phi = -1$ ). This suggests the phantom divide crossing is typically accompanied by the change of the sign of  $c_A^2$ , which leads to the instability of perturbations once the system enters the region  $w_\phi < -1$ . For example in dilatonic ghost condensate model with  $p = -X + ce^{\lambda\phi}X^2$ , we get Eq. (218) and  $c_A^2 = \frac{1-2cY}{1-6cY}$  where  $Y = e^{\lambda\phi}X$ . The cosmological constant boundary corresponds to  $cY = 1/2$ . We find that  $c_A^2$  is negative for  $1/6 < cY < 1/2$  and diverges at  $cY = 1/6$ . The divergence of  $c_A^2$  occurs for  $p_{,X} = 0$ . In scalar-tensor theories ( $\dot{F} \neq 0$ ) the phantom divide ( $w_\phi = -1$ ) does not correspond to the change of the sign of  $c_A^2$ . Hence the perturbations can be stable even in the region  $w_\phi < -1$  [352, 353].

It is worth mentioning that one can calculate the spectrum of density perturbations generated in inflationary cosmology by using the perturbation equation (369) along the line of Ref. [73, 349]. It is really remarkable that the equation for the curvature perturbation reduces to the simple form (369) even for the very general model (343).

### C. Evolution of matter perturbations

We shall study the evolution of perturbations on sub-Hubble scales in the field/fluid system. In particular we wish to derive the equation for matter perturbations defined by  $\delta_m \equiv \delta\rho_m/\rho_m$ . This is important when we place constraints on dark energy from the observation of large-scale galaxy clustering. We assume that the equation of state  $w_m$  is constant.

In the longitudinal gauge ( $B = E = 0$ ), Eqs. (360) and (361) give the following Fourier-transformed equations:

$$\dot{\delta}_m = (1 + w_m) \left( 3\dot{\Psi} - \frac{k}{a}v_m \right), \quad (373)$$

$$\dot{v}_m + (1 - 3w_m)Hv_m = \frac{k}{a} \left( \Phi + \frac{w_m}{1 + w_m}\delta_m \right). \quad (374)$$

Eliminating the  $v_m$  term, we obtain

$$\begin{aligned} \ddot{\delta}_m + H(2 - 3w_m)\dot{\delta}_m + w_m \frac{k^2}{a^2} + (1 + w_m)\frac{k^2}{a^2}\Phi \\ = 3(1 + w_m) \left[ \ddot{\Psi} + (2 - 3w_m)H\dot{\Psi} \right]. \end{aligned} \quad (375)$$

In what follows we shall study the case of a non-relativistic fluid ( $w_m = 0$ ). On scales which are much smaller than the Hubble radius ( $k \gg aH$ ) the contribution of metric perturbations on the RHS of Eq. (375) is neglected, which leads to

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2}{a^2}\Phi \simeq 0. \quad (376)$$

We shall express  $\Phi$  in terms of  $\delta_m$  as the next step. In doing so we use the sub-Horizon approximation in which

the leading terms correspond to those containing  $k^2$  and those with  $\delta_m$ . Then Eq. (356) gives

$$\frac{k^2}{a^2}\Psi \simeq \frac{1}{2F} \left( \frac{k^2}{a^2}\delta F - \delta\rho_m \right). \quad (377)$$

From Eq. (358) one has  $\Psi = \Phi + \delta F/F$ . Hence we obtain

$$\Phi \simeq -\frac{1}{2F} \frac{a^2}{k^2} \rho_m \delta_m - \frac{F_{,\phi}}{2F} \delta\phi. \quad (378)$$

The variation of the scalar-field action in terms of the field  $\phi$  gives the following perturbation equation [349]:

$$\begin{aligned} f_{,X} \left[ \delta\ddot{\phi} + \left( 3H + \frac{\dot{p}_{,X}}{p_{,X}} \right) \delta\dot{\phi} + \frac{k^2}{a^2} \delta\phi - \dot{\phi}(3\dot{\Psi} + \dot{\Phi}) \right] \\ - 2f_{,\phi}\Phi + \frac{1}{a^3} (a^3 \dot{\phi} \delta f_{,X})^\bullet - \delta f_{,\phi} = 0. \end{aligned} \quad (379)$$

Note that this equation can be also obtained from Eqs. (356)-(359). Here  $\delta f$  is given by

$$\delta f = f_{,\phi}\delta\phi + 2p_{,X}\delta X + F_{,\phi}\delta R, \quad (380)$$

where

$$\begin{aligned} \delta R = 2 \left[ -\dot{\xi} - 4H\xi + \left( \frac{k^2}{a^2} - 3\dot{H} \right) \Phi - 2\frac{k^2}{a^2}\Psi \right] \\ \simeq 2\frac{k^2}{a^2}(\Phi - 2\Psi). \end{aligned} \quad (381)$$

Then under the sub-horizon approximation Eq. (379) gives

$$\delta\phi \simeq \frac{F_{,\phi}}{p_{,X}}(\Psi - 2\Phi). \quad (382)$$

Using the relation  $\Psi = \Phi + \delta F/F$  and  $\delta F/F = (F_{,\phi}/F)\delta\phi$ , we find

$$\delta\phi \simeq -\frac{FF_{,\phi}}{Fp_{,X} + 2F_{,\phi}^2}\Phi. \quad (383)$$

From Eqs. (378) and (383) the gravitational potential can be expressed in terms of  $\delta_m$ , as

$$\Phi \simeq -\frac{a^2}{k^2} \frac{Fp_{,X} + 2F_{,\phi}^2}{F(2Fp_{,X} + 3F_{,\phi}^2)} \rho_m \delta_m. \quad (384)$$

Substituting this relation for Eq. (376), we finally obtain the equation for matter perturbations on sub-Hubble scales:

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\text{eff}}\rho_m\delta_m = 0, \quad (385)$$

where

$$G_{\text{eff}} = \frac{Fp_{,X} + 2F_{,\phi}^2}{4\pi F(2Fp_{,X} + 3F_{,\phi}^2)}. \quad (386)$$



For a massless scalar field,  $G_{\text{eff}}$  corresponds to the effective gravitational constant measured by the gravity between two test masses.

Equation (385) was first derived by Boisseau *et al.* [354] in the model  $p = X - V(\phi)$  to reconstruct scalar-tensor theories from the observations of LSS (see also Ref. [355]). We have shown that this can be generalized to a more general model (343). We note that in Einstein gravity ( $F = 1/8\pi G$ ) the effective gravitational constant reduces to  $G$ . Hence we recover the standard form of the equation for dust-like matter perturbations.

In Einstein gravity Eqs. (384) and (385) yield

$$\Phi = -\frac{3a^2}{2k^2}H^2\Omega_m\delta_m, \quad (387)$$

$$\frac{d^2\delta_m}{dN^2} + \left(2 + \frac{1}{H} \frac{dH}{dN}\right) \frac{d\delta_m}{dN} - \frac{3}{2}\Omega_m\delta_m = 0, \quad (388)$$

where  $N = \ln a$ . When  $\Omega_m$  is constant, the solution for Eq. (376) is given by

$$\delta_m = c_+ a^{n_+} + c_- a^{n_-}, \quad (389)$$

where  $c_{\pm}$  are integration constants and

$$n_{\pm} = \frac{1}{4} \left[ -1 \pm \sqrt{1 + 24\Omega_m} \right]. \quad (390)$$

In the matter dominated era ( $\Omega_m \simeq 1$ ) we find that  $n_+ = 1$  and  $n_- = -3/2$ . Hence the perturbations grow as  $\delta_m \propto a$ , which leads to the formation of galaxy clustering. In this case the gravitational potential is constant, i.e.,  $\Phi \propto a^2 \rho_m \delta_m \propto a^0$  from Eq. (387). In the presence of dark energy  $\Omega_m$  is smaller than 1, which leads to the variation of  $\Phi$ . This gives rise to a late-time ISW effect in the temperature anisotropies when the universe evolves from the matter dominated era to a dark energy dominated era.

We should mention that there exist isocurvature perturbations [345, 356, 357, 358, 359] in the field/fluid system, which generally leads to the variation of the curvature perturbation on super-Hubble scales. In order to confront with CMB we need to solve the perturbation equations without using the sub-horizon approximation. We note that a number of authors showed an interesting possibility to explain the suppression of power on largest scales observed in the CMB spectrum by accounting for a correlation between adiabatic and isocurvature perturbations [360, 361, 362].

#### D. Perturbations in coupled dark energy

At the end of this section we shall consider a coupled dark energy scenario in which the field is coupled to the matter fluid with a coupling  $Q$  studied in Sec. VII. The action we study is given by Eq. (221). The perturbation equations in the presence of the coupling  $Q$  are presented

in Ref. [351] in a gauge-ready form. Taking a similar procedure as in the uncoupled case, we obtain [234, 363]

$$\begin{aligned} \frac{d^2\delta_m}{dN^2} + \left(2 + \frac{1}{H} \frac{dH}{dN} + \sqrt{6}Qx\right) \frac{d\delta_m}{dN} \\ - \frac{3}{2}\Omega_m \left(1 + 2\frac{Q^2}{p,x}\right) \delta_m = 0, \end{aligned} \quad (391)$$

where  $x = \dot{\phi}/\sqrt{6}H$  (here we set  $\kappa^2 = 1$ ). Note that the gravitational potential satisfies Eq. (387) in this case as well.

The readers who are interested in the details of the derivation of this equation may refer to the references [363] (see also [364]). When  $Q = 0$ , Eq. (391) reduces to Eq. (388). The presence of the coupling  $Q$  leads to different evolution of  $\delta_m$  compared to the uncoupled case. In what follows we shall study two cases in which analytic solutions can be derived.

##### 1. Analytic solutions in scalar-field matter dominant stage

First we apply the perturbation equation (376) to the coupled quintessence scenario ( $p = X - ce^{-\lambda\phi}$ ) discussed in Sec. IX. The characteristic feature of this model is that there is a possibility to have an intermediate ‘‘scalar-field matter dominated regime ( $\phi$ MDE)’’ [215] characterized by  $\Omega_\phi = 2Q^2/3$  before the energy density of dark energy grows rapidly, see the case (ii) in Fig. 9. The existence of this stage affects the evolution of matter perturbations compared to the case without the coupling  $Q$ .

This transient regime is realized by the fixed point (a) in Table V, which corresponds to

$$x = -\frac{\sqrt{6}}{3}Q, \quad \Omega_\phi = \frac{2}{3}Q^2, \quad w_{\text{eff}} = \frac{2}{3}Q^2. \quad (392)$$

Using the relation

$$\frac{1}{H} \frac{dH}{dN} = -\frac{3}{2}(1 + w_{\text{eff}}), \quad (393)$$

the perturbation equation (391) for the fixed point (392) is given by

$$\frac{d^2\delta_m}{dN^2} + \xi_1 \frac{d\delta_m}{dN} + \xi_2 \delta_m = 0, \quad (394)$$

where

$$\xi_1 = \frac{1}{2} - 3Q^2, \quad \xi_2 = -\frac{3}{2} \left(1 - \frac{3}{2}Q^2\right) (1 + 2Q^2). \quad (395)$$

Here we used  $\Omega_m = 1 - \Omega_\phi$ .

For constant  $\xi_1$  and  $\xi_2$ , the general solution for Eq. (394) is given by Eq. (389) with indices:

$$n_{\pm} = \frac{1}{2} \left[ -\xi_1 \pm \sqrt{\xi_1^2 - 4\xi_2} \right]. \quad (396)$$

For the case (395) we have

$$n_+ = 1 + 2Q^2, \quad n_- = -\frac{3}{2} + Q^2. \quad (397)$$

When  $Q = 0$  this reproduces the result in a matter dominated era discussed in the previous subsection. In the presence of the coupling  $Q$  the perturbations evolve as  $\delta_m \propto a^{1+2Q^2}$ , which means that the growth rate is higher compared to the uncoupled case<sup>5</sup>. Thus the coupling between dark energy and dark matter makes structure formation evolve more quickly.

From Eq. (387) the evolution of the gravitational potential is given by

$$\Phi \propto a^{n_+ - 1 - 3w_{\text{eff}}}, \quad (398)$$

along the fixed point (392). Then from Eqs. (392) and (397) we find that  $\Phi$  is constant. Hence there is no ISW effect by the existence of the  $\phi$ MDE phase. Meanwhile since the effective equation of state given by Eq. (392) differs from the case of  $Q = 0$ , the location of the first acoustic peak is shifted because of the change of an angular diameter distance [215, 366]. In Ref. [365] the coupling is constrained to be  $Q < 0.1$  at a  $2\sigma$  level by using the first year WMAP data.

## 2. Analytic solutions for scaling solutions

It was shown in Ref. [234] that the perturbation equation (391) can be solved analytically in the case of scaling solutions. As we showed in Sec. VII the existence of scaling solutions restricts the Lagrangian of the form  $p = Xg(Xe^{\lambda\phi})$  in Einstein gravity, where  $g$  is an arbitrary function. Then there exists the scaling solution given by Eq. (259) for an arbitrary function  $g(Y)$ . For this scaling solution we also have the following relation

$$Qx = -\frac{\sqrt{6}w_{\text{eff}}}{2}, \quad p_{,X} = \frac{\Omega_\phi + w_{\text{eff}}}{2x^2}, \quad (399)$$

where we have used Eqs. (235), (258) and (259).

Then the equation for matter perturbations (376) is given by Eq. (394) with coefficients:

$$\xi_1 \equiv \frac{1}{2} - \frac{9}{2}w_{\text{eff}}, \quad (400)$$

$$\xi_2 \equiv -\frac{3}{2}(1 - \Omega_\phi) \left( 1 + \frac{6w_{\text{eff}}^2}{\Omega_\phi + w_{\text{eff}}} \right). \quad (401)$$

Since  $w_{\text{eff}}$  and  $\Omega_\phi$  are constants in the scaling regime, we

obtain the analytic solution (389) with indices

$$n_\pm = \frac{1}{4} \left[ 9w_{\text{eff}} - 1 \pm \left\{ (9w_{\text{eff}} - 1)^2 + 24(1 - \Omega_\phi) \left( 1 + \frac{6w_{\text{eff}}^2}{\Omega_\phi + w_{\text{eff}}} \right) \right\}^{1/2} \right]. \quad (402)$$

Remarkably the growth rate of matter perturbations is determined by two quantities  $w_{\text{eff}}$  and  $\Omega_\phi$  only. We stress here that this result holds for any scalar-field Lagrangian which possesses scaling solutions. For a non-phantom scalar field characterized by  $\Omega_\phi + w_{\text{eff}} \equiv \Omega_\phi(1 + w_\phi) > 0$ , Eq. (390) shows that  $n_+ > 0$  and  $n_- < 0$ . Hence  $\delta_m$  (and  $\Phi$ ) grows in the scaling regime, i.e.,  $\delta_m \propto a^{n_+}$ .

In the uncoupled case ( $Q = 0$ ) the effective equation of state is given by  $w_{\text{eff}} = 0$  in Eq. (235). Then we reproduce the indices given by Eq. (390). Since  $0 \leq \Omega_m \leq 1$  in the scaling regime, the index  $n_+$  satisfies  $n_+ \leq 1$  for uncoupled scaling solutions. Equation (402) shows that the index  $n_+$  becomes larger than 1 for  $Q \neq 0$  (i.e.,  $w_{\text{eff}} \neq 0$ ). In Fig. 12 we show the contour plot of  $n_+$  as functions of  $\Omega_\phi$  and  $w_{\text{eff}}$ . The growth rate of perturbations becomes unbounded as we approach the cosmological constant border  $\Omega_\phi + w_{\text{eff}} = 0$ . The large index  $n_+$  is unacceptable from CMB constraints because of a strong ISW effect.

The phantom case corresponds to the parameter range  $\Omega_\phi + w_{\text{eff}} < 0$ . In this case we find that  $n_+$  are either negative real values or complex values with negative real parts. Hence the perturbations decay with damped oscillations. This is understandable, since the repulsive effect of the phantom coupling dissipates the perturbations. In Ref. [234] this phenomenon is called ‘‘phantom damping’’.

The evolution of the gravitational potential  $\Phi$  is also given by Eq. (398). Hence  $\Phi$  is constant for  $n_+ = 3w_{\text{eff}} + 1$ , which corresponds to  $w_{\text{eff}}^\pm = [-2 \pm \sqrt{4 - 3\Omega_\phi}]/3$ . Since  $0 \leq \Omega_\phi \leq 1$  we find  $-1/3 \leq w_{\text{eff}}^+ \leq 0$  and  $-4/3 \leq w_{\text{eff}}^- \leq -1$ . For example we have  $w_{\text{eff}}^+ = -0.207$  and  $w_{\text{eff}}^- = -1.126$  for  $\Omega_\phi = 0.7$ . These values of  $w_{\text{eff}}$  are currently excluded by SN observations, see Fig. 12. Nevertheless it is interesting to find that there exist scaling solutions for which the gravitational potential is exactly constant. We should also mention that values of  $w_{\text{eff}}$  smaller than  $-1$  are allowed if part of the dark matter itself is not coupled [366]. Obviously we require further investigations to constrain scaling dark energy models using observations of the CMB and large-scale structure.

## XII. RECONSTRUCTION OF DARK ENERGY MODELS

We now turn our attention to review the reconstruction of scalar-field dark energy models from observations. This reconstruction is in principle simple for a minimally coupled scalar field with potential  $V(\phi)$  [367, 368, 369, 370, 371]. In fact one can reconstruct

<sup>5</sup> In Ref. [215] there is an error in the sign of  $2Q^2$ . We thank Luca Amendola for pointing this out.

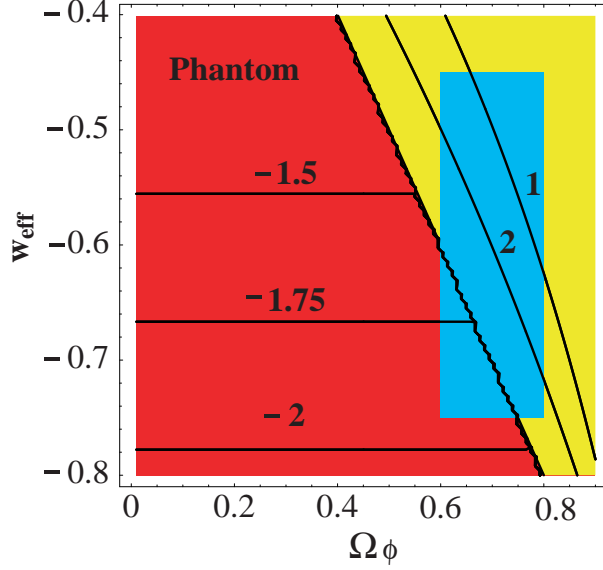


FIG. 12: Contour plot of the index  $n_+$  in terms of the functions of  $\Omega_\phi$  and  $w_{\text{eff}}$ . The numbers which we show in the figure correspond to the values  $n_+$ . In the non-phantom region characterized by  $\Omega_\phi + w_{\text{eff}} > 0$ ,  $n_+$  are always positive. Meanwhile in the phantom region ( $\Omega_\phi + w_{\text{eff}} < 0$ ) with  $w_{\text{eff}} > -1$ ,  $n_+$  take complex values with negative real parts. The real parts of  $n_+$  are plotted in the phantom region. The box (blue in the color version) represents schematically the observational constraints on  $w_{\text{eff}}, \Omega_\phi$  coming from the SN Ia data.

the potential and the equation of state of the field by parametrizing the Hubble parameter  $H$  in terms of the redshift  $z$  [372]. We recall that  $H(z)$  is determined by the luminosity distance  $d_L(z)$  by using the relation (38). This method was generalized to scalar-tensor theories [354, 355, 373],  $f(R)$  gravity [374] and also a dark-energy fluid with viscosity terms [375]. In scalar-tensor theories a scalar field  $\phi$  (the dilaton) is coupled to a scalar curvature  $R$  with a coupling  $F(\phi)R$ . If the evolution of matter perturbations  $\delta_m$  is known observationally, together with the Hubble parameter  $H(z)$ , one can even determine the function  $F(\phi)$  together with the potential  $V(\phi)$  of the scalar field [354].

As we showed in Sec. IX the Lagrangian (286) in the Jordan frame is transformed to the action (287) in Einstein frame with a coupling  $Q$  between the field  $\varphi$  and a barotropic fluid. Hence if we carry out a reconstruction procedure for the action (221) in the Einstein frame, corresponding reconstruction equations can be derived by transforming back to the Jordan frame. Following Ref. [376] we shall provide the recipe of the reconstruction program for the general Lagrangian

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2} + p(X, \phi) \right] + S_m(\phi). \quad (403)$$

We consider the same coupling  $Q$  as we introduced in Sec. VII. In a flat FRW spacetime the field equations for

the action (403) are

$$3H^2 = \rho_m + 2Xp_{,X} - p, \quad (404)$$

$$2\dot{H} = -\rho_m - p_m - 2Xp_{,X}, \quad (405)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = Q(\phi)\rho_m\dot{\phi}. \quad (406)$$

In the case of a non-relativistic barotropic fluid ( $p_m = 0$ ), Eq. (406) can be written in an integrated form

$$\rho_m = \rho_m^{(0)} \left( \frac{a_0}{a} \right)^3 I(\phi), \quad (407)$$

where

$$I(\phi) \equiv \exp \left( \int_{\phi_0}^{\phi} Q(\phi) d\phi \right). \quad (408)$$

Here the subscript 0 represents present values. Using Eq. (29) together with the relation  $\rho_m^{(0)} = 3H_0^2\Omega_m^{(0)}$ , we find that Eq. (407) can be written as

$$\rho_m = 3\Omega_m^{(0)}H_0^2(1+z)^3I(\phi). \quad (409)$$

When  $Q = 0$ , i.e.,  $I = 1$ , we can reconstruct the structure of theory by using Eqs. (404), (405) and (409) if the Hubble expansion rate is known as a function of  $z$ . This was actually carried out for a minimally coupled scalar field with a Lagrangian density:  $p = X - V(\phi)$  [367, 368, 369, 370, 371, 372]. In the presence of the coupling  $Q$ , we require additional information to determine the strength of the coupling. We shall make use of the equation of matter density perturbations for this purpose as in the case of scalar-tensor theories [354, 355].

The equation for matter perturbations on sub-Hubble scales is given by Eq. (391). Let us rewrite Eqs. (404), (405) and (391) by using a dimensionless quantity

$$r(z) \equiv H^2(z)/H_0^2. \quad (410)$$

Then we obtain

$$p = [(1+z)r' - 3r]H_0^2, \quad (411)$$

$$\phi'^2 p_{,X} = \frac{r' - 3\Omega_m^{(0)}(1+z)^2 I}{r(1+z)}, \quad (412)$$

$$\delta_m'' + \left( \frac{r'}{2r} - \frac{1}{1+z} + \frac{I'}{I} \right) \delta_m' - \frac{3}{2}\Omega_m^{(0)} \left( 1 + \frac{2I'^2}{\phi'^2 p_{,X} I^2} \right) \frac{(1+z)I\delta_m}{r} = 0, \quad (413)$$

where a prime represents a derivative in terms of  $z$ . Eliminating the  $\phi'^2 p_{,X}$  term from Eqs. (412) and (413), we obtain

$$I' = \frac{I}{4r(1+z)A} \left[ \delta_m' \pm \sqrt{\delta_m'^2 - 8r(1+z)AB} \right], \quad (414)$$

where

$$A \equiv \frac{3\Omega_m^{(0)}(1+z)\delta_m I}{2r[r' - 3\Omega_m^{(0)}(1+z)^2 I]}, \quad (415)$$

$$B \equiv [r' - 3\Omega_m^{(0)}(1+z)^2 I]A - \delta_m'' - \left( \frac{r'}{2r} - \frac{1}{1+z} \right) \delta_m'. \quad (416)$$

We require the condition  $\delta_m'^2 > 8r(1+z)AB$  for the consistency of Eq. (414).

If we know  $r$  and  $\delta_m$  in terms of  $z$  observationally, Eq. (414) is integrated to give the functional form of  $I(z)$ . Hence the function  $I(z)$  is determined without specifying the Lagrangian density  $p(\phi, X)$ . From Eqs. (411) and (412), we obtain  $p$  and  $\phi'^2 p_{,X}$  as functions of  $z$ . The energy density of the scalar field,  $\rho = \dot{\phi}^2 p_{,X} - p$ , is also determined. Equation (408) gives

$$Q = \frac{(\ln I)'}{\phi'}, \quad (417)$$

which means that the coupling  $Q$  is obtained once  $I$  and  $\phi'$  are known. We have to specify the Lagrangian density  $p(\phi, X)$  to find the evolution of  $\phi'$  and  $Q$ .

The equation of state for dark energy,  $w = p/\rho$ , is given by

$$w = \frac{p}{\dot{\phi}^2 p_{,X} - p} \quad (418)$$

$$= \frac{(1+z)r' - 3r}{3r - 3\Omega_m^{(0)}(1+z)^3 I}. \quad (419)$$

A non-phantom scalar field corresponds to  $w > -1$ , which translates into the condition  $p_{,X} > 0$  by Eq. (418). From Eq. (412) we find that this condition corresponds to  $r' > 3\Omega_m^{(0)}(1+z)^2 I$ , which can be checked by Eq. (419). Meanwhile a phantom field is characterized by the condition  $p_{,X} < 0$  or  $r' < 3\Omega_m^{(0)}(1+z)^2 I$ . Since  $I(z)$  is determined if  $r$  and  $\delta_m$  are known observationally, the equation of state of dark energy is obtained from Eq. (419) without specifying the Lagrangian density  $p(\phi, X)$ . In the next section we shall apply our formula to several different forms of scalar-field Lagrangians.

### A. Application to specific cases

Most of the proposed scalar-field dark energy models can be classified into two classes: (A)  $p = f(X) - V(\phi)$  and (B)  $p = f(X)V(\phi)$ . There are special cases in which cosmological scaling solutions exist, which corresponds to the Lagrangian density (C)  $p = Xg(Xe^{\lambda\phi})$  [see Eq. (233)]. We will consider these classes of models separately.

#### 1. Case of $p = f(X) - V(\phi)$

This includes quintessence [ $f(X) = X$ ] and a phantom field [ $f(X) = -X$ ]. Eq. (412) gives

$$\phi'^2 f_{,X} = \frac{r' - 3\Omega_m^{(0)}(1+z)^2 I}{r(1+z)}. \quad (420)$$

If we specify the function  $f(X)$ , the evolution of  $\phi'(z)$  and  $\phi(z)$  is known from  $r(z)$  and  $I(z)$ . From Eq. (417)

we can find the coupling  $Q$  in terms of  $z$  and  $\phi$ . Eq. (411) gives

$$V = f + [3r - (1+z)r']H_0^2. \quad (421)$$

Now the right hand side is determined as a function of  $z$ . Since  $z$  is expressed by the field  $\phi$ , one can obtain the potential  $V(\phi)$  in terms of  $\phi$ . In the case of Quintessence without a coupling  $Q$ , this was carried out by a number of authors [367, 368, 369, 370, 371, 372]. We have generalized this to a more general Lagrangian density  $p = f(X) - V(\phi)$  with a coupling  $Q$ .

#### 2. Case of $p = f(X)V(\phi)$

The Lagrangian density of the form  $p = f(X)V(\phi)$  includes K-essence [20, 21, 22] and tachyon fields [167]. For example the tachyon case corresponds to a choice  $f = -\sqrt{1 - 2X}$ . We obtain the following reconstruction equations from Eqs. (411) and (412):

$$\phi'^2 \frac{f_{,X}}{f} = \frac{r' - 3\Omega_m^{(0)}(1+z)^2 I}{r(1+z)[(1+z)r' - 3r]H_0^2}, \quad (422)$$

$$V = \frac{[(1+z)r' - 3r]H_0^2}{f}. \quad (423)$$

Once we specify the form of  $f(X)$ , one can determine the functions  $\phi'(z)$  and  $\phi(z)$  from Eq. (422). Then we obtain the potential  $V(\phi)$  from Eq. (423).

#### 3. Scaling solutions

For the Lagrangian density  $p = Xg(Xe^{\lambda\phi})$ , Eqs. (411) and (412) yield

$$Y \frac{g_{,Y}}{g} = \frac{6r - (1+z)r' - 3\Omega_m^{(0)}(1+z)^3 I}{2[(1+z)r' - 3r]}, \quad (424)$$

$$\phi'^2 = \frac{2[(1+z)r' - 3r]}{r(1+z)^2 g}, \quad (425)$$

where  $Y = Xe^{\lambda\phi}$ . If we specify the functional form of  $g(Y)$ , one can determine the function  $Y = Y(z)$  from Eq. (424). Then we find  $\phi'(z)$  and  $\phi(z)$  from Eq. (425). The parameter  $\lambda$  is known by the relation  $Y = (1/2)\dot{\phi}^2 e^{\lambda\phi}$ .

As we showed in Sec. VII the quantity  $Y$  is constant along scaling solutions, in which case the LHS of Eq. (424) is constant. Thus the presence of scaling solutions can be directly checked by evaluating the RHS of Eq. (424) with the use of observational data. We caution however that the above formula needs to be modified if a part of the dark matter is uncoupled to dark energy.

## B. Example of reconstruction

In order to reconstruct dark energy models from observations we need to match supernova data to a fitting function for  $H(z)$ . The fitting functions generally depend upon the models of dark energy [377]. Among a number of fitting functions, the following parametrization for the Hubble parameter is often used [80]:

$$r(x) = H^2(x)/H_0^2 = \Omega_m^{(0)}x^3 + A_0 + A_1x + A_2x^2, \quad (426)$$

where  $x \equiv 1 + z$  and  $A_0 = 1 - A_1 - A_2 - \Omega_m^{(0)}$ . The parametrization (426) is equivalent to the following expansion for dark energy:

$$\rho = \rho_c^{(0)} (A_0 + A_1x + A_2x^2), \quad (427)$$

where  $\rho_c^{(0)} = 3H_0^2$ . The  $\Lambda$ CDM model is included in the above parametrization ( $A_1 = 0$ ,  $A_2 = 0$  and  $A_0 = 1 - \Omega_m^{(0)}$ ).

For a prior  $\Omega_m^{(0)} = 0.3$ , the Gold data set of SN observations gives  $A_1 = -4.16 \pm 2.53$  and  $A_2 = 1.67 \pm 1.03$  [378]. We note that the weak energy condition for dark energy,  $\rho \geq 0$  and  $w = p/\rho \geq -1$ , corresponds to [80]

$$A_0 + A_1x + A_2x^2 \geq 0, \quad A_1 + 2A_2x \geq 0. \quad (428)$$

If we use the best-fit values  $A_1 = -4.16$  and  $A_2 = 1.67$ , for example, we find that the second condition in Eq. (428) is violated today. This means that the field behaves as a phantom ( $w < -1$ ). In the case of a non-phantom scalar field such as quintessence, we need to put a prior  $A_1 + 2A_2x \geq 0$ .

For the moment we have not yet obtained accurate observational data for the evolution of the matter perturbations  $\delta_m(z)$ . Hence the coupling  $Q$  is not well constrained with current observations. In what follows we consider the case without the coupling  $Q$ . Then we only need to use the reconstruction equations (411) and (412) with  $I = 1$ . In this case the equation of state  $w$  of dark energy is determined by Eq. (419) provided that  $r = H^2/H_0^2$  can be parametrized observationally. In Ref. [80] the reconstruction was obtained for the parametrization (426).

We show in Fig. 13 the evolution of  $w$  versus  $z$  for  $\Omega_m^{(0)} = 0.3$ . We find that the equation of state crosses the cosmological constant boundary ( $w = -1$ ) for the best-fit parametrization. Even at the  $2\sigma$  confidence level the crossing to the phantom region ( $w < -1$ ) is allowed. In this figure we do not impose any priors for the coefficients  $A_1$  and  $A_2$ . If dark energy originates from an ordinary scalar field like quintessence, one needs to put a prior  $A_1 + 2A_2x \geq 0$ . This case is also consistent with observations, see Ref. [80]. In Ref. [372] the potential of a quintessence field is reconstructed by using a parametrization different from Eq. (426). For any parametrization of the Hubble parameter, the potential and the kinetic energy of a scalar field can be reconstructed in each model of dark energy by using the formula (411) and (412). See Ref. [379] for a detailed reconstruction of the equation of state of dark energy using

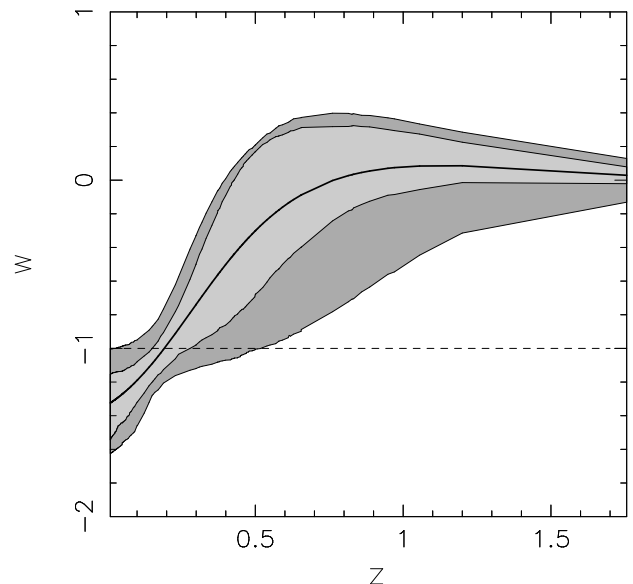


FIG. 13: Evolution of  $w(z)$  versus redshift  $z$  for  $\Omega_m^{(0)} = 0.3$  with the parametrization given by (426). Here the thick solid line corresponds to the best-fit, the light grey contour represents the  $1\sigma$  confidence level, and the dark grey contour represents the  $2\sigma$  confidence. The dashed line corresponds to the  $\Lambda$ CDM model. From Ref. [80].

higher-order derivatives of  $H$  called “state finder” [57], and see [380] where the authors have proposed a new non-parametric method of smoothing supernova data over redshift using a Gaussian kernel, the aim being to reconstruct  $H(z)$  and  $w(z)$  in a model independent manner.

## C. $w = -1$ crossing

The reconstruction of the equation of state of dark energy shows that the parametrization of  $H(z)$  which crosses the cosmological-constant boundary shows a good fit to recent SN Gold dataset [378], but the more recent SNLS dataset favors  $\Lambda$ CDM [381]. This crossing to the phantom region ( $w < -1$ ) is neither possible for an ordinary minimally coupled scalar field [ $p = X - V(\phi)$ ] nor for a phantom field [ $p = -X - V(\phi)$ ]. It was shown by Vikman [382] that the  $w = -1$  crossing is hard to be realized only in the presence of linear terms in  $X$  in single-field models of dark energy. We require nonlinear terms in  $X$  to realize the  $w = -1$  crossing.

This transition is possible for scalar-tensor theories [353], multi-field models [382, 383] (called *quintom* using phantom and ordinary scalar field), coupled dark energy models with specific couplings [384] and string-inspired models [184, 318, 385]<sup>6</sup>. A recent interesting result con-

<sup>6</sup> We also note that loop quantum cosmology [386] allows to realize such a possibility [387].

cerning whether in scalar-tensor theories of gravity, the equation of state of dark energy,  $w$ , can become smaller than  $-1$  without violating any energy condition, has been obtained by Martin *et al.* [388]. In such models, the value of  $w$  today is tied to the level of deviations from general relativity which, in turn, is constrained by solar system and pulsar timing experiments. The authors establish the conditions on these local constraints for  $w$  to be significantly less than  $-1$  and demonstrate that this requires the consideration of theories that differ from the Jordan-Fierz-Brans-Dicke theory and that involve either a steep coupling function or a steep potential.

In this section we shall present a simple one-field model with nonlinear terms in  $X$  which realizes the cosmological-constant boundary crossing and perform the reconstruction of such a model.

Let us consider the following Lagrangian density:

$$p = -X + u(\phi)X^2, \quad (429)$$

where  $u(\phi)$  is a function in terms of  $\phi$ . Dilatonic ghost condensate models [39] correspond to a choice  $u(\phi) = ce^{\lambda\phi}$ . From Eqs. (411) and (412) we obtain

$$\phi'^2 = \frac{12r - 3xr' - 3\Omega_m^{(0)}x^3I}{rx^2}, \quad (430)$$

$$u(\phi) = \frac{2(2xr' - 6r + rx^2\phi'^2)}{H_0^2r^2x^4\phi'^4}. \quad (431)$$

Let us reconstruct the function  $u(\phi)$  by using the parametrization (426) with best-fit values of  $A_1$  and  $A_2$ . We caution that this parametrization is not the same as the one for the theory (429), but this can approximately describe the fitting of observational data which allows the  $w = -1$  crossing.

As we see from Fig. 14 the crossing of the cosmological-constant boundary corresponds to  $uX = 1/2$ , which occurs around the redshift  $z = 0.24$  for the best-fit parametrization. The system can enter the phantom region ( $uX < 1/2$ ) without discontinuous behavior of  $u$  and  $X$ .

We have to caution that the perturbation in  $\phi$  is plagued by a quantum instability when the field behaves as a phantom [39]. Even at the classical level the perturbation is unstable for  $1/6 < uX < 1/2$ , since  $c_A^2$  in Eq. (371) is negative. One may avoid this instability if the phantom behavior is just transient. In fact transient phantom behavior was found in the case of a dilatonic ghost condensate model (see, e.g., Fig. 4 in Ref. [39]). In this case the cosmological-constant boundary crossing occurs again in the future, after which the perturbations become stable.

We found that the function  $u(\phi)$  can be approximated by an exponential function  $e^{\lambda\phi}$  near to the present, although some differences appear for  $z \gtrsim 0.2$ . However the current observational data is still not sufficient to rule out the dilatonic ghost condensate model. We hope that future high-precision observations will determine the functional form of  $u(\phi)$  more accurately.

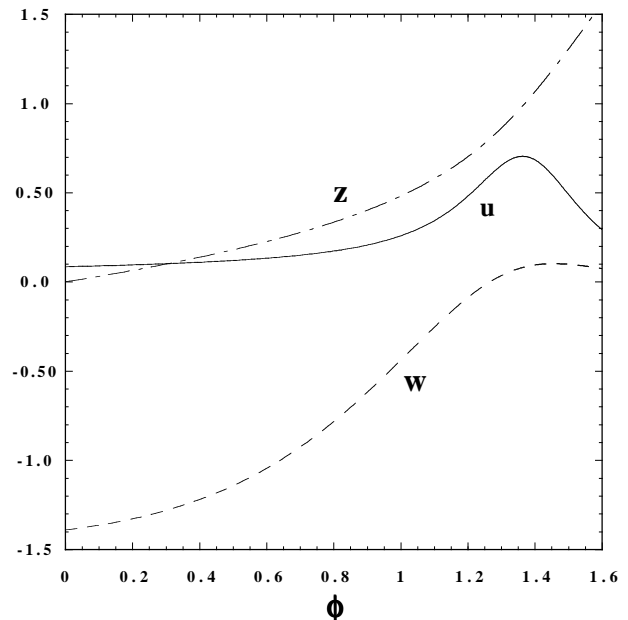


FIG. 14: Reconstruction of generalized ghost condensate model for the parametrization (426) with the best-fit parameters  $A_1 = -4.16$  and  $A_2 = 1.67$ . We show  $u$ ,  $w$  and  $z$  in terms of the function of  $\phi$ . This model allows a possibility to cross the cosmological-constant boundary ( $w = -1$ ).

### XIII. OBSERVATIONAL CONSTRAINTS ON THE EQUATION OF STATE OF DARK ENERGY

In the previous section we provided a set of reconstruction equations for scalar-field dark energy models. However it is distinctly possible (some would say likely), that the origin of dark energy has nothing to do with scalar fields. Fortunately, even in this case we can express the equation of state  $w$  of dark energy in terms of  $r = H^2/H_0^2$ . Let us consider a system of dark energy and cold dark matter which are not directly coupled to each other. Using Eqs. (157) and (158) with the replacement  $\rho_\phi \rightarrow \rho_{\text{DE}}$  and  $p_\phi \rightarrow p_{\text{DE}}$  together with the relation  $\rho_m = \rho_m^{(0)}(1+z)^3$ , we easily find

$$w = \frac{p_{\text{DE}}}{\rho_{\text{DE}}} = \frac{(1+z)r' - 3r}{3r - 3\Omega_m^{(0)}(1+z)^3}, \quad (432)$$

which corresponds to  $I = 1$  in Eq. (419). Hence if observational data is accurate enough to express  $r(z)$  in terms of  $z$ , we obtain  $w(z)$  independent of the model of dark energy. However the parametrization of  $r(z)$  itself depends upon dark energy models and current SN Ia observations are not sufficiently precise to discriminate which parametrizations are favoured.

### A. Parametrization of $w_{\text{DE}}$

Instead of expressing the Hubble parameter  $H$  in terms of  $z$ , one can parametrize the equation of state of dark energy. By using Eq. (157) the Hubble parameter can be written as

$$H^2(z) = H_0^2 \left[ \Omega_m^{(0)}(1+z)^3 + (1 - \Omega_m^{(0)})f(z) \right], \quad (433)$$

where

$$f(z) \equiv \frac{\rho_{\text{DE}}(z)}{\rho_{\text{DE}}^{(0)}} = \exp \left[ 3 \int_0^z \frac{1+w(\tilde{z})}{1+\tilde{z}} d\tilde{z} \right]. \quad (434)$$

Hence  $H(z)$  is determined once  $w(z)$  is parametrized. Then we can constrain the evolution of  $w(z)$  observationally by using the relation (38).

There are a number of parametrizations of  $w(z)$  which have been proposed so far, see for example [389, 390]. Among them, Taylor expansions of  $w(z)$  are commonly used:

$$w(z) = \sum_{n=0} w_n x_n(z), \quad (435)$$

where several expansion functions have been considered [389]:

$$(i) \text{ constant } w : \quad x_0(z) = 1; \quad x_n = 0, \quad n \geq 1, \quad (436)$$

$$(ii) \text{ redshift} : \quad x_n(z) = z^n, \quad (437)$$

$$(iii) \text{ scale factor} : \quad x_n(z) = \left(1 - \frac{a}{a_0}\right)^n = \left(\frac{z}{1+z}\right)^n, \quad (438)$$

$$(iv) \text{ logarithmic} : \quad x_n(z) = [\log(1+z)]^n. \quad (439)$$

Case (i) includes the  $\Lambda$ CDM model. Case (ii) was introduced by Huterer and Turner [391] & Weller and Albrecht [392] with  $n \leq 1$ , i.e.,  $w_{\text{DE}} = w_0 + w_1 z$ . In this case Eq. (433) gives the Hubble parameter

$$H^2(z) = H_0^2 [\Omega_m^{(0)}(1+z)^3 + (1 - \Omega_m^{(0)})(1+z)^{3(1+w_0-w_1)} e^{3w_1 z}]. \quad (440)$$

We can then constrain the two parameters  $w_0$  and  $w_1$  by using SN Ia data. Case (iii) was introduced by Chevallier and Polarski [393] & Linder [394]. At linear order we have  $w(z) = w_0 + w_1 \frac{z}{z+1}$ . Jassal *et al.* [89] extended this to a more general case with

$$w(z) = w_0 + w_1 \frac{z}{(z+1)^p}. \quad (441)$$

For example one has  $w(\infty) = w_0 + w_1$  for  $p = 1$  and  $w(\infty) = w_0$  for  $p = 2$ . Thus the difference appears for larger  $z$  depending on the values of  $p$ . Case (iv) was introduced by Efstathiou [395]. Basically the Taylor expansions were taken to linear order ( $n \leq 1$ ) for the cases

(ii), (iii) and (iv), which means that two parameters  $w_0$  and  $w_1$  are constrained from observations.

A different approach was proposed by Bassett *et al.* [396] and was further developed by Corasaniti and Copeland [397]. It allows for tracker solutions in which there is a rapid evolution in the equation of state, something that the more conventional power-law behavior can not accommodate. This has some nice features in that it allows for a broad class of quintessence models to be accurately reconstructed and it opens up the possibility of finding evidence of quintessence in the CMB both through its contribution to the ISW effect [397, 398] and as a way of using the normalization of the dark energy power spectrum on cluster scales,  $\sigma_8$ , to discriminate between dynamical models of dark energy (Quintessence models) and a conventional cosmological constant model [53, 54].

This *Kink* approach can be described by a 4-parameter parametrization, which is

$$w(a) = w_0 + (w_m - w_0)\Gamma(a, a_t, \Delta), \quad (442)$$

where  $\Gamma$  is the transition function which depends upon  $a$ ,  $a_t$  and  $\Delta$ . Here  $a_t$  is the value of the scale factor at a transition point between  $w = w_m$ , the value in the matter-dominated era, and  $w = w_0$ , the value today, with  $\Delta$  controlling the width of the transition. The parametrization (442) is schematically illustrated in Fig. 15. The transition function used in the papers [389, 397, 399] is of the general form

$$\Gamma(a, a_t, \Delta) = \frac{1 + e^{a_t/\Delta}}{1 + e^{(a_t - a)/\Delta}} \frac{1 - e^{(1-a)/\Delta}}{1 - e^{1/\Delta}}. \quad (443)$$

Its advantage is that it can cope with rapid evolution of  $w$ , something which is difficult to be realized for the case of the Taylor expansions given above.

### B. Observational constraints from SN Ia data

There has been recent interest in how successfully the equation of state of dark energy can be constrained with SN Ia observations. For the Taylor expansion at linear order ( $n \leq 1$ ), Bassett *et al.* [389] found the best fit values shown in Table VIII by running a Markov-Chain Monte Carlo (MCMC) code with the Gold data set [85]. Note that these were obtained by minimizing  $\chi^2 = -2\log \mathcal{L}$ , where  $\mathcal{L}$  is the likelihood value.

Meanwhile the Kink formula (442) gives the best-fit values:  $w_0 = -2.85$ ,  $w_m = -0.41$ ,  $a_t = 0.94$  and  $\ln(\Delta) = -1.52$  with  $\chi^2 = 172.8$ . This best-fit case corresponds to the equation of state which is nearly constant ( $w \sim w_m$ ) for  $z > 0.1$  and rapidly decreases to  $w = w_0$  for  $z < 0.1$ . This behavior is illustrated in Fig. 16 together with the  $2\sigma$  limits of several parametrizations. We find that the best-fit solution passes outside the limits of all three Taylor expansions for  $0.1 \lesssim z \lesssim 0.3$  and  $z \lesssim 0.1$ . It suggests that the Taylor expansions at linear order

Parametrization	$w_0$	$w_1$
Redshift	$-1.30 \pm_{0.52}^{0.43}$	$1.57 \pm_{1.41}^{1.68}$
Scale factor	$-1.48 \pm_{0.64}^{0.57}$	$3.11 \pm_{3.12}^{2.98}$
Logarithmic	$-1.39 \pm_{0.57}^{0.50}$	$2.25 \pm_{2.15}^{2.19}$

TABLE VIII: Best fits values of  $w_0$  and  $w_1$  for several different Taylor expansions at linear order. Error bars correspond to the  $1\sigma$  confidence level. From Ref. [389].

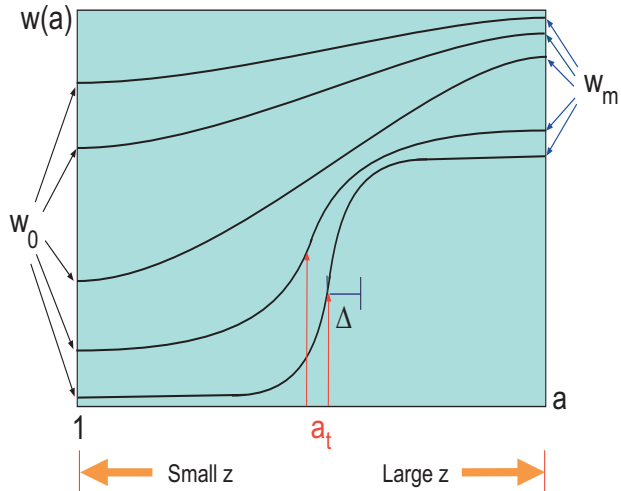


FIG. 15: Schematic illustration of the equation of state of dark energy for the kink parametrization (442).

are not sufficient to implement the case of such rapid evolution of  $w(z)$ . Note the general similarities with the results in Ref. [80] shown in Fig. 13.

In Ref. [389] it was found that the redshift  $z_c$  at which the universe enters an accelerating stage strongly depends upon the parameterizations of  $w(z)$ . The  $\Lambda$ CDM model corresponds to  $z_c = 0.66 \pm_{0.11}^{0.11}$ , which is consistent with the estimation (47). One has  $z_c = 0.14 \pm_{0.05}^{0.14}$  in redshift parametrization [ $w(z) = w_0 + w_1 z$ ] and  $z_c = 0.59 \pm_{0.21}^{8.91}$  in scale-factor parametrization [ $w(z) = w_0 + w_1 z / (1 + z)$ ]. While these large differences of  $z_c$  may be used to distinguish the cosmological constant from dynamical dark energy models, this also casts doubt on the use of standard two-parameter parametrizations in terms of  $w_0$  and  $w_1$ .

If we include higher-order terms ( $n \geq 2$ ) in the Taylor expansions (435), the above problems can be alleviated to some extent. In this case it was found by Bassett *et al.* [389] that the allowed ranges of  $w_0$  are shifted toward smaller values with a maximum likelihood  $w_0 \sim -4$ . In addition huge values of  $w_1 \sim 50$  and  $w_2 \sim -100$  are allowed.

The above results show that observational constraints on the equation of state of dark energy are sensitive to the parametrization of it and that we require at least three parameters to address a wide range of the variation of  $w(z)$ . In Ref. [389] it was found that the  $\chi^2$  for the best-

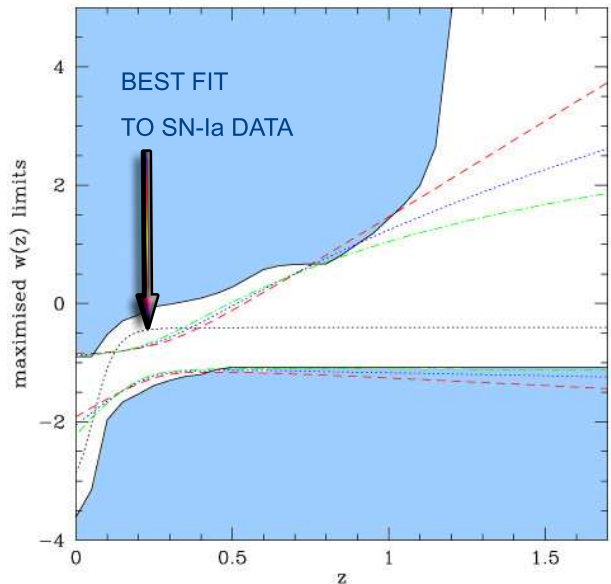


FIG. 16: Evolution of the equation of state of dark energy for the best-fit Kink parametrization. The maximized limits on  $w(z)$  are shown for (a) redshift (red dashed line), (b) scale-factor (green dash-dotted), (c) logarithmic (blue-dotted), and (d) Kink (solid black lines) parametrizations. From Ref. [389].

fit Kink parametrization is lowest compared to the values in the linear Taylor expansions. This situation changes if we account for the second-order term ( $n = 2$ ); then the redshift parametrization gives the lowest  $\chi^2$ .

A question then arises: How many dark energy parameters do we need to describe the dark energy dynamics? This may be addressed by using the Akaike information criterion (AIC) and Bayesian Information criterion (BIC) [65, 400] (see also Refs. [401]). These two criteria are defined as:

$$\text{AIC} = -2 \ln \mathcal{L} + 2k_p, \quad (444)$$

$$\text{BIC} = -2 \ln \mathcal{L} + k_p \ln N. \quad (445)$$

Here  $\mathcal{L}$  is the maximum value of the likelihood,  $k_p$  is the number of parameters and  $N$  is the number of data points. The optimal model minimizes the AIC or BIC. In the limit of large  $N$ , AIC tends to favour models with more parameters while BIC more strongly penalizes them (since the second term diverges in this limit). BIC pro-



vides an estimate of the posterior evidence of a model assuming no prior information. Hence BIC is a useful approximation to a full evidence calculation when we have no prior on the set of models. Bassett *et al.* found that the minimum value of BIC corresponds to the  $\Lambda$ CDM model [389]. This general conclusion has been confirmed in the recent work of Ref. [402]. It is interesting that the simplest dark energy model with only one parameter is preferred over other dynamical dark energy models. This situation is similar to early universe inflation in which single-field models are preferred over multi-field models from two information criteria [403]. Ending this subsection on a cautionary note, Corasaniti has recently emphasised how extinction by intergalactic gray dust introduces a magnitude redshift dependent offset in the standard-candle relation of SN Ia [404]. It leads to overestimated luminosity distances compared to a dust-free universe and understanding this process is crucial for an accurate determination of the dark energy parameters.

### C. Observational constraints from CMB

Let us consider observational constraints arising from the CMB. The temperature anisotropies in CMB are expanded in spherical harmonics:  $\delta T/T = \Sigma a_{lm} Y_{lm}$ . The CMB spectrum,  $C_l \equiv \langle |a_{lm}|^2 \rangle$ , is written in the form [70]

$$C_l = 4\pi \int \frac{dk}{k} \mathcal{P}_{\text{ini}}(k) |\Delta_l(k, \eta_0)|^2, \quad (446)$$

where  $\mathcal{P}_{\text{ini}}(k)$  is an initial power spectrum and  $\Delta_l(k, \eta_0)$  is the transfer function for the  $l$  multipoles of the  $k$ -th wavenumber at the present time  $\eta_0$  (here we use conformal time:  $\eta \equiv \int a^{-1} dt$ ). The initial power spectrum is nearly scale-invariant, which is consistent with the prediction of an inflationary cosmology.

The dynamical evolution of dark energy affects the CMB temperature anisotropies in at least two ways. First, the position of the acoustic peaks depends on the dark energy dynamics because of the fact that an angular diameter distance is related to the form of  $w(z)$ . Second, the CMB anisotropies are affected by the ISW effect.

In order to understand the effect of changing the position of acoustic peaks, let us start with the constant equation of state  $w$  of dark energy. The presence of dark energy induces a shift by a linear factor  $s$  in the  $l$ -space positions of the acoustic peaks. This shift is given by [405]

$$s = \sqrt{\Omega_m^{(0)}} D, \quad (447)$$

where  $D$  is an angular diameter distance which is written as

$$D = \int_0^{z_{\text{dec}}} \frac{dz}{\sqrt{\Omega_m^{(0)}(1+z)^3 + \Omega_{\text{DE}}^{(0)}(1+z)^{3(1+w)}}}, \quad (448)$$

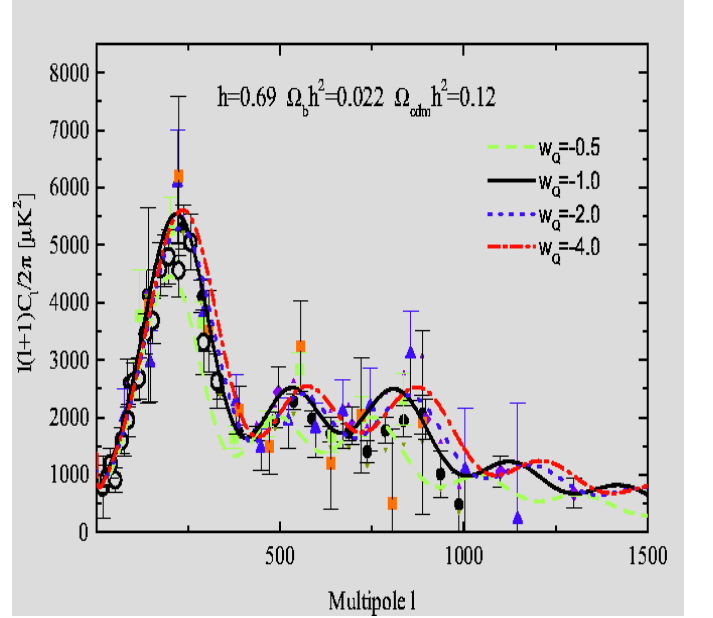


FIG. 17: The shift of the power spectrum for the equation of state  $w = -0.5, -1.0, -2.0, -4.0$  with  $h = 0.69$ ,  $\Omega_{\text{CDM}}^{(0)} = 0.252$  and  $\Omega_b^{(0)} = 0.046$ . The spectrum shifts toward larger  $l$  for smaller  $w$ . From Ref. [405].

where  $z_{\text{dec}}$  is the redshift at decoupling. The shift of the power spectrum is proportional to  $sl$ . In Fig. 17 we show a CMB angular power spectrum with the relative density in cold dark matter  $\Omega_{\text{CDM}}^{(0)} = 0.252$  and that in baryons  $\Omega_b^{(0)} = 0.046$ , for various values of  $w$ . As we decrease  $w$ , the power spectrum is shifted toward smaller scales (i.e., larger  $l$ ).

The transfer function  $\Delta_l(k, \eta_0)$  in Eq. (446) can be written as the sum of the contribution coming from the last scattering surface and the contribution from the ISW effect. This ISW contribution is given by [406]

$$\Delta_l^{\text{ISW}}(k) = 2 \int d\eta e^{-\tau} \frac{d\Phi}{d\eta} j_l[k(\eta - \eta_0)], \quad (449)$$

where  $\tau$  is the optical depth due to scattering of photons,  $\Phi$  is the gravitational potential, and  $j_l$  are the Bessel functions. As we showed in Sec. XI the gravitational potential is constant in the matter-dominated period, which means the absence of the ISW effect. However the presence of dark energy leads to a variation of  $\Phi$ , which gives rise to the ISW effect. This is especially important for large-scale perturbations corresponding to  $l \lesssim 20$ . In particular coupled dark energy models can have a strong impact on the CMB spectrum.

There have been a number of papers placing constraints on the equation of state of dark energy by combining the CMB data sets (WMAP1) together with SN Ia and LSS [51, 89, 405, 407, 408, 409, 410, 412, 413, 414, 415, 416]. Melchiorri *et al.* [405] studied the case of constant  $w$  and found that the combined analysis of

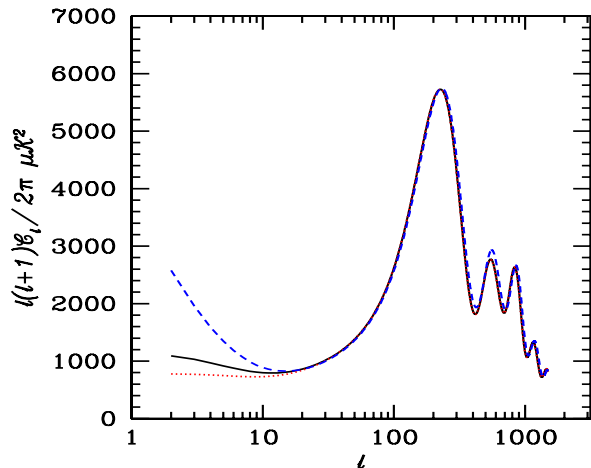


FIG. 18: CMB angular power spectra for three different models without dark energy perturbations. The solid line, dotted line and dashed line correspond to (a)  $w = -1$ ,  $\Omega_m^{(0)} = 0.3$ ,  $\Omega_b^{(0)} = 0.05$ ,  $H_0 = 65 \text{ kms}^{-1}\text{Mpc}^{-1}$ , (b)  $w = -0.6$ ,  $\Omega_m^{(0)} = 0.44$ ,  $\Omega_b^{(0)} = 0.073$ ,  $H_0 = 54 \text{ kms}^{-1}\text{Mpc}^{-1}$ , and (c)  $w = -2.0$ ,  $\Omega_m^{(0)} = 0.17$ ,  $\Omega_b^{(0)} = 0.027$ ,  $H_0 = 84 \text{ kms}^{-1}\text{Mpc}^{-1}$ , respectively. When  $w = -2$  there is a large contribution to low multipoles from the ISW effect. From Ref. [410].

CMB (WMAP1), HST, SN Ia and 2dF data sets gives  $-1.38 < w < -0.82$  at the 95 % confidence level with a best-fit model  $w = -1.05$  and  $\Omega_m^{(0)} = 0.27$ . We note that this result was obtained by neglecting perturbations in the dark energy component. Pogosian *et al.* [416] analysed the way that future measurements of the ISW effect could constrain dynamical dark energy models as a function of redshift,  $w(z)$ . Introducing a new parameterization of  $w$ , the mean value of  $w(z)$  as an explicit parameter, they argue that it allows them to separate the information contained in the estimation of the distance to the last scattering surface (from the CMB) from the information contained in the ISW effect.

Weller and Lewis [410] studied the contribution of dark energy perturbations with constant  $w$ . In Fig. 18 the CMB power spectrum is plotted in the case where dark energy perturbations are neglected for  $w = -1$ ,  $w = -0.6$  and  $w = -2$ . This shows that the ISW effect is significant for  $w < -1$ , whereas there is a small contribution to the low multipoles for  $w > -1$ . The situation changes in the presence of dark energy perturbations. When  $w$  is larger than  $-1$ , the inclusion of dark energy perturbations increases the large scale power (see Fig. 19). For  $w < -1$  dark energy fluctuations partially cancel the large contribution from the different evolution of the background via matter perturbations. As is clearly seen from Figs. 18 and 19, the inclusion of perturbations in the dark energy component increases the degeneracies. In fact the combined analysis of the WMAP1 (first year), ACBAR and CBI data together with a prior from BBN and HST shows that even the values  $w < -1.5$  are al-

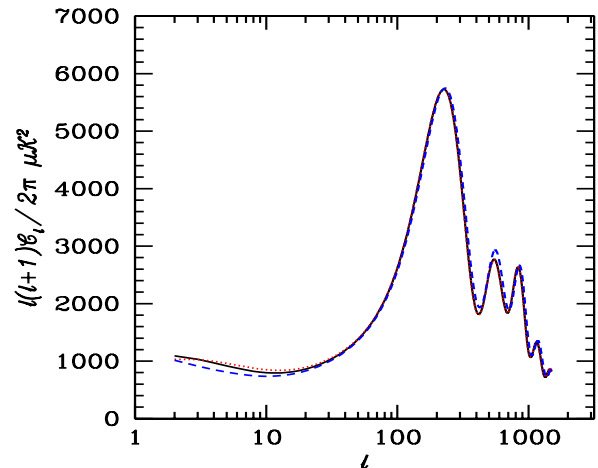


FIG. 19: CMB angular power spectra for three different models with dark energy perturbations. Each line corresponds to the same model parameters as in Fig. 18. From Ref. [410].

lowed if we take into account dark energy fluctuations [410]. When SN Ia data are added in the analysis, Weller and Lewis [410] obtained a constraint:  $w = -1.02 \pm 0.16$  at  $1\sigma$  level for the speed of sound  $c_s^2 = 1$ . This result does not change much even allowing for different values of the speed of sound, see Fig. 20.

A similar conclusion is reached in [411], where the authors investigate the possibility of constraining dark energy with the ISW effect recently detected by cross-correlating the WMAP1 maps with several LSS surveys, concluding that current available data put weak limits on a constant dark energy equation of state  $w$ . In fact they find no constraints on the dark energy sound speed  $c_s^2$ . For quintessence-like dark energy ( $c_s^2=1$ ) they find  $w < -0.53$ . Hopefully, better measurements of the CMB-LSS correlation will be possible with the next generation of deep redshift surveys and this will provide independent constraints on the dark energy which are alternative to those usually inferred from CMB and SN-Ia data.

Let us next consider the case of a dynamically changing  $w$ . If the equation of state changes from  $w > -1$  to  $w < -1$ , the perturbations become unstable when the system crosses a cosmological constant boundary. This problem can be alleviated in the presence of non-adiabatic pressure perturbations. In fact it was shown in Refs. [418] that the phantom divide crossing can be realized in multiple scalar field models.

A number of authors [51, 89, 90, 413] placed constraints on the dynamical evolution of dark energy by using several parametrizations of  $w(z)$  or  $\rho(z)$ . When the phantom divide crossing occurs, these results should be regarded as speculative since the evolution of dark energy perturbations around  $w = -1$  was not fully addressed. For a complete analysis we need to take into account non-adiabatic perturbations which makes dark energy gravitationally stable.

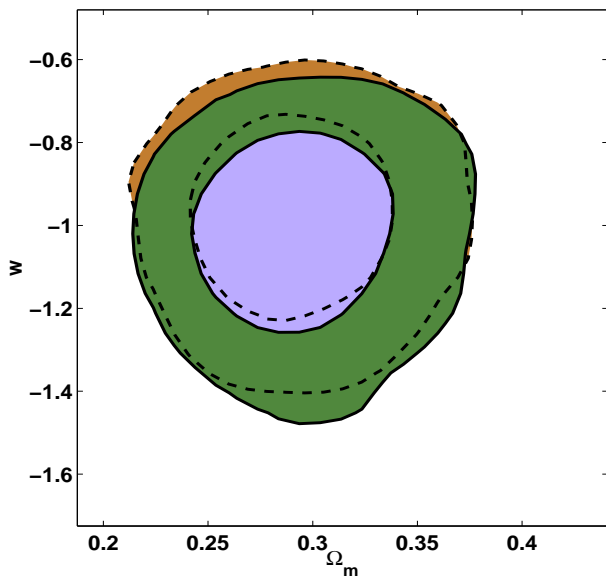


FIG. 20: Likelihood contours for  $w$  and  $\Omega_m^{(0)}$  in the case of constant equation of state of dark energy at the 68% and 95% confidence levels. This is a combined analysis of CMB (WMAP1), 2dF, SN Ia, HST and BBN data sets. The solid line corresponds to  $c_s^2 = 1$ , whereas the dashed line to marginalizing over  $c_s^2$ . From Ref. [410].

Corasaniti *et al.* [51] restricted the models to those with  $w(z) \geq -1$  in order to avoid the instability of perturbations. The authors used the Kink parametrization (442) with (443) and performed the likelihood analysis by varying four dark energy parameters ( $w_0, w_m, a_t, \Delta$ ) and six cosmological parameters ( $\Omega_{\text{DE}}^{(0)}, \Omega_b^{(0)} h^2, h, n_S, \tau, A_S$ ). The total likelihood is taken to be the product of each data set (CMB, SN-Ia and LSS)

$$\chi_{\text{tot}}^2 = \chi_{\text{WMAP1}}^2 + \chi_{\text{SN Ia}}^2 + \chi_{\text{2dF}}^2. \quad (450)$$

The 2dF data does not provide strong constraints on dark energy beyond those obtained using CMB + SN-Ia data set. The total  $\chi_{\text{tot}}^2$  of this model is 1602.9, whereas the best fit  $\Lambda$ CDM model has  $\chi_{\text{tot}}^2 = 1605.8$ . The total number of degrees of freedom is 1514, which shows that none of the fits are very good. This is mainly due to the WMAP1 data. Corasaniti *et al.* evaluated AIC and BIC defined in Eqs. (444) and (445), and found that the quintessence models have an AIC of 1622.9 and the  $\Lambda$ CDM model of 1617.8. This means that the  $\Lambda$ CDM model is favoured over the quintessence model. This property also holds when the BIC criterion is used.

In the case of quintessence, the best-fit dark energy parameters are given by  $w_m = -0.13$ ,  $a_t = 0.48$ ,  $w_0 = -1.00$  and  $\Delta = 0.06$ . This corresponds to a transition in which  $w(z)$  does not vary much for  $z > 2$  ( $w(z) \sim w_m = -0.13$ ) and rapidly changes around  $z = 1$  toward  $w_0 = -1.00$ . Models with  $w_m \geq 0$  for  $z > 1$  with fast transition at  $z \leq 1$  are ruled out. This is because the models with

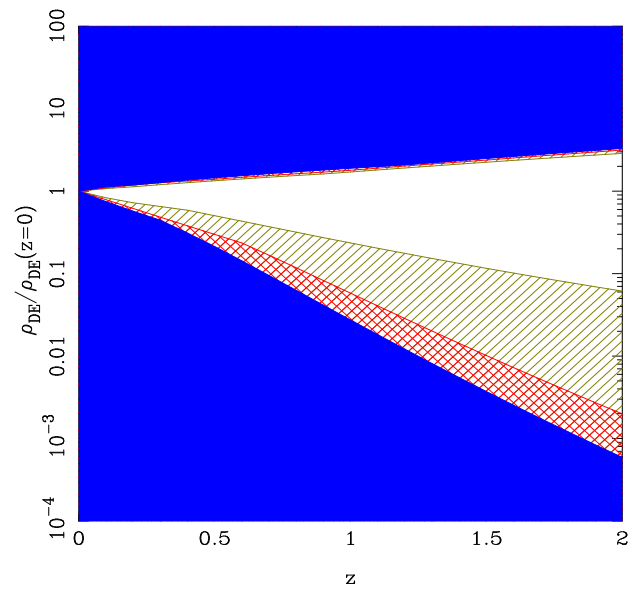


FIG. 21: The variation of dark energy density is shown as a function of redshift for the parameterisation (441) with  $p = 1$ . This is the combined constraints from WMAP1 and SNLS data. The green/hatched region is excluded at 68% confidence limit, red/cross-hatched region at 95% confidence level and the blue/solid region at 99% confidence limit. The white region shows the allowed range of variation of dark energy at 68% confidence limit. The phantom like models have  $\rho_{\text{DE}}(z)/\rho_{\text{DE}}(z=0) < 1$ . The allowed values of  $w_0$  at 95% confidence limit for this parameterisation are  $-1.89 < w_0 < -0.61$  (with SNLS data),  $-1.64 < w_0 < -0.42$  (with WMAP1 data) and  $-1.46 < w_0 < -0.81$  when we combine the SNLS and WMAP1 data. The values for  $w_1$  for these data sets are constrained to lie in the range  $-4.82 < w_1 < 3.3$  (SNLS),  $-3.09 < w_1 < 1.32$  (WMAP1) and  $-0.99 < w_1 < 1.04$  (combined). This clearly shows that the WMAP1 data is more effective in constraining the equation of state parameters are compared to the supernova data. From Ref. [419].

transition at  $z < 10$  with  $w_m > -0.1$  lead to a non-negligible dark energy contribution at decoupling which is strongly constrained by CMB. Then perfect tracking behaviour for which  $w = 0$  during the matter era with late time fast transition from tracking to acceleration are disfavoured. On the other hand models with approximate tracking behaviour slowly varying equation of state with  $w_0 < -0.8$  and  $w_m > -0.1$  are consistent with data. These include quintessence models with inverse power law potential [10, 16], supergravity inspired potentials [275] and off tracking quintessence models [420]. We note that models of late-time transitions [421] (see also [422]) have a similar property to the best-fit model.

The recent results published in Ref. [92], using a different data set can be seen in Fig 21. We note that dark energy perturbations are not taken into account in their analysis. Figure 21 shows that everything is perfectly consistent with a true non evolving cosmological constant. The fact that different data sets have been used (e.g., Gold SN in Ref. [80] versus SNLS in Ref. [92]),

as well as different priors, such as the value of  $\Omega_m^{(0)}$  and the parametrisation for  $w(z)$  could well lead to different conclusions. Basically it is still too early to say whether observations prefer varying  $w$  or constant  $w$  at present.

An interesting alternative approach to parameterising dark energy has been proposed in [423] where they develop a phenomenological three parameter fluid description of dark energy which allows them to include an imperfect dark energy component on the large scale structure. In particular in addition to the equation of state and the sound speed, they allow a nonzero viscosity parameter for the fluid. It means that anisotropic stress perturbations are generated in the dark energy, something which is not excluded by the present day cosmological observations. They also investigate structure formation of imperfect fluid dark energy characterized by an evolving equation of state, concentrating on unified models of dark energy with dark matter, such as the Chaplygin gas or the Cardassian expansion, with a shear perturbation included.

#### D. Cross-correlation Tomography

An interesting approach for measuring dark energy evolution with weak lensing has been proposed by Jain and Taylor [424]. They developed a cross-correlation technique of lensing tomography. The key concept they were able to use, was that the variation of the weak lensing shear with redshift around massive foreground objects such as bright galaxies and clusters depends solely on ratios of the angular diameter distances. By using massive foreground halos they can compare relatively high, linear shear values in the same part of the sky, allowing them to effectively eliminate the dominant source of systematic error in cosmological weak lensing measurements.

They estimate the constraints that deep lensing surveys with photometric redshifts can provide on the  $\Omega_{DE}$ , the equation of state parameter  $w$  and  $w' \equiv dw/dz$ . They claim that the accuracies on  $w$  and  $w'$  are:  $\sigma(w) \simeq 0.02 f_{\text{sky}}^{-1/2}$  and  $\sigma(w') \simeq 0.05 f_{\text{sky}}^{-1/2}$ , where  $f_{\text{sky}}$  is the fraction of sky covered by the survey and  $\sigma(\Omega_{DE}) = 0.03$  is assumed in the marginalization. When this cross-correlation method is combined with standard lensing tomography, which possess complementary degeneracies, Jain and Taylor argue that it will allow measurement of the dark energy parameters with significantly better accuracy than has previously been obtained [424].

In [425] constraints on quintessence models where the acceleration is driven by a slow-rolling scalar field are investigated, focusing on cosmic shear, combined with supernovae Ia and CMB data. Based on earlier theoretical work developed in [426], the authors combine quintessence models with the computation of weak lensing observables, and determine several two-point shear statistics with data that includes, for the first time, the "gold set" of supernovae Ia, the WMAP-1 year data and

the VIRMOS-Desart and CFHTLS-deep and -wide data for weak lensing. In doing so, it is the first analysis of high-energy motivated dark energy models that uses weak lensing data, and allows for the exploration of larger angular scales, using a synthetic realization of the complete CFHTLS-wide survey as well as next space-based missions surveys. In other words it opens up the possibility of predicting how future wide field imagers can be expected to perform.

#### E. Constraints from baryon oscillations

In addition to SN Ia, CMB and LSS data, the recently observed baryon oscillations in the power spectrum of galaxy correlation functions also constrain the nature of dark energy [427] (see also Refs [429, 431]). The universe before the decoupling consists of a hot plasma of photons, baryons, electrons and dark matter. The tight coupling between photons and electrons due to Thomson scattering leads to oscillations in the hot plasma. As the universe expands and cools, electrons and protons combine into atoms making the universe neutral. The acoustic oscillations then cease but become imprinted on the radiation as well as on the baryons and should be seen in the spectrum of galaxy correlations today.

The detection of imprints of these oscillations in the galaxy correlation function is difficult as the signal is suppressed by the fractional energy density of baryons which is about 4% of the total cosmic budget. Thus a large volume of the universe is required to be surveyed in order to detect the signature. Recently, the imprints of baryon oscillations were observed by the Sloan Digital Sky Survey [427]. A peak in the correlation function was found around  $100h^{-1}\text{Mpc}$  separation. With this finding it has been possible to measure the ratio of the distances at redshifts  $z = 0.35$  and  $z = 1089$  to a high accuracy.

From the CMB radiation it is possible to constrain the angular diameter distance at a redshift  $z = 1089$  for fixed values of  $\Omega_m^{(0)}h^2$  and  $\Omega_b^{(0)}h^2$ . In the case of a flat model with a cosmological constant  $\Lambda$ , this distance depends only on the energy fraction of  $\Lambda$ . These measurements therefore can be used to constrain  $\Omega_\Lambda^{(0)}$  or  $\Omega_m^{(0)}$  to good precision. The consideration of the flat model with an unknown equation of state  $w \neq -1$  provides us a 2-dimensional parameter space [for instance  $(\Omega_\Lambda^{(0)}, w)$ ] which requires more information in addition to the CMB acoustic scale. With recent detection of baryon oscillations, we have the possibility to accurately constrain one more parameter, say, the equation of state or  $\Omega_K^{(0)}$ . In the case of constant  $w$ , Eisenstein *et al.* found that  $w = -0.80 \pm 0.18$  and  $\Omega_m^{(0)} = 0.326 \pm 0.037$  [427], which gives an independent confirmation of dark energy. For another approach to dark energy including the input of baryon oscillations see [428].

The measurements of baryon oscillations, however, can say nothing about the dynamics of dark energy at

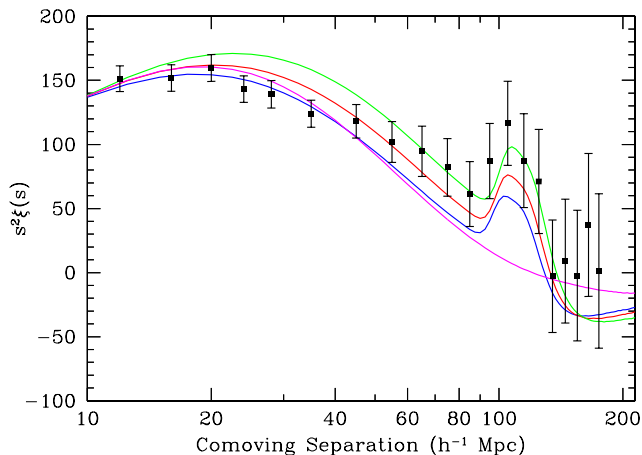


FIG. 22: Large-scale redshift-space correlation function (multiplied by the square of the separation  $s$ ) from the SDSS survey. From top to bottom the models are  $\Omega_m^{(0)} h^2 = 0.12, 0.13, 0.14$  and  $0.105$  with  $\Omega_b^{(0)} h^2 = 0.024$ . The bottom one corresponds to a pure cold dark matter model, which does not have an acoustic peak. Meanwhile there exist acoustic peaks around  $100h^{-1}$  Mpc in other models. From Ref. [427].

present. For that, the dynamical equation of state  $w(z)$  would require additional information coming from LSS such as the observation on baryon oscillations at higher values of the redshift which is one of the dreams of future missions of LSS studies. Finally, a word of caution. Forcing  $w$  to be equal to a constant can lead to bias, thereby hiding the actual dynamics of dark energy. Presumably, future surveys of large scale structure at other redshifts or perhaps more ambitious measurements of  $H(z)$  at different values of  $z$  will provide vital information for establishing the nature of dark energy [429]. In an interesting approach using the current astronomical data and based on the use of the Bayesian information criteria of model selection, Szydlowski *et al.* have analysed a class of models of dynamical dark energy, arriving at their top ten accelerating cosmological models [432]. The interested reader, wishing to learn more about the observational status of dark energy may want to look at the recent lectures of Perivolaropoulos [433].

#### XIV. THE FATE OF A DARK ENERGY UNIVERSE—FUTURE SINGULARITIES

In this section we shall discuss the future singularities which can in principle appear in a dark energy universe. When the equation of state of dark energy is less than  $-1$ , the universe reaches a Big Rip singularity within a finite time. In this case the null energy condition

$$\rho + p \geq 0, \quad (451)$$

is violated. Barrow [434] showed that a different type of future singularity can appear at a finite time even when

the strong energy condition

$$\rho + 3p \geq 0, \quad \rho + p \geq 0, \quad (452)$$

is satisfied (see also Refs. [436]). This sudden future singularity corresponds to the one in which the pressure density  $p$  diverges at  $t = t_s$  but the energy density  $\rho$  and the scale factor  $a$  are finite.

There exist a number of different finite-time singularities in a dark energy universe. The future-singularities can be classified into the following five classes [384]:

- Type I (“Big Rip”) : For  $t \rightarrow t_s$ ,  $a \rightarrow \infty$ ,  $\rho \rightarrow \infty$  and  $|p| \rightarrow \infty$
- Type II (“sudden”) : For  $t \rightarrow t_s$ ,  $a \rightarrow a_s$ ,  $\rho \rightarrow \rho_s$  and  $|p| \rightarrow \infty$
- Type III : For  $t \rightarrow t_s$ ,  $a \rightarrow a_s$ ,  $\rho \rightarrow \infty$  and  $|p| \rightarrow \infty$
- Type IV : For  $t \rightarrow t_s$ ,  $a \rightarrow a_s$ ,  $\rho \rightarrow 0$ ,  $|p| \rightarrow 0$  and higher derivatives of  $H$  diverge.
- Type V : For  $t \rightarrow t_s$ ,  $a \rightarrow \infty$ ,  $\rho \rightarrow \rho_s$ ,  $p \rightarrow p_s$  and higher derivatives of  $H$  diverge.

Here  $t_s$ ,  $a_s$ ,  $\rho_s$  and  $p_s$  are constants with  $a_s \neq 0$ . The type I corresponds to the Big Rip singularity [82], whereas the type II corresponds to the sudden future singularity mentioned above. The type III singularity has been discovered in the model of Ref. [437] and is different from the sudden future singularity in the sense that  $\rho$  diverges. The type IV is found in Ref. [384] for the model described below. This also includes the case when  $\rho$  ( $p$ ) or both of them tend to approach some finite values while higher derivative of  $H$  diverge. The type V is called a “quiescent singularity” that appears in braneworld models [435].

In what follows we shall describe some concrete models which give rise to the above singularities. Let us consider the equation of state of dark energy which is given by

$$p = -\rho - f(\rho), \quad (453)$$

where  $f(\rho)$  is a function in terms of  $\rho$ . We note that this type of an equation of state may be related to bulk viscosity [438]. The function  $f(\rho)$  characterizes the deviation from a  $\Lambda$ CDM cosmology. Nojiri and Odintsov [437] proposed the function of the form  $f(\rho) \propto \rho^\alpha$  and this case was studied in detail in Ref. [439]. For the equation of state (453) with  $f(\rho) \neq 0$  the continuity equation (159) is written in an integrated form as:

$$a = a_0 \exp\left(\frac{1}{3} \int \frac{d\rho}{f(\rho)}\right), \quad (454)$$

where  $a_0$  is constant. In the absence of any barotropic fluid other than dark energy, the Hubble rate satisfies Eq. (9) with  $K = 0$ . Then we obtain the following relation

$$t = \int \frac{d\rho}{\kappa \sqrt{3\rho f(\rho)}}. \quad (455)$$

In what follows we shall study the properties of future singularities for several choices of the function  $f(\rho)$  establishing the relation between the singularities and the behavior of  $f(\rho)$ .

### A. Type I and III singularities

Type I and III singularities appear when the function  $f(\rho)$  is given by

$$f(\rho) = A\rho^\alpha, \quad (456)$$

where  $A$  and  $\alpha$  are constants. Let us consider a situation in which  $\rho$  goes to infinity with positive  $\alpha$ . From Eqs. (454) and (455) we have

$$a = a_0 \exp \left[ \frac{\rho^{1-\alpha}}{3(1-\alpha)A} \right], \quad (457)$$

and

$$t = t_s + \frac{2}{\sqrt{3\kappa A}} \frac{\rho^{-\alpha+1/2}}{1-2\alpha}, \quad \text{for } \alpha \neq \frac{1}{2}, \quad (458)$$

$$t = t_s + \frac{\ln \rho}{\sqrt{3\kappa A}}, \quad \text{for } \alpha = \frac{1}{2}, \quad (459)$$

where  $t_s$  is constant.

When  $\alpha > 1$ , the scale factor is finite even for  $\rho \rightarrow \infty$ . When  $\alpha < 1$  we find  $a \rightarrow \infty$  ( $a \rightarrow 0$ ) as  $\rho \rightarrow \infty$  for  $A > 0$  ( $A < 0$ ). If  $\alpha > 1/2$  the energy density  $\rho$  diverges in the finite future or past ( $t = t_s$ ). On the other hand, if  $\alpha \leq 1/2$ ,  $\rho$  diverges in the infinite future or past.

Since the pressure is given by  $p \sim -\rho - A\rho^\alpha$ ,  $p$  always diverges when  $\rho$  becomes infinite. The equation of state of dark energy is

$$w = \frac{p}{\rho} = -1 - A\rho^{\alpha-1}. \quad (460)$$

When  $\alpha > 1$  one has  $w \rightarrow +\infty$  ( $w \rightarrow -\infty$ ) as  $\rho \rightarrow \infty$  for  $A < 0$  ( $A > 0$ ). Meanwhile when  $\alpha < 1$ ,  $w \rightarrow -1 + 0$  ( $-1 - 0$ ) for  $A < 0$  ( $A > 0$ ).

From the above argument, one can classify the singularities as follows:

1.  $\alpha > 1$ :

There exists a type III singularity.  $w \rightarrow +\infty$  ( $-\infty$ ) if  $A < 0$  ( $A > 0$ ).

2.  $1/2 < \alpha < 1$ :

There is a type I future singularity for  $A > 0$ . When  $A < 0$ , one has  $a \rightarrow 0$  as  $\rho \rightarrow \infty$ . Hence if the singularity exists in the past (future), we may call it Big Bang (Big Crunch) singularity.  $w \rightarrow -1 + 0$  ( $-1 - 0$ ) if  $A < 0$  ( $A > 0$ ).

3.  $0 < \alpha \leq 1/2$ :

There is no finite future singularity.

When  $\alpha < 0$ , it was shown in Ref. [439] that the type II singularity appears when  $\rho$  approaches 0. In the next subsection we shall generalize this to a more general case.

### B. Type II singularity

Let us consider the function

$$f(\rho) = C(\rho_0 - \rho)^{-\gamma}, \quad (461)$$

where  $C$ ,  $\rho_0$  and  $\gamma$  are constants with  $\gamma > 0$ . We study the case in which  $\rho$  is smaller than  $\rho_0$ . In the limit  $\rho \rightarrow \rho_0$ , the pressure  $p$  becomes infinite because of the divergence of  $f(\rho)$ . The scalar curvature  $R$  also diverges since  $R = 2\kappa^2(\rho - 3p)$ . The equation of state of dark energy is

$$w = -1 - \frac{C}{\rho(\rho_0 - \rho)^\gamma}. \quad (462)$$

Hence,  $w \rightarrow -\infty$  for  $C > 0$  and  $w \rightarrow \infty$  for  $C < 0$  as  $\rho \rightarrow \rho_0$ .

From Eq. (454) the scale factor is given by

$$a = a_0 \exp \left[ -\frac{(\rho_0 - \rho)^{\gamma+1}}{3C(\gamma+1)} \right], \quad (463)$$

which means that  $a$  is finite for  $\rho = \rho_0$ . Since the Hubble rate  $H \propto \sqrt{\rho}$  is nonsingular,  $\dot{a}$  remains finite. On the other hand Eq. (12) shows that  $\ddot{a}$  diverges for  $\rho \rightarrow \rho_0$ . By using Eq. (455) we find the following relation around  $\rho \sim \rho_0$ :

$$t \simeq t_s - \frac{(\rho_0 - \rho)^{\gamma+1}}{\kappa C \sqrt{3\rho_0}(\gamma+1)}, \quad (464)$$

where  $t_s$  is an integration constant. Then we have  $t = t_s$  for  $\rho = \rho_0$ . The above discussion shows that the function  $f(\rho)$  in Eq. (461) gives rise to the type II singularity. We note that the strong energy condition (452) is satisfied for  $C < 0$  around  $\rho = \rho_0$ . This means that the sudden singularity appears even in the case of a non-phantom dark energy ( $w > -1$ ).

This type II singularity always appears when the denominator of  $f(\rho)$  vanishes at a finite value of  $\rho$ . The model (456) with negative  $\alpha$  is a special case of the model (461) with  $\rho_0 = 0$ .

### C. Type IV singularity

In Ref. [384] it was shown that the type IV singularity can appear in the model given by

$$f(\rho) = \frac{AB\rho^{\alpha+\beta}}{A\rho^\alpha + B\rho^\beta}, \quad (465)$$

where  $A$ ,  $B$ ,  $\alpha$  and  $\beta$  are constants. We note that this model also gives rise to the type I, II, III singularities [384].

In what follows we shall study the case with  $\alpha = 2\beta - 1$ . Then Eqs. (454) and (455) are integrated to give

$$a = a_0 \exp \left\{ -\frac{1}{3} \left[ \frac{\rho^{-\alpha+1}}{(\alpha-1)A} + \frac{\rho^{-\beta+1}}{(\beta-1)B} \right] \right\}, \quad (466)$$



and

$$\begin{aligned} & \frac{2}{4\beta-3}\rho^{-\frac{4\beta-3}{2}} + \frac{2A}{(2\beta-1)B}\rho^{-\frac{2\beta-1}{2}} \\ & = -\sqrt{3}\kappa A(t-t_s) \equiv \tau, \end{aligned} \quad (467)$$

where  $t_s$  is an integration constant. Equation (467) is valid for  $\beta \neq 1$ ,  $\beta \neq 3/4$ , and  $\beta \neq 1/2$ .

Let us consider the case with  $0 < \beta < 1/2$ . In this case the pressure density behaves as

$$p \sim -\rho - B\rho^\beta, \quad \text{when } \rho \sim 0. \quad (468)$$

Equation (467) shows that  $t \rightarrow t_s$  as  $\rho \rightarrow \rho_0$ . Then from Eq. (468) one has  $p \rightarrow 0$  and  $\rho \rightarrow 0$  as  $t \rightarrow t_s$ . By using Eqs. (466) and (467) we obtain the following relation around  $t = t_s$ :

$$\ln(a/a_0) \propto \tau^s, \quad s = 1 - \frac{1}{2\beta-1}. \quad (469)$$

Hence the scale factor is finite ( $a = a_0$ ) at  $t = t_s$ .

From Eq. (469) we find that  $s > 2$  for  $0 < \beta < 1/2$ , which means that  $H$  and  $\dot{H}$  are finite. However  $d^n H/dt^n$  diverges for  $n > -1/(2\beta-1)$  as long as  $s$  is not an integer. This corresponds to the type IV singularity in which higher-order derivatives of  $H$  exhibit divergence even if  $a$ ,  $\rho$  and  $p$  are finite as  $t \rightarrow t_s$ . In this case  $w \rightarrow +\infty$  ( $-\infty$ ) for  $B < 0$  ( $B > 0$ ).

Thus we have shown that the equation of state given by Eq. (453) has a rich structure giving rise to four types of sudden singularities. We note that there are other types of equation of state which lead to the singularities mentioned above, see Refs. [434, 436]. In the presence of a bulk viscosity  $\zeta$  the effective pressure density is given by  $p_{\text{eff}} = p - 3\zeta H$  [440]. This was generalized to a more general inhomogeneous dark energy universe in Ref. [441]. Such inhomogeneous effects can change the type of singularities discussed in this section. See Refs. [442] for other interesting aspects of future singularities.

Finally, we should mention that the model studied in Ref. [115] provides an alternative mechanism for the emergence of future singularities, see Ref. [443] for details.

## XV. DARK ENERGY WITH HIGHER-ORDER CURVATURE CORRECTIONS

In the previous section, we saw that a dark energy universe with singularities is typically associated with the growth of the curvature of the universe. For the models we have been considering, the type I, II, III singularities lead to the divergence of the Ricci scalar  $R$  at finite time. In such circumstances we expect that the effect of higher-order curvature terms can be important around singularities [384, 437, 444, 445, 446, 447, 448, 449, 450]. This may moderate or even remove the singularities.

Let us consider the following action with a correction term  $\mathcal{L}_c^{(\phi)}$ :

$$\begin{aligned} S = \int d^D x \sqrt{-g} & \left[ \frac{1}{2} f(\phi, R) - \frac{1}{2} \zeta(\phi) (\nabla\phi)^2 - V(\phi) \right. \\ & \left. + \xi(\phi) \mathcal{L}_c + \mathcal{L}_\rho^{(\phi)} \right], \end{aligned} \quad (470)$$

where  $f$  is a generic function of a scalar field  $\phi$  and the Ricci scalar  $R$ .  $\zeta$ ,  $\xi$  and  $V$  are functions of  $\phi$ .  $\mathcal{L}_\rho^{(\phi)}$  is the Lagrangian of a perfect fluid with energy density  $\rho$  and pressure density  $p$ . The barotropic index,  $w \equiv p/\rho$ , is assumed to be constant. In general the fluid can couple to the scalar field  $\phi$ . We note that the action (470) includes a wide variety of gravity theories such as Einstein gravity, scalar-tensor theories and low-energy effective string theories. In what follows we shall consider two types of higher-order correction terms and investigate the effects on the future singularities.

### A. Quantum effects from a conformal anomaly

Let us first study the effect of quantum effects in four dimensions by taking into account the contribution of the conformal anomaly as a backreaction. We shall consider the case of a fixed scalar field without a potential in which the barotropic fluid  $\mathcal{L}_\rho$  is responsible for dark energy, i.e.,  $f = R$ ,  $\zeta = 0$ ,  $V = 0$  and  $\xi = 1$  in Eq. (470).

The conformal anomaly  $T_A$  takes the following form [384, 437, 444]:

$$T_A = b_1 \left( F + \frac{2}{3} \square R \right) + b_2 G + b_3 \square R, \quad (471)$$

where  $F$  is the square of a 4-dimensional Weyl tensor,  $G$  is a Gauss-Bonnet curvature invariant, which are given by

$$F = (1/3)R^2 - 2R_{ij}R^{ij} + R_{ijkl}R^{ijkl}, \quad (472)$$

$$G = R^2 - 4R_{ij}R^{ij} + R_{ijkl}R^{ijkl}. \quad (473)$$

With  $N$  scalar,  $N_{1/2}$  spinor,  $N_1$  vector fields,  $N_2$  ( $= 0$  or  $1$ ) gravitons and  $N_{\text{HD}}$  higher derivative conformal scalars, the coefficients  $b_1$  and  $b_2$  are given by

$$b_1 = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{\text{HD}}}{120(4\pi)^2}, \quad (474)$$

$$b_2 = -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{\text{HD}}}{360(4\pi)^2}. \quad (475)$$

We have  $b_1 > 0$  and  $b_2 < 0$  for the usual matter except for higher derivative conformal scalars. We note that  $b_2$  can be shifted by a finite renormalization of the local counterterm  $R^2$ , so  $b_2$  can be arbitrary.

The conformal anomaly is given by  $T_A = -\rho_A + 3p_A$  in terms of the corresponding energy density  $\rho_A$  and the pressure density  $p_A$ . Using the continuity equation

$$\dot{\rho}_A + 3H(\rho_A + p_A) = 0, \quad (476)$$

$T_A$  can be expressed as

$$T_A = -4\rho_A - \dot{\rho}_A/H. \quad (477)$$

This then gives the following expression for  $\rho_A$ :

$$\begin{aligned} \rho_A &= -\frac{1}{a^4} \int dt a^4 H T_A \\ &= -\frac{1}{a^4} \int dt a^4 H \left[ -12b_1 \dot{H}^2 + 24b_2 (-\dot{H}^2 + H^2 \dot{H} + H^4) \right. \\ &\quad \left. - (4b_1 + 6b_3) \left( \ddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H} \right) \right]. \quad (478) \end{aligned}$$

In Ref. [451] a different form of  $\rho_A$  was obtained by requiring that the quantum corrected energy momentum tensor  $T_{A\mu\nu}$  has the form as  $T_{A\mu\nu} = (T_A/4)g_{\mu\nu}$  in the conformal metric case rather than assuming the conservation law (476).

Now, we are considering a universe with a dark energy fluid and quantum corrections. Then the Friedmann equation is given by

$$3H^2 = \kappa^2 (\rho + \rho_A). \quad (479)$$

Since the curvature is large around the singularity, we may assume  $(3/\kappa^2)H^2 \ll |\rho_A|$ . This gives  $\rho_A \sim -\rho$ , which reflects the fact that the conformal anomaly can give rise a negative energy density coming from higher-order curvature terms.

$$\begin{aligned} &\dot{\rho} + 4H\rho \\ &= H \left[ -12b_1 \dot{H}^2 + 24b_2 (-\dot{H}^2 + H^2 \dot{H} + H^4) \right. \\ &\quad \left. - (4b_1 + 6b_3) \left( \ddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H} \right) \right]. \quad (480) \end{aligned}$$

One can understand whether the singularities may be moderated or not by using this equation. We consider a dark energy fluid with an equation of state given by Eq. (453). As an example we study the case of the type II (sudden) singularity for the model (461). From Eq. (464) the evolution of the energy density  $\rho$  around the singularity is described by

$$\rho \sim \rho_0 - \left[ \kappa C \sqrt{3\rho_0} (\gamma + 1) (t_s - t) \right]^{\frac{1}{\gamma+1}}. \quad (481)$$

In the absence of quantum corrections the Hubble parameter is given by

$$H \sim \kappa \sqrt{\frac{\rho_0}{3}} \left\{ 1 - \frac{1}{2\rho_0} \left[ \kappa C \sqrt{3\rho_0} (\gamma + 1) (t_s - t) \right]^{\frac{1}{\gamma+1}} \right\}. \quad (482)$$

Although  $H$  is finite at  $t = t_s$ ,  $\dot{H}$  diverges there because of the condition:  $0 < 1/(\gamma + 1) < 1$ .

The situation is different if we include quantum corrections. Let us assume the following form of  $\rho$  around  $t = t_s$ :

$$\rho = \rho_0 + \rho_1 (t_s - t)^\nu, \quad (483)$$

where  $\nu$  is a positive constant. The continuity equation (11) gives  $H = \dot{\rho}/3f(\rho)$  for the equation of state (453), thereby giving

$$H \sim \frac{\nu\rho_1^{1+\gamma}}{3(-C)} (t_s - t)^{-1+\nu(1+\gamma)}. \quad (484)$$

Here  $\nu(1 + \gamma)$  is positive. Picking up the most singular terms in Eq. (480) around the singularity, we find

$$\dot{\rho} \sim -6 \left( \frac{2}{3}b_1 + b_3 \right) H\ddot{H}. \quad (485)$$

Then substituting Eqs. (483) and (484) for Eq. (485), we obtain  $\nu = 4/(2\gamma + 1)$  and

$$\rho = \rho_0 + \rho_1 (t_s - t)^{\frac{4}{2\gamma+1}}, \quad (486)$$

$$H \propto (t_s - t)^{\frac{2\gamma+3}{2\gamma+1}}. \quad (487)$$

This shows that both  $H$  and  $\dot{H}$  are finite because  $(2\gamma + 3)/(2\gamma + 1)$  is larger than 1. Hence quantum effects work to prevent the type II singularity. When the quantum correction becomes important, this typically works to provide a negative energy density  $\rho_A$  which nearly cancels with the energy density  $\rho$  of dark energy. This is the reason why the Hubble rate does not diverge in such a case. It was shown in Ref. [384] that the type I and III singularities can be moderated as well in the presence of quantum corrections. This property also holds for scalar-field dark energy models [444]. Thus quantum effects can work to make the universe less singular or completely nonsingular.

## B. String curvature corrections

We now turn our attention to study the effect of higher-order corrections [452] in low-energy effective string theory in the presence of a dark energy fluid (see Ref.[453] for cosmological relevance of strings and branes). In this case the field  $\phi$  in the action (470) corresponds to either the dilaton or another modulus. At tree level the potential of the field  $\phi$  vanishes, so we include the  $\alpha'$ -order quantum corrections of the form:

$$\mathcal{L}_c = a_1 R_{ijkl} R^{ijkl} + a_2 R_{ij} R^{ij} + a_3 R^2 + a_4 (\nabla\phi)^4, \quad (488)$$

where  $a_i$  are coefficients depending on the string model one is considering. The Gauss-Bonnet parametrization ( $a_1 = 1$ ,  $a_2 = -4$  and  $a_3 = 1$ ) corresponds to the ghost-free gravitational Lagrangian, which we shall focus on below. See Ref. [448] for the cosmological dynamics in the case of other parametrizations. The reader should note that the expansion does not include the quantum loop expansion which is governed by the string coupling constant, as these have not been fully determined. It could well be that these would also play an important role in any dynamics, but we have to ignore them for this argument.



For a massless dilaton field the action (470) is given by [164]

$$F = -\zeta = e^{-\phi}, \quad V = 0, \quad \xi = \frac{\lambda}{2} e^{-\phi}, \quad (489)$$

where  $\lambda = 1/4, 1/8$  for the bosonic and heterotic string, respectively, whereas  $\lambda = 0$  in the Type II superstring. The choice of  $\xi$  corresponds to the tree-level correction. In general the full contribution of  $n$ -loop corrections is given by  $\xi(\phi) = \sum C_n e^{(n-1)\phi}$ , with coefficients  $C_n$ .

Generally moduli fields appear whenever a submanifold of the target spacetime is compactified with radii described by the expectation values of the moduli themselves. In the case of a single modulus (one common characteristic length) and heterotic string ( $\lambda = 1/8$ ), the four-dimensional action corresponds to [454]

$$F = 1, \quad \zeta = 3/2, \quad a_4 = 0, \quad \xi = -\frac{\delta}{16} \ln[2e^\phi \eta^4(ie^\phi)], \quad (490)$$

where  $\eta$  is the Dedekind function and  $\delta$  is a constant proportional to the 4D trace anomaly.  $\delta$  depends on the number of chiral, vector, and spin-3/2 massless supermultiplets of the  $N = 2$  sector of the theory. In general it can be either positive or negative, but it is positive for the theories which do not have too many vector bosons present. Again the scalar field corresponds to a flat direction in the space of nonequivalent vacua and  $V = 0$ . At large  $\phi$  the last equation can be approximated as

$$\xi_{,\phi} \approx \xi_0 \sin h\phi, \quad \xi_0 \equiv \frac{1}{24} \pi \delta. \quad (491)$$

As shown in Ref. [455] this is a very good approximation to the exact expression (490).

In Ref. [448] cosmological solutions based on the action (470) without a potential ( $V = 0$ ) were discussed in details for three cases—(i) fixed scalar field ( $\dot{\phi} = 0$ ), (ii) linear dilaton ( $\dot{\phi} = \text{const}$ ), and (iii) logarithmic modulus ( $\dot{\phi} \propto 1/t$ ). For case (i) we obtain geometrical inflationary solutions only for  $D \neq 4$ . Case (ii) leads to pure de-Sitter solutions in the string frame, but this corresponds to a contracting universe in the Einstein frame. These solutions are unrealistic when we apply to dark energy scenarios. In what follows we shall focus on cosmological solutions in case (iii) in four dimensions ( $D = 4$ ). We assume that the dilaton is stabilized by some non-perturbative mechanism.

In general the field  $\phi$  can be coupled to a barotropic fluid. We choose the covariant coupling  $Q$  introduced in Sec. VII. Then the energy density  $\rho$  of the dark energy fluid satisfies

$$\dot{\rho} = \left[ -3H(1+w) + Q\dot{\phi} \right] \rho. \quad (492)$$

We also obtain the equations of motion for the modulus

system:

$$\dot{H} = \frac{2\dot{\rho} + 3\dot{\phi}\ddot{\phi} - 48H^3\ddot{\xi}}{12H(1+12H\dot{\xi})}, \quad (493)$$

$$\ddot{\phi} = 16\frac{d\xi}{d\phi}H^2(H^2 + \dot{H}) - 3H\dot{\phi} - \frac{2}{3}Q\rho. \quad (494)$$

Let us search for future asymptotic solutions with the following form

$$H \sim \omega_1 t^\beta, \quad \phi \sim \phi_0 + \omega_2 \ln t, \quad \xi \sim \frac{1}{2}\xi_0 e^{\phi_0} t^{\omega_2}, \quad (495)$$

$$\rho \sim \rho_0 t^{Q\omega_2} \exp\left[-\frac{3(1+w)\omega_1}{\beta+1}t^{\beta+1}\right], \quad \beta \neq -1, \quad (496)$$

$$\rho \sim \rho_0 t^\alpha, \quad \beta = -1, \quad (497)$$

where  $\omega_1$  and  $\omega_2$  are real values of constants, and

$$\alpha \equiv Q\omega_2 - 3(1+w)\omega_1. \quad (498)$$

Substituting Eqs. (495), (496) and (497) into Eqs. (493) and (494), we can obtain a number of asymptotic solutions depending on the regimes we are in [448]. Among them the following two solutions are particularly important.

### 1. Solution in a low-curvature regime

The solution which appears in a low-curvature regime in which the  $\xi$  terms are subdominant at late times, is characterized by

$$\beta = -1, \quad \omega_2 < 2, \quad (499)$$

together with the constraints

$$\omega_1 = \frac{1}{3} - \frac{2Q\rho_0 t^{\alpha+2}}{9\omega_2}, \quad (500)$$

$$3\omega_2^2 = 12\omega_1^2 - 4\rho_0 t^{\alpha+2}, \quad (501)$$

$$\alpha = Q\omega_2 - 3(1+w) \leq -2. \quad (502)$$

This corresponds to the solution ‘ $A_\infty$ ’ in Ref. [454] and describes the asymptotic solution of the tree-level system ( $\delta = 0$ ).

### 2. Solution in a asymptotically flat-space regime

This solution appears in a situation where some of the  $\xi$  terms contribute to the dynamics, and is given by

$$\beta = -2, \quad \omega_2 = 5, \quad Q \leq -2/5, \quad (503)$$

$$\omega_1^3 = \frac{1}{24c_1\xi_0 e^{\phi_0}} (15 - 2Q\rho_0 t^{5Q+2}), \quad (504)$$

for a non-vanishing fluid. We note that this is different from the high-curvature solution in which the  $\xi$  terms completely dominate the dynamics [448]. The solution corresponds to ‘ $C_\infty$ ’ in Ref. [454] and describes an asymptotically flat universe with slowly expanding (or contracting) scale factor. In fact an expanding solution is given by  $a(t) \sim a_0 \exp(-\omega_1/t)$ , which exhibits superinflation as  $t \rightarrow -0$ .

These solutions can be joined to each other if the coupling constant  $\delta$  is negative [454]. There exists an exact solution for Eqs. (493) and (494), but this is found to be unstable in numerical simulations of Ref. [448]. In the asymptotic future the solutions tend to approach the low-curvature one given by Eq. (499) rather than the others, irrespective of the sign of the modulus-to-curvature coupling  $\delta$ .

Let us consider the case in which a phantom fluid ( $w < -1$ ) is present together with the modulus string corrections. Equation (502) shows that the condition for the existence of the low-curvature solution (499) is not satisfied for  $Q = 0$  and  $w < -1$ . However the presence of the coupling  $Q$  can fulfill this condition. This suggests that the Big Rip singularity may be avoided when the modulus field  $\phi$  is coupled to dark energy.

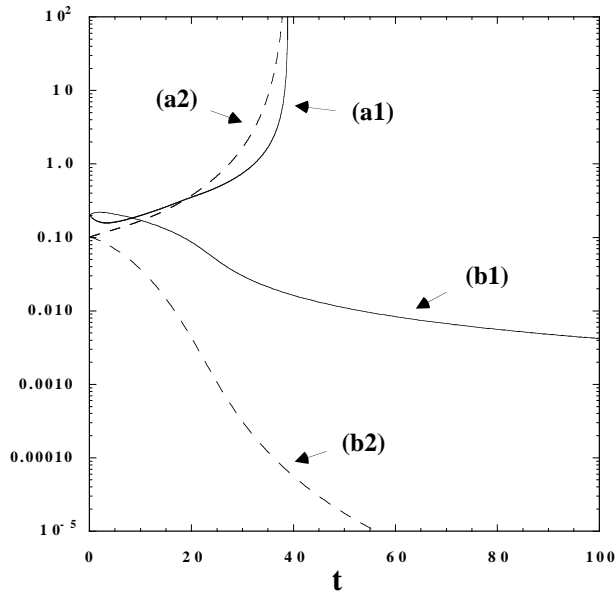


FIG. 23: Evolution of  $H$  and  $\rho$  with  $\xi_0 = -2$ ,  $w = -1.1$  for (a)  $Q = 0$  and (b)  $Q = -5$ . We choose initial conditions as  $H_i = 0.2$ ,  $\phi_i = 2.0$  and  $\rho_i = 0.1$ . The curves (a1) and (b1) represent the evolution of  $H$  for  $Q = 0$  and  $Q = -5$ , respectively, while the curves (a2) and (b2) show the evolution of  $\rho$  for corresponding  $Q$ .

In Fig. 23 we show the evolution of  $H$  and  $\rho$  with negative  $\delta$  for (a)  $Q = 0$  and (b)  $Q = -5$  in the presence of the phantom fluid with  $w = -1.1$ . Although the solution approaches a Big Rip singularity for  $Q = 0$ , this singularity is actually avoided for  $Q = -5$ . In the latter case the universe approaches the low-curvature solution given by Eq. (499) at late times. Since the asymptotic values of  $\omega_1$  and  $\omega_2$  are  $\omega_1 = 1/3$  and  $\omega_2 = 2/3$  from Eqs. (500) and (501), the condition (502) for the existence of low-curvature solution is  $Q < 3(w - 1)/2 = -3.15$ . Numerical calculations show that the Big Rip singularity can be avoided for a wide range of initial conditions [448].

When  $Q > 0$  the condition (502) is not satisfied for  $\omega_2 > 0$ . However it is numerically found that the sys-

tem approaches the low-energy regime characterized by  $\omega_1 = 1/3$  and  $\omega_2 = -2/3$  [448]. This negative value of  $\omega_2$  means that the Big Rip singularity may be avoided even for positive  $Q$ . In fact  $H$  and  $\rho$  decrease when the condition (502) is satisfied in the asymptotic regime.

When  $\delta > 0$ , there is another interesting circumstance in which the Hubble rate decreases but the energy density of the fluid increases [448]. This corresponds to the solution in which the growing energy density  $\rho$  can balance with the GB term ( $\rho \approx 24H^3\xi$  in the Friedmann equation). Hence the Big Rip singularity does not appear even when  $w < -1$  and  $Q = 0$ .

The above discussion shows that for a restricted class of modulus-type string corrections there exists the possibility of avoiding the Big Rip singularity. We also note that recent development of loop quantum cosmology allows us to avoid several future singularities discussed in Sec. XIV [456].

## XVI. COSMIC ACCELERATION FROM MODIFIED GRAVITY AND OTHER ALTERNATIVES TO DARK ENERGY

The contribution of the matter content of the universe is represented by the energy momentum tensor on the right hand side of Einstein equations, whereas the left hand side is represented by pure geometry. There are then two ways to give rise to an accelerated expansion: (i) either by supplementing the energy momentum tensor by an exotic form of matter such as a cosmological constant or scalar field; (ii) by modifying the geometry itself. The geometrical modifications can arise from quantum effects such as higher curvature corrections to the Einstein Hilbert action. In the previous section we have used such curvature corrections to avoid future singularities in the presence of a dark energy fluid. In this section we are interested in whether it is possible to obtain an accelerated expansion driven by geometrical terms alone.

It is well known that the quadratic term in  $R$  leads to an inflationary solution in the early universe [457]. In this model the effective potential in the Einstein frame vanishes at a potential minimum, in which case we can not have a late time accelerated expansion of the universe. However, it was pointed out in Refs. [25, 26] that late time acceleration can be realized by terms containing inverse powers of the Ricci scalar added to Einstein Hilbert action<sup>7</sup>. However the original model ( $\mathcal{L} \propto 1/R$ ) is not compatible with solar system experiments [459] and possess instabilities [460, 461] (see Refs. [27, 462] for recent reviews). It was argued by Nojiri and Odintsov [463] that the situation could be remedied by adding a counterterm term proportional to  $R^2$  in the action (see

<sup>7</sup> We note that inflationary solutions in such cosmological models were already studied in 1993 in Ref. [458].

also Refs. [464], however also see [465] for a different take on the problem).

Another interesting approach which can avoid the above mentioned problem is provided by Palatini formalism [466, 467, 468, 469, 470]. The Palatini formalism leads to differential equations of second order even in presence of non-linear terms in  $R$  in the gravitational action and is free from the problem of instabilities [466, 467]. A variety of different aspects of  $f(R)$  gravity and associated cosmological dynamics is discussed in Ref. [471]. An interesting possibility of obtaining late time acceleration from modified Gauss-Bonnet gravity is discussed in Ref. [472].

The other exiting possibility of obtaining accelerated expansion is provided by theories with large extra dimensions known as braneworlds. Being inspired by string theory, our four dimensional spacetime (*brane*) is assumed to be embedded in a higher dimensional *bulk* spacetime. In these scenarios all matter fields are confined on the brane whereas gravity being a true universal interaction can propagate into the anti de Sitter bulk. In Randall-Sundrum (RS) braneworld [300] the Einstein equations are modified by high energy corrections [301, 302], but this modification is generally not thought to be important for late-time cosmology (However, see [299] for interesting possibilities). The situation is reversed in the braneworld model of Dvali-Gabadadze-Porrati (DGP) [24] in which the brane is embedded in a *Minkowski* bulk. They differ from the RS brane world by a curvature term on the brane (see Ref. [473] for review and Refs. [474] for related works). Unlike the RS scenario, in DGP braneworld, gravity remains four dimensional at short distances but can leak into the bulk at large distances leading to infrared modifications to Einstein gravity. In the DGP model there is a cross-over scale around which gravity manifests these higher-dimensional properties. This scenario is a simple one parameter model which can account for the current acceleration of the universe provided the cross-over scale is fine tuned to match observations.

In this section we shall briefly describe these two approaches for obtaining the current acceleration of the universe from modified theories of gravity.

### A. $f(R)$ gravities

Let us start with an action [25, 26]

$$S = \int d^4x \sqrt{-g} f(R), \quad (505)$$

where  $f(R)$  is an arbitrary function in terms of  $R$ . By varying the action (505) with respect to the metric leads

to the following field equations

$$G_{\mu\nu} = \left( \frac{\partial f}{\partial R} \right)^{-1} \left[ \frac{1}{2} g_{\mu\nu} \left( f - \frac{\partial f}{\partial R} R \right) + \left\{ \nabla_\mu \nabla_\nu \frac{\partial f}{\partial R} - \square \left( \frac{\partial f}{\partial R} \right) g_{\mu\nu} \right\} \right], \quad (506)$$

where  $G_{\mu\nu}$  is an Einstein tensor.

Equation (506) looks complicated but can acquire a simple form after a conformal transformation

$$g_{\mu\nu}^{(E)} = e^{2\omega} g_{\mu\nu}, \quad (507)$$

where  $\omega$  is a smooth and positive function of space time coordinates. Here ' $E$ ' denotes the metric in the Einstein frame. From Eqs. (506) and (507) we find that the Einstein tensor in the  $g_{\mu\nu}^{(E)}$  metric can be written as [165]

$$G_{\mu\nu}^{(E)} = \left( \frac{\partial f}{\partial R} \right)^{-1} \left[ \frac{1}{2} g_{\mu\nu} \left( f - \frac{\partial f}{\partial R} R \right) + \left\{ \nabla_\mu \nabla_\nu \frac{\partial f}{\partial R} - \square \left( \frac{\partial f}{\partial R} \right) g_{\mu\nu} \right\} \right] - 2(\nabla_\mu \nabla_\nu \omega - \square \omega g_{\mu\nu}) + 2\nabla_\mu \omega \nabla_\nu \omega + (\nabla \omega)^2 g_{\mu\nu}. \quad (508)$$

If we choose the conformal factor of the form

$$2\omega = \ln \left[ 2\kappa^2 \left| \frac{\partial f}{\partial R} \right| \right], \quad (509)$$

we find that the term  $(\partial f / \partial R)^{-1} \nabla_\mu \nabla_\nu (\partial f / \partial R)$  cancels with the term  $-2\nabla_\mu \nabla_\nu \omega$  in Eq. (508). In this case  $\omega$  behaves like a scalar field  $\phi$ , which is defined by

$$\kappa\phi \equiv \sqrt{6}\omega = \frac{\sqrt{6}}{2} \ln \left[ 2\kappa^2 \left| \frac{\partial f}{\partial R} \right| \right]. \quad (510)$$

Then the action in the Einstein frame is given by  $S_E = \int d^4x \sqrt{-g_E} \mathcal{L}$  with Lagrangian density

$$\mathcal{L} = \frac{1}{2\kappa^2} R(g_E) - \frac{1}{2} (\nabla_E \phi)^2 - U(\phi), \quad (511)$$

where

$$U(\phi) = (\text{sign}) e^{-\frac{2\sqrt{6}}{3}\kappa\phi} \left[ \frac{(\text{sign})}{2\kappa^2} R e^{\frac{\sqrt{6}}{3}\kappa\phi} - f \right], \quad (512)$$

and  $(\text{sign}) = (\partial f / \partial R) / |\partial f / \partial R|$ .

We now consider the modified gravity action given by [25, 26]

$$f(R) = \frac{1}{2\kappa^2} \left[ R - \mu^{2(n+1)} / R^n \right], \quad n > 0, \quad (513)$$

where  $\mu$  is a parameter with units of mass. From Eq. (512) the effective potential in Einstein frame is

$$U(\phi) = \mu^2 M_{\text{pl}}^2 \frac{n+1}{2n} n^{\frac{1}{n+1}} e^{-\frac{2\sqrt{6}}{3}\kappa\phi} (e^{\frac{\sqrt{6}}{3}\kappa\phi} - 1)^{\frac{n}{n+1}}, \quad (514)$$

where we used the relation (510). This potential has a maximum at  $\kappa\phi = 2(n+1)/(n+2)$  and has the following asymptotic form:

$$U(\phi) \propto \exp\left(-\frac{\sqrt{6}\kappa}{3} \frac{n+2}{n+1} \phi\right), \quad \kappa\phi \gg 1. \quad (515)$$

Taking note that the potential (182) leads to the power-law expansion (105), we find that the evolution of the scale factor in the Einstein frame is given by

$$a_E \propto t_E^p, \quad p = \frac{3(n+1)^2}{(n+2)^2}. \quad (516)$$

When  $n = 1$  one has  $p = 4/3$ , which corresponds to an accelerated expansion. The power-law index  $p$  increases for larger  $n$  and asymptotically approaches  $p \rightarrow 3$  as  $n \rightarrow \infty$ .

We note that scale factor  $a$  and cosmic time  $t$  in the Jordan frame are related to those in Einstein frame via the relation  $a = e^{-\kappa\phi/\sqrt{6}} a_E$  and  $dt = e^{-\kappa\phi/\sqrt{6}} dt_E$ . Since the field  $\phi$  is given by  $\kappa d\phi/dt_E = \sqrt{2p}/t_E$  for the potential (515), we find that the evolution of scale factor in the Jordan frame is

$$a \propto t^q, \quad q = \frac{(2n+1)(n+1)}{(n+2)}. \quad (517)$$

From Eq. (19) this corresponds to the effective equation of state:

$$w_{\text{DE}} = -1 + \frac{2(n+2)}{3(2n+1)(n+1)}. \quad (518)$$

When  $n = 1$  we have  $q = 2$  and  $w_{\text{DE}} = -2/3$ . WMAP in concert with other observations have really begun to constrain the current value of the dark energy equation of state, although it does depend on the priors. For instance, in a flat universe, the combination of WMAP3 and the Supernova Legacy Survey (SNLS) gives  $w_{\text{DE}} = -0.97_{-0.09}^{+0.07}$ , whereas even if we do not include a prior of a flat universe, then by combining WMAP3 with large scale structure and supernova data we obtain  $w_{\text{DE}} = -1.06_{-0.08}^{+0.13}$  at the  $2\sigma$  level [61]. It follows that the  $n = 1$  case is outside of observational bounds. However the model is compatible with observations when  $n \geq 2$ . We note that the  $n \rightarrow \infty$  limit corresponds to the equation of state of cosmological constant ( $w_{\text{DE}} = -1$ ). The effects of modification should become important only at late times, which requires the tuning of the energy scale  $\mu$ .

It was pointed out by Chiba [459] that theories of the type (513) give the Brans-Dicke parameter  $\omega_{\text{BD}} = 0$ , which contradicts with the constraint of solar-system experiments ( $\omega_{\text{BD}} > 40000$  [475]). This means that the field  $\phi$  couples to matter with a comparable strength as gravity. Dolgov and Kawasaki [460] showed that a non-linear gravitational action (513) suffers from serious instabilities which lead to a dramatic change of gravitational fields associated with any gravitational bodies.

Nojiri and Odintsov [463] have argued that this problem can be alleviated by adding a counter term  $R^n$  to the modified action with appropriate coefficients. However, there is some debate as to whether this can actually work. In [465], Navarro and Acoleyen argue that for this mechanism to work, it relies on a particular value for the background scalar curvature and that if it deviates from this background value, as will happen in the Solar System, the mass of this scalar field decreases again to a value  $m \sim H$ , hence we would observe corrections to Einstein gravity.

In Ref. [476] the authors show that in all  $f(R)$  theories that behave as a power of  $R$  at large or small  $R$  the scale factor during the matter dominated stage evolves as  $a \propto t^{1/2}$  instead of  $a \propto t^{2/3}$ , except for Einstein gravity (see also Ref. [477]). This means that these cases are incompatible with cosmological observations such as WMAP. The absence of the standard matter dominated era also holds for the model given by (513). It would be of interest to find  $f(R)$  dark energy models in which a matter dominated epoch exists before the late-time acceleration.

Another interesting way to tackle the problem is provided by the so-called Palatini formalism [466, 467]. In this formalism the action is varied with respect to the metric and connection by treating them as independent field variables. In the case of the Einstein Hilbert action this method leads to the same field equations as the one derived from a standard variation principle. However when the action includes nonlinear functions of the Ricci scalar  $R$ , the two approaches give different field equations. An important point is that the Palatini formalism provides second-order field equations, which are free from the instability problem mentioned above. It was pointed out by Flanagan [468] that even in the Palatini formalism matter fields of the standard model at an energy scale of order  $10^{-3}$  eV can have interactions, thus the model (513) may be excluded by particle physics experiments. This is based on the argument that minimally coupled fermions are included in the Jordan frame and that transforming to Einstein frame induces additional interactions between matter fields. Vollick [469] argued that the equivalence between the two frames discussed by Flanagan is not physical but mathematical. The physical interpretation of the difference of the frames is a thorny subject, which we will not enter in detail. Setting these subtleties aside we shall proceed with the discussion of the observational constraints on  $f(R)$  gravity theories.

Amarzguioui *et al.* [478] tried to place constraints on  $f(R)$  gravity models with the Palatini formalism using several observational data sets (see also Ref. [479]). They parameterized the gravity Lagrangian of the form

$$f(R) = R \left[ 1 + \alpha \left( -\frac{R}{H_0^2} \right)^{\beta-1} \right], \quad (519)$$

where  $\alpha$  and  $\beta$  are dimensionless constants. Using the combined analysis of SN Ia, CMB, baryon oscillations

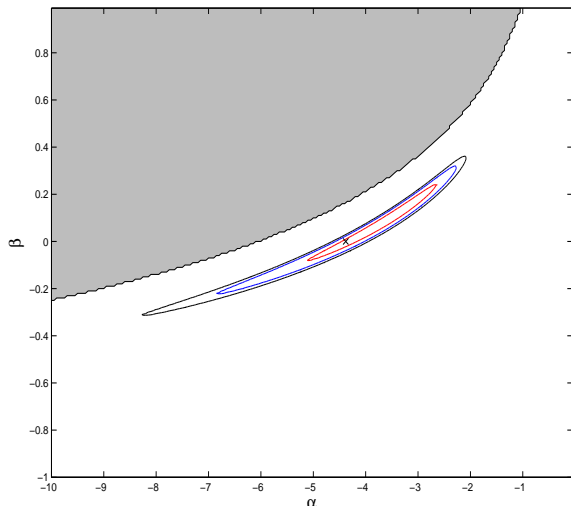


FIG. 24: The  $1\sigma$ ,  $2\sigma$  and  $3\sigma$  contours drawn using the compilation of data sets from SN Ia, CMB, baryon oscillations and LSS observations. The  $\Lambda$ CDM model ( $\alpha = -4.38$ ,  $\beta = 0$ ) is marked with the cross. The grey part represents the region which is excluded observationally. From Ref. [478].

and LSS data sets, the best fit values of the model parameters were found to be  $(\alpha, \beta) = (-3.6, 0.09)$ . As is clearly seen in Fig. 24, the  $\beta = -1$  case is ruled out observationally. The allowed values of  $\beta$  exist in the narrow range:  $|\beta| < 0.2$ . Hence  $f(R)$  gravity models do not exhibit any significant observational preference compared to the GR case ( $\beta = 0$ ).

### B. DGP model

Let us now discuss a DGP braneworld model [24, 480] which can also lead to an accelerated expansion. We consider a brane embedded in a 5-dimensional Minkowski bulk described by the action

$$S = -\frac{M_5^3}{2} \int d^5 X \sqrt{-g} R_5 - \frac{M_{\text{pl}}^2}{2} \int d^4 x \sqrt{-h} R_4 + \int d^4 x \sqrt{-h} \mathcal{L}_m + S_{\text{GH}}, \quad (520)$$

where  $g_{ab}$  is the metric in the bulk and  $h_{\mu\nu}$  is the induced metric on the brane.  $\mathcal{L}_m$  is the matter Lagrangian confined to the brane. The second term containing the 4-dimensional Ricci scalar on the brane is an extra piece appearing in the DGP model in contrast to the RS scenario. Such a term can be induced by quantum effects in the matter sector on the brane. The last term  $S_{\text{GH}}$  is a Gibbons-Hawking boundary term necessary for the consistency of the variational procedure and leads to the Israel junction conditions.

The ratio of the two scales, namely, the 4-dimensional Planck mass  $M_{\text{pl}}$  and its counterpart  $M_5$  in the 5-dimensional bulk, defines a cross over scale

$$r_c = \frac{M_{\text{pl}}^2}{2M_5^3}. \quad (521)$$

For characteristic length scales much smaller than  $r_c$ , gravity manifests itself as four dimensional theory whereas at large distances it leaks into the bulk making the higher dimensional effects important. Across the crossover scale  $r_c$ , the weak-field gravitational potential behaves as

$$\Phi \sim \begin{cases} r^{-1} & \text{for } r < r_c, \\ r^{-2} & \text{for } r > r_c. \end{cases} \quad (522)$$

We are interested in a situation in which the cross over occurs around the present epoch. In this case  $r_c$  is the same order as the present Hubble radius  $H_0^{-1}$ , which corresponds to the choice  $M_5 = 10\text{-}100$  MeV.

In the FRW brane characterized by the metric (1) we obtain the following modified Hubble equation [481]

$$H^2 + \frac{K}{a^2} = \left( \sqrt{\frac{\rho}{3M_{\text{pl}}^2} + \frac{1}{4r_c^2}} + \frac{1}{2r_c} \right)^2, \quad (523)$$

where  $\rho$  is the total cosmic fluid energy density on the brane which satisfies the standard conservation equation (11). For a flat geometry ( $K = 0$ ) we find that Eq. (523) reduces to

$$H^2 - \frac{\epsilon}{r_c} H = \frac{\rho}{3M_{\text{pl}}^2}, \quad (524)$$

where  $\epsilon = \pm 1$ . When the Hubble length  $H^{-1}$  is much smaller than the distance scale  $r_c$ , i.e.  $H^{-1} \ll r_c$ , the second term on the left hand side of Eq. (524) is negligible relative to the first term, thus giving the Friedmann equation,  $H^2 = \rho/3M_{\text{pl}}^2$ . The second term in Eq. (524) becomes important on scales comparable to the crossover scale ( $H^{-1} \gtrsim r_c$ ). Depending on the sign of  $\epsilon$  we have two different regimes of the DGP model. When  $\epsilon = +1$ , Eq. (524) shows that in a CDM dominated situation characterized by  $\rho \propto a^{-3}$  the universe approaches the de Sitter solution

$$H \rightarrow H_\infty = \frac{1}{r_c}. \quad (525)$$

Thus we can have an accelerated expansion at late times without invoking dark energy. In order to explain acceleration now we require that  $H_0$  is of order  $H_\infty$ , which means that the cross-over scale approximately corresponds to the present Hubble radius ( $r_c \sim H_0^{-1}$ ). We stress here that this phenomenon arises in DGP from the the gravity leakage at late times. In other words it is not due to the presence of a negative pressure fluid but rather to the weakening of gravity on the brane.

When  $\epsilon = -1$  and  $H^{-1} \gg r_c$  the second term in Eq. (524) dominates over the first one, which gives

$$H^2 = \frac{\rho^2}{36M_5^6}. \quad (526)$$

This is similar to the modified FRW equations in RS cosmology at high energy. However this does not give rise to an accelerated expansion unless we introduce dark energy on the brane. Hence in what follows we shall concentrate on the case of positive  $\epsilon$ .

The FRW equation (524) can be written in the form

$$H(z) = H_0 \left[ \sqrt{\Omega_{r_c}^{(0)}} + \sqrt{\Omega_{r_c}^{(0)} + \Omega_m^{(0)}(1+z)^3} \right], \quad (527)$$

where  $\Omega_m^{(0)}$  is the matter density parameter and

$$\Omega_{r_c}^{(0)} \equiv \frac{1}{4r_c^2 H_0^2}. \quad (528)$$

Setting  $z = 0$  in Eq. (527), we get the normalization condition

$$\Omega_{r_c}^{(0)} = \frac{(1 - \Omega_m^{(0)})^2}{4}. \quad (529)$$

Deffayet *et al.* [483] placed observational constraints coming from SN Ia and CMB (WMAP1) data sets. When only SN Ia data [1] is used in likelihood analysis, the best fit values were found to be  $\Omega_m^{(0)} = 0.18_{-0.06}^{+0.07}$  and  $\Omega_{r_c}^{(0)} = 0.17_{-0.02}^{+0.03}$ . If we include CMB data sets [482], it was shown in Ref. [483] that larger values of  $\Omega_m^{(0)}$  are allowed. In particular a concordance model with  $\Omega_m^{(0)} = 0.3$  is consistent with both SN Ia and CMB (WMAP1) data sets. The cross-over scale was constrained to be  $r_c \sim 1.4H_0^{-1}$ . We caution that the analysis in Ref. [483] made use of the observational data before the WMAP1 data appeared. Updated observational constraints on the DGP model have been carried out by a number of authors [184, 484].

Recently, in Ref. [485], Sawicki and Carroll looked at the evolution of cosmological perturbations on large scales in the DGP model. They found that at late times, perturbations enter a DGP regime with an increase in the effective value of Newton's constant because the background density diminishes. This in turn leads to a suppression of the integrated Sachs-Wolfe effect, which has the effect of making the DGP gravity fit the WMAP1 data better than conventional  $\Lambda$ CDM. This conclusion has been questioned in [486] where it is argued that the authors of [485] are using an inconsistent assumption for the truncation of the 5D perturbations. More precisely, their ansatz leads to the breakdown of the 4D Bianchi identity, making their results for the suppressions of the integrated Sachs-Wolfe effect as being unreliable.

In [485], the authors also found a significantly worse fit to supernova data and the distance to the last-scattering surface in the pure DGP model as compared to the  $\Lambda$ CDM model, concluding that  $\Lambda$ CDM overall provides

the best fit. A similar conclusion appears to be reached in [487] and [488], where the two groups have also tried to constrain the DGP model using SN Ia data and the baryon acoustic peak in the Sloan Digital Sky Survey. In Fig. 25 we show observational contour bounds together with the constraint relation (529) in a flat universe in the DGP model. This was obtained by using Supernova Legacy Survey (SNLS) data [91] and recent results of baryon oscillations [427], which shows that the original DGP model discussed above is ruled out at  $3\sigma$  level [487]. The analysis of Ref. [488] which is the combined analysis of SN Ia Gold data [85] and baryon oscillations [427], for a spatially flat cosmology ( $K = 0$ ), shows that the model is allowed at the  $2\sigma$  level. Figure 26 shows the analysis of SN Ia Gold data [85] and baryon oscillations [427] with  $\Omega_K$  being varied [489]. The flat DGP model is marginally on the border of the  $2\sigma$  contour bound. Clearly the results are sensitive to which SN Ia data are used in the analysis. Thus SN Ia observations alone are not yet reliable enough to reach a definite conclusion.

Both the  $\Lambda$ CDM and DGP models can describe the current acceleration of the universe provided that  $\Lambda \sim H_0^2$  and  $r_c \sim H_0^{-1}$ . The degeneracy can be broken using LSS data as the two models predict different evolution of density perturbations. The comprehensive treatment of perturbations in DGP model is still an open problem. A possible solution to this problem and its future perspectives were discussed in details in Ref. [486]. We should also mention that apart from the fine tuning of the cross-over scale, the DGP model is plagued with an instability problem related to ghosts and strong couplings. Thus the model deserves further investigations perhaps along the lines suggested in Ref. [490].

Finally we should mention that a generalization of the DGP model was proposed in Ref. [184]. The model contains additional free parameters but exhibits an intriguing possibility of transient phantom phase in the presence of a non-zero cosmological constant on the brane [184, 491].

### C. Dark energy arising from the Trans-Planckian Regime

A novel approach to addressing the issue of the origin of the dark energy is to link it to another unknown, that of the transplanckian regime, or what are the observable effects of physics occurring in the early Universe on length scales below the Planck scale, or energies well above the Planck scale? In [492] the authors model the transplanckian regime by replacing the usual linear dispersion relation  $w^2(k) = k^2$  with a one-parameter family of smooth non-linear dispersion relations which modify the dispersion relation at very short distances. In particular motivated by arguments from superstring duality (see [493] for a justification of the argument), they choose an Epstein function

$$w^2(k) = k^2 \left( \frac{\epsilon_1}{1+e^x} + \frac{\epsilon_2 e^x}{1+e^x} + \frac{\epsilon_3 e^x}{(1+e^x)^2} \right), \quad (530)$$

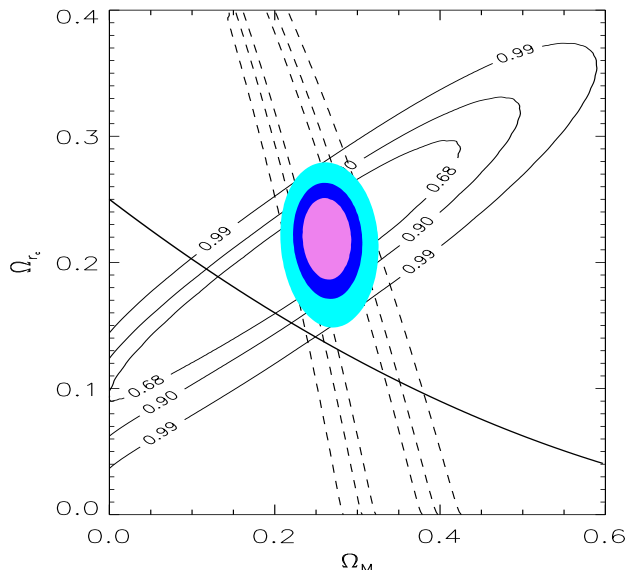


FIG. 25: The allowed parameter space in the  $\Omega_m^{(0)} - \Omega_{r_c}^{(0)}$  plane in the DGP braneworld model with  $\Omega_K = 0$  from a combined analysis of the first year SNLS data [91] and the baryon oscillation data [427]. The thick solid line shows the constraint relation (529) in a flat universe. The solid thin contours correspond to the allowed parameter regions at the  $1\sigma$ ,  $2\sigma$  and  $3\sigma$  confidence levels coming from the SNLS data. The dashed lines represent the corresponding regions from the baryon oscillation peak. The colored contours show the result of the combination of both data-sets. From Ref. [487].

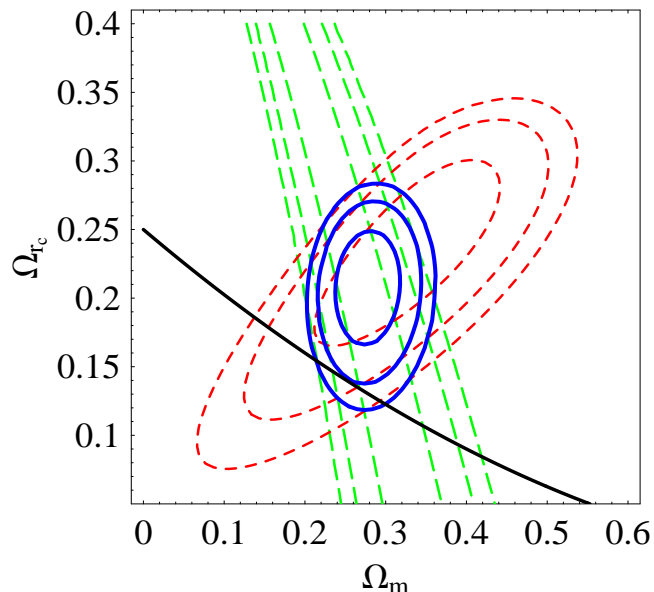


FIG. 26: The allowed parameter space in the  $\Omega_m^{(0)} - \Omega_{r_c}^{(0)}$  plane in DGP braneworld model from a combined analysis of the SN Ia Gold data set [2] and the baryon oscillation data [427]. In this figure  $\Omega_K$  is varied in the likelihood analysis. From Ref. [489].

where  $x = (k/k_c)^{1/\beta}$ .  $\beta$  is the constant determining the rate of expansion in the inflating universe given by  $a(\eta) \propto |\eta|^{-\beta}$  where the scale factor is evaluated in conformal time, and  $k_c$  is the wavenumber where the frequency reaches a maximum. The constants satisfy  $\epsilon_2 = 0$ ,  $\epsilon_1/2 + \epsilon_3/4 = 1$ , giving a one parameter (say  $\epsilon_1$ ) family of functions [492].

A particular feature of the family of dispersion functions they choose is the production of ultralow frequencies at very high momenta  $k > m_{\text{pl}}$ , and there are a range of ultralow energy modes (of very short distances) that have frequencies equal or less than the current Hubble rate  $H_0$ , known as the *tail* modes. These modes are still frozen today due to the expansion of the universe. Calculating their energy today, the authors argue that the *tail* provides a strong candidate for the *dark energy* of the universe. In fact during inflation, their energy is about 122-123 orders of magnitude smaller than the total energy, for any random value of the free parameter in the modified dispersion relations. The exact solutions of the system show that the CMBR spectrum is that of a (nearly) black body, and that the adiabatic vacuum is the only choice for the initial conditions. In a nice follow up paper, Bastero-Gil and Mersini-Houghton investigate a more general class of models and show how demanding they satisfy both SN1a and CMBR data severely constrains the viability of these models, the most important constraint coming from the CMBR [494].

#### D. Acceleration due to the backreaction of cosmological perturbations

The role of gravitational backreaction in inflating cosmologies has a long history [495]. It was pioneered by in a series of papers by Tsamis and Woodard [496, 497, 498, 499] who investigated the quantum gravitational back-reaction on an initially inflating, homogeneous and isotropic universe and showed that the role of long wavelength gravitational waves back-reacting on an inflationary background, was to slow the rate of inflation.

In [500, 501] the authors derive the effective gauge-invariant energy-momentum tensor for cosmological perturbations and use it to study the influence of perturbations on the behaviour of the Friedmann background in inflationary Universe scenarios. In particular they found that the back reaction of cosmological perturbations on the background can become important at energies below the self-reproduction scale. For the cases of scalar metric fluctuations and gravitational waves in chaotic inflation, the backreaction resulting from the effective gauge-invariant energy-momentum tensor is such that for long wavelength scalar and tensor perturbations, the effective energy density is negative and counteracts any pre-existing cosmological constant. This then leads the authors to speculate that gravitational back-reaction may lead to a dynamical cancellation mechanism for a bare cosmological constant, and yield a scaling fixed point in

the asymptotic future in which the remnant cosmological constant satisfies  $\Omega_\Lambda \sim 1$  [502].

More recently, in a series of papers, Rasanen [503], Barausse *et al.* [504] and Kolb *et al.* [505, 506], have returned to the basic idea of the backreaction being important. They have explored the possibility that the observed acceleration of the universe has nothing to do with either a new form of dark energy, or a modification of gravity. Rather it is due to the effect of the backreaction of either super or sub-horizon cosmological perturbations. By considering the effective Friedmann equations describing an inhomogeneous Universe after smoothing (for a derivation see [507, 508]), they argued that it is possible to have acceleration in our local Hubble patch even if the fluid elements themselves do not individually undergo accelerated expansion.

The time behavior of the regularized general-relativistic cosmological perturbations possesses an instability which occurs in the perturbative expansion involving sub-Hubble modes. The above authors interpret this as acceleration in our Hubble patch originating from the backreaction of cosmological perturbations on observable scales. The conclusion has raised a considerable amount of interest and criticism [509, 510] (see also Refs. [511]). Ishibashi and Wald [510] have argued that it is not plausible for acceleration to arise in general relativity from a back-reaction effect of inhomogeneities in our universe, unless there is either a cosmological constant or some form of dark energy. Basically the fact our universe is so well described by a FLRW metric perturbed by Newtonian mechanics implies the back-reaction of inhomogeneities on the dynamics of the universe is negligible. Moreover, they argue that the acceleration of the scale factor may accelerate in these models without there being any physically observable consequences of this acceleration. It has been argued that the no-go theorems due to Hirata and Seljak [509] do not hold for the case of Refs. [500, 501] where there is a large positive bare cosmological constant which dominates the dynamics [495].

In an interesting recent paper, Buchert *et al* [512], have demonstrated there exists a correspondence between the kinematical backreaction and more conventional scalar field cosmologies, with particular potentials for their ‘morphon field’. For example, they argue that it is possible reinterpret, say, quintessence scenarios by demonstrating that the physical origin of the scalar field source can be ascribed to inhomogeneities in the Universe. Through such a correspondence they explain the origin of dark energy as emerging from the morphon fields. The averaged cosmology is characterized by a weak decay (quintessence) or growth (phantom quintessence) of kinematical fluctuations, fed by ‘curvature energy’ that is stored in the averaged 3-Ricci curvature.

Although the idea of sub-horizon perturbations in a conventional cosmology driving the current acceleration may not be flavour of the month, in many ways it would be great if this idea was to work out, it would allow us

to live in a universe where gravity is conventional, there is no negative-pressure fluid out there waiting to be discovered, and no cosmological constant needed. Unfortunately the Universe looks like it has not been so obliging.

## XVII. CONCLUSIONS

The question of the nature of the dark energy that is driving the observed acceleration of the Universe today is without doubt one of the most exciting and challenging problems facing physicists and astronomers alike. It is at the heart of current astronomical observations and proposals, and is driving the way particle theorists are trying to understand the nature of the early and late universe. It has led to a remarkable explosive surge in publications over the past few years. For example over 900 papers with the words “Dark Energy” in the title have appeared on the archives since 1998, and nearly 800 with the words “Cosmological constant” have appeared.

Writing a review on the subject has been a daunting task, it is just impossible to properly do justice to all the avenues of investigation that people have ventured down. Instead we have concentrated on a subset of all the work that has gone on, trying to link it wherever possible to the other works. In particular we have decided to take seriously the prospect that the dark energy may be dynamical in origin, and so have performed quite a thorough investigation into both the nature of the cosmological constant in string theory, as well as the nature of Quintessence type scenarios. This has allowed us to compare many models which are in the literature and to point out where they are generally fine tuned and lacking motivation. Unfortunately it is a problem that faces many such scalar field inspired scenarios. On the odd occasion where a really promising candidate field seems to have emerged, we have said so.

Alongside the modification due to the presence of new sources of energy momentum in Einstein’s equations, another route we have explored is to allow for the possibility that Einstein’s equations themselves require some form of modification, in other words the geometry part of the calculation needs rethinking. Although there is no reason as of yet to believe this is the case, it is perfectly possible and so we have spent some time looking at alternatives to Einstein gravity as a source of the current acceleration today. As we have mentioned, there is more that we have not dealt with, than we have. For example we have not addressed the issues related to the holographic approach [513] and other observational aspects about dark energy, such as gravitational lensing which can serve as an important probe of dark energy [56].

We should also mention recent developments related to Bekenstein’s relativistic theory of modified Newton dynamics (MOND) [514]. Bekenstein’s theory is a multi-field theory which necessarily contains a vector and a scalar field apart from a spin two field— so called tensor-vector-scalar theory (TeVeS) [515] (see the review of



Sanders and references therein [516], as well as the recent detailed work of Skordis [517]). Since TeVeS contains a scalar field, it is natural to ask whether this theory can account for late-time acceleration and inflation. Recently efforts have been made to capture these two important aspects of cosmological dynamics in the frame work of TeVeS [518, 519, 520]. However, these investigations are at the preliminary level at present.

In the context of modified gravity models there has recently been some interesting work which can be related to MOND [465, 521]. The authors have proposed a class of actions for the spacetime metric that introduce corrections to the Einstein-Hilbert Lagrangian depending on the logarithm of some curvature scalars, as opposed to power-law corrections discussed earlier in the review. For some choices of these invariants the models are ghost free and modify Newtonian gravity below a characteristic acceleration scale given by  $a_0 = c\mu$ , where  $c$  is the speed of light and  $\mu$  is a parameter of the model that also determines the late-time Hubble constant  $H_0 \sim \mu$ . The model has a massless spin two graviton, but also a scalar excitation of the spacetime metric whose mass depends on the background curvature. Although almost massless in vacuum, the scalar becomes massive and effectively decouples close to any source leading to the recovery of an acceptable weak field limit at short distances. The classical “running” of Newton’s constant with the distance to the sources and gravity is easily enhanced at large distances by a large ratio opening up the possibility of building a model with a MOND-like Newtonian limit that could explain the rotation curves of galaxies without introducing Dark Matter using this kind of actions. Perhaps advances in our ability to perform solar and stellar system tests of the cosmological constant, will allow us to discriminate different models for  $\Lambda$ .

On the observational front, to many people’s frustration, pretty much everything seems perfectly consistent with the true cosmological constant being the source of the acceleration, but of course we are not really sure (well some of us aren’t anyway) why it has the value it does have, or why it should be coming to dominate so recently. However, there are a number of exciting observational proposals on the horizon (including solar and stellar system tests of the cosmological constant [522, 523]) which if they come up trumps may well provide us with vital information about the nature and magnitude of the cosmological constant today.

They include the Dark Energy Survey (DES) [524], a proposed optical-near infrared survey of 5000 sq. deg of the South Galactic Cap. It will allow for the measurement of the dark energy and dark matter densities and the dark energy equation of state through: galaxy clusters, weak gravitational lensing tomography, galaxy angular clustering, and supernova distances. The beauty of this is that the methods constrain different combinations of the cosmological model parameters and are subject to different systematic errors.

A second proposed mission which has generated a lot of

excitement is SNAP [525]. It seeks to place constraints on the dark energy using two distinct methods, first through obtaining more and deeper Type Ia SN, and the second through weak gravitational lensing, which relies on the coherent distortions in the shapes of background galaxies by foreground mass structures. Once again, the two methods for probing dark energy are highly complementary with error contours from the two methods that are largely orthogonal.

A third proposed mission (which is funded!) is the Planck CMB satellite which, although probably not having the sensitivity to measure any evolution in the dark energy equation of state, should be able to tell us whether or not it is a true cosmological constant with  $w = -1$ , or whether  $w$  is different from that value. Such a result if it proved the case would be as dramatic as evidence for evolution in the dark energy. What form of matter would be giving us such a result?

Recently the suggestions that Gamma Ray Bursters may actually be excellent standard candles have been revisited, with some interesting tentative initial conclusions [97]. The significance of such a result, if true, is hard to underestimate. GRB’s are some of the brightest objects in the universe and so can be seen much further than Type Ia Supernovae. In principle they could be seen out to redshifts of around  $z \sim 10$ , which would allow us to have a much more detailed Hubble diagram, and to probe more accurately whether there is evidence of evolution in the dark energy equation of state. Although the error bars are still large, the initial evidence actually suggests that for GRB’s out to a redshift of 6, the Hubble diagram is best fit with a dynamical equation of state, as opposed to a cosmological constant. It may not be statistically significant, but what the heck its a fun and tantalising way to end this review!

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