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Newtonian Time Essential to Astronomy.*

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" Let us now praise famous men."

THE HISTORICAL PROBLEM.

WHEN Newton undertook the unprecedented task of exploring the dynamics of the motions of the heavenly bodies, sparsely scattered in the vast depths of cosmical space, the preliminary and novel problem in front of him was naturally to settle their bearings. He formally established the principle that, though there may be one essential space, or æther, all spatial frames of reference are equally permitted in material dynamics provided their motion relative to this standard frame and so to one another is one of uniform translation. Thereby, for example, he was justified in referring the internal motions of the planetary system to the sun as if it were a fixed centre, although it is in motion with velocity then unknown: going even deeper, he showed how the apparent regular revolution of the directions of the stars could be proved, by experiment, to include them within the range of a frame of the dynamics constructed locally so as to be appropriate to the events on the earth as a whole.[1] But he curtly laid down the postulate that there is one universal reckoning of time. He even went on, in another context in more speculative vein, to assign to this physical absolute time a transcendental significance,[2] which has made it a battle-ground of philosophers from Leibniz onward. On these fundamental topics no degree of precision can be excessive.

The next essential progress in general ideas relevant to permitted frames of reference for dynamical science came from James Thomson, the engineer brother of Lord Kelvin, as late as 1884.[3] He recognised that the totality of such permitted frames form an interchangeable group: he invented for them the name 'frames of inertia,' which has been recently supplanting the term Galileian co-ordinates introduced in the relativity theory. He even propounded the fundamental question: given three systems of bodies in independent motion in the same space, under what conditions is it possible to discover a moving frame of reference in that space (and therefore a group of frames) relative to which their motions are all uniform translatory? The very idea of a frame of reference, as something requiring a name, seems to have originated with his direct and acute, but sometimes tedious, thought. He recognised the problem of simultaneity, but could not solve it: that had to wait for the modern group-theory. About the same time (1870) C. Neumann published discussions, that attracted far wider attention, in which motions are framed by being made relative to 'the body Alpha.'

In the higher theoretical astronomy rotating frames of reference have however been practically in use ever since Newton, but more formally since the brilliant initiative of Euler in referring the dynamics of a rotating solid body to itself.

The simplicity of the Newtonian frames of space and absolute time was disturbed practically, by the fact that our celestial knowledge comes by rays of light which suffer delay in transmission. Principles of correction, taking cognisance also of the motion of the observer, were introduced by the great practical astronomer Bradley, during Newton's lifetime, which have remained adequate until this day.

But close tests on the propagation of the light itself, initiated by Michelson (1881) after an idea broached by Maxwell, indicated that things pass as if the velocity of light were practically the same absolute constant within all local environments, unaffected by the directions and magnitudes of their convections through space. This result, then unexpected and even paradoxical, became the stimulus to modern projects of complete local relativity, based on a general postulate (1900) that each local scheme of events is, so far as relates to the atoms of matter, self-contained. Its theoretical proof, up to the second order of approximation even within the atoms if only their structure is electric, rested on a principle of correspondence which had in Lorentz's hands already (1892) given an easy explanation of the aberration of light and other first-order effects. Such theories, necessarily only approximate for application within the atoms, culminated in impressive practical form (1915) in the inclusion of gravitation by Einstein, leading to his astronomical predictions and their subsequent verification.

The confirmation (1919) of Einstein's prediction of the deviation of rays of light in the sun's gravitational field, at once placed these later developments of the dynamics of relativity in a privileged position, and made their critical study imperative. Later the eclipse result has been enhanced by a gradual consensus of expert opinion regarding the alleged gravitational influence on the atomic periods of free vibration: but the original close

* Communicated to NATURE early in February 1927.

[1] Cf. the weighty scholium at the end of the *Definitiones* in the "Principia," where Newton lays down the necessity of a formal foundation of *Axiomata sive Leges motus.*

[2] Cf. the condensed theological metaphysic of the final *Scholium Generale*; also the Queries at the end of the "Opticks."

[3] *Proc. R. S. Edin.*: "Collected Papers," pp. 379-403.

explanation, by the theory, of the outstanding small residue of the advance of planetary perihelia has become less cogent on account of uncertainties of the astronomical problem.

Could Orbits be Isolated Geodesic Curves?

These predictions all rested on a postulate that in a planetary system of orbits, as represented symbolically in an Einstein-Minkowski fourfold continuum, each gravitational orbit is separately on its own account a quasi-geodesic curve, like a line of shortest length. But it has been held (1922) to be intrinsically impossible to reason on the basis that the warping of this auxiliary fourfold continuum is determined by the representative orbits within it, while at the same time each orbit is determined by the warping of the fourfold. The influences have to be treated as mutual: the necessary intrinsic invariance of Nature, holding amid all the accidents of arbitrary modes of reference, which at first pointed to the geodesic postulate as a suitable choice, must thus belong, as the invariant Hamiltonian Minimal Action does in the actual dynamical astronomy, to the system as a whole, not to the orbits separately. When, however, the orbital problem was examined [4] in the representative fourfold on this basis of a single mathematical Action function, originating in a field-Action, binding into a unity all the mutual influences, it appeared that the constant of gravitation had to enter into the expression for the Einsteinian absolute symbolic interval of fourfold space-time in the form γ instead of 2γ, leading to smaller potency of gravitation on light so that the two predicted optical effects of gravitation must each be halved.

The only criticism of this result of which I am aware was a remark to the effect that, if this is really so, then there must be something wrong with the application of the method of Minimal Action. Yet that method has continued to be, without substantial modification, the essential and eminently convenient foundation of the general fourfold mathematical theory of relativity. The astronomical verifications in agreement with the Einsteinian values, as they became generally accepted, have thus pointed insistently for several years to the necessity of finding some way of reconciliation of this alleged fundamental discrepancy in the theory. It was indeed clear enough that the rôle of astronomical time in the formulæ, as taken over from Einstein's own adaptation for his geodesic postulate, was unsatisfactory; but nothing better presented itself, though a slight error here involves great consequences. Various attempts at reconciliation, essayed by the writer during the last four years, have shown themselves inadmissible. It is easy, for example, to solve the problem, which is simply that of fitting gravitation within the domain of the optical and electrical relativity postulate, by means of a scalar potential of gravitation, introduced alongside the fourfold vector potential that is the expression of the electric and optical field: but that course, otherwise apparently unexceptionable, does not suggest any interaction between gravitation and light. Thus the actual astronomical detection of such an influence, as stimulated originally by Einstein's formulation, and carried through in the first instance in eclipse expeditions guided by Davidson and by Eddington with the help of the Astronomer Royal in the preparation, has now come to control the whole discussion in its own right.

[4] *Philosophical Magazine*, Jan. 1923, pp. 243-256.

Dynamical Time Inherent in the Intrinsic Structure of Matter, not in Formal Frames.

The reconciliation of this discrepancy that will now be put forward is held to place the modern mathematical theory of relativity on its intelligible physical foundation. An actual astronomer is not an isolated solitary flying particle of the abstract mathematical theory. In his annual journeys at high speed round the extensive circumference of the earth's orbit, he takes his instruments and his landscape of reference along with him, without which he would be helpless: more important still, he takes his astronomical records extending over centuries of time, which have to be permanently intelligible and consistent to all brother astronomers, actual and potential, at all times and all places and moving in concert with their environments with all conceivable speeds. There must be something of the absolute involved here. The solution now offered is that our astronomer's own local space and time *are* absolute Newtonian space and time: that, unless we turn back from and reject the whole *corpus* of modern spectroscopic atomic doctrine, it is not possible to wriggle out of Newton's *dictum* of the necessity in dynamics of a universal continuous measure of time. Nor indeed do the creators of modern relativity attempt to do so, except in so far as many of them have become uncomfortable under the Einsteinian universal invariance of the time-space auxiliary fourfold interval, which as here interpreted involves and even imposes absolute time.

In the Einsteinian form of fourfold Action it is the modification of the time-like variable that dominates almost completely, as regards the actual universe with its slow motions, the forms of the symbolic tracks, because t enters into ds in the form cdt: when we have learned how to transform the symbolic track into an actual orbit in space and time with close accuracy as regards t, a much more rough adjustment will suffice spatially. Hence the importance of the utmost precision in the time-specification.

The extension of the environments of the astronomers, as enshrined in frames of the local standard, or so-called natural, space and time universal in character and the same for the environments of them all, into wider regions, is troubled by the fact that their only practical means of distant communication is by rays of light or other electric signals, which are delayed in transmission and affected by the motions of the observers. As regards astronomical observations that difficulty was discovered and surmounted

as early as 1727 by Bradley, who, especially by his fundamental principle of aberration, founded and even perfected the science of practical astronomy. But not merely an approximate practical consistency; an absolute mathematical fit is required, though possibly of a kind which may never attain to verification, if the accumulated work of astronomers is to be intelligible in ideal completeness to intellectual scrutiny, notwithstanding every variety of position and motion of possible observers and recorders and the complexities of light-transmission, of all which complications an ideal foundation for the records of astronomical history must in theory take exact cognisance.

It is held that the demand for universal intelligibility thus introduced, though of a generality which transcends any special *corpus* of physical science and so may appear at first sight to be unfruitful as being metaphysical, does point to a necessary and attainable actual construct within this domain of knowledge, that of astronomy and physics. Knowledge is rendered possible because, though the intrinsic times of all observers everywhere have to be identical because the local atoms are absolute in relation to their environments, yet the epochs or origins from which these times are measured in the various localities do not come into consideration at all, only intervals of time being concerned in the astronomical dynamics. It is this indetermination *à priori* of epochs, in conjunction with the absorption of all possible convections of local frames into simple invariance of the fourfold interval, that confers the requisite flexibility, permitting all the identical local reckonings of space and durations of time, of all observers everywhere in all ages, to be consolidated precisely within one wider auxiliary conglomerated fourfold scheme of mathematical representation.

The course of an astronomer's life, by the interpretation here advanced of the natural demand for relativity, works itself out of necessity under the sustaining local influences, is relative to his sub-permanent natural surroundings — this being indeed a type of postulate already recognised in mathematical form for similar reasons in the more minute cosmos of the atomic theory of elastic bodies. Our astronomer thus cannot know whether his environment is careering through space or not; that question must be meaningless to him, except so far as it is opened out by probable inference from the messages coming to his marvellously adapted sense of vision by light from the distant regions of the cosmos. In any case it is only his local smoothed-out frame of convenient reference, in which the occurrences that affect him are set, that can in any scheme of feasible simplicity thus be regarded as careering through space. He can and does construct his own Newtonian dynamical science of the motions observable within his local frame, often involving large relative velocities, even including local so-called centrifugal forces; and he could do this effectively though with far more trouble without calling upon light at all, just as if he and his associates were blind. Unless indeed the motion of the suitable local frame relative to adjacent permitted frames proves to be so erratic as to involve conspicuous differences of speed in its own adjacent parts, such as are called rotation: and then the remedy is to reduce the importance of the rotation, as can be done by increasing, by subdivision of frames, the fineness of the meshes of the analysis, so diminishing the extent of the local landscape which is the differential element, effectively uniform, with which a quasi-geometrical theory operates. Rotation of frames of reference, modifying velocities to an extent depending on the effective radius of the frame, appears on this view as a secondary and avoidable feature: in illustration, the rotation of the local element has added no term directly to the usual dynamical equations for a differential element of volume of a strained elastic solid.

The fact that the Lorentz transformation has to relate to convection without rotation is thus not now a harmful restriction. Moreover, on our view the influence of rotation can in any case be confined within the local frames, for it can be taken to belong to the dynamics of the bodies in the frame and not to the frame itself: a law of centrifugal force need not now belong to an unlimited frame and so extend away in increasing intensity, to be compensated at infinity. The original Newtonian argument from a spinning bucket, or in equivalent actuality the Foucault pendulum or gyro-compass, can demonstrate with the aid of light that the relative invariant property expressing fixity of gyrostatic directions within the frame, which the internal local Newtonian dynamics involves and can discover locally in self-contained manner, does actually extend outside as far as the fixed stars, which thus prove to be included within the region of spatial extension over which the play of physical theory is effectively Newtonian or uniform.

The directional fixations, unlike time, are thus local, not absolute; and the relevant question is, how far does the frame, for which they can be taken practically to subsist, extend: thus removing an insistent but, like unbounded centrifugal force, an unnecessary absolute. Or conversely, if we adopt as the expression of the natural notion of relativity, this idea that the convenient frames of reference of dynamics cannot be expected to extend uniformly to infinity or further than they can be tested, it points naturally towards trying a variable continuum, in which indirectly, as one may even say, Einstein discovered a suitable symbolic representation for an invariant gravitation. On the other hand, our immediate personal domain of local dynamics of matter can be so small that optical relativity merges practically in Newtonian relativity, the delays and other complications of light becoming negligible: but there is nothing to exclude another wider uniform frame, regarded as flitting across it, whose contents may be when so desired referred to it instead of their own frame, by aid of the Lorentz transformation which conserves the optical invariance all over it. This is the essential feature emphasised here, that the relations of pure relativity are concerned only with the specifications of localities in space and time,

and prescribe the modes of adjustment and continuation of the group of the permissible local frames of reference: that the mutual dynamics of the masses existing in each local frame proceeds independently by Newtonian principles, if, as in Nature is actually the case, they arise in a landscape practically permanent as regards the larger features each within its own most convenient frame.

For example, on this formulation of the natural notion of relativity, an astronomer will be under no temptation to undertake a violent journey through the cosmos in order "to return on the previous night" as has been said and so rejuvenate the activities of his life, any more than he will be tempted to go to the antipodes in order to stand on his head.

Only relations are directly accessible to our knowledge, and in this domain most conspicuously of all. No progress has yet been made, any more than in Newton's day, in unravelling the essential nature of gravitation. No reason can be assigned why it is just as intense as we find it, or why it exists at all. What can be established by the Einsteinian verified predictions is that, being in some unknown way an essential feature of the physical universe, it does fall into line, as one had anticipated that it ought to do, with electrodynamics and optics in obeying the principle of local relativity, the regional self-containedness of local material phenomena, as above elaborated.

Dynamics in a Two-Dimensional Cosmos.

The analogy, and contrast, drawn from the Gaussian intrinsic geometry on a surface of variable curvature, by which these beautiful recent developments in a fourfold mathematical theory of universal relations in space and time were doubtless suggested, can be pushed a considerable way in this direction. We consider now not an individual explorer, but a whole population, whose activities are confined close to the surface on which they subsist, their thin cosmos of which alone they are conscious. Each community have their local world of events occurring in their local landscape on this sheet, and may construct, with a view to express the orderly succession of these events, their own local science of dynamics and physics. This science is conveniently laid down for each of the groups within a local frame of reference, which is flat or Euclidean, and can be envisaged as a uniform reference-lattice of straight lines; also in a local scale of time regarded abstractly as flowing uniformly, while determined practically by the prominent uniformities of recurrence in their phenomena. But when their region is extended too far, a misfit in their simple scheme rapidly develops; for the surface of their activities curves round, as we outsiders know but the inhabitants do not.

A theory of a wider cosmos cannot, however, arise at all unless the local populations possess some means of communication with one another. Let us consider a forest of threads stretched tight over their surface, and imagine that messages can be transmitted along them from each group to the others: these threads are so far the analogues of our suitable paths for transmission of rays of light. Without them, all these local frames of flat geometries and their related dynamics in natural time would subsist, but in isolation; without a very refined use of them there could be no kind of relation established between standard axes defining the expression of directions, in any two of the local regions, just as there is no overt relation between the epochs of the absolute times in our local regions also internally self-contained. But such inter-connexions can be worked up for them by constructing an auxiliary universal geodesic geometry (for example, spherical trigonometry) resting for its expression on the stretched threads, a conglomerate scheme which includes and also sums up all the local geometries. Though unbounded for them, it may even be restricted essentially, unknown to the inhabitants, to a finite, even possibly cyclic, domain as a complete frame for their cosmos, if the surface expressing our external, but to them abstract or symbolic and difficultly explorable, frame for the expression of their experiences closes up like an ellipsoid or solid ring: thereby relieving essentially the indetermination of the infinite. Their local plane geometries are here implied, that is, are postulated, to be all identical in type and scale —also their local experiences to be of the same type and therefore the time in which they are naturally set—that is, they are absolute in their dynamics.

This type of scheme involves, and is secured in its widest generality by, the analytic postulate, after Gauss and Riemann, that the squared interval of length on their surface, expressible as a quadratic function of the differentials of any two functions defining any lattice of coordinate curves determining position on the surface, is transferable all over it without change of value after the manner of a terrestrial surveying chain, is in fact mathematically invariant and absolute. Yet really all that our present science of differential geometry is usually capable of dealing with is gradual transfer, explored step by step on the basis of local differential equations. Here the rays of light are more potent than the threads or the geometry, for they can translate directly a finite interval of time, expressed in absolute atomic periods, across an almost unlimited range in the cosmos.

This is something beyond geometry, which imposes itself on the formal pseudo-geometric continuum determined by the postulated fourfold invariant interval. This latter, as already remarked, is able to create the fourfold scheme, because it can condense local frames, with all speeds of convection, within one element. How far this extraneous physical feature, the presence of light, tightens up the pseudo-geometric scheme we do not now stop to inquire; except to remark that it accentuates the absoluteness of time, as *sui generis*, already suggested and indeed provided for by the different sign of the relevant term in the expression for the quadratic interval itself. Nor will we consider the more recent problem set by Weyl and Eddington and followed up by Einstein, which really is how far a much wider type of frame for events, imposing

only local relation between vector displacements of points, can be constructed so as to involve in itself, be consistent locally, and beyond by continuation, with the different and more tangible kind of foundation provided by postulation of invariant intervals such as distance—for it would seem, as has been already remarked, to be too loosely hung together to be identical with a geometric frame even combined with an arbitrary electric field, though such frames may well subsist inside it.

Other Elucidations and Contrasts.

There may be some instruction also in an analogy of our Newtonian Absolute Time T with the Kelvin Available Energy, A, in thermodynamics. At each temperature of a material system under discussion, intervals of A are measured as definite amounts of absolute energy; which is in fact the essence of the perpetual-motion postulate of Carnot, there the analogue of the present one of a local absolute time. But in passing to a different temperature of the material system the origin or epoch of measurement of A has been lost, and it becomes necessary to include A as a feature in a wider auxiliary theory, introducing a new and universal formal quantity, the entropy of Clausius, in addition to energy and of different essence. But alternatively we can usually do without any such general scheme by relying, as is familiar in special problems, on special isolated cycles of change resting directly on the Carnot-Clausius principle, our general knowledge of Nature supplying the relevant foundation—just as here we might perhaps reason directly on the relations between the local Newtonian frames instead of merging them in the one fourfold consolidation which their law of mutual correspondence permits.

We may also perhaps illustrate in another different manner. The pseudo-spatial construct of Einstein-Minkowski has given us an inkling as to how the cosmos of discrete events in history may be laid out, as it would be present to the sensorium (to adopt Newton's term) of a divinity who would have knowledge of all occurrences, but necessarily in an incoherent manner; for coherence is foreign to direct awareness of the totality of things, being the compensation permitted to imperfect knowledge, to some degree artificially and arbitrarily, by relating the succession of events in some kind of co-ordinated fourfold index, or frame, of reference. This gradual development of coherence in the range of our experience in space and time, which is scientific method, must run into some calculus of representation, in this generalised problem provided by the multiple algebra of tensors. The mathematical theory, brilliant as it is, soon indicates that this consolidated point of view, apparently losing all distinction between past and future, would transcend human intellectual ability to develop, except in its very rudiments; at any rate until mankind have learned to deal with a formal pseudo-space in four dimensions with the same intuitive familiarity that they now cope with ordinary space in three dimensions.

On the other hand, the actual science of astronomy is an affair not merely of going on "observing coincidences of point-events" in the heavens, but of accumulation of coherent records extending over centuries, permanent human documents in which all the new observations must find absolutely consistent places. There can be no science without memory and without records. For it to be humanly feasible to find out, for example, from minute discussion in the long-continued records, whether there really is a very small residue, unexplained by Newtonian theory, in the precession of the apse of the planet Mercury, must demand, if even only as a practical measure, some definite way of reckoning duration that is independent of the accidents of place and time and motion of observers; just as the instinct of Newton briefly postulated. And when in the setting out of a timeless cosmic history in a fourfold transcendental scheme, our special human knowledge, in space and time, acquired in marvellous manner from the advance of optical science, that an atom say of hydrogen must be taken to be intrinsically the same whether it be in a seething star or on the earth, so is absolute, obtrudes itself into this hitherto merely abstract historical spread, and demands, what is indeed a very small fraction of what it involves, an absolute universal (Newtonian) time for itself in its own material environment, determined by its own permanent vibrational properties, science must eagerly grasp at this revelation as in fact bringing the Promethean fire down from heaven, and rendering, as the philosophers say, human knowledge possible in this domain.

If the present course of argument is valid, the essential proof hitherto that this inference from the absoluteness of the atoms is in fact the right one, is just the confirmation in actuality of the Einstein predictions as to the gravitational influence on light, especially on the period of vibration, which, as one has been forced to hold for the last four years, would otherwise rest on a foundation largely accidental. Without that direct astronomical verification, insistence nowadays on absolute local times might well seem an anachronism.

If we could contemplate history like gods, looking equally before and after, generalised matter and energy might conceivably present themselves, as the elaborated mathematical representation in the fourfold implies, each as a tensor having as many as ten essential components: and they might be perhaps within limits identical and interchangeable. But within the frame of structure of the absolute standard thought that is valid and exchangeable amongst the human race, there have been at length recognised these wonderful absolute atoms existing as matter in their own rights without any sign of ageing or decay, being still in Maxwell's phrase "the foundation stones of the material universe"; and there has also been gradually acquired an idea of conserved energy which it has been useful to postulate as likewise universal and fundamental: and these two stand out as independent features of the foundations on which physical science builds; though related by the circumstance that the electric field belonging

to an atom is proved mathematically to add to its inertia, otherwise naturally an absolute possession, by virtue of the energy of that field.

When two atoms approach and, by overlap of their fields, release in the form of radiation some of these intrinsic field-energies then partially superposed, a proportionate part of the inertia of the atoms goes away along with it, subtracted from their translational energies when the atoms become separated again unless indeed they have suffered permanent internal change in the process. This works out consistently: energy of electric activity, even of free radiation, carries inertia with it, and thus carries momentum too. But the assertion that the whole of the mass of an ultimate atom is energy would remain a barren though permissible[5] figure of speech, unless it means to assert a postulate that mass is somehow practically all separable electric and other field-mass arising from *motional* energy of merely formal nuclei with separable æthereal fields. This would compete with the idea of an electron as a field of *static* strain-energy essentially and irrevocably locked together around a centre. A development by Minimal Action, as indicated in what follows, can provide room for both kinds of mass, the intrinsic and the exchangeable, within the postulate of absoluteness of the atoms.

Gravitation remains Unexplained.

The original case for the postulate that orbits are separately geodesic curves in the gravitational auxiliary fourfold was based, very forcefully, on the Newtonian identity of mass and weight, which otherwise remains a challenge, as an unexplained universal feature of matter. According to the marvellously precise results of Eötvös, working with his modification of the Michell-Cavendish torsion-balance, the force of gravity is found to be the same per unit inertial mass as exerted on all kinds of matter, this being actually verified beyond the order of 10^{-7} which is nearly as far as the order of the observed optical and electric relativities, namely 10^{-8}. If we reason on the basis of the Faraday type of concept, thinking however of a pre-ordained unchanging field of gravitational activity, then a collection of bodies composed of various kinds of matter move down the field in company, though with acceleration, not separating if they start with the same speed, almost as if they drifted in a current of fluid. If this field of gravitation is interpreted as merely a warp in a frame of space and time, this would naturally be regarded as a modification of the inertia which makes every free body describe, after the manner of Newton's First Law of Motion, on its own account alone, a representative path in a space-time auxiliary fourfold, determined by the purely quasi-geometrical property of minimal intrinsic quasi-length: and that is the gist of the geodesic postulate. But the other side of this consolidation of inertia with amenity to gravitation presents itself, when we reflect that a field of gravitation can vary, and so should have an assignable origin: which Newton found in an influence emanating from all the masses in the field, yet of a type of exceeding simplicity functioning just as if it were simple attraction according to his inverse-square law.

The Einstein mathematical theory can determine, at any rate as here amended, by self-consistent algebra in an abstract space-time fourfold extension, as a domain symbolically enshrining paths and masses of the bodies, a warp which proves to be necessarily restricted to a special type—just what is needed to represent gravitation. It has still to assume, however, after Newton and without explanation, that the mass which operates in this different function, that of creating the field, also is identical with the inertia mass. One may indeed reply that the geodesic postulate says nothing about mass: but it does determine a track along which the entity that determines the gravitational warp of the fourfold continuum is distributed, much as a magnetic field is determined by the tracks of electric currents,—a track which as above cannot be both direct cause and direct effect.

Though indeed the direct astronomical evidence to verify this identity of attracting mass with inertial mass may not as yet be very exact, the principle reposes firmly on a basis of its own, different from that of the Newton-Eötvös result; namely Newton's consideration that the accelerating influences between two masses rigidly connected together must be exactly balanced if motion is not to increase spontaneously without limit and so destroy any steady order of Nature. The argument from Eötvös carries only part of the way: this Newtonian doctrine of conservation of momentum is essential as well. But the latter is mutual dynamics of bodies, rather than geodesic geometry of an isolated track in a warped pseudo-space. It involves mutual relation between all the bodies in the field, bodies which also create the field: and this concurs with the previous conclusion that the path of each single body can hardly be regarded, except by an argument moving round in a circle, as a minimal inertial track in a continuum which that track itself has a share in modifying or even creating.

A study, in the light of this point of view, of the crisp exposition of principles in the first four sections of Einstein's earliest formal exposition of 1916 shows how easily it could have been turned round into the present direction. His 'natural' or absolute time, here adapted into the forefront of physical theory, is there practically put away in favour of a fourfold variable, transcending space and time,[6] and necessary to the auxiliary algebraic tensor theory, which is no more time than entropy is energy. It was natural in advance to presume that the symbolic path of a body, at any rate of an infinitesimal particle, was expressible as a minimal inertial track as if in a pre-determined continuum, until it appears as here maintained that this could not conform to actual gravitation as one mutual force between two bodies.

[5] But only up to the second order.

[6] It seems to have imposed itself originally in a very interesting exploring discussion, 'on the influence of gravitation on light,' *Ann. der Physik* (1911), § 3, English translation, p. 105, prior to the opening out of the general problem.

The discipline known as the mathematical theory of relativity would on the present view become a condensed census of the mutual necessary relations of the group of permitted frames of reference, in which investigators variously situated in space, time, and motion may conveniently and consistently formulate the physics of the local worlds, of infinite variety, to which they belong: but the cosmos is far more than its frame or even than our most far-reaching yet still superficial analysis of it within that frame.

It is a modified Newtonian physical foundation that has here been summarily set out. After all, if the present view justifies itself, it will be no derogation to the brilliant scheme of mathematical co-ordination of the general relations of gravitational and electric fields worked out by Einstein, and indeed not pressed by him (originally at any rate) to any transcendental issues. And on any view we have to admit that gravitation, like the great bulk of the rational exact principles of uniformity discovered by science in Nature, physical and biological, still in essence continues to lie beyond our scrutiny.

Relation of Absolute Time to the Auxiliary Fourfold Hypercosmos.

It is becoming widely recognised—it seems now to be Einstein's own considered opinion—that an absolute transferable interval of space-time, the analogue of a universal measuring chain, cannot be avoided, however eager be the quest for an unconditional relativity of physical knowledge. It is here maintained that the difference in sign gives the time-element a footing independent from space in that symbolic absolute interval: in a sort of analogy to this, the general fourfold invariant tensor has been recognised recently as involving fundamentally, for the abstract relativity theory, two independent tensor *data*, a symmetric one and a skew one. When absolute time is thus taken out, the remaining part of the interval, giving the foundation for spatial determinations, is to be associated with an absolute lattice-frame locally uniform or Euclidean.

The pseudo-spatial auxiliary fourfold continuum of Einstein is made up by fitting together, so to say by dovetailing, the sub-groups consisting of all the local convected frames of reference (frames of inertia) which are suitable to enshrine the local Newtonian dynamics of ordinary experience belonging to the various environments in the cosmos. These variously convected local frames of reference, adapted for each domain of experience, but all condensed into one element in the auxiliary fourfold, which is effectively differential in the sense of differential geometry, may be presumed, in the search for a *locus* for gravitation, not to fit together into one universal group symbolically extending over the whole fourfold pseudo-space, without some straining of each of them, such as would cumulate sensible effects at places further removed from the centres of their localities. An essential feature is that this warping is here put on the frame, in preference to an equivalent warping of the laws of the local Newtonian dynamics that the frame enshrines. It is now well recognised that it might be put on either: the frame might be taken to be absolutely uniform throughout the universe, at the cost of complicating the expression of the experience formulated within it.

The essence of relativity as a practical proposition is that a scheme of knowledge is relative to some suitable frame of reference: it may be a matter of facility of exposition how much of universal relations is put into the frame, and how much is treated as belonging to the local dynamics of the system of bodies whose phenomena are referred to that frame. This is the key to the present treatment: everywhere there is the convenient frame of reference: it is relative to that basic frame, not to one another, that the local events are most simply and concisely expressed: such local frames, convenient and practically indispensable, have to be grouped together, consolidated into one compound scheme in order that scientific knowledge gained in the locality, in space and time, of one of them should be transferable into the localities of the others. That becomes possible, because all local frames, however differently convected, can be consolidated within one element of the fourfold by the Lorentz transformation, so that, the convections being thus disposed of, only space-time distributions remain to be fitted together, by strain if that proves to be necessary for the convenient expression of Nature.

The choice, however, of the local frames has here developed into more than a matter of convenience. The thorough absoluteness of the symbolic Einsteinian interval, involving, as here asserted, absolute time as well as a universal measuring chain for all local spaces, now working in concert with the marvellous practical instrument we have come to possess for exploring material systems far outside our own dynamical environment, allows us not indeed to envisage directly the universal symbolic composite frame which subsumes all the actual local ones, but to carry through experimental tests for the formal validity of any attractive mathematical representation thereof that we may be able to contrive. The rays of light enable us actually to transfer directly the intervals of t, one of the fourfold auxiliary set of co-ordinates in this composite universal frame, being that one which is more specially related to time, from one part of the universe to another.

The differentials (δt) of this co-ordinate that are associated with the ray-pulsations stand locally in connexion with intervals of absolute time (δT) which are intrinsic for all environments, by a relation that may involve local gravitation among other things: it is the spectrometer that can provide the test whether this constructive process for communication with distant systems is in actuality verified,—and that in a way that Newton could hardly even have imagined. First the absolute period of vibration (ΔT) of the solar hydrogen atom is to be transferred into an equivalent interval $\kappa\Delta t$ ($=\Delta T$) in the universal auxiliary co-ordinate t of the minimised fourfold; then the very same absolute period, namely ΔT, of a terrestrial hydrogen atom may be observed,

and also can be transferred into the equivalent interval $\kappa'\Delta t'$ expressed in this co-ordinate t by the ratio κ' appropriate to terrestrial instead of solar environment; thus giving $\kappa\Delta t = \kappa'\Delta t'$. As it is the auxiliary period Δt that is transmitted by undulation along the ray and received into the spectrometer, that instrument functions *as if* it were measuring directly two different actual periods $\Delta T/\kappa$ and $\Delta T'/\kappa'$. For vibrating atoms at rest in the solar frame $\kappa = (1 - 2V_s/c^2)^{\frac{1}{2}}$ where V_s is the gravitational potential of the sun: for a terrestrial atom κ' is practically unity. This agrees with the Einstein prolongation, now only *apparent* not intrinsic, of the periods of the solar radiation. If the atom were moving with speed v in the solar frame of reference, instead of being at rest, the Lorentz transformation as *infra* would seem to impose another factor $\epsilon^{-\frac{1}{2}}$ or $1 - \frac{1}{2}v^2/c^2$, on the period, which is in the opposite direction but wholly swamped by the Doppler effect.

This feature, that every interval Δt is conveyed without change all along the representative ray-path of the auxiliary fourfold, is based on the nature of the undulation along a ray; it rests in theory on the circumstance that the electrodynamic optical equations, as transferred into the invariant form inherent in this composite fourfold (*xyzt*), are satisfied in undulatory manner, not exactly but to adequate approximation, by making the variable t enter in the expression of the solutions only through a universal periodic factor e^{ipt}. It is the absoluteness of the hydrogen or other atom, steadily maintained through all vicissitudes of place and motion even doubtless of acceleration of its environment, referred to its own frame that accompanies it, for which internally the translatory motion means nothing, that provides a universal natural measure of an absolute flux of time T, namely that of any one of its intrinsic types of vibration: it is the spectrometer alone that in quite recent times has become available to extend our local dynamical experience practically and directly into a universal scheme of intervals of time everywhere absolute, such as is involved symbolically in the universality of the Einstein space-time measuring interval.

In terms of universal Newtonian time thus acquired, the permanent results of astronomy have to be expressed: for its accumulated records must be in accord with the local measuring appliances, pendulums, gyrostats, marks of local direction, position, etc., on which observers wherever they be in space and time and however convected have to rely. This postulate of local frames of reference in space and time, enshrining their material dynamical contents, everywhere Newtonian and absolutely identical just because dead matter and its manifestations are presumed to be identical in type everywhere, is already latent in the preconception of a *continuous* differential geometry applicable to the fourfold: that the frames can be thus consolidated therein, conveniently and manageably to sufficient approximation, along mathematical methods such as the one that Einstein has opened out, is from the present point of view consistent with our fundamental requirement that as knowledge proves to be possible and enduring, the modes of its acquisition and record by all potential observers of natural phenomena, however variously situated in the cosmos, must be consistent among themselves.

Theoretical Coherence is ensured under Least Action.

It remains to indicate briefly how these ideas modify essentially the mathematical analysis and its interpretation, without unduly disturbing Einstein's famous three tests, postponing for the present other issues that incidentally arise.

We base the whole development, after Einstein, Hilbert, and Lorentz, and recently Whittaker who has attempted a striking electrodynamic generalisation, with ideal security and simplicity, on one invariant scalar integral, extended over the fourfold conglomerated representative continuum in which the symbolic history, past and future, of the cosmos is spread out, namely,

$$A = \int (G + L + \lambda) d\tau.$$

That integral is the closest available analogue, in this mixed (*xyzt*) continuum, of the Action integral of Lagrange and Green framed in ordinary space with independent time, as applied to develop the laws of undulatory disturbance whether in a continuous elastic material body or in an æthereal optical medium. Here L is the analogue of the Action-density of the elastic or electric field; λ is a constant which may be needed to satisfy the restrictions of our point of view, the analogue of a pressure which was familiar in early optical theory as necessary to adjust across an interface undulations purely transverse like those of light, occurring in a medium supposed to be incompressible; G is a multiple of the Riemann scalar curvature-invariant which confers just the suitable intrinsic freedom when the basic fourfold is now contemplated after Einstein as non-uniform, while its element of extension $d\tau$ then involves the familiar scalar factor $\sqrt{g}$.

This formulation of a compound Action-density, which is to become expressive of universal dynamical history, is first to be minimised as regards its distribution, with reference to variation of structure of the fourfold pseudo-space, as in the end controlled by its material contents expressed symbolically by their historic tracks therein. This variation leads to structural differential equations of that fourfold, thus conferring on it a definite character of the Riemannian geometric type. Within the pseudo-spatial connecting structure thus established, the Action can now be integrated by parts, leading to boundary terms along with a different spatial integral: it may be that the spatial integral vanishes, and this will occur under quadratic conditions. The boundary for such integration in the fourfold consists, in an atomic theory, of the surfaces of very thin filaments enclosing the historic tracks of the atoms and electrons which alone are out of bounds: the boundary integral is reducible practically to line-

integrations along those tracks. The Action A may thus, on account of the structure of the field as already settled by its own variation, be expressible as made up of parts associated with the material system alone as represented by the historic tracks of the atoms or finite masses. It is now to be further minimised[7] with respect to variation of these historic tracks, that is, of the representation of the history of matter: that will lead to the expression, suitable to the fourfold, of the dynamical equations of interaction of the atoms, in fact to a *symbolic* dynamics of the matter that is present in the field, now latent, which itself arises from that matter as nuclei or mathematical singularities of suitable type, in accordance with the laws determined by the previous variation in the field itself: all this mutual complexity being held firmly in check by its origin from a single Action formula.

THE GUIDANCE OF CLASSICAL ELECTRIC THEORY.

This process is directly suggested, and has even been guided, by cognate Maxwellian electric theory:[8] the special case of a system of linear electric currents in space affords a close analogue of the historic linear tracks of atoms. In that theory the energy of the currents—there a kinetic part T and a potential part W—is postulated to reside, possibly as strain and motion, in the dynamical field of the interconnecting medium or æther. An Action-density L in this medium, or kinetic potential $-L$, is formulated analytically, being equal to $T-W$. The Action $\int L d\tau' dt$, $d\tau'$ being spatial, thus expressed by local elements, is minimised over all the field, thereby adjusting the electrodynamic field to a coherent structure, presented in the form of its Maxwellian absolute circuital equations. With this field-structure, so determined, the energy, and the Action, become condensible into expressions in terms of the currents alone that are the sources of the field, in the form of line-integrals involving their circuits: the result as regards the kinetic part is represented by F. Neumann's energy formula, on Amperean lines, for linear currents in terms of the ancient current elements such as ιds, namely a mutual energy

$$T = \Sigma \int\int \frac{1}{r_{pq}} \iota_p ds_p \iota_q ds_q \cos(ds_p ds_q).$$

Incidentally and more precisely, when the current is expressed in ultimate form as made up of electrons, the complete Action, as it has now to be, in form suitable to the fourfold, is

$$A = \Sigma \int\int \frac{1}{r_{pq}} (e_p dx_p e_q dx_q + e_p dy_p e_q dy_q + e_p dz_p e_q dz_q - c^2 e_p e_q dt_p dt_q);$$

for this form is invariant (except in one feature as regards r_{pq} which leads to a different story that would now carry us too far) and involves the two sets of independent variables $(x_p y_p z_p t_p)$ and $(x_q y_q z_q t_q)$ in the fourfold, expressing position of each pair of interacting electrons but not any one universal time t. Reverting to Maxwellian theory, the forces between the bodies carrying the electric currents are then determined by further variation applied to the Neumann formula, now with respect to change of form of their circuits, in time, by variation with respect to position without further reference to the field, still in the background but already previously settled: and that in fact constitutes the Neumann energy-theory. If we wish to avoid specifications in terms of any concept of mutual forces, this process of minimising a scalar Action by variation can lead direct to the complete set of equations of motion of the conductors that carry the currents.

One essential point the analogy brings out prominently. It will not do, after Hilbert and most writers on this subject, merely to add some suitable invariant form of line-integral to the field Action A in order to express the interactions of the material atoms. The Faraday-Maxwell doctrine is that the energy, or in complete general dynamics the invariant Action, all resides in the field, there being no other; coherently, in that the field adjusts itself minimally to laws that permit this energy, other than free energy expressed as radiation, to be associated permanently with its sources, the atoms or electrons constituting the currents, and thus to be treated as belonging to them.

We have to determine what form the corresponding generalised Action, thus reduced to material form, ought to take in the fourfold problem. The Neumann formula suggests a double integral extended along each pair of the circuits or historic tracks. Under certain limitations (now postponed) it is reducible to a single integral, analogous to the Maxwellian kinetic form involving the electric potential ($FGHIV$), namely

$$\Sigma \int e_p (F_p dx_p + G_p dy_p + H_p dz_p - c^2 V_p dt),$$

involving with Maxwell dt not dt_p. There arises also naturally here, in regard to the fourfold, a local part deriving from mutual activities of the nature of self-induction between the sub-filaments of the cylinder-track of the moving electron, which adds a term $\mu_p ds_p$, where μ_p is a constant expressive of the familiar electric inertia; this part is directly suggestive of the modified Einsteinian intrinsic inertial form now to be introduced, to which an electric part like the above is to be added as the reduced expression of electrodynamics.

PREVIOUS ANALYSIS BY ACTION REVEALED DISCREPANCY WITH FACT.

In the previous effort towards this type of gravitational theory, already referred to (*Phil. Mag.*, Jan. 1923), a linear form of integral was assumed for this reduced Action, as being the nearest analogue of the Einstein geodesic form which connects itself so readily with dynamical orbits. This restricted the choice to the only available type of invariant linear form, which is

$$A = \Sigma \int -cm_p ds_p;$$

where locally in each frame ds^2 reduces by suitable change of co-ordinates to the absolute intrinsic standard form $c^2 dT^2 - d\sigma^2$ where $d\sigma^2$ is Euclidean, while all over the fourfold it is expressed con-

[7] This is not the procedure for determining *actual* orbits in space and time. This reduced Action has the requisite invariance, from its mode of formation: and it can now be transferred from the timeless fourfold into a frame in space and time as *infra*, when it becomes the ordinary dynamical Action of the planetary system as referred in space and time to that frame.

[8] Cf. "Æther and Matter" (1900), Ch. vi, especially §§ 56-9.

tinuously in terms of universal co-ordinates by an invariant quadratic differential.

We now introduce the Schwarzschild expression for the spatial-gravitational field as modified by the symbolic track of a mass m moving with changes of its speed restricted to be very slow compared with the velocity of light, namely

$$ds^2 = c^2\left(1 - \frac{km}{c^2 r}\right)dt^2 - \left(1 - \frac{km}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.$$

Or better for the problem of several interacting planets, the earlier but less exact Einstein expression

$$ds^2 = c^2\left(1 - \frac{km}{c^2 r}\right)dt^2 - \left(1 + \frac{km}{c^2 r}\right)(dx^2 + dy^2 + dz^2),$$

which is spatially isotropic and so adapted to superposition of disturbing influences. Our direct immediate concern is the question whether, in order to express gravitation within the scheme if that proves to be possible, k must be identified with its absolute constant γ, or with 2γ, the first alternative halving the gravitational effects on light as originally predicted. With sufficient accuracy for this purpose we have for the field of a system of planets the collective isotropic form

$$ds_p = c dt_p \left\{1 - \Sigma\frac{km_q}{c^2 r_{pq}} - \left(1 + \Sigma\frac{km_q}{c^2 r_{pq}}\right)\frac{v_p^2}{c^2}\right\}^{\frac{1}{2}},$$

$$v^2 = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2},$$

from which by further approximation in which the square root is expanded, keeping only the most important terms, we obtain for the Action the expression

$$A = \int \Sigma dt_p \left(\tfrac{1}{2} m_p v_p^2 + m_p \Sigma \tfrac{1}{2}\frac{km_q}{r_{pq}} + \ldots\right) + \text{constant}.$$

Now compare this with the ordinary dynamical Action from which by itself alone the entire system of equations of motion of any planetary astronomical system is derivable, namely,

$$A' = \int (T - W)dt = \int \Sigma dt(\tfrac{1}{2} m_p v_p^2 + \tfrac{1}{2} m_p V_p), \quad V_p = \Sigma\frac{\gamma m_q}{r_{qp}},$$

where V_p is the gravitational potential of the other masses at m_p. The former expression involves a plurality of independent co-ordinates t_p as free variables, one belonging to each body on its historic track: the latter has the one time-variable t. The former variables must somehow be reduced to a single one, in order to be expressive of the progress of history from stage to stage.[9] In default of anything better, we previously, following the usual tacitly adopted course, simply substituted the same dt for each of these independent differentials dt_p; as the result we had

$$A = \int \Sigma dt\left(\tfrac{1}{2} m_p v_p^2 + \tfrac{1}{2}\frac{k}{\gamma} m_p V_p\right).$$

On that pure assumption it proves to be possible to establish the necessary agreement between A and A', merely by equating k to γ: whereas the geodesic postulate that each orbit is determined by itself alone from its own equation $\delta\int ds_p = 0$, with the same identification of dt_p with dt, would obviously, with Einstein, make k equal to 2γ.

Thus by use of the single invariant mutual Action, instead of the plurality of geodesic forms which as we hold are only spuriously invariant, both optical effects of gravitation were, on adopting perforce the usual procedure, reduced to half their observed values.

The weak link in the argument was recognised at the time, and has been already here sharply indicated. As nothing better suggested itself, the usual course was followed by replacing the independent co-ordinates t_p, t_q, . . . expressive in part of the positions of the bodies on their tracks in the fourfold, by one universal time-variable t. In another aspect, instead of conducting a general variation of position in the fourfold, it was restricted to displacements confined to hypersurfaces transverse to t; which is not an invariant or intrinsic process, because t may belong to any slicing whatever of the fourfold.

Newtonian Time now introduced.

We now claim to be in a position to do better, as the Newtonian absolute time T has been acquired. The t_p is the variable of the type of time that is appropriate to the material system m_p referred to the convenient frame of our problem in which all the local systems are taken as moving, this one with velocity v_p. It can be placed in relation to T, which is the time-variable appropriate to its own frame, changing relative to the others, in which m_p is always at rest, by a Lorentz transformation expressed at each stage, if x_p is taken for brevity along v_p, by equations of type ("Æther and Matter" (1900), p. 174: Lorentz, *Proc. Amsterdam Acad.* (1904): Einstein, *Ann. der Physik* (1905))

$$\delta X = \epsilon^{\frac{1}{2}}(\delta x - v\delta t), \quad \epsilon = \left(1 - \frac{v^2}{c^2}\right)^{-1},$$

$$\delta T = \epsilon^{\frac{1}{2}}\left(\delta t - \frac{v}{c^2}\delta x\right).$$

Here, now introducing subscripts, δX_p, δT belong to the particular planet's own frame, δx_p, δt_p belong for each planet to the solar frame to which the motions of all the planets have to be referred in a theory which includes them all. It is the transfer from δt_p to the universal δT, which is none other than the (reversed) familiar very slight relativity shrinkage of time, of order 10^{-8}, that rises to be an essential feature for orbits because δt always occurs with a factor c. The transfer is effected by the relation $\delta t_p = \epsilon_p^{-\frac{1}{2}}\delta T$; provided we can ignore the addition $-v_p/c^2 . \delta x_p$ to δt_p in the formula, which expresses a change of epoch of T with locality analogous in Lorentz's striking comparison to change in civil terrestrial time for different longitudes. If our local standard frame of inertia, that is, such that dynamics of its material content is practically uniform and Newtonian, is not of too great extent, this omission makes no practical difference: the theory has however to remain approximate, though abundantly covering actuality, in this as in other respects, leaving the practical issues for systems involving speeds of a higher order than planetary still a riddle.

We can now introduce into the Action this unique absolute time T as subsisting intrinsically within every moving system m_p. But it may still be objected that t, as usually introduced in place of

[9] It is just the negation of this consideration that has led to deposing time into an accident of place and motion, and so to abolishing history.

t_p, t_q . . . could be regarded as the absolute time on the sun, to which the motions of all the planets are thereby referred. This remark may suggest a contradiction in our processes : but it is removed by reflecting that though T is measured in the same unit on the sun and on a planet, being absolute, yet it is measured from different epochs, which we cannot specify, and which do not keep constant for the succession of elements of T because the planet's own frame of T is continually changing relative to that of the sun. The only δT that is a continuous differential of a function T, for the environment of the planet in which the process of variation is carried through, is by Newtonian postulation that one which constantly belongs to it, with its optically vibrating atoms, when referred to a frame continuously carried along as part of itself. It is this T that is the unambiguous cumulating time of the records of the astronomers, who work each in his own essential local landscape in which, as in all others, it inheres.

Reverting after these explanations to our analysis, by a simple substitution for dt^2 within the radical, we now have immediately, using isotropic forms for ds^2, with sufficient approximation [10]

$$A=\Sigma\int -c^2m_p\left\{\left(1-\Sigma\frac{km_q}{c^2r_{pq}}\right)\left(1-\frac{v_p^2}{c^2}\right)dT^2-\left(1+\Sigma\frac{km_q}{c^2r_{pq}}\right)\frac{d\sigma_p^2}{c^2}\right\}^{\frac{1}{2}},$$

$$=\Sigma\int -c^2m_pdT\left(1-\Sigma\frac{km_q}{c^2r_q}-\frac{2}{c^2}v_p^2+\ldots\right)^{\frac{1}{2}},$$

as $dt_p=\left(1-\frac{v_p^2}{c^2}\right)^{\frac{1}{2}}dT$, $d\sigma_p^2=dx_p^2+dy_p^2+dz_p^2=v_p^2dT^2$,

the product terms of higher order under the radical in A now, be it noted, cancelling out. Comparison with the Newtonian Action now restores k to the value 2γ as with Einstein ; but the result may still differ (see, however, footnote *infra*) by second order terms from the Newtonian Action in a way which might affect sensibly the secular advances of the perihelia of the planets.

We claim in fact to have discovered how to transfer, with sufficient approximation, the reduced form of Action, constructed in the conglomerated fourfold in order to secure the requisite invariance, into the solar frame in space and time of the planets, which is competent to contain it and will lead to the development of dynamical astronomy by further variation in the usual manner with respect to the planetary positions.

Introduction of the factor transforming from the co-ordinate dt_p to a universal time dT has thus affected the kinetic energy of each body as it occurs in the reduced integrand of the Action with a factor 2, together with minor changes : and to restore large-scale agreement with the astronomical Action A' the potential energy has to be affected with the same factor, which requires us to make k equal to 2γ instead of γ. This new inconvenient common factor 2 would then be absorbed by amending the original Action form to $A=\Sigma\int -\frac{1}{2}cm_pds_p$, a change that will presently prove to be essential as regards the relation of mass to energy.

[10] If a track p is regarded as made up of filaments which influence one another the summation $\Sigma km_q/c^2r_{pq}$ ought to be increased by a constant k_p which is a sort of gravitational self-induction like that of an electric current on itself. This would function as a field-addition to mass, did it not cancel out as occurring in two places with opposite signs : there is no field-inertia of gravitation.

But a point arises here that demands close scrutiny, as it may be a main source of obscurity. Reduced to standard local form the transferable invariant ds is expressed by $ds^2=c^2dt^2-d\sigma^2$; as ds is thus cdt for a frame in which the $d\sigma^2$ of a track locally vanishes, it has been customary in the theory to call $c^{-1}ds$ standard time. If this is meant to be the same dT as the present interval of absolute time, it involves the relation

$$dT=c^{-1}ds=\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}dt=\epsilon^{-\frac{1}{2}}dt,$$

whereas our present result, derived from an immediate comparison of frames, is the direct opposite one, $dt=\epsilon^{-\frac{1}{2}}dT$. The justification for it is, briefly, that the Minkowskian fourfold determined by invariance of ds is not a frame of reference for matter in motion ; its element of extension is rather a complex of local frames with all speeds of convection. Neither matter nor motion is discernible within the fourfold : it is on the present view a necessary consolidation of the relations of the permitted (locally inertial) frames, in any one of which the actual world can be framed as bodies in motion. But it bears traces of this composite origin, from groups of local frames with all possible motions of convection, in the historic tracks which are indirectly symbolic of the movement of bodies as exhibited in any of the merged constituent frames.

The $c^{-1}ds$, though it has been called standard time, has nothing to do with any moving material system : it is to the local material system itself in its own frame, as the essential *datum*, not to any or all of the other extraneous permitted frames of reference in their various convections relative to it, that the absolute time belongs. Though perhaps a relation between the epochs of T in two local systems is determinable ideally by counting the alternations in t along the undulatory ray connecting them, yet as time progresses the ray rapidly changes owing to their motions and no practicable relation could ensue. Or, expressed differently, the frame of the planet is constantly changing relative to the solar frame, which prevents a steady difference of time-epochs between their frames.

When we come, however, to determine the deflection of a ray passing near the sun it is the spatial gravitational field adjacent to the sun with which we are concerned. Referring the rays to the frame of the sun, the electrodynamic theory shows that the coefficient of dt^2 in the expression for ds^2 expresses with adequate approximation the square of velocity of light near the sun, while the curvature of the ray is by Huygens' principle the local gradient of velocity along its normal.[11] Thus the Einstein deduction for the deviation of the ray holds good with his coefficient 2γ as now restored, but on a different foundation, in place of γ.

Relation of Electric Mass to Energy demands Newtonian Time.

The point has not yet been considered that the time-like co-ordinate t in the fourfold is not unique ; for it is one of the impressive features in general relativity-theory that any mode of slicing of the

[11] This needs closer elucidation, now postponed.

hypercosmic fourfold may equally well express it. But what we are concerned with in our actual approximations is the simplified case which Levi-Civita has called Einsteinian statics, in which ds^2 involves no products of dt with other differentials: the ambitions of the wider abstract theory are scarcely practicable, and may be excessive. The motions of all local systems which have speeds small compared with light are then represented by historic tracks, now in a real (x, y, z, ct) fourfold, which are almost parallel: and the co-ordinates t_p, t_q, ... of the various systems are suitably measured along some direction for t nearly parallel to them all. It is this simplification which makes an approximate development of the fourfold scheme practicable for actual astronomy, while leaving its relations to actuality unsolved for higher speeds.

It will be noted that in circumstances where the gravitational potential is negligible, the direct inertial part of the Action, as now expressed by

$$A = \ldots + \Sigma\int -\tfrac{1}{2}cm_p ds_p,$$

where $$ds_p = cdt_p\left(1-\frac{v_p^2}{c^2}\right)^{\frac{1}{2}}, \quad dt_p = \left(1-\frac{v_p^2}{c^2}\right)^{\frac{1}{2}}dT,$$

becomes $$A = \ldots + \Sigma\int -\tfrac{1}{2}c^2 m_p\left(1-\frac{v_p^2}{c^2}\right)^{\frac{1}{2}}dt_p$$

$$= \ldots + \int\{\text{const.} + \Sigma\tfrac{1}{2}m_p v_p^2\}dT,$$

when each system is referred to its own intrinsic absolute time; thus indicating that essential mass of an atom, as distinct from inertia derived from an attached field to some degree separable, is an intrinsic constant m_p unaffected by its relation to frames of reference, as naturally it ought to be.[12] It would only be electric field-mass that is affected by its velocity in the frame, being related to field-energy relative to the frame.

Moreover the latter relation, fundamental though it be, is *not* substantiated at all except on the present Newtonian scheme. For an electric system it is common ground that the relevant reduced Action, of fourfold invariant type, must be of form, again with the new adjusting factor $\frac{1}{2}$,

$$\tfrac{1}{2}\Sigma e_p\int(F_p dx_p + G_p dy_p + H_p dz_p - c^2 V_p dt_p),$$

where $$F_p = \Sigma e_q \dot{x}_q/r_{qp}, \quad V_p = \Sigma e_q/r_{qp},$$

For a static electric system it is thus, closely,

$$-\tfrac{1}{2}\int\Sigma c^2 e_p V_p dt_p, \text{ becoming } -\int W dt,$$

where W is its electrostatic energy, when all its electrons are travelling together so that t_p is the same for all. On introducing Newtonian time, but not otherwise, this is

$$-\int W\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}dT,$$

which is the familiar form of Action indicative of varying inertia according to a law which makes the energy $W(1-v^2/c^2)^{-\frac{1}{2}}$, thus showing its dependence on the translatory motion. For example, to our second order of approximation this Action is

$$\int\left(-W + \tfrac{1}{2}\frac{W}{c^2}v^2 + \ldots\right)dT,$$

indicating an increase of inertia of amount W/c^2.

[12] In a field of large uniform gravitational potential V_P an increase of inertia of m_p depending on V_P/c^2 is simulated: but that point of view is confused. The Action is the sole arbiter, which here determines the total result of the field as orbital motions.

Conclusion.

The conservation of Newtonian absolute time for local material systems, as supplied by their intrinsic vibrating atoms of matter, within the Einstein mathematical method of fitting the existence of gravitation into the optical relativity, has thus led to his value of the displacement of spectral lines, also as we have seen to his value for the ray-deflection, but not to his result for the precessions of planetary perihelia.[13]

It is noticeable how little use of the mathematical tensor theory is required for the general argument on the scope of relativity: it here provides only the form of ds^2 for the gravitational field within the solar system, as affected by each mass that is present, and that form is required only to a rough approximation, in order to identify a representation of gravitation within the formulation by ds^2 of the invariant fourfold, though closer calculation is necessary in order to determine the deflection of actual rays and the precessions of planetary perihelia.

Finally it will be observed that the present scheme has to be throughout an approximate one, leading as is claimed in a natural manner to the two optical effects of gravitation as now widely accepted. It works with frames of ordinary space and time, correlated in an auxiliary mathematical fourfold. It may be regarded as the continuation of the previous approximate scheme of electric equivalence of frames, of long ago, "Æther and Matter" (1900), Chapters vi, xi, which at that date covered adequately all the verified facts of optical and electrodynamic relativity, and justified the postulate of its universal validity at any rate up to the second order: without noticing, however, as Lorentz afterwards discovered by an equivalent independent electrodynamic formulation, that the formulæ on which it was based are obviously valid without approximation for the Maxwellian field; though valid only problematically, if at all, within the domains of the atoms, except to the second order of approximation there regarded as imposed on that account. The hypothesis of unrestricted exactness, regardless of structure in the atoms, has eventuated in wide and profound formulations towards hypercosmic schemes, which constitute the modern mathematical theory of relativity. The physical interpretations here advanced rest of course on the most brilliant Einsteinian fourfold device for involving gravitation within optical local relativity; that seems to have now become justified,—but as here urged when it is reconstructed into more consistent and Newtonian setting—by the astronomical tests, as the right type of mathematical formulation; in contrast with modes, one for example making use of a scalar gravitational potential in the fourfold, that might otherwise be in competition with it.

[13] When the approximation to the reduced Action for the Sun-Mercury system is carried to the second order, it appears that its form comes out of the type of the Newtonian Action for an elliptic orbit. If this is confirmed, then on the present theory there would be no influence on the secular progress of the perihelion: which, amid the astronomical complications, and in face of the smallness of the outstanding residue, can scarcely now be regarded as an unsatisfactory conclusion. The masses have to be predominantly intrinsic, as in the text.

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