Proof of Noether's Theorem for Forces defined by Conservative/Non-conservative Vector Fields

Consider a Lagrangian that is a function of only the field of a conservative force ϕ and its derivatives.

$$\mathcal{L} = \mathcal{L}(\phi, \partial_{\mu_i}\phi)$$

Along a path of least action, slight variations from the path are defined by

$$\delta\, \mathcal{L}(\phi, \partial_{\mu_i}\phi) = \partial\mathcal{L}\, \delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu_i}\phi)}\delta\left(\partial_{\mu_i}\phi\right) = 0$$

$$= \partial\mathcal{L}\, \delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu_i}\phi)}\partial_{\mu_i}(\delta\phi) = 0$$

where the order of partial derivatives commutes due to the Clairut-Schwarz theorem. The second derivative can be re-written as

$$\delta\, \mathcal{L}(\phi, \partial_{\mu_i}\phi) = \partial\mathcal{L}\, \delta\phi + \partial_{\mu_i}\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu_i}\phi\right)}\delta\, \phi\right] - \delta\, \phi\partial_{\mu_i}\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu_i}\phi\right)}\right] = 0$$

Collecting terms, we have the following

$$\delta\, \mathcal{L}(\phi, \partial_{\mu_i}\phi) = \delta\phi\left[\partial\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu_i}\phi\right)}\right] + \partial_{\mu_i}\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu_i}\phi\right)}\delta\, \phi\right] = 0$$

For a conservative force, $\partial\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu_i}\phi\right)} = 0$, so the second term $\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu_i}\phi\right)}\delta\, \phi = constant$. This corresponds to a conserved current.

For a non-conservative force, $\partial\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu_i}\phi\right)} = f$, so the second term corresponds to the potential defining the non-conservative force.