ON THE ORIGIN OF THE ANOMALOUS PRECESSION OF MERCURY'S PERIHELION

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Abstract. Firstly, we recover an ancient work of Gerber in 1898 as a precursor of the retarded theories, see [17]. In that paper Gerber gave an explanation of the anomalous precession of Mercury's perihelion in terms of a velocity–dependent potential. In the present paper an explanation of the anomalous precession of Mercury's perihelion is given in terms of a simple retarded potential, which, at first orders, coincides with Gerber's potential, and which agrees with the author's previous works [20, 21].

1. Introduction

The problem of the anomalous precession of Mercury's perihelion appeared in 1859 when the French astronomer Le Verrier observed that the perihelion of the planet Mercury precesses at a slightly faster rate than the one that can be accounted by Newtonian mechanics with the distribution of masses of the solar system well–known until then. This discovery began different lines of investigation to explain the new phenomenon. One of the explanations was the existence of a new planet that would explain the anomaly in Mercury's orbit within the context of Newton's laws. Other lines of investigation considered the modification or re–interpretation of Newton's law of gravitation so that it would give Mercury's precession with the known distribution of masses of the solar system. For a complete description of the historical development of the problem see [24, 25].

Einstein found that the extra precession unavoidably arises from the fundamental principles of General Relativity. The general problem of

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the integration of the Einstein equations, given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor and $T_{\mu\nu}$ is the stress–energy tensor, is extremely difficult and the determination of the explicit solutions is only possible in a restricted number of cases. One of the most important is the Schwarzschild solution for the case of a point–like or spherical and homogeneous mass and with the assumption that the limit values at infinity for the $g_{\mu\nu}$ are the galilean values.

In [24] the way to determine the relativistic prediction of the anomalous precession of Mercury's perihelion from the Schwarzschild solution is given in a very comprehensible and clear form.

2. Gerber's theory

At the end of the 19th century, theoretical physicists were investigating possibilities for the modifications of Coulomb inverse–square law. For instance, Gauss and Weber introduced velocity–dependent potential to represent the electromagnetic field, consistent with the finite propagation speed of changes in the field. Several physicists proposed different gravitational potentials based on finite propagation speed in order to account for Mercury's orbital precession (see for instance [29, 30] for a review of these proposals).

In fact, this line of investigation goes back to the works of Laplace [22] in 1805 where it was presented as a correction of Newtonian force produced by the particle m_1 in m_2, which moves with velocity v the expression

$$\mathbf{F} = -G m_1 m_2 \left(\frac{\mathbf{r}}{r^3} + \frac{\mathbf{v}}{h} \right).$$

Here h is the finite propagation speed. But this work did not find echo practically until around 1880, when a series of papers to estimate the gravitational finite propagation speed started to appear. A brief list of authors that used the hypothesis of the finite propagation speed is Th. von Oppolzer (1883), J. von Hepperger (1889), R. Lehmann–Filhes (1894), K. Schwarzschild (1900), H. Minkowski (1908), H. Poincaré (1908), W. Ritz (1909). In other works different forms for the gravitational potential were proposed; we may mention H. von Seeliger (1895) and C. Neumann (1896). Under the influence of the electrodynamical development made by F. Neumann (1845), W. Weber (1846) and B. Riemann (1861), some authors began to think in modifying Newton's law adding terms which depend on the speeds of the involved bodies, see for instance [39]. In 1870 F.G. Holzmuller [16] proposed a law of gravitation of the same form as the electrodynamic Weber's law, given by

$$F = \frac{Gm_1m_2}{r^2}\left(1 - \frac{\dot{r}^2}{h^2} + \frac{2r\ddot{r}}{h^2}\right).$$

Later, F. Tisserand [36] used this law to study the anomalous precession of Mercury's perihelion and he explained only 14.1 arc seconds per century. In the same way O. Liman (1886) and M. Lévi (1890), proposed a law of gravitation of the same form as the electrodynamic Riemann's law, given by

$$F = \frac{Gm_1m_2}{r^2}\left(1 - \frac{(\dot{r}_1 - \dot{r}_2)^2}{h^2}\right),$$

where r_1 and r_2 are the position vectors of the particles m_1 and m_2, respectively. The Riemann–Liman–Lévi law explained only 28 arc seconds per century of the anomalous precession of Mercury's perihelion. Finally, M. Lévi, by means of a purely formal development, found a force law that led to the observed exact value of the anomalous precession of Mercury's perihelion. The theories to explain the form of the proposed law forces are based, in general, on analogy between electromagnetism and gravitation known as gravitational field with a gravitoelectric component and with a gravitomagnetic component, [1, 23]. In the next section, we will see that all these laws are, in fact, based on the developments until certain order of a retarded potential. These lines of research were abandoned after Einstein's Relativity theory. Modifications to Newton's law of gravitation have recently reappeared in the context of Mordehai Milgrom theory (MOND theory) as an alternative to the dark matter and galaxies rotation curves problem, [26]. Moreover, Jacob D. Bekenstein has recently develop a relativistic MOND which resolves the problems of the classical MOND theory. A tensor–vector–scalar field (TeVeS) theory which has the classical MOND and Newtonian limits under the proper circumstances is given in [2, 3, 4, 5].

One of the first velocity–dependent potential used was

$$V(r,\dot{r}) = -\frac{m}{r}\frac{1}{\left(1 - \frac{\dot{r}}{c}\right)},$$

where a finite propagation speed is incorporated into the law of gravity substituting the retarded radial distance for the present distance. This velocity–dependent potential predicts only one third of the observed value for the anomalous precession of Mercury's perihelion, [25].

A German school teacher named Paul Gerber proposed in 1898 a velocity–dependent potential that predicts exactly the observed value for the anomalous precession of Mercury's perihelion, see [17, 18]. In [25] it is concluded with a speculative re–construction of a semi–classical line of reasoning by which it is actually possible to derive Gerber's potential, albeit in a way that evidently never occurred to Gerber. The proposed Gerber's velocity–dependent potential is

$$V(r,\dot{r}) = -\frac{m}{r}\frac{1}{\left(1-\frac{\dot{r}}{c}\right)^2}. \tag{1}$$

which depends not only on the radial distance from the gravitational mass but also on the derivative (with respect to time) of that distance. The force law associated to this velocity–dependent potential is

$$\begin{aligned} f &= \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{r}}\right) - \frac{\partial V}{\partial r} \\ &= -\frac{m}{r^2}\left(1-\frac{\dot{r}}{c}\right)^{-4}\left(\frac{6r\ddot{r}}{c^2} - \frac{2\dot{r}}{c}\left(1-\frac{\dot{r}}{c}\right) + \left(1-\frac{\dot{r}}{c}\right)^2\right). \end{aligned}$$

and the expansion of this expression in powers of $\dot{r}/c$, gives

$$f = -\frac{m}{r^2}\left(1 - \frac{3\dot{r}^2}{c^2} + \frac{6r\ddot{r}}{c^2} - \frac{8\dot{r}^3}{c^3} + \frac{24r\dot{r}\ddot{r}}{c^3} - \dots\right). \tag{2}$$

In [25], it is showed that the Gerber's velocity–dependent potential (1) results in elliptical orbits that precess by the same amounts as predicted by General Relativity (to the lowest order of approximation), and, of course, this fact agrees with the observed precession rates for the perihelia of the planets, including Mercury. The question, then, is whether we can justify the use of this particular velocity–dependent potential rather than the Newtonian potential $V = -m/r(t)$. Moreover, in [25] it is also shown that although General Relativity and Gerber's potential predict the same first–order precession, the respective equations of motion are not identical, even at the first non–Newtonian level of approximation. One of the objectives in Gerber's works, taking into account the assumption of a finite propagation speed, was to infer the speed of gravity from observations of the solar system. The open question is if gravity and light move at the same speed, which is still today on discussing, see [40] and references therein. In the introduction of the Gerber's paper [20], Ernst Gehrcke concludes:

> Whether and how the theory of Gerber can be merged with the well–known electromagnetic equations into a

new unified theory is a difficult problem, which still awaits a solution.

3. A simple retarded potential

Action at distance in Newtonian physics is replaced by finite propagation speeds in classical post–Newtonian physics. As a result, the differential equations of motion in Newtonian physics are replaced by functional differential equations, where the delay associated with the finite propagation speed is taken into account. Newtonian equations of motion, with post–Newtonian corrections, are often used to approximate the functional differential equations, see, for instance, [6, 7, 8, 9, 19, 33, 34]. In [20] a simple atomic model based on a functional differential equation which reproduces the quantized Bohr atomic model was presented. The unique assumption was that the electrodynamic interaction has finite propagation speed, which is a consequence of the Relativity theory. A straightforward consequence of the theory developed in [20], and taking into account that gravitational interaction has also a finite propagation speed, is that the same model is applicable to the gravitational 2–body problem. In [21] a simple gravitational model based on a functional differential equation which gives a gravitational quantification and an explanation of the modified Titius–Bode law is described. In the following an explanation of the anomalous precession of Mercury's perihelion is given in terms of a simple retarded potential, which, at first orders, coincides with the Gerber's potential.

The most straightforward way of incorporating a finite propagation speed into the law of gravity is to simply substitute the current distance for the retarded radial distance. Therefore, we consider the simplest retarded potential

$$V = -\frac{m}{r(t-\tau)}, \tag{3}$$

where $r(t)$ denotes the instantaneous position vector of the test particle, at time t, and τ is the delay, so that $r(t-\tau)$ is the retarded position of the test particle. In fact this retarded potential depends on the position vector but also on the velocity vector $\dot{r}$, on the acceleration vector $\ddot{r}$ an so on. The appearance of a delay implies all these dependences in the potential. From the retarded potential (3) we will obtain, in a theoretical point of view, the equation of motion of the particle. This equation will be a functional differential equation. The functional differential equations of motion are in general difficult and, often impossible, to be expressed in a form amenable to analysis. Thus, in order to obtain useful dynamical predictions from realistic models, we frequently replace the functional differential equations of motion by approximations that are ordinary or partial differential equations, [6]. In our case, if we develop the retarded potential (3) in powers of τ (up to second order in τ), we obtain

$$V \approx -\frac{m}{r}\left[1+\frac{\dot{r}}{r}\,\tau+\left(\frac{\dot{r}^2}{r^2}-\frac{\ddot{r}}{2\,r}\right)\,\tau^2\right], \tag{4}$$

To develop some easier calculations we can reject on the right hand side of expression (4) the term with $\ddot{r}$ (in fact this term is negligible and only gives terms of higher order). Hence, at this approximation, we obtain the velocity–dependent potential

$$V \approx -\frac{m}{r}\left[1+\frac{\dot{r}}{r}\,\tau+\frac{\dot{r}^2}{r^2}\,\tau^2\right], \tag{5}$$

In a first approximation, the delay τ must be equal to r/c (the time that the field uses to go from Mercury to the Sun at the speed of the light) and according with the theories developed in [20, 21], we introduce a new constant g in the delay and hence, $\tau = g\,r/c$. Introducing this expression of the delay in (6) we have

$$V \approx -\frac{m}{r}\left[1+g\frac{\dot{r}}{c}+g^2\frac{\dot{r}^2}{c^2}\right]. \tag{6}$$

On this basis, that is, with this velocity–dependent potential function (6), the gravitational force law is given by substituting the potential function (6) into equation

$$f \;=\; \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{r}}\right)-\frac{\partial V}{\partial r}=-\frac{m}{r^2}\left(1-\frac{g^2\dot{r}^2}{c^2}+\frac{2g^2 r\ddot{r}}{c^2}\right).$$

It is easy to see that if we fix $g=\sqrt{3}$, we obtain the same radial force, at first orders, that gives Gerber's potential, see the expression of the force (2). In fact, we have constructed a potential that, varying g, predicts $2g^2\pi m/(Lc^2)$ as non–Newtonian advance of orbital perihelia per revolution, where m is the Sun's mass, L is the semi–latus rectum of the orbit, and c is the speed of the light. Note that for $g=1$, it results in a value which is one third of the observed value, so it predicts only 14.1 arc seconds per century for the precession of Mercury's perihelion. The problem of the retarded potential (3) is that it can account for the anomalous precession of the Mercury's perihelion precisely by adjusting a free parameter of the theory. In the following we give a retarded potential which gives an explanation of the anomalous precession of the Mercury's perihelion without adjusting any free parameter of the theory. We will see that this new retarded potential also coincides, at first orders, with Gerber's one.

We now consider a small modification of the retarded potential (3), given by

$$V = -\frac{m}{r(t-\tau)}\frac{r(t)}{r(t-\tau)}, \tag{7}$$

where the modification consists on dividing the retarded potential (3) by the quotient $r(t-\tau)/r(t)$. This quotient represents the ratio of the distance between the masses when the potential was "emitted" with respect to the distance between the masses at the present instant. We can think that the retarded potential (3) was obtained from the Newtonian potential $V = -m/r(t)$ of the form

$$V = -\frac{m}{r(t)\frac{r(t-\tau)}{r(t)}} = -\frac{m}{r(t-\tau)},$$

and the quotient $r(t-\tau)/r(t)$ is the corrective factor to obtain the retarded distance. This corrective factor is applied because the potential must propagate from the source to the location of the particle in question. In the same way we can think that the retarded potential (7) is obtained from the Newtonian potential $V = -m/r(t)$ by:

$$V = -\frac{m}{r(t)\frac{r(t-\tau)}{r(t)}\frac{r(t-\tau)}{r(t)}} = -\frac{m}{r(t-\tau)}\frac{r(t)}{r(t-\tau)},$$

In the same way that in the Neumann's theories [28] we conceive the potential essentially as information being transmitted from place to place, and we assume a finite speed for the propagation of this information.

As it is described in [25], a particle create a potential, whose value depends not only on the emitting particle, but also on the receiving particle. Therefore, the information must come back from the receiving particle to the emitting particle. Thus, we ought to regard an elementary interaction not as a one–way exchange, but as a two–way round–trip. In fact, in a similar way that the transactional interpretation of quantum mechanics where the basic element of the transactional is an emitter–absorber transaction through the exchange of advanced and retarded waves, as first described by Wheeler and Feynman, [37, 38]. Hence, we must apply the corrective factor twice in the initial potential.

In fact the correct expression of the retarded potential, taking into account that the information must do a two–way round–trip and that

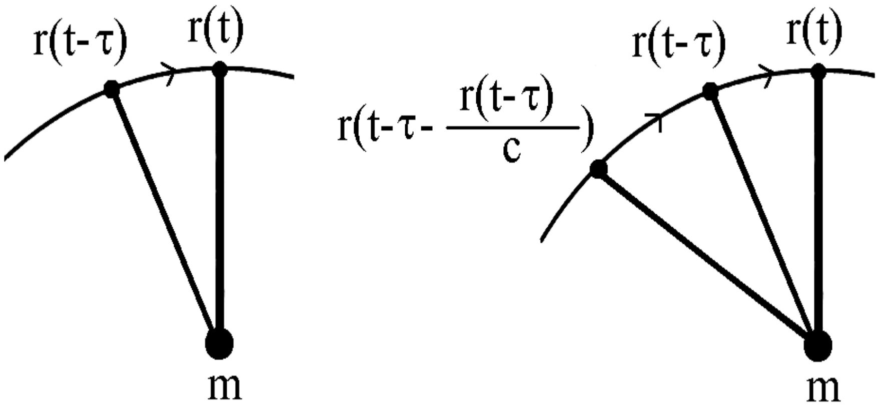

FIGURE 1. The retarded position of the test particle.

$\tau = r(t)/c$, is

$$V = -\frac{m}{r(t-\tau-\frac{r(t-\tau)}{c})}. \tag{8}$$

where $r(t-\tau-r(t-\tau)/c)$ is the distance between the masses when the potential was "emitted" to go from the emitting particle to the receiving particle and come back, see the second graphic of Fig. 1. To find the retarded potential (7) as approximation of the retarded potential (8) we take into account that for a small τ we have that

$$r(t)\, r(t-\tau-\frac{r(t-\tau)}{c}) \approx (r(t-\tau))^2 \,.$$

Therefore, for a small τ we obtain

$$V = -\frac{m}{r(t-\tau-\frac{r(t-\tau)}{c})} \approx -\frac{m}{r(t-\tau)}\frac{r(t)}{r(t-\tau)}.$$

Hence, the correct retarded potential is (8), but this functional potential is difficult to be expressed in a form amenable to analysis. Therefore, we use the approximation (7) whose physical interpretation and use is totally justified. In fact, the retarded potential (7) is a generalization of the Gerber's potential. The Gerber's potential is the particular case when the velocity of the test particle is constant, i.e., when $\ddot{r} = 0$. In [25] a physical explanation (albeit in a way that evidently never occurred to Gerber) of the form of the Gerber's potential is given.

Now we are going to see that the retarded potential (7) gives an explanation of the anomalous precession of Mercury's perihelion because coincides, at first orders, with the force law associated to Gerber's one.

If we develop the retarded potential (7) in powers of τ (up to second order in τ), we obtain

$$V \approx -\frac{m}{r}\left[1+\frac{2\dot{r}}{r}\,\tau+\left(\frac{3\dot{r}^2}{r^2}-\frac{\ddot{r}}{r}\right)\,\tau^2\right], \tag{9}$$

To develop some easier calculations we can reject, as before, on the right hand side of expression (9) the term with $\ddot{r}$ (in fact this term is negligible and only gives terms of higher order). Hence, at this approximation, we obtain the velocity–dependent potential

$$V \approx -\frac{m}{r}\left[1+\frac{2\dot{r}}{r}\,\tau+\frac{3\dot{r}^2}{r^2}\,\tau^2\right], \tag{10}$$

In a first approximation, the delay τ must be equal to r/c (the time that the field uses to goes from Mercury to the Sun at the speed of the light) according with the theories developed in [20, 21]. Introducing this expression of the delay in (10) we have:

$$V \approx -\frac{m}{r}\left[1+\frac{2\dot{r}}{c}+\frac{3\dot{r}^2}{c^2}\right]. \tag{11}$$

With this velocity–dependent potential function (11), the gravitational force law is given by substituting the potential function (11) into the equation:

$$f \;=\; \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{r}}\right)-\frac{\partial V}{\partial r}=-\frac{m}{r^2}\left(1-\frac{3\dot{r}^2}{c^2}+\frac{6r\ddot{r}}{c^2}\right).$$

Hence, we obtain (without fixing any parameters) the same radial force, at first orders, that gives Gerber's potential, see (2).

In fact, it is straightforward to see that, at first orders, the retarded potential (7) and Gerber's potential coincide. If we develop the retarded potential (7) we have

$$\begin{aligned} V = -\frac{m}{r(t-\tau)}\frac{r(t)}{r(t-\tau)} &= -\frac{m}{r(t)-\dot{r}(t)\tau+\ldots}\cdot\frac{r(t)}{r(t)-\dot{r}(t)\tau+\ldots} \\ &= \frac{m}{r(t)(1-\frac{\dot{r}(t)}{r(t)}\tau+\ldots)}\cdot\frac{1}{1-\frac{\dot{r}(t)}{r(t)}\tau+\ldots}. \end{aligned}$$

Now substituting the delay $\tau = r/c$ we obtain

$$V=-\frac{m}{r(t)(1-\frac{\dot{r}(t)}{c}+\ldots)}\cdot\frac{1}{1-\frac{\dot{r}(t)}{c}+\ldots}.$$

Therefore, at first orders, the retarded potential (7) has the form

$$V=-\frac{m}{r(t)\left((1-\frac{\dot{r}(t)}{c})^2+\ldots\right)}.$$

4. Concluding remarks

The anomalous precession of Mercury's perihelion can be explained by taking into account the second order in the delay of the retarded potential (7) which is an approximation of the retarded potential (8). It is still necessary to see if the prediction for the deflection of electromagnetic waves grazing the Sun using this potential coincide with the value given by General Relativity, assuming a plausible application of such potential to the propagation of electromagnetic waves. We hope to give an answer to this question in a future work. On the other hand, the introduction of dissipation and limit cycles through these retarded potentials is a first step in the right direction [20, 21]. In a more complex system, deterministic chaos would appear. It is well–known that the geometry of deterministic chaos is, in general, a fractal. In this way, towards the end of eighties, fractal space–time physics, the theory of scale relativity and E–infinity theory were introduced by G. Ord [31], K. Svozil [35], L. Nottale [28] and M.S. El Naschie [12], all of them independently, although based on the original ideas of A. Einstein and D. Bohm, as well as a fractal space–filling curve proposal by R. Feynman, [13]. The aim of these works is to establish the deterministic chaos origin of quantum mechanics and the gravitational quantization at large scales, [14, 15].

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