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# SOME ELECTROMAGNETIC PARADOXES

BY

PARRY MOON [1] AND DOMINA EBERLE SPENCER [2]

## 1. INTRODUCTION

Electromagnetic theory is undoubtedly one of the most imposing theoretical edifices reared by the mind of man. It is a work of genius—of Maxwell, Hertz, Lorenz, Lorentz, Poincaré, Einstein. Yet it does not form a logical whole, as evidenced by numerous paradoxes.

Classical theory is built up of pieces, each admirable in its way but not fitting too comfortably into the whole. Sometimes we pretend that flux lines have physical existence: we visualize them as rubber bands pulling on conductors or as filaments that are cut by moving wires. At other times we are unable to decide whether these flux lines are moving or stationary, and we fall back on Maxwell's equations which express the phenomena in terms of the rather abstract vectors **E**, **D**, **B**, **H**. At still other times, we find it desirable to use retarded potentials. Einstein's relativity may also be introduced.

Unfortunately, these various methods of handling a problem do not always lead to the same conclusion. Hence the paradoxes! If one knows the correct result, he can usually find a part of classical theory that will give him the answer. And he can always justify his choice by some plausible argument. But an equally logical argument can be found for other approaches that may give erroneous results. As Ko-Ko says, "Here's a state of things, here's a how-de-do!"

In the following pages, we consider a number of paradoxes. In each case, the problem is stated and a solution is given. This solution is in accordance with classical theory but does not agree with experiment. It is the kind of solution that might be obtained by a scrupulous but inexperienced student. The reader may amuse himself by finding a classical approach that gives the correct answer.

Each problem is also solved by employing the interparticle method developed by Coulomb, Ampère, Gauss, and Ritz. This method, unlike the Maxwell formulation, can be summarized (1)[3] in *a single equation and this equation gives unambiguous results.* In the new electrodynamics, no paradoxes have been found.

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[3] The boldface numbers in parentheses refer to the references appended to this paper.

## 2. THE NEW ELECTRODYNAMICS

The non-Maxwellian approach subsumes the whole of electrodynamics under the single equation,

$$\mathbf{F}_2/Q_2 = \mathbf{a}_r \frac{Q_1}{4\pi\epsilon c^2 r^2}\left[\mathbf{v}\cdot\mathbf{v} - \tfrac{3}{2}(\mathbf{a}_r\cdot\mathbf{v})^2\right] - \frac{Q_1}{4\pi\epsilon c^2 r}\frac{d}{dt}\mathbf{v}(t - r/c) - \mathbf{a}_r \frac{1}{4\pi\epsilon}\frac{\partial}{\partial r}\left[\frac{1}{r}Q_1(t - r/c)\right]. \quad (1)$$

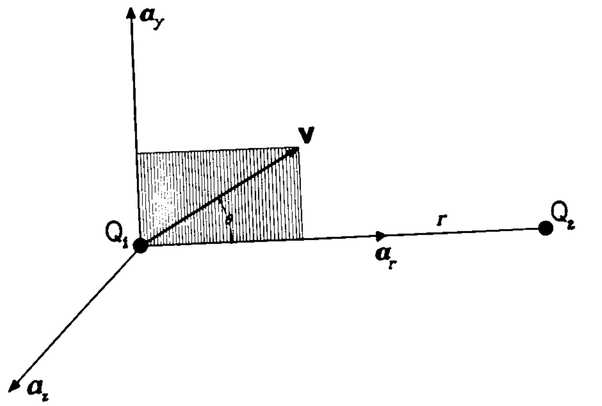


FIG. 1. Two charged particles.

The notation is indicated in Fig. 1. The unit vector $\mathbf{a}_r$ points from $Q_1$ to $Q_2$. Angles are measured counter-clockwise from $\mathbf{a}_r$. Vector $\mathbf{v}$ is the *relative velocity of* $Q_1$ *with respect to* $Q_2$. Equation 1 applies at all ordinary velocities: in fact, it is reasonably accurate up to approximately half the velocity of light. At higher velocities, corrections must be applied.

Note that the magnetic-field concept is completely eliminated and **B**, **H**, and $\mu$ do not occur. The usual "magnetic" forces and "flux-cutting emf.'s" are handled by the first term of Eq. 1 which depends on the *relative velocity* of charges. The second term depends on the *acceleration* of charges, while the third term is related to Maxwell's displacement current and includes the Coulomb force as a special case (1). For moderate frequencies and distances, retardation may be neglected and Eq. 1 reduces to

$$\mathbf{F}/Q_2 = \mathbf{a}_r \frac{Q_1}{4\pi\epsilon r^2}\left(\frac{v}{c}\right)^2\left[1 - \tfrac{3}{2}\cos^2\theta\right] - \mathbf{a}_a \frac{Q_1}{4\pi\epsilon c^2 r}\frac{dv}{dt} + \mathbf{a}_r \frac{Q_1}{4\pi\epsilon r^2}. \quad (1a)$$

Equations 1 and 1*a* apply to the force between charges $Q_1$ and $Q_2$, Fig. 1. These equations are easily extended (1) to give the force between current elements or between a current element and a charge. For a current element $I_1 d\mathbf{s}_1$ and a charge $Q_2$ (Fig. 2), the first term of Eq. 1 gives

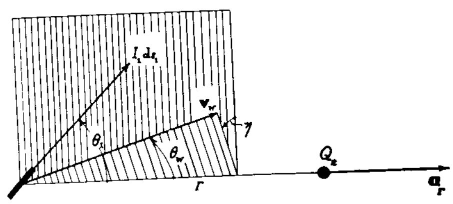


FIG. 2. Current element and charged particle.

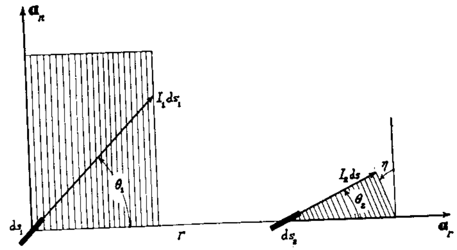


FIG. 3. Two current elements.

$$d\mathbf{F}/Q_2 = \mathbf{a}_r \frac{|I_1|}{4\pi\epsilon c^2 r^2}[2(\mathbf{a}_r \times \mathbf{v}_w)\cdot(\mathbf{a}_r \times d\mathbf{s}_1) - (\mathbf{a}_r\cdot\mathbf{v}_w)(\mathbf{a}_r\cdot d\mathbf{s}_1)]$$

$$= \mathbf{a}_r \frac{|I_1|v_w ds_1}{4\pi\epsilon c^2 r^2}[2\sin\theta_w \sin\theta_1 \cos\eta - \cos\theta_w \cos\theta_1]. \quad (2)$$

Here the sense of $d\mathbf{s}_1$ is the direction of the current, and $\mathbf{v}_w$ is the velocity of the material wire with respect to $Q_2$. Equation 1 also yields a term directly proportional to the drift velocity $\mathbf{v}_1$ of the electrons (2). Apparently, this additional force is always too small to be measured, so it will be ignored in this paper.

The first term of Eq. 1 can be applied also to two current elements (Fig. 3). This Ampère force is

$$d^2\mathbf{F} = -\mathbf{a}_r \frac{|I_1||I_2|}{4\pi\epsilon c^2 r^2}[2(\mathbf{a}_r \times d\mathbf{s}_1)\cdot(\mathbf{a}_r \times d\mathbf{s}_2) - (\mathbf{a}_r\cdot d\mathbf{s}_1)(\mathbf{a}_r\cdot d\mathbf{s}_2)]. \quad (3)$$

This force is independent of the relative velocity of the material elements. Equation 3 is a rigorous deduction from the first part of Eq. 1, with no terms neglected.

The remainder of the paper is devoted to a consideration of a few of the paradoxes that occur in conventional electromagnetic theory.

### 3. ELECTRON BEAMS

*Statement*

Parallel electron guns produce two identical electron beams in a long, evacuated tube. Initial velocity of the electrons with respect to the laboratory is $\mathbf{v}_0$. What is the magnitude and direction of the force on the electrons?

*A Classical Solution*

The argument may be stated rather neatly in the form of a classical syllogism:

(*a*) A stream of electrons is equivalent to a current (Rowland's experiment).
(*b*) Parallel currents in the same direction attract each other, as proved experimentally by the behavior of parallel, current-carrying wires.
(*c*) Therefore, the electron beams will attract each other and will tend to converge.

The current in one beam is

$$I = - NA|Q_e|v,$$

where

$N$ = number of free electrons per unit volume,
$A$ = cross-sectional area of the beam,
$|Q_e|$ = magnitude of electronic charge.

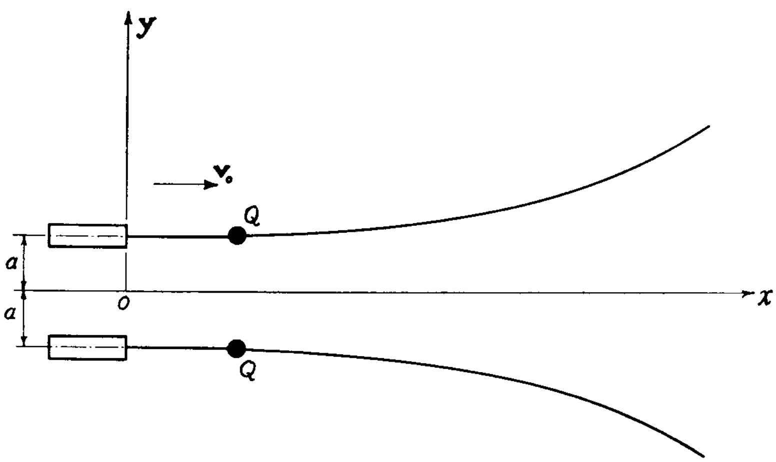


FIG. 4. Electron beams. Electrons are emitted at $x = 0$ with velocity $\mathbf{v}_0$ in the $x$-direction.

This current produces a magnetic flux density at the other beam,

$$B = \frac{\mu_0 I}{2\pi a},$$

where $a$ is the distance between the axes of the beams. The resulting force per unit length is

$$F/l = BI = \frac{\mu_0 I^2}{2\pi a} = \frac{N^2 A^2 |Q_e|^2}{2\pi \epsilon c^2 a} v^2. \qquad (4)$$

The force is perpendicular to $\mathbf{v}$ and tends to pull the beams together; a result that is certainly not in accord with experiment.

For other solutions, the reader is referred to *Electrical Engineering* (3), where a long and inconclusive discussion reminds one of Stephen Leacock's rider who "leaped on his horse and dashed madly off in all directions."

*Electrodynamic Solution*

Consider two charged particles moving in parallel paths in the $x$-direction (Fig. 4). Each particle has a charge $Q$ and is at the same distance $x$ from the origin. The velocity is $\mathbf{v}_0$ with respect to the laboratory, but the *relative* velocity of one charge with respect to the other is zero. Thus Eq. 1 reduces to the Coulomb equation, and the force on the upper particle is

$$\mathbf{F} = \mathbf{a}_y \frac{Q^2}{4\pi\epsilon r^2}. \tag{5}$$

For a coordinate system moving with the charges, *the force is an ordinary electrostatic repulsion which is exactly the same as if the charges were stationary.* The electron beams of Fig. 4 will therefore tend to diverge, as found in practice.

The force, Eq. 5, will accelerate the particles in the $y$-direction:

$$F = m\frac{d^2y}{dt^2}$$

or

$$y = a + \frac{F}{m}t^2.$$

But $x = v_0 t$; so the path of the upper particle referred to a stationary coordinate system, is a parabola:

$$y = a + (F/mv_0^2)x^2. \tag{6}$$

Thus the problem is a very simple one, requiring only a knowledge of elementary mechanics and of Coulomb's equation. Classical treatment, on the other hand, introduces puzzling questions such as the presence or absence of a magnetic field, depending on the motion of the observer. Such questions do not arise in the new electrodynamics, nor is Einstein's relativity needed.

Equations 1 and 6 do not apply for $v \to c$ without additional correction terms. Strictly speaking, corrections should be made also for the velocity and acceleration in the $y$-direction. This modification is easily accomplished by use of the first and second terms of Eq. 1; but since both terms include $c^2$ in the denominator, their effect is ordinarily negligible. For the electron-beam problem, the Coulomb repulsions of all electrons in the beams should be considered. But this would not change the foregoing general conclusion that *the force is merely a Coulomb repulsion and is never the attraction obtained with parallel currents.*

### 4. LONGITUDINAL MOTION

*Statement*

A flexible conductor $AA'$, Fig. 5, carries a direct current $I_1$. Near $A$ is a long, straight conductor $C$ carrying a steady current $I_2$. The

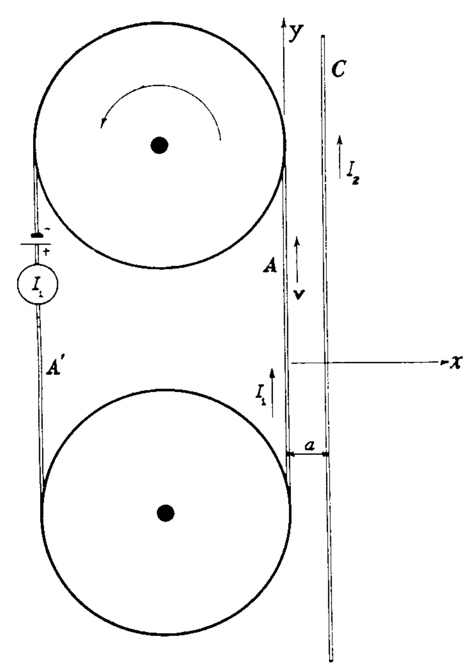


FIG. 5. Determine the force between conductors $A$ and $C$ when $A$ is in uniform longitudinal motion.

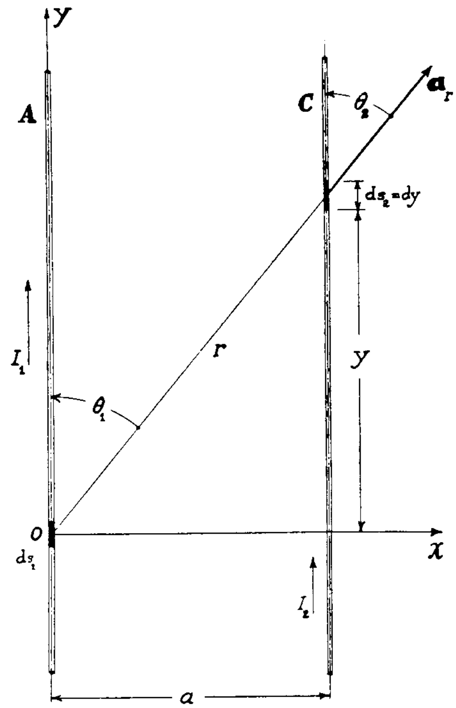


FIG. 6. Analysis of Fig. 5.

distance $a$ is small compared with the other dimensions, so the effects of the return conductors may be neglected. What is the force per unit length, exerted on conductor $C$?

*A Classical Solution*

First consider $AA'$ stationary. The magnetic flux density at $C$, caused by current $I_1$, is

$$B = \frac{\mu_0 I_1}{2\pi a}.$$

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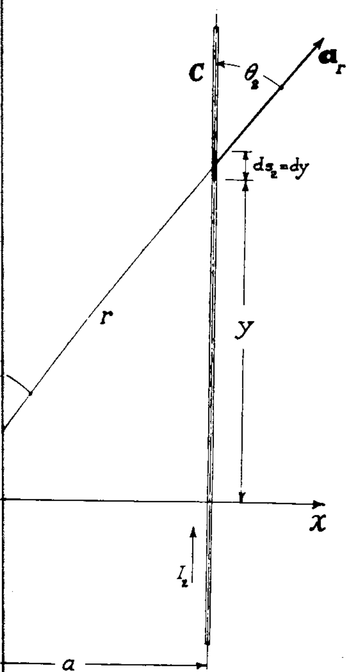


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The force on $C$, according to Faraday's relation, is

$$F_2 = BlI_2$$

or

$$F_2/l = \frac{\mu_0 I_1 I_2}{2\pi a} = \frac{I_1 I_2}{2\pi\epsilon c^2 a}. \tag{7}$$

The current in $A$ is

$$I_1 = N_1 A_1 |Q_e| v_1,$$

where the symbols are as in Section 3 and $v_1$ is the magnitude of the drift velocity of the electrons in $A$.

Now allow $A$ to move longitudinally at velocity $\mathbf{v}$. According to Rowland's experiment, moving charges constitute a current, so the *effective* current is not $I_1$ but is

$$\begin{aligned} I_1' &= N_1 A_1 |Q_e| (v_1 - v) \\ &= I_1 \left( \frac{v_1 - v}{v_1} \right). \end{aligned}$$

Replacing $I_1$ in Eq. 7 by $I_1'$, we obtain

$$F_2/l = \frac{I_1 I_2}{2\pi\epsilon c^2 a} \left( \frac{v_1 - v}{v_1} \right). \tag{8}$$

Since $v$ may easily exceed $10^4 v_1$, we can obtain with the moving conductor a force 10,000 times as great as that obtained with the stationary conductor. What is wrong with this prediction?

*Electrodynamic Solution*

It is easily proved that Eq. 3 is independent of conductor velocity. Thus the force is independent of $\mathbf{v}$ and is obtained by integration of Eq. 3. From Fig. 6, $\theta_1 = \theta_2 = \theta$, $\eta = 0$, and the force on element $ds_2$ is

$$\begin{aligned} d\mathbf{F}_2 &= -\mathbf{a}_x \frac{|I_1||I_2| ds_2}{4\pi\epsilon c^2} \int_{-\infty}^{\infty} \frac{[2\sin^2\theta - \cos^2\theta]\sin\theta}{r^2} dy \\ &= \mathbf{a}_x \frac{|I_1||I_2| ds_2}{2\pi\epsilon c^2 a}. \end{aligned}$$

The force is independent of the position of $ds_2$ in the $y$-direction, so the force per unit length on $C$ is

$$\mathbf{F}_2/l = \mathbf{a}_x \frac{|I_1||I_2|}{2\pi\epsilon c^2 a}, \tag{9}$$

which agrees with Eq. 7.

The only difference between the classical and the electrodynamic solution is that Eq. 7 may leave one with doubts as to possible effects of the motion of the conductor, while Eq. 1 leads directly and conclusively to Eq. 9, *irrespective of longitudinal motion.*

## 5. INDUCED EMF.

*Statement*

A moving conductor $AA'$, Fig. 7, carries a steady current $I_1$. A wire $W$ is provided with ballistic galvanometer $G$ and switch $S$. When

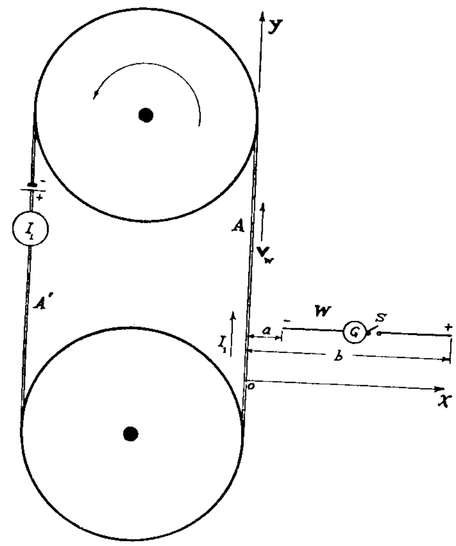


FIG. 7. Determine the induced emf. in $W$, caused by the longitudinal motion of conductor $A$.

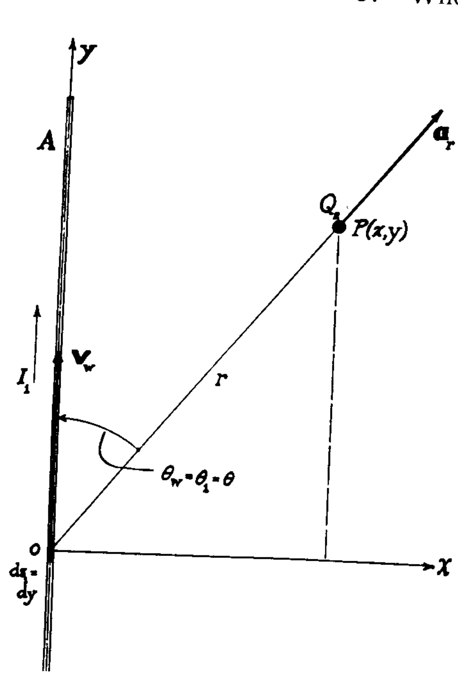


FIG. 8. Analysis of Fig. 7.

$A$ is in uniform motion, $S$ is closed. If an emf. is induced, there will be a momentary deflection of $G$. Is there an induced emf. in $W$?

*A Classical Solution*

At a given distance $x$ from conductor $A$, the magnetic flux density is constant, independent of $y$. Thus the vectors **B** along $W$ are independent of time, irrespective of the motion of $A$. Consequently $\partial\mathbf{B}/\partial t = 0$ and there can be no induced emf., either by transformer action or by flux cutting. This conclusion is also in accordance with Faraday's idea that flux lines are not dragged along by the longitudinal motion of a conductor.



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*A Classical Solution*

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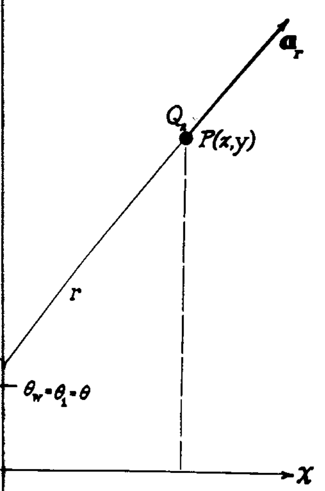


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*Electrodynamic Solution*

The force per unit charge at a point $(x,y)$, Fig. 8, is obtained by using Eq. 2:

$$d\mathbf{F}_2/Q_2 = \mathbf{a}_r \frac{I_1 v_w ds_1}{4\pi\epsilon c^2 r^2}\left[2\sin^2\theta - \cos^2\theta\right].$$

The total effect of conductor $A$ is obtained by integration. The $y$-component of $\mathbf{F}_2$ is zero, leaving

$$\mathfrak{F}_x = F_x/Q_2 = \frac{|I_1| v_w}{4\pi\epsilon c^2}\int_{-\infty}^{\infty}\left[2\sin^2\theta - \cos^2\theta\right]\sin\theta\,\frac{dy}{r^3}$$

$$= \frac{|I_1| v_w}{2\pi\epsilon c^2 x}. \qquad (10)$$

The emf. induced in $W$ is therefore

$$V = \int_a^b \mathfrak{F}_x dx = \frac{I_1 v_w}{2\pi\epsilon c^2}\ln\,(b/a), \qquad (11)$$

which is certainly not zero as predicted by the foregoing classical theory.

### 6. MOVING BRIDGE

*Statement*

A metal bridge $A$ floats in two mercury troughs $H$ and $H'$ (Fig. 9). The circuit carries a direct current $I$. In what direction will the bridge move? This is one of Carl Hering's celebrated experiments (4).

*A Classical Solution*

It is a well-known experimental fact that a circuit carrying a current tends to expand. For instance, a loop of flexible wire will tend to assume a circular form as the result of electromagnetic forces. In accordance with this principle, *the bridge will move to the right.*

To check this conclusion, consider the direction of the magnetic vector **B**. With current as shown in Fig. 9, the entire circuit produces within the loop a magnetic flux that is perpendicular to the diagram and directed into the paper. The force on the bridge is given by the firmly established relation,

$$\mathbf{F} = l\mathbf{I} \times \mathbf{B}.$$

But if **I** in the bridge is downward and **B** is into the paper, **F** *must be to the right for any position of the bridge.* This conclusion does not agree with experiment.

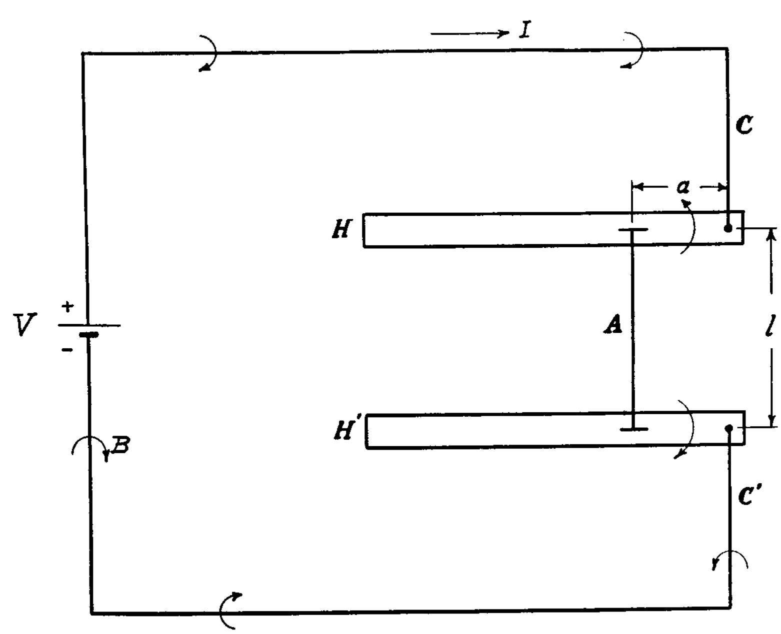


FIG. 9. Motion of metal bridge A floating on mercury troughs $H$, $H'$.

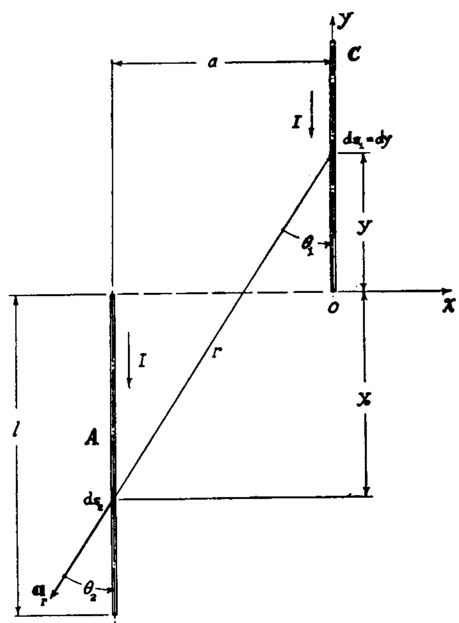


FIG. 10*a*. Analysis of forces caused by conductor $C$, Fig. 9.

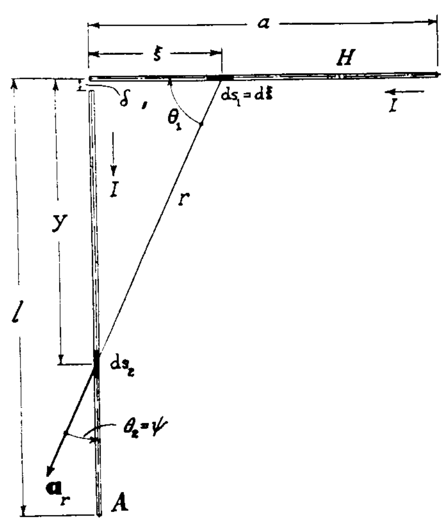


FIG. 10*b*. Analysis of forces caused by conductor $H$, Fig. 9.



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*Electrodynamic Solution*

The solution requires merely the integration of Eq. 3 around the circuit. This integration is not troublesome in any case, but the ideas can be presented most simply by letting the left part of the circuit expand until its effect on $A$ is negligible. We need consider then only the forces caused by two long vertical conductors $C$ and $C'$ and by the portions of the mercury that are carrying current.

For the conductor $C$, Fig. 10*a*, $\eta = 0$ and $\theta_1 = \theta_2 = \theta$, so Eq. 3 becomes

$$d^2\mathbf{F}_2 = -\,\mathbf{a}_r \frac{I^2 ds_1 ds_2}{4\pi\epsilon c^2 r^2}\left[2\sin^2\theta - \cos^2\theta\right].$$

The force contains both $x$ and $y$ components, but integration over equal, long conductors $C$ and $C'$ will cancel the $y$-component. Thus the total force on $d\mathbf{s}_2$, caused by conductor $C$, is

$$d\mathbf{F}_2 = \mathbf{a}_x \frac{I^2 ds_2}{4\pi\epsilon c^2}\int_0^\infty \frac{2a^3 - a(y + y_0)^2}{r^5}\,dy$$

$$= \mathbf{a}_x \frac{I^2 ds_2}{4\pi\epsilon c^2 a}\left[1 - \frac{y_0^3 + 2y_0 a^2}{(a^2 + y_0^2)^{\frac{3}{2}}}\right].$$

The force on the bridge, caused by current in $C$ and $C'$, is

$$\mathbf{F}_2 = \mathbf{a}_x \frac{I^2}{2\pi\epsilon c^2 a}\int_0^l \left[1 - \frac{y_0^3 + 2y_0 a^2}{(a^2 + y_0^2)^{\frac{3}{2}}}\right] dy_0$$

$$= \mathbf{a}_x \frac{I^2}{2\pi\epsilon c^2 \lambda}\left[1 - \frac{1}{(1 + \lambda^2)^{\frac{1}{2}}}\right], \qquad (12)$$

where $\lambda = a/l$. *This force on the bridge is in the positive x-direction and tends to pull the conductor to an equilibrium position at $x = 0$.*

Equation 12 applies only to the effect of conductors $C$ and $C'$. Now take a mercury trough, Fig. 10*b*. Let $\eta = 0$, $\theta_2 = \psi$, $\theta_1 = -(\pi/2 - \psi)$. Then Eq. 3 becomes

$$d^2\mathbf{F}_2 = \mathbf{a}_r \frac{3I^2 ds_1 ds_2}{4\pi\epsilon c^2 r^2}\sin\psi\cos\psi.$$

Evidently the $y$-components for $H$ and $H'$ will cancel, leaving only

$$d^2\mathbf{F}_2 = -\,\mathbf{a}_r \frac{3I^2 ds_1 ds_2}{4\pi\epsilon c^2 r^2}\sin^2\psi\cos\psi.$$

The force on $d\mathbf{s}_2$, caused by $H$, is

$$d\mathbf{F}_2 = -\,\mathbf{a}_x \frac{3I^2 ds_2}{4\pi\epsilon c^2}\int_0^a \frac{y\xi^2}{r^5}\,d\xi$$

$$= -\,\mathbf{a}_x \frac{I^2 ds_2}{4\pi\epsilon c^2}\,\frac{a^3}{y(y^2+a^2)^{\frac{3}{2}}}.$$

The total force on the bridge, produced by current in $H$ and $H'$, is therefore

$$\mathbf{F}_2 = -\,\mathbf{a}_x \frac{I^2 a^3}{2\pi\epsilon c^2}\int_\delta^l \frac{dy}{y(y^2+a^2)^{\frac{3}{2}}}.$$

Since $\delta \ll a$,

$$\mathbf{F}_2 \cong -\,\mathbf{a}_x \frac{I^2}{2\pi\epsilon c^2}\left[\ln\left(\frac{2a}{\delta}\,\frac{1}{\lambda+(1+\lambda^2)^{\frac{1}{2}}}\right) - \left(1 - \frac{\lambda}{(1+\lambda^2)^{\frac{1}{2}}}\right)\right]. \qquad (13)$$

The force caused by current in the mercury elements is always in the negative $x$-direction. The value of the bracket in Eq. 13 is generally in the neighborhood of $+3$ or $+4$. The exact value depends on detailed conditions at the right-angled corners, an analysis of which is not needed here.

The total force on the bridge is expressed as the sum of Eqs. 12 and 13, or

$$\mathbf{F}_2 = -\,\mathbf{a}_x \frac{I^2}{2\pi\epsilon c^2}\left\{\ln\left(\frac{2a}{\delta}\,\frac{1}{\lambda+(1+\lambda^2)^{\frac{1}{2}}}\right) - \frac{1}{\lambda}\left[1+\lambda-(1+\lambda^2)^{\frac{1}{2}}\right]\right\}. \qquad (14)$$

For comparatively small displacements $a$, the first term of Eq. 14 predominates and the bridge moves to the left, as found experimentally by Hering (4). But for large displacements, the bridge may move to the right until it reaches an equilibrium position determined by the relation,

$$\ln\left(\frac{2a}{\delta}\,\frac{1}{\lambda+(1+\lambda^2)^{\frac{1}{2}}}\right) = \frac{1}{\lambda}\left[1+\lambda-(1+\lambda^2)^{\frac{1}{2}}\right]. \qquad (15)$$

For instance, if the logarithm has the value 4, then

$$4\lambda = 1 + \lambda - (1+\lambda^2)^{\frac{1}{2}}$$

or

$$\lambda = a/l = \tfrac{3}{4}.$$

Thus, in this example, the bridge will move to the left until its displacement is $\frac{3}{4}$ of its length. The bridge will remain in this equilibrium position.

### 7. UNIPOLAR GENERATOR

*Statement*

A copper disk $D$ (Fig. 11) rotates on its axis $A$ at uniform angular velocity $\Omega$. Direct current $I_1$ in a stationary loop $W$ produces a magnetic field. Contact is made to the disk at $A$ and $C$, and a bifilar conductor connects these brushes to an electrostatic voltmeter $V$.

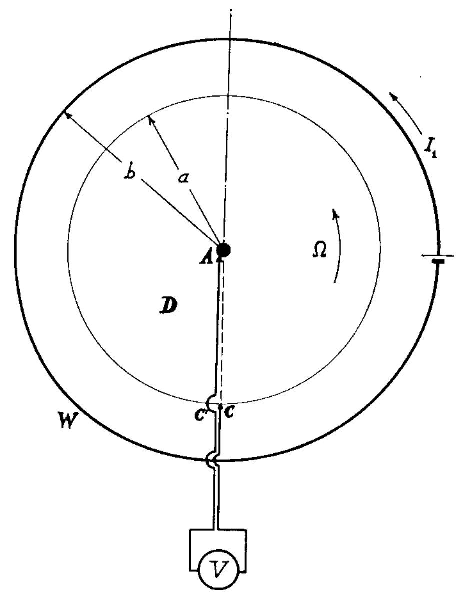


FIG. 11. Unipolar generator. Magnetic field produced by direct current in loop $W$; voltage is generated by rotation of disk $D$.

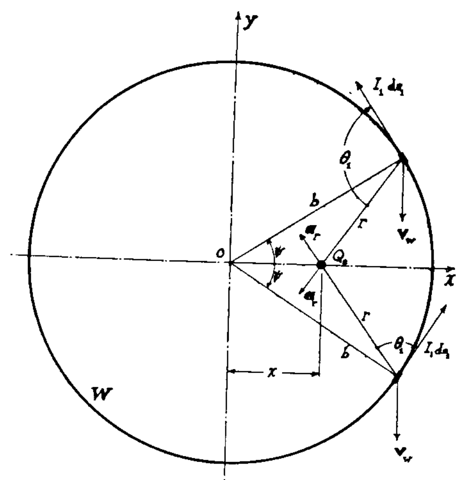


FIG. 12. Analysis of Fig. 11.

(*a*) What is the emf. indicated by the voltmeter?
(*b*) If $D$ and $W$ now rotate in the same direction and at the same angular velocity, what is the emf.?
(*c*) The disk and voltmeter are stationary and only $W$ rotates. What is the emf.?

*A Classical Solution*

Start with Maxwell's equation,

$$\text{curl } \mathbf{E} = -\,\partial\mathbf{B}/\partial t,$$

which is regarded as a cornerstone of electromagnetics, true under all circumstances. Integration over *any closed path* $L$ enclosing an area $\mathcal{a}$ gives

$$\int_{\mathcal{a}} \operatorname{curl} \mathbf{E} \cdot d\mathfrak{A} = \oint_L \mathbf{E} \cdot d\mathbf{s} = V = -\int_{\mathcal{a}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathfrak{A}.$$

For our closed path, we choose the dotted line $AC$ in the disk and the fixed metallic circuit $CVA$. But the flux density $\mathbf{B}$, produced by the steady current $I_1$, is certainly time-invariant, so $\partial \mathbf{B}/\partial t = 0$ and

$$V = -\int_{\mathcal{a}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathfrak{A} = 0. \tag{16}$$

It is hard to see how anything could be more completely zero than this emf. If the leads to $V$ are directly above each other, there is no flux through the circuit $ACVA$ in any case; if the leads are very close to each other, negligible flux is linked even if they are not above each other; and finally, any flux that does link the circuit is constant, so $\partial \Phi/\partial t = 0$. This conclusion applies to all three cases. But experiment shows an induced emf. in cases ($a$) and ($b$).

Suppose that we abandon Maxwell's equation and employ the flux-cutting relation,

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} \quad \text{or} \quad V = Blv.$$

Each element of the metal disk cuts through the magnetic field at velocity $v = \Omega r$, so

$$V = \int_0^a E_r dr = \Omega \int_0^a rB(r)\, dr. \tag{17}$$

Equation 17 gives the same emf. as that produced by a radial wire, and this result agrees with experiment for case ($a$).

In ($b$), however, there is no relative motion between wire and disk, so there is no flux cutting and $V = 0$. Experiment shows, however, that the emf. is the same as in ($a$). In ($c$), there is again relative motion between the wire and the disk, so the emf. should be given by Eq. 17. Actually, there is no emf.

*Electrodynamic Solution*

Equation 2 expresses the force per unit charge acting at any point in the disk because of a current element in $W$. It is convenient to take the elements in pairs at $\pm\psi$ as shown in Fig. 12. Each element produces a force in the direction of its $\mathbf{a}_r$. The $y$-components cancel, leaving only an $x$-component:

$$d\mathfrak{F} = -\mathbf{a}_x \frac{|I_1|v_w}{2\pi\epsilon c^2} \frac{b\,d\psi}{r^4} [bx \cos^2\psi + 3bx - 2(b^2 + x^2)\cos\psi](b\cos\psi - x).$$