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CLS1943804: Journal of the Franklin Institute.

WR: (ISSN 0016-0032 / Pergamon Press [etc.] /)

Requested: 2011-04-10 18:52

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Citation: Issue: v.260() 1955; Article: Some Electromagnetic Paradoxes / Moon, Parry; Pages: 373-395

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SOME ELECTROMAGNETIC PARADOXES

BY

PARRY MOON [1] AND DOMINA EBERLE SPENCER [2]

1. INTRODUCTION

Electromagnetic theory is undoubtedly one of the most imposing theoretical edifices reared by the mind of man. It is a work of genius—of Maxwell, Hertz, Lorenz, Lorentz, Poincaré, Einstein. Yet it does not form a logical whole, as evidenced by numerous paradoxes.

Classical theory is built up of pieces, each admirable in its way but not fitting too comfortably into the whole. Sometimes we pretend that flux lines have physical existence: we visualize them as rubber bands pulling on conductors or as filaments that are cut by moving wires. At other times we are unable to decide whether these flux lines are moving or stationary, and we fall back on Maxwell's equations which express the phenomena in terms of the rather abstract vectors **E**, **D**, **B**, **H**. At still other times, we find it desirable to use retarded potentials. Einstein's relativity may also be introduced.

Unfortunately, these various methods of handling a problem do not always lead to the same conclusion. Hence the paradoxes! If one knows the correct result, he can usually find a part of classical theory that will give him the answer. And he can always justify his choice by some plausible argument. But an equally logical argument can be found for other approaches that may give erroneous results. As Ko-Ko says, "Here's a state of things, here's a how-de-do!"

In the following pages, we consider a number of paradoxes. In each case, the problem is stated and a solution is given. This solution is in accordance with classical theory but does not agree with experiment. It is the kind of solution that might be obtained by a scrupulous but inexperienced student. The reader may amuse himself by finding a classical approach that gives the correct answer.

Each problem is also solved by employing the interparticle method developed by Coulomb, Ampère, Gauss, and Ritz. This method, unlike the Maxwell formulation, can be summarized (1)[3] in *a single equation and this equation gives unambiguous results*. In the new electrodynamics, no paradoxes have been found.

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[3] The boldface numbers in parentheses refer to the references appended to this paper.

2. THE NEW ELECTRODYNAMICS

The non-Maxwellian approach subsumes the whole of electrodynamics under the single equation,

$$\mathbf{F}_2/Q_2 = \mathbf{a}_r \frac{Q_1}{4\pi\epsilon c^2 r^2}\left[\mathbf{v}\cdot\mathbf{v} - \tfrac{3}{2}(\mathbf{a}_r\cdot\mathbf{v})^2\right] - \frac{Q_1}{4\pi\epsilon c^2 r}\frac{d}{dt}\mathbf{v}(t - r/c) - \mathbf{a}_r \frac{1}{4\pi\epsilon}\frac{\partial}{\partial r}\left[\frac{1}{r} Q_1(t - r/c)\right]. \quad (1)$$

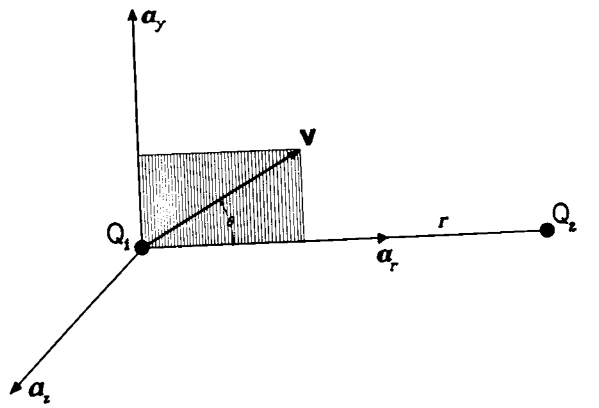

FIG. 1. Two charged particles.

The notation is indicated in Fig. 1. The unit vector $\mathbf{a}_r$ points from Q_1 to Q_2. Angles are measured counter-clockwise from $\mathbf{a}_r$. Vector $\mathbf{v}$ is the *relative velocity of* Q_1 *with respect to* Q_2. Equation 1 applies at all ordinary velocities: in fact, it is reasonably accurate up to approximately half the velocity of light. At higher velocities, corrections must be applied.

Note that the magnetic-field concept is completely eliminated and **B**, **H**, and μ do not occur. The usual "magnetic" forces and "flux-cutting emf.'s" are handled by the first term of Eq. 1 which depends on the *relative velocity* of charges. The second term depends on the *acceleration* of charges, while the third term is related to Maxwell's displacement current and includes the Coulomb force as a special case (1). For moderate frequencies and distances, retardation may be neglected and Eq. 1 reduces to

$$\mathbf{F}/Q_2 = \mathbf{a}_r \frac{Q_1}{4\pi\epsilon r^2}\left(\frac{v}{c}\right)^2\left[1 - \tfrac{3}{2}\cos^2\theta\right] - \mathbf{a}_a \frac{Q_1}{4\pi\epsilon c^2 r}\frac{dv}{dt} + \mathbf{a}_r \frac{Q_1}{4\pi\epsilon r^2}. \quad (1a)$$

Equations 1 and 1a apply to the force between charges Q_1 and Q_2, Fig. 1. These equations are easily extended (1) to give the force between current elements or between a current element and a charge. For a current element $I_1 d\mathbf{s}_1$ and a charge Q_2 (Fig. 2), the first term of Eq. 1 gives

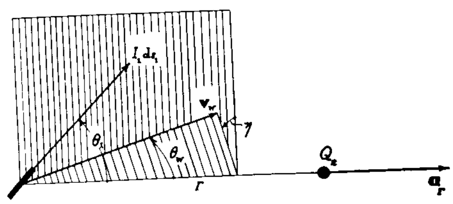

FIG. 2. Current element and charged particle.

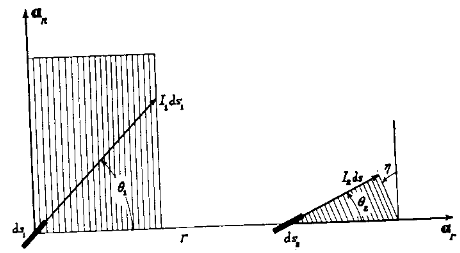

FIG. 3. Two current elements.

$$d\mathbf{F}/Q_2 = \mathbf{a}_r \frac{|I_1|}{4\pi\epsilon c^2 r^2}[2(\mathbf{a}_r \times \mathbf{v}_w)\cdot(\mathbf{a}_r \times d\mathbf{s}_1) - (\mathbf{a}_r\cdot\mathbf{v}_w)(\mathbf{a}_r\cdot d\mathbf{s}_1)]$$

$$= \mathbf{a}_r \frac{|I_1| v_w ds_1}{4\pi\epsilon c^2 r^2}[2 \sin\theta_w \sin\theta_1 \cos\eta - \cos\theta_w \cos\theta_1]. \quad (2)$$

Here the sense of $d\mathbf{s}_1$ is the direction of the current, and $\mathbf{v}_w$ is the velocity of the material wire with respect to Q_2. Equation 1 also yields a term directly proportional to the drift velocity $\mathbf{v}_1$ of the electrons (2). Apparently, this additional force is always too small to be measured, so it will be ignored in this paper.

The first term of Eq. 1 can be applied also to two current elements (Fig. 3). This Ampère force is

$$d^2\mathbf{F} = -\mathbf{a}_r \frac{|I_1||I_2|}{4\pi\epsilon c^2 r^2}[2(\mathbf{a}_r \times d\mathbf{s}_1)\cdot(\mathbf{a}_r \times d\mathbf{s}_2) - (\mathbf{a}_r\cdot d\mathbf{s}_1)(\mathbf{a}_r\cdot d\mathbf{s}_2)]. \quad (3)$$

This force is independent of the relative velocity of the material elements. Equation 3 is a rigorous deduction from the first part of Eq. 1, with no terms neglected.

The remainder of the paper is devoted to a consideration of a few of the paradoxes that occur in conventional electromagnetic theory.

3. ELECTRON BEAMS

Statement

Parallel electron guns produce two identical electron beams in a long, evacuated tube. Initial velocity of the electrons with respect to the laboratory is $\mathbf{v}_0$. What is the magnitude and direction of the force on the electrons?

A Classical Solution

The argument may be stated rather neatly in the form of a classical syllogism:

(*a*) A stream of electrons is equivalent to a current (Rowland's experiment).
(*b*) Parallel currents in the same direction attract each other, as proved experimentally by the behavior of parallel, current-carrying wires.
(*c*) Therefore, the electron beams will attract each other and will tend to converge.

The current in one beam is

$$I = -NA|Q_e|v,$$

where

N = number of free electrons per unit volume,
A = cross-sectional area of the beam,
$|Q_e|$ = magnitude of electronic charge.

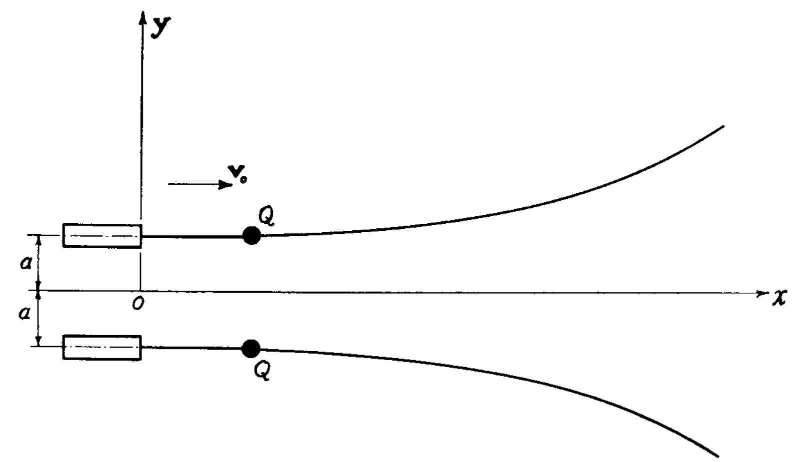

FIG. 4. Electron beams. Electrons are emitted at $x = 0$ with velocity $\mathbf{v}_0$ in the x-direction.

This current produces a magnetic flux density at the other beam,

$$B = \frac{\mu_0 I}{2\pi a},$$

where a is the distance between the axes of the beams. The resulting force per unit length is

$$F/l = BI = \frac{\mu_0 I^2}{2\pi a} = \frac{N^2A^2|Q_e|^2}{2\pi\epsilon c^2 a} v^2. \tag{4}$$

The force is perpendicular to **v** and tends to pull the beams together; a result that is certainly not in accord with experiment.

For other solutions, the reader is referred to *Electrical Engineering* (3), where a long and inconclusive discussion reminds one of Stephen Leacock's rider who "leaped on his horse and dashed madly off in all directions."

Electrodynamic Solution

Consider two charged particles moving in parallel paths in the x-direction (Fig. 4). Each particle has a charge Q and is at the same distance x from the origin. The velocity is $\mathbf{v}_0$ with respect to the laboratory, but the *relative* velocity of one charge with respect to the other is zero. Thus Eq. 1 reduces to the Coulomb equation, and the force on the upper particle is

$$\mathbf{F} = \mathbf{a}_y \frac{Q^2}{4\pi\epsilon r^2}. \tag{5}$$

For a coordinate system moving with the charges, *the force is an ordinary electrostatic repulsion which is exactly the same as if the charges were stationary.* The electron beams of Fig. 4 will therefore tend to diverge, as found in practice.

The force, Eq. 5, will accelerate the particles in the y-direction:

$$F = m \frac{d^2y}{dt^2}$$

or

$$y = a + \frac{F}{m} t^2.$$

But $x = v_0 t$; so the path of the upper particle referred to a stationary coordinate system, is a parabola:

$$y = a + (F/mv_0^2)x^2. \tag{6}$$

Thus the problem is a very simple one, requiring only a knowledge of elementary mechanics and of Coulomb's equation. Classical treatment, on the other hand, introduces puzzling questions such as the presence or absence of a magnetic field, depending on the motion of the observer. Such questions do not arise in the new electrodynamics, nor is Einstein's relativity needed.

Equations 1 and 6 do not apply for $v \rightarrow c$ without additional correction terms. Strictly speaking, corrections should be made also for the velocity and acceleration in the y-direction. This modification is easily accomplished by use of the first and second terms of Eq. 1; but since both terms include c^2 in the denominator, their effect is ordinarily negligible. For the electron-beam problem, the Coulomb repulsions of all electrons in the beams should be considered. But this would not change the foregoing general conclusion that *the force is merely a Coulomb repulsion and is never the attraction obtained with parallel currents.*

4. LONGITUDINAL MOTION

Statement

A flexible conductor AA', Fig. 5, carries a direct current I_1. Near A is a long, straight conductor C carrying a steady current I_2. The distance a is small compared with the other dimensions, so the effects of the return conductors may be neglected. What is the force per unit length, exerted on conductor C?

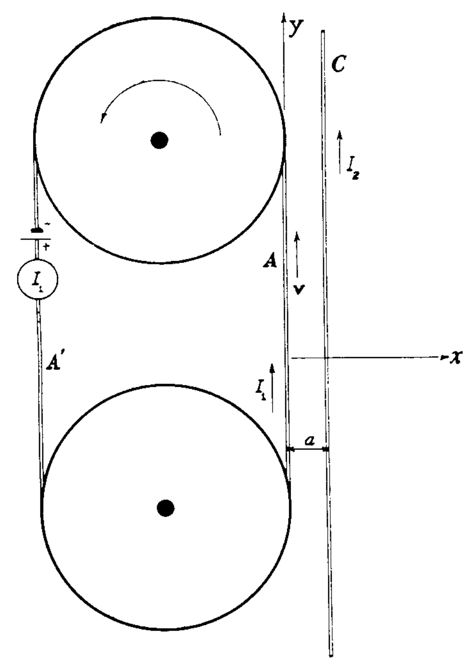

FIG. 5. Determine the force between conductors A and C when A is in uniform longitudinal motion.

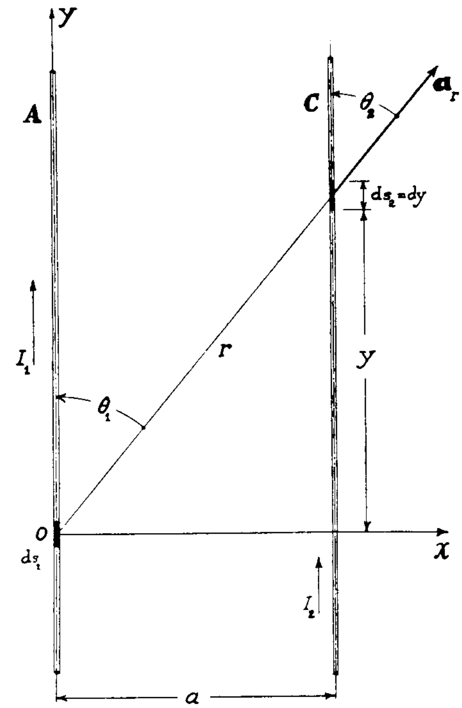

FIG. 6. Analysis of Fig. 5.

A Classical Solution

First consider AA' stationary. The magnetic flux density at C, caused by current I_1, is

$$B = \frac{\mu_0 I_1}{2\pi a}.$$

The force on C, according to Faraday's relation, is

$$F_2 = BlI_2$$

or

$$F_2/l = \frac{\mu_0 I_1 I_2}{2\pi a} = \frac{I_1 I_2}{2\pi\epsilon c^2 a}. \tag{7}$$

The current in A is

$$I_1 = N_1 A_1 |Q_e| v_1,$$

where the symbols are as in Section 3 and v_1 is the magnitude of the drift velocity of the electrons in A.

Now allow A to move longitudinally at velocity $\mathbf{v}$. According to Rowland's experiment, moving charges constitute a current, so the *effective* current is not I_1 but is

$$I_1' = N_1 A_1 |Q_e| (v_1 - v)$$
$$= I_1 \left(\frac{v_1 - v}{v_1}\right).$$

Replacing I_1 in Eq. 7 by I_1', we obtain

$$F_2/l = \frac{I_1 I_2}{2\pi\epsilon c^2 a}\left(\frac{v_1 - v}{v_1}\right). \tag{8}$$

Since v may easily exceed $10^4 v_1$, we can obtain with the moving conductor a force 10,000 times as great as that obtained with the stationary conductor. What is wrong with this prediction?

Electrodynamic Solution

It is easily proved that Eq. 3 is independent of conductor velocity. Thus the force is independent of $\mathbf{v}$ and is obtained by integration of Eq. 3. From Fig. 6, $\theta_1 = \theta_2 = \theta$, $\eta = 0$, and the force on element ds_2 is

$$d\mathbf{F}_2 = -\mathbf{a}_x \frac{|I_1||I_2|ds_2}{4\pi\epsilon c^2}\int_{-\infty}^{\infty}\frac{[2\sin^2\theta - \cos^2\theta]\sin\theta}{r^2}\,dy$$
$$= \mathbf{a}_x \frac{|I_1||I_2|ds_2}{2\pi\epsilon c^2 a}.$$

The force is independent of the position of ds_2 in the y-direction, so the force per unit length on C is

$$\mathbf{F}_2/l = \mathbf{a}_x \frac{|I_1||I_2|}{2\pi\epsilon c^2 a}, \tag{9}$$

which agrees with Eq. 7.

The only difference between the classical and the electrodynamic solution is that Eq. 7 may leave one with doubts as to possible effects of the motion of the conductor, while Eq. 1 leads directly and conclusively to Eq. 9, *irrespective of longitudinal motion.*

5. INDUCED EMF.

Statement

A moving conductor AA', Fig. 7, carries a steady current I_1. A wire W is provided with ballistic galvanometer G and switch S. When

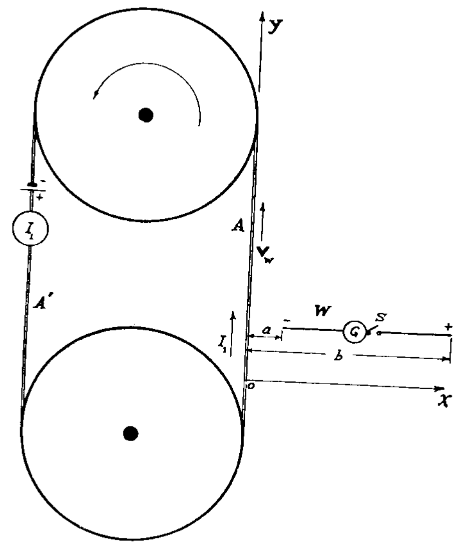

FIG. 7. Determine the induced emf. in W, caused by the longitudinal motion of conductor A.

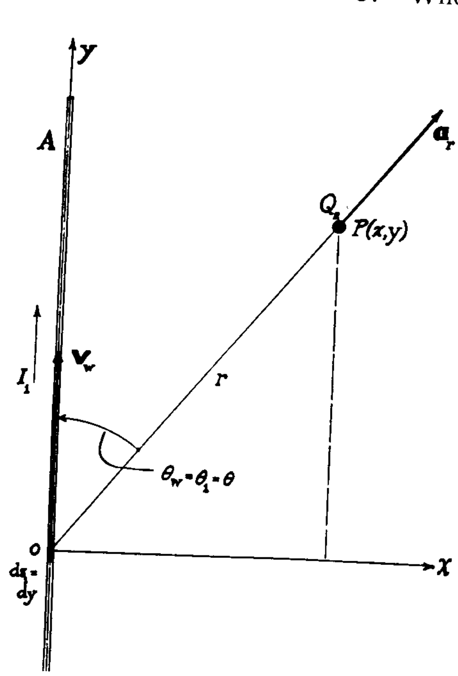

FIG. 8. Analysis of Fig. 7.

A is in uniform motion, S is closed. If an emf. is induced, there will be a momentary deflection of G. Is there an induced emf. in W?

A Classical Solution

At a given distance x from conductor A, the magnetic flux density is constant, independent of y. Thus the vectors **B** along W are independent of time, irrespective of the motion of A. Consequently $\partial\mathbf{B}/\partial t = 0$ and there can be no induced emf., either by transformer action or by flux cutting. This conclusion is also in accordance with Faraday's idea that flux lines are not dragged along by the longitudinal motion of a conductor.

Electrodynamic Solution

The force per unit charge at a point (x,y), Fig. 8, is obtained by using Eq. 2:

$$d\mathbf{F}_2/Q_2 = \mathbf{a}_r \frac{I_1 v_w ds_1}{4\pi\epsilon c^2 r^2}\left[2\sin^2\theta - \cos^2\theta\right].$$

The total effect of conductor A is obtained by integration. The y-component of $\mathbf{F}_2$ is zero, leaving

$$\mathfrak{F}_x = F_x/Q_2 = \frac{|I_1|v_w}{4\pi\epsilon c^2}\int_{-\infty}^{\infty}\left[2\sin^2\theta - \cos^2\theta\right]\sin\theta\,\frac{dy}{r^3}$$

$$= \frac{|I_1|v_w}{2\pi\epsilon c^2 x}. \qquad (10)$$

The emf. induced in W is therefore

$$V = \int_a^b \mathfrak{F}_x dx = \frac{|I_1|v_w}{2\pi\epsilon c^2}\ln(b/a), \qquad (11)$$

which is certainly not zero as predicted by the foregoing classical theory.

6. MOVING BRIDGE

Statement

A metal bridge A floats in two mercury troughs H and H' (Fig. 9). The circuit carries a direct current I. In what direction will the bridge move? This is one of Carl Hering's celebrated experiments (4).

A Classical Solution

It is a well-known experimental fact that a circuit carrying a current tends to expand. For instance, a loop of flexible wire will tend to assume a circular form as the result of electromagnetic forces. In accordance with this principle, *the bridge will move to the right.*

To check this conclusion, consider the direction of the magnetic vector $\mathbf{B}$. With current as shown in Fig. 9, the entire circuit produces within the loop a magnetic flux that is perpendicular to the diagram and directed into the paper. The force on the bridge is given by the firmly established relation,

$$\mathbf{F} = l\mathbf{I} \times \mathbf{B}.$$

But if $\mathbf{I}$ in the bridge is downward and $\mathbf{B}$ is into the paper, $\mathbf{F}$ *must be to the right for any position of the bridge.* This conclusion does not agree with experiment.

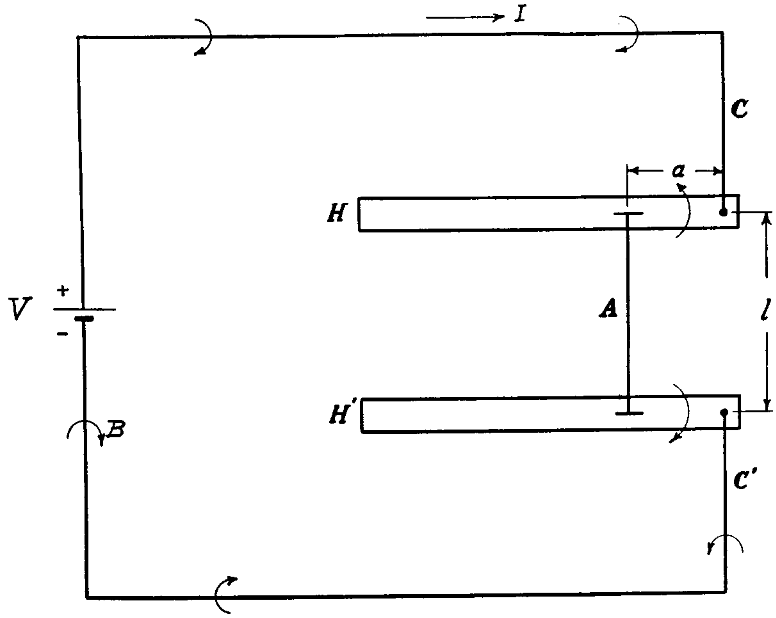

FIG. 9. Motion of metal bridge A floating on mercury troughs H, H'.

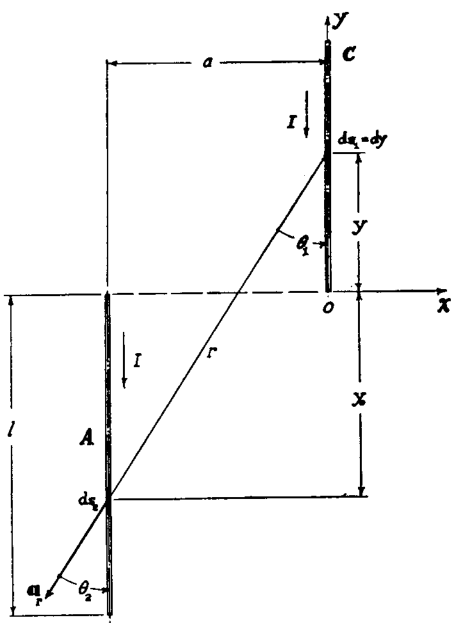

FIG. 10a. Analysis of forces caused by conductor C, Fig. 9.

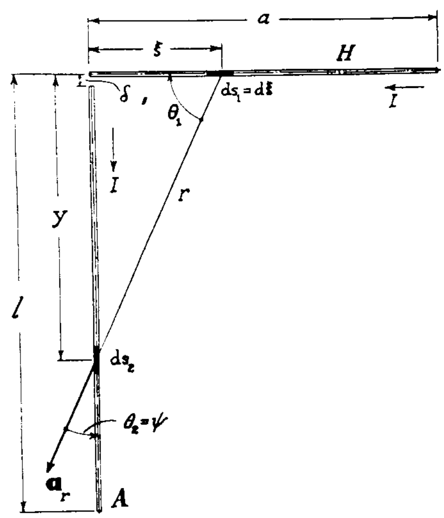

FIG. 10b. Analysis of forces caused by conductor H, Fig. 9.

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Electrodynamic Solution

The solution requires merely the integration of Eq. 3 around the circuit. This integration is not troublesome in any case, but the ideas can be presented most simply by letting the left part of the circuit expand until its effect on A is negligible. We need consider then only the forces caused by two long vertical conductors C and C' and by the portions of the mercury that are carrying current.

For the conductor C, Fig. 10*a*, $\eta = 0$ and $\theta_1 = \theta_2 = \theta$, so Eq. 3 becomes

$$d^2\mathbf{F}_2 = -\,\mathbf{a}_r \frac{I^2 ds_1 ds_2}{4\pi\epsilon c^2 r^2}\left[2\sin^2\theta - \cos^2\theta\right].$$

The force contains both x and y components, but integration over equal, long conductors C and C' will cancel the y-component. Thus the total force on $d\mathbf{s}_2$, caused by conductor C, is

$$d\mathbf{F}_2 = \mathbf{a}_x \frac{I^2 ds_2}{4\pi\epsilon c^2}\int_0^\infty \frac{2a^3 - a(y+y_0)^2}{r^5}\,dy$$

$$= \mathbf{a}_x \frac{I^2 ds_2}{4\pi\epsilon c^2 a}\left[1 - \frac{y_0^3 + 2y_0a^2}{(a^2+y_0^2)^{3/2}}\right].$$

The force on the bridge, caused by current in C and C', is

$$\mathbf{F}_2 = \mathbf{a}_x \frac{I^2}{2\pi\epsilon c^2 a}\int_0^l \left[1 - \frac{y_0^3 + 2y_0a^2}{(a^2+y_0^2)^{3/2}}\right] dy_0$$

$$= \mathbf{a}_x \frac{I^2}{2\pi\epsilon c^2\lambda}\left[1 - \frac{1}{(1+\lambda^2)^{1/2}}\right], \qquad (12)$$

where $\lambda = a/l$. *This force on the bridge is in the positive x-direction and tends to pull the conductor to an equilibrium position at* $x = 0$.

Equation 12 applies only to the effect of conductors C and C'. Now take a mercury trough, Fig. 10*b*. Let $\eta = 0$, $\theta_2 = \psi$, $\theta_1 = -(\pi/2 - \psi)$. Then Eq. 3 becomes

$$d^2\mathbf{F}_2 = \mathbf{a}_r \frac{3I^2 ds_1 ds_2}{4\pi\epsilon c^2 r^2}\sin\psi\cos\psi.$$

Evidently the y-components for H and H' will cancel, leaving only

$$d^2\mathbf{F}_2 = -\,\mathbf{a}_r \frac{3I^2 ds_1 ds_2}{4\pi\epsilon c^2 r^2}\sin^2\psi\cos\psi.$$

The force on $d\mathbf{s}_2$, caused by H, is

$$d\mathbf{F}_2 = -\mathbf{a}_x \frac{3I^2 ds_2}{4\pi\epsilon c^2}\int_0^a \frac{y\xi^2}{r^5}\,d\xi$$

$$= -\mathbf{a}_x \frac{I^2 ds_2}{4\pi\epsilon c^2}\,\frac{a^3}{y(y^2+a^2)^{3/2}}.$$

The total force on the bridge, produced by current in H and H', is therefore

$$\mathbf{F}_2 = -\mathbf{a}_x \frac{I^2 a^3}{2\pi\epsilon c^2}\int_\delta^l \frac{dy}{y(y^2+a^2)^{3/2}}.$$

Since $\delta \ll a$,

$$\mathbf{F}_2 \cong -\mathbf{a}_x \frac{I^2}{2\pi\epsilon c^2}\left[\ln\left(\frac{2a}{\delta}\,\frac{1}{\lambda+(1+\lambda^2)^{1/2}}\right) - \left(1 - \frac{\lambda}{(1+\lambda^2)^{1/2}}\right)\right]. \quad (13)$$

The force caused by current in the mercury elements is always in the negative x-direction. The value of the bracket in Eq. 13 is generally in the neighborhood of +3 or +4. The exact value depends on detailed conditions at the right-angled corners, an analysis of which is not needed here.

The total force on the bridge is expressed as the sum of Eqs. 12 and 13, or

$$\mathbf{F}_2 = -\mathbf{a}_x \frac{I^2}{2\pi\epsilon c^2}\left\{\ln\left(\frac{2a}{\delta}\,\frac{1}{\lambda+(1+\lambda^2)^{1/2}}\right) - \frac{1}{\lambda}\left[1+\lambda-(1+\lambda^2)^{1/2}\right]\right\}. \quad (14)$$

For comparatively small displacements a, the first term of Eq. 14 predominates and the bridge moves to the left, as found experimentally by Hering (4). But for large displacements, the bridge may move to the right until it reaches an equilibrium position determined by the relation,

$$\ln\left(\frac{2a}{\delta}\,\frac{1}{\lambda+(1+\lambda^2)^{1/2}}\right) = \frac{1}{\lambda}\left[1+\lambda-(1+\lambda^2)^{1/2}\right]. \quad (15)$$

For instance, if the logarithm has the value 4, then

$$4\lambda = 1 + \lambda - (1+\lambda^2)^{1/2}$$

or

$$\lambda = a/l = \tfrac{3}{4}.$$

Thus, in this example, the bridge will move to the left until its displace-

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Statement

A copper disk D
velocity Ω. Direct

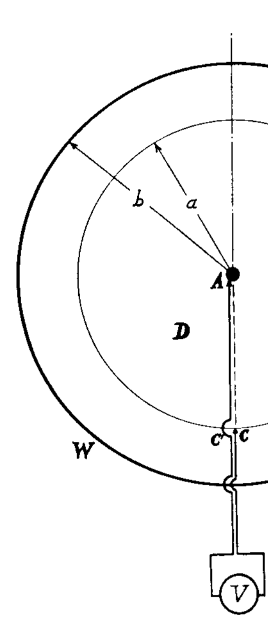

FIG. 11. Unipolar ge
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A Classical Solution

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ment is $\frac{3}{4}$ of its length. The bridge will remain in this equilibrium position.

7. UNIPOLAR GENERATOR

Statement

A copper disk D (Fig. 11) rotates on its axis A at uniform angular velocity Ω. Direct current I_1 in a stationary loop W produces a magnetic field. Contact is made to the disk at A and C, and a bifilar conductor connects these brushes to an electrostatic voltmeter V.

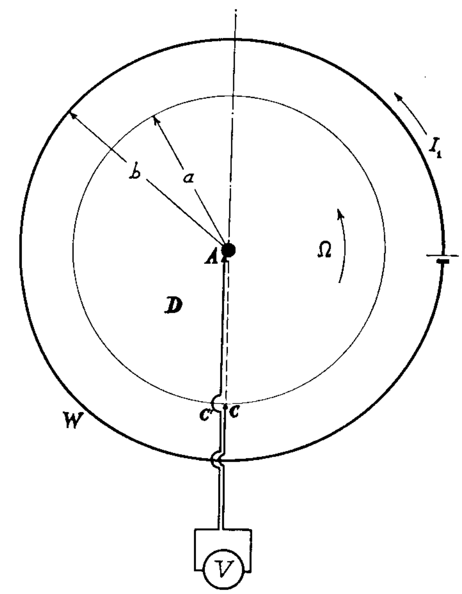

FIG. 11. Unipolar generator. Magnetic field produced by direct current in loop W; voltage is generated by rotation of disk D.

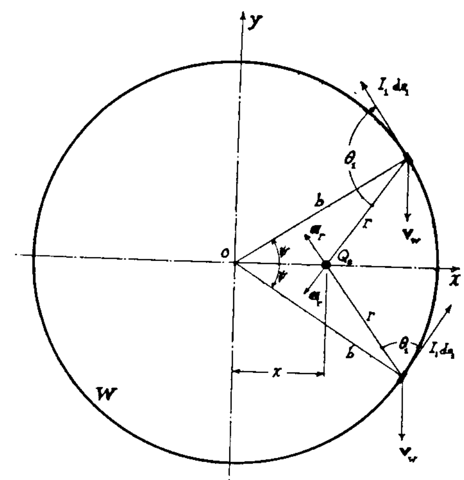

FIG. 12. Analysis of Fig. 11.

(*a*) What is the emf. indicated by the voltmeter?
(*b*) If D and W now rotate in the same direction and at the same angular velocity, what is the emf.?
(*c*) The disk and voltmeter are stationary and only W rotates. What is the emf.?

A Classical Solution

Start with Maxwell's equation,

$$\text{curl } \mathbf{E} = -\,\partial \mathbf{B}/\partial t,$$

which is regarded as a cornerstone of electromagnetics, true under all circumstances. Integration over *any closed path* L enclosing an area $\mathfrak{a}$ gives

$$\int_{\mathfrak{a}} \operatorname{curl} \mathbf{E}\cdot d\mathfrak{A} = \oint_L \mathbf{E}\cdot d\mathbf{s} = V = -\int_{\mathfrak{a}} \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathfrak{A}.$$

For our closed path, we choose the dotted line AC in the disk and the fixed metallic circuit CVA. But the flux density $\mathbf{B}$, produced by the steady current I_1, is certainly time-invariant, so $\partial\mathbf{B}/\partial t = 0$ and

$$V = -\int_{\mathfrak{a}} \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathfrak{A} = 0. \tag{16}$$

It is hard to see how anything could be more completely zero than this emf. If the leads to V are directly above each other, there is no flux through the circuit $ACVA$ in any case; if the leads are very close to each other, negligible flux is linked even if they are not above each other; and finally, any flux that does link the circuit is constant, so $\partial\Phi/\partial t = 0$. This conclusion applies to all three cases. But experiment shows an induced emf. in cases (a) and (b).

Suppose that we abandon Maxwell's equation and employ the flux-cutting relation,

$$\mathbf{E} = \mathbf{v}\times\mathbf{B} \quad \text{or} \quad V = Blv.$$

Each element of the metal disk cuts through the magnetic field at velocity $v = \Omega r$, so

$$V = \int_0^a E_r dr = \Omega \int_0^a rB(r)\, dr. \tag{17}$$

Equation 17 gives the same emf. as that produced by a radial wire, and this result agrees with experiment for case (a).

In (b), however, there is no relative motion between wire and disk, so there is no flux cutting and $V = 0$. Experiment shows, however, that the emf. is the same as in (a). In (c), there is again relative motion between the wire and the disk, so the emf. should be given by Eq. 17. Actually, there is no emf.

Electrodynamic Solution

Equation 2 expresses the force per unit charge acting at any point in the disk because of a current element in W. It is convenient to take the elements in pairs at $\pm\psi$ as shown in Fig. 12. Each element produces a force in the direction of its $\mathbf{a}_r$. The y-components cancel, leaving only an x-component:

$$d\mathfrak{F} = -\mathbf{a}_x \frac{|I_1|v_w}{2\pi\epsilon c^2}\frac{b\,d\psi}{r^4}\left[bx\cos^2\psi + 3bx - 2(b^2+x^2)\cos\psi\right](b\cos\psi - x).$$

So at any point in the disk, *electrons are acted on by a radial force.*

(*a*) Integration from 0 to π gives the effect of the complete current loop:

$$\mathfrak{F} = \mathbf{a}_x \frac{|I_1|v_w}{2\pi\epsilon c^2}\left[\frac{E(k)}{b-x} + \frac{K(k)}{b+x}\right], \tag{18}$$

where E and K are complete elliptic integrals of modulus

$$k = \frac{2(bx)^{\frac{1}{2}}}{b+x}.$$

Equation 18 is positive for all points within the disk. Thus all electrons in the disk are acted on by a force that urges them toward the center, and an emf. is indicated by the voltmeter. The polarity is as shown in Fig. 11 and the voltage is

$$V = \int_R^a \mathfrak{F} dr, \tag{19}$$

where R is the radius of the shaft. In the derivation of Eq. 18, only the Ampère force produced by unaccelerated motion of electrons in W has been considered. Equation 1*a* gives also other forces caused by velocity and acceleration of displaced electrons in the disk and by acceleration v_1^2/b of electrons in W. Under ordinary circumstances, these contributions are negligible.

(*b*) D and W now rotate at the same angular velocity (Fig. 11). Except for negligible velocity and acceleration terms, the emf. induced in the disk is zero in accordance with Eq. 18. The emf. induced in the bifilar conductor CV is also zero. But the conductor AC' is in motion with respect to W, so an emf. is induced in it. This emf. is the same as obtained in (*a*), which agrees with experiment.

(*c*) Rotation of W, with everything else stationary, induces an emf. in the conductor AC' but it induces an equal and opposite emf. in the disk, so $V = 0$.

8. FLEXIBLE LOOP

Statement

A loop of flexible wire W_1 (Fig. 13*a*) carries a steady current I_1. Attached to W_1 is a second wire W_2 which is connected to an electrostatic voltmeter V. The composite loop is deformed in an arbitrary way. What is the induced emf.?

To make the problem more amenable to calculation, consider the special case of Fig. 13*b*. The loops have been greatly expanded in the y-direction so that we have two long parallel conductors W_1 of radius a whose axes are separated by distance ξ, $(\xi \gg a)$. The conductor at

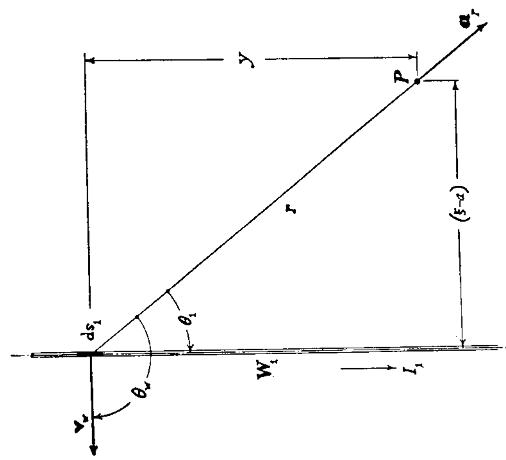

FIG. 13c. Analysis of Fig. 13b.

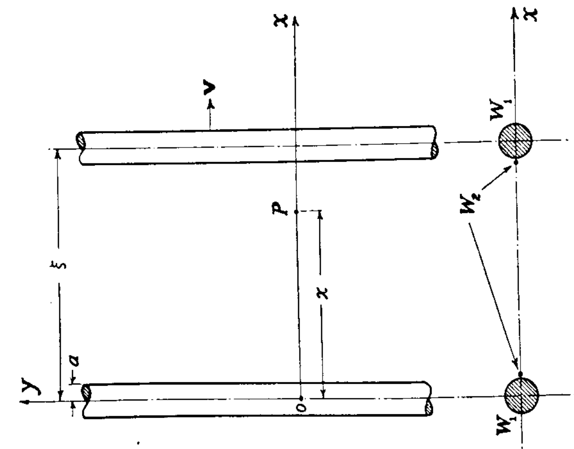

FIG. 13b. Special case of Fig. 13a.

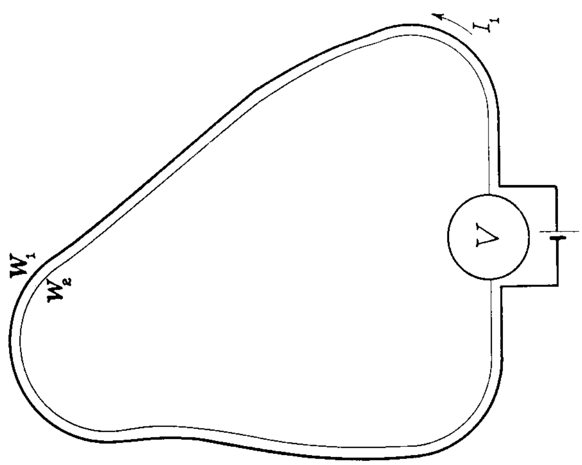

FIG. 13a. Double flexible loop. Is an emf. induced by deforming the loop?

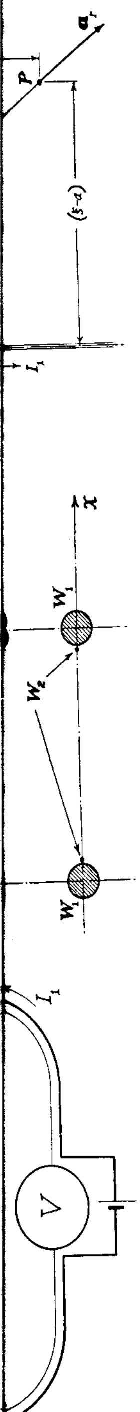

FIG. 13c. Analysis of Fig. 13b.

FIG. 13b. Special case of Fig. 13a.

FIG. 13a. Double flexible loop. Is an emf. induced by deforming the loop?

$x = 0$ is fixed, while the conductor at $x = \xi$ is moving in the x-direction at constant velocity $\mathbf{v}$. Cemented to the inside of each W_1 is an enameled wire W_2 of small diameter, which is connected to the voltmeter.

A Classical Solution

Since there are no moving contacts to cause possible ambiguity, we use Maxwell's equation,

$$\text{curl}\ \mathbf{E} = -\ \partial\mathbf{B}/\partial t$$

or

$$V = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathfrak{A}. \tag{20}$$

The flux density at an arbitrary point P is

$$B(x,\ \xi) = \frac{I}{2\pi\epsilon c^2}\frac{\xi}{x(\xi - x)},$$

or

$$\frac{\partial B}{\partial t} = \frac{\partial B}{\partial \xi}\frac{d\xi}{dt} = -\frac{Iv}{2\pi\epsilon c^2}\frac{1}{(\xi - x)^2}.$$

Therefore, according to Eq. 20, the induced emf. per unit length in the y-direction is

$$V = -\int_a^{\xi - a}\frac{\partial B}{\partial t}\,dx = \frac{Iv}{2\pi\epsilon c^2 a}\left(\frac{\xi - 2a}{\xi - a}\right). \tag{21}$$

In the usual case of $a \ll \xi$, the bracket becomes essentially unity and the induced emf. per unit length approaches infinity as $a \rightarrow 0$!

Electrodynamic Solution

Consider the left current-carrying conductor and a point P in the right voltmeter wire (Fig. 13*c*). Here $\theta_w = \pi/2 + \theta_1$, $\cos\theta_w = -\sin\theta_1$, $\sin\theta_w = \cos\theta_1$, and Eq. 2 becomes

$$d\mathfrak{F}_2 = \mathbf{a}_r\frac{3|I_1|v_w ds_1}{4\pi\epsilon c^2 r^2}\cos\theta_1\sin\theta_1.$$

The total force per unit charge, caused by the left conductor, is

$$\mathfrak{F} = \mathbf{a}_y\frac{3|I_1|v_w(\xi - a)}{2\pi\epsilon c^2}\int_0^\infty\frac{y^2dy}{r^5} = \frac{|I_1|v_w}{2\pi\epsilon c^2(\xi - a)}.$$

Thus the induced emf. in the two wires W_2, per unit y-length, is

$$V = \frac{|I_1| v_w}{\pi \epsilon c^2 (\xi - a)}, \tag{22}$$

which does not agree with Eq. 21 but yields a reasonable voltage for any value of a.

9. BLONDEL EXPERIMENT (5)

Statement

Two V-pulleys D and D' of insulating material are arranged on parallel shafts at some distance from each other, Fig. 14. One end of a

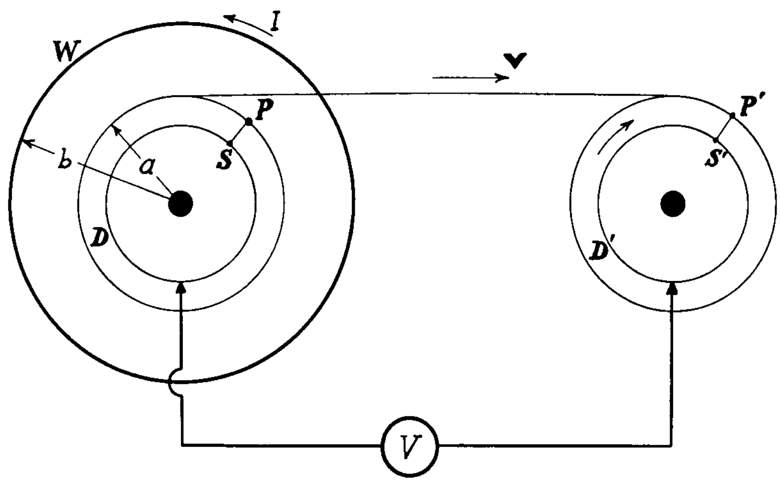

FIG. 14. The Blondel experiment. Conductor is unwrapped from pulley D, thus changing the flux linkages. Is an emf. induced in the circuit?

long flexible wire is soldered to P and is wound in the groove of the pulley. The other end of the wire is soldered to P'. Points P and P' are connected to slip rings S and S', which actually have the same diameter as the pulleys. A magnetic field is produced in D by direct current I in a circular loop W which is in the plane of D.

When D' is turned in the direction of the arrow, thereby unwinding the coil on D, what emf. is indicated by V?

A Classical Solution

The pulley D' is so far from W that the increase in its number of turns produces no emf. The wire that is unwound from D is moving longitudinally through a magnetic field, so no emf. is produced in it. The lower part of the circuit (including V) is stationary and produces no emf. We can therefore concentrate on the coil wound on D.

A fundamental principle of circuit theory (6) is that

$$V = -\frac{d\Lambda}{dt},$$

where $\Lambda = N\Phi$ = flux linkages. In this case, Φ is constant, so the induced emf. is

$$V = -\Phi\frac{dN}{dt} = \frac{\Phi v}{2\pi a}. \qquad (23)$$

But experiment shows that *the emf. is zero.*

Electrodynamic Solution

Equation 18 expresses the force per unit charge at any point within the loop W, caused by rotation about the axis. There are only two possible seats of induced emf.:

(*a*) In the circular loops of wire on D,
(*b*) In the straight wire joining the two pulleys.

According to Section 7, the force on an electron is radial if the motion is a rotation. Thus no emf. is induced in the circular loops. Similarly, one can prove by use of Eq. 2 that the force on electrons in the straight wire is perpendicular to the direction of motion. Thus neither (*a*) nor (*b*) contributes anything and the voltmeter reads zero.

10. CULLWICK EXPERIMENT (7)

Statement

A long, straight wire C carries a direct current I_1 (Fig. 15). Coaxial with the wire is a steel cylindrical ring R with radii a and b and permeability μ. Sliding contacts A and B are connected to a stationary voltmeter V. When the ring moves upward at uniform velocity $\mathbf{v}$, an emf. is induced in the circuit. Will this emf. be different if the steel ring is replaced by a brass ring of the same size?

A Classical Solution

At any radius r, the magnetic field strength is

$$H = \frac{I_1}{2\pi r},$$

and $B_0 = \mu_0 H$ in air, $B = \mu H$ in the steel. When the ring moves upward by distance Δz, the flux linked by the circuit is increased by

$$\Delta\Phi = \Delta z\left[\int_a^b B dr - \int_a^b B_0 dr\right] = \Delta z\frac{(\mu - \mu_0)I_1}{2\pi}\ln(b/a).$$

In accordance with Maxwell's equations, the induced emf. is

$$V = -\frac{d\Phi}{dt} = -\frac{d\Phi}{dz}\frac{dz}{dt}$$

or

$$V = -\frac{I_1 v}{2\pi}(\mu - \mu_0)\ln b/a. \tag{24}$$

According to Maxwell's equations, therefore, the brass ring ($\mu = \mu_0$) gives zero emf. while the steel ring gives an emf. that depends on permeability. Experiment shows (7), however, that *an emf. is induced in both cases and this emf. is independent of permeability.*

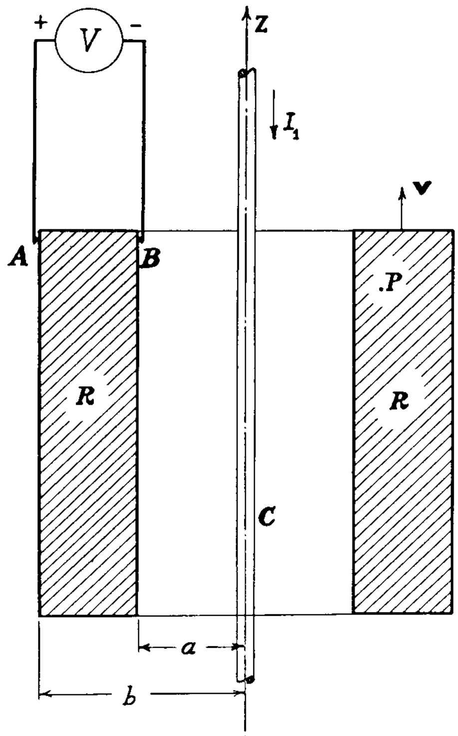

FIG. 15. The Cullwick experiment. Steel tube R moves upward, thus changing flux linkages in voltmeter circuit.

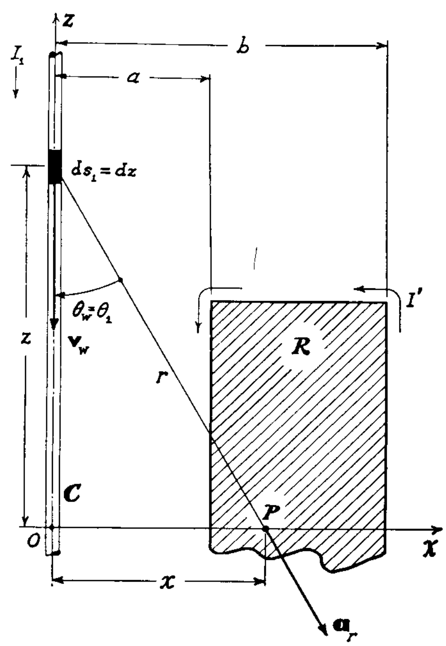

FIG. 16. Analysis of Fig. 15.

Electrodynamic Solution

Consider an arbitrary point P within the material of R and moving with it. The relative velocity $\mathbf{v}_w$ of the wire with respect to R is in the negative z-direction (Fig. 16). From Eq. 2,

$$d\mathfrak{F}_2 = \mathbf{a}_r \frac{|I_1|v_w ds_1}{4\pi\epsilon c^2 r^2}(2\sin^2\theta_1 - \cos^2\theta_1),$$

and the total effect of the vertical wire is

$$\mathfrak{F} = \mathbf{a}_x \frac{|I_1|v_w}{2\pi\epsilon c^2} \int_0^\infty (2\sin^3\theta_1 - \cos^2\theta_1 \sin\theta_1) \frac{dz}{r^2}$$
$$= \mathbf{a}_x \frac{|I_1|v_w}{2\pi\epsilon c^2 x}.$$

Thus the voltage appearing everywhere in the ring between inside and outside surfaces is

$$V = \frac{|I_1|v_w}{2\pi\epsilon c^2} \int_a^b \frac{dx}{x} = \frac{|I_1|v_w}{2\pi\epsilon c^2} \ln(b/a). \tag{25}$$

There are no eddy currents. The field strength $\mathfrak{F}$ is independent of the angle about the axis and independent of z.

With a steel tube, we must also include the effect of induced magnetism. Because of domain orientation in the material, there will be Amperian currents in the xz-plane. The total effect of these currents is represented by a fictitious current sheet I' which is directed upward over the entire outer surface of the ring and downward over the inner surface. The effect of this current is determined by use of Eq. 2. But since P is stationary with respect to the ring, $\mathbf{v}_w = 0$ and the Amperian currents contribute nothing to the force on an electron at P. Thus Eq. 25 expresses the entire induced emf. and shows that this emf. is *independent of the material of the ring.* The polarity is as shown in Fig. 15.

11. LOOP OF ZERO RESISTANCE

Statement

A large number of metallic elements and alloys become superconducting when the temperature is reduced below a critical value. Critical temperatures for a few elements are lead 7.22° K, mercury 4.15°, tin 3.73°, aluminum 1.20°.

A loop L_2 (Fig. 17) is cooled below the critical temperature and thus has zero resistance. The loop L_1 is at room temperature and is excited with alternating current,

$$I(t) = \sqrt{2}\, I_1^* e^{i\omega t}.$$

What is the current in L_2?

Classical Treatment

We apply two of Maxwell's equations:

$$\operatorname{curl} \mathbf{E}^* = -i\omega \mathbf{B}^*, \tag{26}$$

$$\mathbf{E}^* = \mathbf{J}^* \mathfrak{R}, \tag{27}$$

where $\mathbf{J}^*$ is rms. current density and $\mathfrak{R}$ is resistivity. Surely these equations are applicable, since there are no moving contacts and there is no motion between L_1 and L_2. We employ also the universally accepted definition

$$V^* = \oint \mathbf{E}^* \cdot d\mathbf{s}. \tag{28}$$

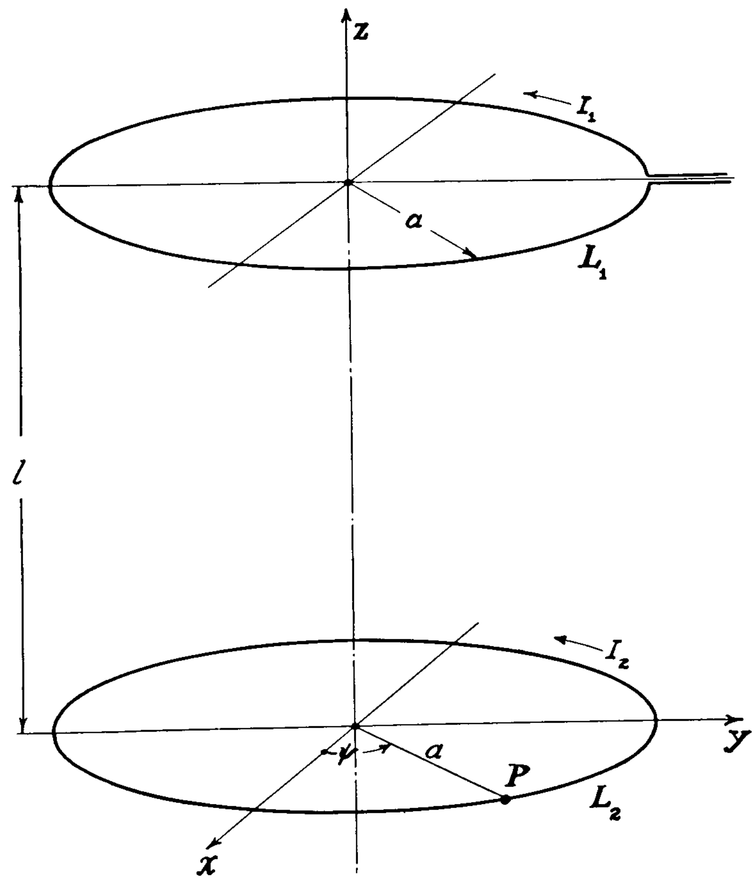

FIG. 17. To find I_2 in a zero-resistance loop L_2, caused by a given alternating current I_1 in L_1.

From Eqs. 26 and 28,

$$V^* = -i\omega\Phi^*, \tag{29}$$

where Φ^* is the total rms. flux linking L_2. From Eqs. 27 and 28,

$$V^* = a\int_0^{2\pi} E^* d\psi = 2\pi a J^* \mathfrak{R}. \tag{30}$$

But $\mathfrak{R} \doteq 0$, so $V^* = 0$ in Eqs. 29 and 30. Therefore $\Phi^* = 0$ and no flux can link loop L_2.

Does this not mean that $I_2^* = 0$? The answer is obvious from circuit theory but is not quite so easy to grasp on a field-theory basis.

Electrodynamic Solution

A current element $I_1\,ds_1$ in the upper conductor of Fig. 17 produces a force on electrons in L_2. This force is obtained from the acceleration term of Eq. 1*a*. Integration around L_1 gives the total force per unit charge at P. According to a previous paper (1, p. 374)

$$\mathfrak{F} = \mathbf{F}/Q_2 = -\mathbf{a}_\psi \omega I_1^* \frac{(4a^2+l^2)^{\frac{1}{2}}}{4\pi\epsilon c^2}\left[\left(\frac{4a^2+2l^2}{4a^2+l^2}\right)K(k) - 2E(k)\right], \quad (31)$$

where K and E are complete elliptic integrals with modulus

$$k = 2a/(4a^2+l^2)^{\frac{1}{2}}.$$

The current in loop L_2 also produces an acceleration force on electrons at P:

$$\mathfrak{F} = -\,\mathbf{a}_\psi \omega I_2^* \frac{a}{2\pi\epsilon c^2}[K(1) - 2E(1)]. \quad (32)$$

Additional forces, such as those caused by the fact that electrons are moving in circular paths rather than straight lines, are negligible as in Section 7. So the only appreciable forces are those given by Eqs. 31 and 32.

Since the resistivity of L_2 is zero, the total $\mathfrak{F}$ must be zero. Equating the sum of Eqs. 31 and 32 to zero, we obtain a definite value of current in the second loop:

$$I_2^* = -\,I_1^* \frac{(4a^2+l^2)^{\frac{1}{2}}}{2a}\left[\frac{\left(\frac{4a^2+2l^2}{4a^2+l^2}\right)K(k) - 2E(k)}{K(1) - 2E(1)}\right]. \quad (33)$$

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New Aircraft Servo Motor.—A new miniature d.c. servo-motor, capable of withstanding a high potential of 1500 volts and of responding to field currents of 0.0075 amp., has been announced by the General Electric Company's Specialty Component Motor Department. Suited for aircraft applications on blowers, actuators, tuners and similar devices, the new motor will operate equally well at sea level or at 50,000-ft. altitude and within temperature limits of −65°F. to 165°F. It is also used as a prime mover in gun direction computer systems, electronic devices, and can be modified for other aircraft and missile applications.

Weighing less than eight ounces, the motor may be wound as a straight shunt machine when required. It is reversed by a small double-pole, double-throw relay and is operated by separate excitation of the armature and differentially wound-shunt fields, with the signal differential between the field structures utilized as its control.

The new G-E d.c. servo motor, designated 5BBY13DJ7, draws maximum armature current of 0.8 amps. from a 28 volt line. It is rated at 0.002 hp. at 6500 rmp., and can be geared to speeds as low as 130 rpm.

Telephone Fire and Police Alarms.—"Fire!" or "Police!"—the cry for help can now be telephoned directly by the public from street corners. A new telephone system, developed at Bell Telephone Laboratories, may eventually become as commonplace as the familiar red fire alarm boxes in which a handle is pulled to send out a telegraphic signal. A number of cities, including Omaha, Neb., Indianapolis, Ind., Miami, Fla., Syracuse, N. Y. and Sioux Falls, S. D., have approved the use of the system. Omaha already has it in operation.

The new system enables firemen or policemen to talk directly with the person placing the alarm. They can immediately determine the exact location of the emergency, its extent and the best type of equipment to dispatch.

Outdoor telephone sets encased in brightly painted cast aluminum housings are mounted at street corners. The spring door can be quickly flung back and the receiver snatched from its hook to summon aid in any emergency.

In a typical installation, when a call is made, a light flashes on a console at alarm headquarters until the call is answered. The light shows the box number and its location. There are no switches or dials; the caller is in direct contact. The console light remains on after the dispatcher answers until released by the operator so he knows where the alarm is coming from even if the person reporting the emergency is too excited to talk.

If the call is for police, the fire department operator immediately transfers the call to the police department switchboard. Calls may be relayed from one switchboard to another.

An automatic voice recorder flips on when the call comes in to take the caller's report. Another recorder can be provided which marks the time and box location.

An optional feature of the system which may be provided is a special bypass switch which enables foot patrolmen to report directly to police headquarters without having calls go through the fire department.

All reporting lines are under continuous electrical test which signals the operator at the console and the telephone company in the event of damage to the circuits.

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