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An Introduction to The Lie–Santilli Isotopic Theory

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Lie's theory in its current formulation is linear, local and canonical. As such, it is not applicable to a growing number of non-linear, non-local and non-canonical systems which have recently emerged in particle physics, superconductivity, astrophysics and other fields. In this paper, which is written by a physicist for mathematicians, we review and develop a generalization of Lie's theory proposed by the Italian-American physicist R. M. Santilli back in 1978 when at the Department of Mathematics of Harvard University and today called *Lie–Santilli isotheory*. The latter theory is based on the so-called *isotopies* which are non-linear, non-local and non-canonical maps of any given linear, local and canonical theory capable of reconstructing linearity, locality and canonicity in certain generalized spaces and fields. The emerging Lie–Santilli isotheory is remarkable because it preserves the abstract axioms of Lie's theory while being applicable to non-linear, non-local and non-canonical systems. After reviewing the foundations of the Lie–Santilli isoalgebras and isogroups, and introducing seemingly novel advances in their interconnections, we show that the Lie–Santilli isotheory provides the invariance of all infinitely possible (well-behaved), non-linear, non-local and non-canonical deformations of conventional Euclidean, Minkowskian or Riemannian invariants. We also show that the non-linear, non-local and non-canonical symmetry transformations of deformed invariants are easily computable from the linear, local and canonical symmetry transforms of the original invariants and the given deformation. We then briefly indicate a number of applications of the isotheory in various fields. Numerous rather fundamental and intriguing, open mathematical and physical problems are indicated during the course of our analysis.

1. Introduction

1.1. *Limitations of Lie's theory*

As it is well-known, *Lie's theory* has permitted outstanding achievements in various disciplines. Nevertheless, in its current conception [30] and realization (see, e.g., [13] for a physical treatment and [15] for a mathematical presentation), Lie's theory is *linear, local-differential and canonical-Hamiltonian.* As such, it possesses clear limitations.

An illustration is provided by the historical distinction introduced by Lagrange [29], Hamilton [14] and others between the *exterior dynamical problems* in vacuum and the *interior dynamical problems* within physical media. Exterior problems consist of particles which can be effectively approximated as being point-like while moving within the homogeneous and isotropic vacuum under action-at-a-distance interactions (such as a space-ship in a stationary orbit around Earth). The point-like character of particles permits the exact validity of conventional local-differential

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topologies (e.g., the Zeeman topology in special relativity); the homogeneity and isotropy of space then allow the exact validity of the geometries underlying Lie's theory (such as the Riemannian geometry); and the action-at-a-distance interactions assures their representation via a potential with consequential canonical character.

Interior problems consists of extended, and therefore deformable particles moving within inhomogeneous and anisotropic physical media, with action-at-a-distance as well as contact-resistive interactions (such as a space-ship during re-entry in Earth's atmosphere). In the latter case the forces are of local-differential type (e.g., potential forces acting on the centre-of-mass of the particle) as well as of non-local-integral type (e.g., requiring an integral over the surface of the body), thus rendering inapplicable conventional local-differential topologies; the inhomogeneity and anisotropy of the medium imply the inapplicability of conventional geometries for their quantitative treatment; while contact–resistive interactions violate Helmholtz's conditions for the existence of a potential (the *conditions of variational selfadjointness* [49]), thus implying the non-canonical character of interior systems.

We can therefore say that Lie's theory in its conventional linear, local and canonical formulation is *exactly valid* for all exterior dynamical problems, while it is *inapplicable* (and not 'violated') for the more general interior dynamical problems on topological, geometrical, analytic and other grounds.

1.2. *The need for a suitable generalization of Lie's theory*

Lie's theory is currently applied to non-linear, non-local and non-canonical systems via their simplification into more treatable forms, e.g., via the expansion of non-local-integral terms into power series in the velocities and then the transformation of the system into a co-ordinate frame in which it admits a Hamiltonian via the Lie–Koening Theorem or, equivalently, via a Darboux map [49].

However, assuming that a given non-local-integral interior system admits a local-differential non-Hamiltonian approximation, the transformations of a non-Hamiltonian system into a Hamiltonian form are necessarily (non-canonical and) non-linear. This implies the known fact that the Darboux chart is not realizable in actual experiment and, if used, it implies the necessary abandonment of conventional relativities, evidently because the transformed frames are highly non-inertial (see [49] for technical details). This establishes the need for a suitable generalization of Lie's theory which is applicable to local-differential non-Hamiltonian systems *in the co-ordinates of their experimental verification.* Only after achieving such a theory the use of co-ordinate transformations may acquire practical significance.

Moreover, non-linear, non-local and non-Hamiltonian interior systems cannot be, in general, consistently reduced or transformed into linear, local and Hamiltonian ones. An illustration exists in gravitation. The distinction between exterior and interior gravitational problems was in full use in the early part of this century (see, e.g., Schwarzschild's *two* papers, the first celebrated paper [74] on the exterior problem and the second little known paper [75] on the interior problem). The distinction was then kept in early well-written treatises in the field (see, e.g., [4, 38]). The distinction was then progressively abandoned up to the current treatment of all gravitational problems, whether interior or exterior, via the same local-differential Riemannian geometry.

The above trend was based on the belief that interior dynamical problems within physical media can be effectively reduced to a collection of exterior problems in vacuum (e.g., the reduction of a space-ship during re-entry in our atmosphere to its elementary constituents moving in vacuum).

It is important for this paper to know that the above reduction is mathematically impossible. For instance, the so-called *No-Reduction Theorems* [54] prohibit the reduction of a macroscopic interior system (such as satellite during re-entry) *with a monotonically decreasing angular momentum*, to a finite collection of elementary particles each one with a *conserved angular momentum*, and *vice versa.*

On geometrical grounds, gravitational collapse and other interior gravitational problems are not composed of ideal points, but instead of a large number of extended and hyperdense particles (such as protons, neutrons and other particles) in conditions of total mutual penetration, as well as of compression in large numbers into small regions of space. This implies the emergence of a structure which is arbitrarily non-linear (in co-ordinates and velocities), non-local-integral (in various quantities) and non-Hamiltonian (variationally non-self-adjoint).

Additional insufficiencies of the current formulation of Lie's theory and of its underlying geometries exist for the characterization of antimatter, e.g., because of the lack of a suitable (e.g., antiautomorphic) map which permits the characterization of antimatter, first, at the classical-astrophysical level, and then at the level of its elementary constituents.

Similar occurrences have recently emerged in astrophysics, superconductivity, theoretical biology and other disciplines. These occurrences establish *the need for a generalization of the conventional Lie theory which is directly applicable* (i.e., applicable without approximation or transformations) to *non-linear, integro-differential and variationally non-self-adjoint systems* for the characterization of *matter.* The theory should then possess a suitable antiautomorphic map for the effective characterization of antimatter.

1.3. *Santilli's isotopies and isodualities of Lie's theory*

In a seminal memoir [47] written in 1978 when at the Department of Mathematics of Harvard University under support from the U.S. Department of Energy, the Italian–American scholar Ruggero Maria Santilli proposed a step-by-step generalization of the conventional Lie theory specifically conceived for non-linear, integro-differential and non-canonical equations. The generalized theory was subsequently studied by Santilli in Refs. [48–72], as well as by a number of mathematicians and theoreticians, and it is today called *Lie–Santilli isotopic theory* or *isotheory* (see papers [1, 2, 6, 11, 12,16–23, 25, 32, 33, 35–37, 40–43], monographs [3, 24, 31, 76] and additional references quoted therein).

A main characteristic of the Lie–Santilli isotheory, which distinguishes it from all other possible generalizations, is its 'isotopic' character intended (from the Greek meaning of the word) as the capability of preserving the original Lie axioms. More specifically, Santilli's isotopies are maps of any given linear, local and canonical structure into its most general possible non-linear, non-local and non-canonical forms which are capable of reconstructing linearity, locality and canonicity in generalized isospaces and isofields within a fixed system of local co-ordinates.

The latter property is remarkable, mathematically and physically, inasmuch as it permits the preservation of the abstract Lie theory and the transition from exterior to interior problems via a more general *realization* of the same theory within the fixed frame of the experimenter.

Another main characteristic of the Lie–Santilli isotheory is that of admitting a novel antiautomorphic map, called *isoduality*, which has resulted to be equivalent to charge conjugation, thus being effective for the first characterization on record of antimatter at the *classical* level with a consequential operator image.

It should be indicated that Santilli [47] submitted his isotopic theory *as a particular case* of a yet more general theory today called *Santilli's Lie-admissible theory* or *Lie–Santilli genotopic theory*, where the term *genotopic* is used (in its Greek meaning) to 'induce configuration', and interpreted in the sense of violating the original Lie axioms, yet *inducing* covering Lie-admissible axioms.

More recently, the Lie–Santilli isotopic and genotopic theories have resulted to be particular cases of yet more general formulations of *hyperstructural* type with a unit [73], thus resulting in a hierarchy of methods of increasing complexity for the representation of physical or biological systems with progressively more complex structures [61].

Finally, Santilli [52, 53, 59, 61] has shown that all preceding theories admit a novel anti-automorphic map he called *isoduality* particularly suited for the characterization of antimatter, which cannot be formulated in conventional mathematics because it requires a generalization of the basic unit.

This paper, written by a theoretical physicist, is devoted to the Lie–Santilli isotheory. A study of the broader Lie–Santilli geno- and hypertheories are contemplated as future works.

In section 2 we outline the methodological foundations of the theory. The istopies of Lie's theory are presented in section 3 jointly with new developments, such as a study of the transition form the Lie–Santilli isogroups to the corresponding isoalgebras. As an illustration of the capabilities of the isotheory, we prove its 'direct universality' in gravitation, that is, the achievement of the symmetries of all possible gravitational metrics (universality), directly in the frame of the experimenter (direct universality). A number of fundamental open mathematical problems will be identified during the course of our analysis.

A comprehensive mathematical presentation of the Lie–Santilli isotheory up to 1992 is available in monograph [76]. A historical perspective is available in monograph [31]. Recent mathematical studies on isomanifolds (today called *Tsagas–Sourlas isomanifolds*) have been conducted in Ref. [77] which also provides a topological complement of the algebraic studies of this paper.

2. Isotopies and isodualities of numbers, fields, differential calculus, metric spaces, differential geometries, functional analysis, classical and quantum mechanics

Lie's theory is the embodiment of the virtual entirety of contemporary mathematics by encompassing: the theory of numbers; differential and exterior calculus; vector and metric spaces; geometry, algebra and topology; functional analysis; and other disciplines. Santilli's isotopies of Lie's theory require the isotopic lifting of *all* these mathematical methods. In this section we shall identify the basic isotopies and isodualities which are necessary for a correct formulation of the Lie–Santilli isotheory, by referring to the quoted literature for more detailed treatments.

2.1. *Isotopies and isodualities of the unit*

Santilli's fundamental step from which all generalized formulations can be uniquely derived is the generalization of the unit I of the current formulation of Lie's theory into a quantity $\hat{I}$ of the same dimension of I, but with unrestricted functional dependence of its elements in the local co-ordinates x, their derivatives of arbitrary order with respect to an independent variable $t, \dot{x}, \ddot{x}, \ldots$ as well as any needed additional quantity [47, 49b, 61a],

$$I \to \hat{I} = \hat{I}(x, \dot{x}, \ddot{x}, \ldots). \tag{2.1}$$

The *isotopies* [47] occur when $\hat{I}$ preserves all the topological characteristics of I, such as nowhere-degeneracy, Hermiticity and positive-definiteness. The *genotopies* [47] occurs when $\hat{I}$ is non-Hermitean (e.g., real-valued but non-symmetric), while the *hyperstructures* [73] occur when $\hat{I}$ is a finite or infinite (and ordered or non-ordered) *set* of generally non-Hermitean quantities.

In conventional Lie's theory, the systems are identified via the sole knowledge of the Hamiltonian H. In the Lie–Santilli isotheory, the identification of the systems requires the knowledge of *two* generally different quantities, the Hamiltonian H and the generalized unit $\hat{I}$.

Isotopic methods have resulted to be effective for the direct representation of closed-isolated systems of particles with conventional interactions represented with H plus internal non-local and non-Hamiltonian interactions represented with $\hat{I}$ and time-reversal invariant centre-of-mass trajectories (from the property $\hat{I} = \hat{I}^\dagger$). The genotopic methods apply for the direct representation of open-non-conservative, non-local and non-Hamiltonian systems in irreversible conditions (from the property $\hat{I} \neq \hat{I}^\dagger$). The hyperstructural methods are significant for quantitative representations of the more complex biological systems.

Once the unit is generalized, there is the natural emergence of the map [52, 53, 59],

$$\hat{I} \to \hat{I}^{\mathrm{d}} = -\hat{I}, \tag{2.2}$$

called *isoduality* which provides an antiautomorphic image of all formulations based on $\hat{I}$. When properly formulated within the context of Hilbert spaces, the above map has resulted to be equivalent to charge conjugation [61b], thus permitting a representation of systems of antiparticles beginning at the classical level, the first known to this author, which then persists under quantization.

The above liftings were preliminarily classified by this author [22] into:

Class I. (generalized units that are smooth, bounded, non-degenerate, Hermitean and positive definite, characterizing the isotopies properly speaking);

Class II. (the same as Class I although $\hat{I}$ is negative-definite, characterizing isodualities);

Class III. (the union of classes I and II);

Class IV. (Class III plus singular isounits); and

Class V. (Class IV plus unrestricted generalized units, e.g., realized via discontinuous functions, distributions, lattices, etc.).

All isotopic structures identified below also admit the same classification which will be omitted for brevity. In this paper we shall generally study isotopies of Classes I and II, at times treated in a unified way via those of Class III whenever no ambiguity arises. Santilli's isotopies of Classes IV and V are vastly unexplored at this writing.

2.2. *Isotopies and isodualities of fields*

Lie's theory is constructed over ordinary fields $F(a, +, \times)$ hereon assumed to be of characteristic zero (the fields of real $\Re$ complex C and quaternionic numbers Q) with generic elements a, addition $a_1 + a_2$, multiplication $a_1 a_2 := a_1 \times a_2$, additive unit 0, $a + 0 = 0 + a \equiv a$, and multiplicative unit I, $a \times I = I \times a \equiv a, \forall a, a_1, a_2 \in F$.

The Lie–Santilli isotheory is based on a generalization of the very notion of numbers and, consequently of fields first introduced by Santilli at the *Conference on Differential Geometric Methods in Mathematical Physics* held in Clausthal, Germany, in 1980. A first rudimentary treatment appeared in Santilli's joint paper with the (mathematician) H.C. Myung [39] of 1982. Comprehensive studies were then conducted by Santilli in the following years (see paper [59] for a mathematical presentation and monographs [61] for extensive physical applications).

Consider a Class I lifting of the unit I of F, $I \to \hat{I}$, with $\hat{I}$ being *outside* the original set, $\hat{I} \notin F$. In order for $\hat{I}$ to be the left and right unit of the new theory, it is necessary to lift the conventional associative multiplication *ab into*, the so-called *isomultiplication* [47]

$$ab := a \times b \Rightarrow a * b := a \times \hat{T} \times a = a\hat{T}b, \quad \hat{T} = \text{fixed}, \tag{2.3}$$

where the quantity $\hat{T}$ is called the *isotopic element*. Whenever $\hat{I} = \hat{T}^{-1}$, $\hat{I}$ is the correct left and right unit of the theory, $\hat{I} * a = a * \hat{I} = a, \forall a \in F$, in which case (only) $\hat{I}$ is called the *isounit*. In turn, the liftings $I \to \hat{I}$ and $\times \to *$, imply the generalization of fields into the Class I structures

$$\hat{F}_{\mathrm{I}} = \{(\hat{a}, +, *) \,|\, \hat{a} = a \times \hat{I};\ a = n, c, q \in F; \times \to * = \times \hat{T} \times ; \hat{I} = \hat{T}^{-1}\}, \tag{2.4}$$

called *isofields*, with elements $\hat{a} \in \hat{F}$ called *isonumbers* [59]. It is instructive to verify that the above isofields satisfy all conventional axioms of ordinary fields as necessary for the lifting $F \to \hat{F}_{\mathrm{I}}$ to be an isotopy (see [59] for details).

All conventional operations among numbers are evidently generalized in the transition from numbers to isonumbers. In fact, we have:

$$a + b \to \hat{a} + \hat{b} = (a + b)\hat{I}, \quad a \times b \to \hat{a} * \hat{b} = \hat{a} \times \hat{T} \times \hat{b} = (a \times b) \times \hat{I} = (ab)\hat{I},$$

$$a^{-1} \to \hat{a}^{-\hat{I}} = a^{-1} \times \hat{I}, \quad a/b = c \to \hat{a}\,\hat{/}\,\hat{b} = \hat{c}, \quad \hat{c} = c \times \hat{I} = c\hat{I},$$

$$a^{1/2} \to \hat{a}^{\hat{\frac{1}{2}}} = a^{1/2}\hat{I}^{1/2},$$

etc. Thus, conventional squares $a^2 = a \times a = aa$ have no meaning under isotopy and must be lifted into the *isosquare* $\hat{a}^{\hat{2}} = \hat{a} * \hat{a} = a^2\hat{I}$. The *isonorm* is

$$\hat{\uparrow}\hat{a}\hat{\uparrow} = (\bar{a} \times a)^{1/2} \times \hat{I} = |a| \times \hat{I} = |a| \times \hat{I} \in \hat{F}, \tag{2.5}$$

where $\bar{a}$ denotes the conventional conjugation in F and $|a|$ is the conventional norm. Note that the *isonorm is positive-definite* (for isofields of Class I), as a necessary condition for isotopies.

The isotopic character of the lifting $1 \to \hat{I}$ is confirmed by the fact that the isounit $\hat{I}$ verifies all axioms of 1,

$$\hat{I} * \hat{I} * \cdots * \hat{I} \equiv \hat{I}, \quad \hat{I}\,\hat{/}\,\hat{I}, \equiv \hat{I}, \quad \hat{I}^{\hat{\frac{1}{2}}} \equiv \hat{I}, \text{ etc.}$$

The *isodual isofields* [59] are the antihomomorphic image of $\hat{F}(\hat{a}, +, *)$ induced by the map $\hat{I} \rightarrow \hat{I}^{d} = -\hat{I}$ and are given by the Class II structures

$$\hat{F}^{d}_{II} = \{(\hat{a}^{d}, +, *^{d}) \mid \hat{a}^{d} = \bar{a} \times \hat{I}^{d};\ a = n, c, q \in F; * \rightarrow *^{d} = \times \hat{T}^{d} \times \hat{T}^{d} = -\hat{T}, \hat{I}^{d} = -\hat{I}\}, \tag{2.6}$$

in which the elements $\hat{a}^{d} = \bar{a} \times \hat{I}^{d}$ are called *isodual isonumbers.* For real numbers we have $n^{d} = -n$, for complex numbers we have $c^{d} = -\bar{c}$, where $\bar{c}$ is the ordinary complex conjugate, and for quaternions in matrix representation we have $q^{d} = -q^{\dagger}$, where $\dagger$ is the Hermitean conjugate. Note that the conjugation of a complex number is $(n + i \times m)^{d} = n^{d} + i^{d} \times m^{d} = -n + (-\bar{i})(-\times)(-m) = -n + im$. The *isodual isosum* is given by $\hat{a}^{d} + \hat{b}^{d} = (\bar{a} + \bar{b}) \times \hat{I}^{d}$, while the *isodual isomultiplication* is given by [59]

$$\hat{a}^{d} *^{d} \hat{b}^{d} = \hat{a}^{d} \times \hat{T}^{d} \times \hat{b}^{d} = -\hat{a}^{d} \times \hat{T} \times \hat{b}^{d} = (\bar{a} \times \bar{b}) \times \hat{I}^{d}.$$

An important property is that the *norm of isodual isofields is negative-definite* because it is characterized by [59]

$$\lceil \hat{a}^{d} \rceil^{d} = |\bar{a}| \times \hat{I}^{d} = -\lceil \hat{a} \rceil. \tag{2.7}$$

The latter property has non-trivial implications. For instance, it implies that *physical quantities defined on an isodual isofield, such as time, energy, etc., are negative-definite.* For these reasons isodual theories provide a novel and intriguing characterization of antimatter [61].

Note also that, *as a necessary condition for isotopies (isodualities) all isofields* $\hat{F}_{I}(\hat{a}, +, *)$ *(isodual isofields* $\hat{F}^{d}_{II}(\hat{a}^{d}, +, *^{d})$*) are isomorphic (antiisomorphic) to the original field* $F(a, +, *)$. The reader should be aware that the distinction between real, complex and quaternionic numbers is lost under isotopies because all possible numbers are unified by the isoreals owing to the freedom in the generalized unit [26].

Recall that the set of imaginary numbers *does not* constitute a field, evidently because not closed under the multiplication. On the contrary, Santilli's isofields $\hat{F}(\hat{n}, +, *)$ with $\hat{n} = n \times i$, isounit $\hat{I} = i$ and n real do indeed verify the axioms for a field as one can readily verify. Note that the imaginary unit i is *isoselfdual*, i.e., invariant under isoduality, $i^{d} = -\bar{i} \equiv i$.

We also recall [59] that the lifting $a \rightarrow \hat{a} = a \times \hat{I}$ is *necessary* for $\hat{F}_{I}(\hat{a}, +, *)$ to preserve the axioms of $F(a, +, *)$ whenever the isounit $\hat{I}$ is not an element of the original field. On the contrary, when $\hat{I} \in F(a, +, *)$ there is no need to lift the numbers and we shall write $\hat{F}_{I}(a, +, *)$. In physical applications, the isounit is generally *outside* the original field and actually possesses a non-linear as well as integral dependence on the local variables and their derivatives. This implies that the 'numbers' used in the Lie–Santilli isotheory generally have an *integral* structure.

As an example, the isounit used by Animalu [1] for the representation of the Cooper pair in superconductivity is given by

$$\hat{I} = I\, e^{tN \int d^{3}x \psi_{\downarrow}^{\dagger}(r) \phi_{\uparrow}(r)}, \tag{2.8}$$

where t represents time, N is a constant, and $\psi_{\downarrow}$ and $\phi_{\uparrow}$ are the wave functions of the two electrons of the Cooper pair in singlet coupling of their spin. *Animalu's isounit* (2.8) therefore represents the *non-local-integral* contributions due to the wave overlapping of the two electrons in the Cooper pairs. Such contributions, since they are of contact type, are variationally non-self-adjoint and, therefore, they should be represented with anything possible, *except* the Hamiltonian. In Santilli's isotopies there are therefore represented with the isounit.

In particular, Animalu has shown that the lifting of the conventional Coulomb interactions characterized by isounit (2.8) produces an *attraction* among the *identical* electrons of the Cooper pair, as experimentally established in superconductivity. Note that when the overlapping of the wavepackets is no longer appreciable (e.g., at large mutual distances), the integral in the exponent of (2.8) is null and the isounit $\hat{I}$ recovers the conventional unit I. Conventional fields $F(a, +, \times)$ are used for large distances among the electrons, while isofields $\hat{F}(\hat{a}, +, *)$ with isounit (2.8) are used when the wave-overlapping of the electrons is appreciable. Other examples of isounits will be provided later on.

We also recall Santilli's [59] more general *genofields*, which are characterized first by an isotopy of conventional fields, and then by an ordering of the isomultiplications, one to the right $\hat{a} > \hat{b} = \hat{a} \times \hat{R} \times \hat{b}$ and one to the left $\hat{a} < \hat{b} = \hat{a} \times \hat{S} \times \hat{b}$, $\hat{R} \neq \hat{S}$ which are different among themselves, yet they are commutative when the original field is commutative, $\hat{a} > \hat{b} = \hat{b} > \hat{a}$, $\hat{a} > \hat{b} = \hat{b} > \hat{a}$, $\hat{a} > \hat{b} \neq \hat{a} < \hat{b}$. In this case we have a *genounit to the right*, $\hat{I}^{>} = \hat{R}^{-1}$, and a *genounit to the left*, ${}^{<}\hat{I} = \hat{S}^{-1}$ which are usually interconnected via a conjugation, e.g., $\hat{I}^{>} = (\hat{I}^{>})^{\dagger}$. The important property is that all abstract axioms of a field are verified per *each* ordered isomultiplication, thus yielding a *genofield to the right* $\hat{F}^{>}(\hat{a}^{>}, +, >)$ and a *genofield to the left* ${}^{<}\hat{F}({}^{<}\hat{a}, +, <)$ which are at the foundation of Santilli's Lie-admissible theory [61]. The *hyperfields to the right and to the left* emerge when $\hat{R}$ and $\hat{S}$ are *sets* of generally non-Hermitean quantities [73].

We finally recall Santilli's [59] still more general liftings characterized by the generalization of the sum $+$ and related additive unit 0, e.g., $+ \rightarrow \hat{+} = + \hat{K} +$, $0 = \hat{K} \neq 0$, $K \in F(a \hat{+} b = a + \hat{K} + b)$ called *pseudo-isotopies*, which *do not* preserve the axioms of a field (in fact, closure under the distributive law is not verified under the conventional $\times$ or isotopic $*$ multiplication and the addition $\hat{+}$). Thus, *pseudo-isofields are not fields.* For these and other reasons (e.g., the general divergence of the exponentiation), applications in physics and biology are restricted to iso-, geno- and hyper-fields, while the pseudoiso- and pseudogeno- and pseudohyper-fields have a mere mathematical interest at this writing.

The care needed in inspecting and appraising the Lie–Santilli isotheory can be pointed out from these introductory lines. In fact, familiar statements such as 'two multiplied by two equals four' are correct for the conventional Lie theory, but they have no mathematical meaning for the Lie–Santilli isotheory because they lack the identification of the assumed unit and multiplication. In fact, for $\hat{I} = 3$, $\hat{2} * \hat{2} = 12$. Similarly, care must be expressed before claiming that a number is *prime* or not. In fact, Santilli [59] has shown that non-prime numbers can become prime under a proper selection of the unit.

Our current knowledge of *Santilli's theory of isonumbers* includes the lifting of all conventional numbers (real, complex and quaternionic numbers, plus the isotopies of octonions [59]) into the following four classes used in this paper: (A) *ordinary numbers* with unit 1; (B) *isonumbers* with isounits of Class I, $\hat{I} > 0$; (C) *isodual numbers* with isodual unit $\hat{I}^{d} = -1$; (D) *isodual isonumbers* with isodual isounits of Class II, $\hat{I}^{d} < 0$. In this paper we shall therefore have *four* different types of real numbers, complex numbers and quaternions, at times unified in the isonumbers of Class III, excluding generalizations of Classes IV and V.

Despite the above advances, studies on the isonumber theory are their initiation and so much remains to be done. To begin, the entire conventional number theory (including all familiar theorems on factorization, primes, etc.) can be subjected to an isotopies of Classes I, II or III. Moreover, we have the birth of new numbers without counterpart in the current number theory, such as the isonumbers of Class IV (with singular isounits) and Class V (with distributions or discontinuous functions as isounits). The above liftings then admit antiautomorphic images under isoduality which are also absent in the conventional number theory. In turn, all the preceding generalizations can be subjected to further enlargements via the differentiation of the multiplications to the right and to the left, and then yet more general formulations via the multivalued hyperstructures.

One can begin to understand the vastity of the Lie–Santilli isotheory as compared to the conventional formulation of Lie's theory by nothing that the above hierarchy of fields implies a corresponding hierarchy of Lie-isotopic theories.

2.3. *Isotopies and isodualities of the differential calculus.*

The next important mathematical discovery by Santilli is an axiom-preserving integro-differential generalization of the conventional local-differential calculus called *isodifferential calculus*, first presented in a systematic way in the recent papers [71] although it is implicit in preceding works (e.g., [58, 61]).

Consider a set of functions $f(x)$, $g(x)$, ..., on an N-dimensional space $S(x, \mathfrak{R})$ with local chart $x = \{x^k\}, k = 1, 2, \ldots, N$, over the reals $\mathfrak{R}(n, +, \times)$. Let dx^k and $\partial_k = \partial/\partial x^k$ be the conventional differential and derivative on S, respectively.

Consider now the set of functions $f(x)$, $g(x)$, ..., this time, on an N-dimensional isospace $\hat{S}(x, \hat{\mathfrak{R}}), x = \{x^k\}, k = 1, 2, \ldots, N$, defined over the Class I isofield $\hat{\mathfrak{R}}(\hat{n}, +, *)$ (where we shall drop hereon the subscript I for simplicity), with N-diensional positive-definite isounit $\hat{I} = \hat{I}^\dagger = (\hat{I}^j_i) = (\hat{I}^i_j) = \hat{T}^{-1} = (T^i_j)^{-1} = (T^j_i)^{-1} > 0$ whose elements possess a generally non-linear-integral dependence on all local quantities and their derivatives with respect to an independent variable t, $\hat{I}(x, \dot{x}, \ddot{x}, \ldots)$. *Santilli's isodifferential calculus* is characterized by the *isodifferential*

$$\hat{d}x^k = \hat{I}^k_i \, dx^i, \tag{2.9}$$

with corresponding *isoderivative*

$$\hat{\partial}_k = \frac{\hat{\partial}}{\hat{\partial}x^k} = \hat{T}^i_k \frac{\partial_i}{\partial x^i} = \hat{T}^i_k \partial_1 . \tag{2.10}$$

under the condition that all conventional operations and properties of the ordinary differential calculus are lifted into their axiom-preserving isotopic form, e.g.,

$$\hat{d}f(x) = \hat{\partial}_k f \times \hat{d}x^k = \hat{T}^i_k \partial_i f \hat{I}^k_j dx^j, \quad \hat{d}^2 x^k = \hat{d} \times \hat{d}x^k = \hat{I}^k_i \hat{I}^k_j dx^i dx^j,$$

$$\hat{\partial}^2_k = \hat{\partial}_k \times \hat{\partial}_k = \hat{T}^i_k \hat{T}^j_k \partial_i \partial_j, \text{ etc.},$$

where there is no sum over the repeated k-index.

A hidden condition is that, starting with a set of functions over an isofield $\hat{\mathfrak{R}}(\hat{n}, +, *)$ with isounit $\hat{I}$, the operations of isodifferentiation and isoderivatives must preserve the original unit for consistency. This condition remains generally unidentified in the conventional calculus because the preservation of the unit follows from its constancy, $\partial_k I = 0$. For the case of a generalized unit with the same functional dependence as that of the functions, the condition of preservation of the unit must be added to the calculus to prevent the transition from the original set of functions defined with respect to $\hat{I}$ to a new set of functions defined over a new unit $\hat{I}'$.

As an example, the definition of the isodifferential

$$\hat{\mathrm{d}}x^k = \mathrm{d}(\hat{I}^k_i x^i) = (\mathrm{d}\,\hat{I}^k_i)\,x^i + \hat{I}^k_i\,\mathrm{d}x^i = \hat{I}'^k_i\,\mathrm{d}x^i, \quad \hat{I}'^k_i = (\partial_1 \hat{I}^k_m)\,x^m + \hat{I}^k_i$$

would imply the alteration of the isounit $\hat{I}$. In turn, the occurrence would have serious drawbacks in applications, such as lack of invariance of perturbative series.

Santilli's isodifferential calculus does verify the condition of preserving the basic isounit, although the question whether realizations (2.9) and (2.10) are unique has not been explored until now. Note also the mutual compatibility of isoforms (2.9) and (2.10).

The lifting of the integral calculus follows quite simply from the above isodifferential forms. We here limit ourselves to indicate that an *indefinite isointegral* defined as the operation inverse of the isodifferential is given by

$$\hat{\int}\hat{\mathrm{d}}x = \int \hat{T}\hat{I}\,\mathrm{d}x = \int \mathrm{d}x = x, \quad \text{i.e.,} \quad \hat{\int} = \int \hat{T}. \tag{2.11}$$

Note that the isodifferential calculus is one of the simplest possible forms of *integro-differential calculus*, in the sense that each operation has a differential contribution characterized by d or ∂ and an integral component characterized by T or $\hat{I}$, respectively.

Despite its simplicity, the isodifferential calculus has far reaching mathematical and physical implications. Mathematically, it permits a step-by-step generalization of conventional local-differential geometries into covering integro-differential geometries. Physically, the isocalculus permits a generalization of classical and quantum mechanics as well as of their interconnecting map (quantization), as outlined below.

The *isodual isodifferential calculus* is the antiautomorphic image of the preceding one characterized by the isodual isotopic element $\hat{T}^{\mathrm{d}} = -\hat{T} < 0$ or isodual isounit $\hat{I}^{\mathrm{d}} = -\hat{I} < 0$ and it is defined on the *isodual isospace* $\hat{S}^{\mathrm{d}}(\hat{x}, \hat{\mathfrak{R}}^{\mathrm{d}})$ defined over the Class II isodual isofield $\hat{\mathfrak{R}}^{\mathrm{d}}$ (where, again, the subscript II has been dropped for simplicity). Note that the isodifferential calculus and its isodual can be unified into that of Class III.

The *genodifferential calculus* [61] occurs when the Hermiticity condition on the isounit is relaxed, $\hat{I} \neq \hat{I}^\dagger$. As such, the operation of differentiation itself acquires a structural ordering, namely, we have two different genoderivatives $\hat{\partial}^> f(x)$ and $f(x)\,{}^<\hat{\partial}$ defined for the corresponding units $\hat{I}^> = \hat{I}$ and ${}^<\hat{I} = \hat{I}^\dagger$ which are naturally set to represents the 'arrow of time'. This indicates that the genodifferential calculus is significant to represent *irreversible processes*. The *hyperdifferential calculus* has not been explored at this writing, to our best knowledge.

2.4. *Isospaces, isogeometries and their isoduals*

Santilli's third important discovery presented for the first time in paper [51] of 1983 (see also the recent paper [71] and the comprehensive treatment [61]) is the isotopic lifting of conventional, N-dimensional, metric (or pseudo-metric) spaces and related geometries. Consider a metric space $S(x, g, \mathfrak{R})$ with local co-ordinates $x = \{x^k\}$ and (nowhere singular, real valued and symmetric) metric $g = (g_{ij})$ over the reals $\mathfrak{R}(n, +, \times)$. Its infinite class of isotopic images over the isoreals, called *isospaces* (hereon assumed for simplicity to be of Class I) is given by structures [51]

$$\hat{S}(\hat{x}, \hat{g}, \hat{\mathfrak{R}}): \hat{g} = \hat{T} g,\ \hat{I} = \hat{T}^{-1},\ \hat{x}^{\hat{2}} = (\hat{x}^t \hat{g} \hat{x}) \hat{I} \in \mathfrak{R}(\hat{n}, +, *),$$

$$\hat{x} = \{\hat{x}^k\} \equiv \{x^k\}, \hat{x}_k = \{\hat{g}_{ki} \hat{x}^i\} \neq x_k, \tag{2.12}$$

where $\hat{g} = Tg$ is called the *isometric.* The above liftings are necessary for compatibility with the isotopies of the unit $I \to \hat{I}$, of the product $\times \to *$ and of the field $\mathfrak{R} \to \hat{\mathfrak{R}}$. From hereon we shall adopt Santilli's convention [71] of using symbols with a 'hat' to represent quantities computed in isospaces while ordinary symbols represent quantities computed in the original spaces.

Despite their simplicity, isospaces have far reaching implications. In fact, they imply that *the same abstract axioms of conventional spaces (such as the Euclidean, Minkowskian or Riemannian spaces) admit unrestricted functional dependence of the metric.* As an illustration, the conventional metric $g(x)$ of a Riemannian space $R(x, g, \mathfrak{R})$ is believed to be restricted to the sole dependence on the local co-ordinates x. Santilli has shown that the same Riemannian axioms permit an unrestricted functional dependence of the metric $\hat{g}(x, \dot{x}, \ddot{x}, \ldots)$. While Riemannian spaces $R(x, \hat{g}, \mathfrak{R})$ are ideally suited for *exterior* gravitational problems, the *Riemann–Santilli isospaces* $\hat{\mathfrak{R}}(x, \hat{g}, \hat{\mathfrak{R}})$ are ideally suited for the treatment of *interior* gravitational problems with a nonlinearity in the velocities, integral structure and variationally non-self-adjoint character (section 1).

This remarkable result is due to the construction of the isospaces via the deformation of the metric $g \to \hat{g} = Tg$ while jointly lifting the original unit in the *inverse* amount, $I \to \hat{I} = \hat{T}^{-1}$, under which *isospaces* $\hat{S}(\hat{x}, \hat{g}, \hat{\mathfrak{R}})$ (*isodual isospaces* $\hat{S}^{\mathrm{d}}(\hat{x}, \hat{g}^{\mathrm{d}}, \hat{\mathfrak{R}}^{\mathrm{d}})$) *are locally isomorphic (antiisomorphic) to the original spaces* $S(x, g, \mathfrak{R})$.

Additional salient properties of isospaces are the *preservation of the original dimensionality and of the original basis (except for renormalization factors)* [61a].

Via the use of the isotopies of fields, differential calculus and vector spaces, Santilli's has constructed step-by-step, non-local-integral isotopies and isodualities of conventional geometries on metric (or pseudo-metric) spaces. Their most salient application is the geometrization of interior physical media, that is, the geometrization of the departures from empty space caused by matter.

The isogeometries most important for physical applications are (see [61a, b] for details):

(A) *Santilli's isoeuclidean geometry* of Class I on three-dimensional isospaces $\hat{E}(\hat{x}, \hat{\delta}, \hat{\mathfrak{R}})$, $\hat{\delta} = \hat{T}\delta = (\hat{T}_i^k \delta_{kj})$, $\delta = (\delta_{ij}) = \text{diag.}\ (1, 1, 1)$, over the isoreals $\mathfrak{R}(\hat{n}, +, *)$ with a 3×3-dimensional isounit which, being positive-definite, can always be diagonalized into the form

$$\hat{I} = \text{diag.}\ (b_1^{-2}, b_2^{-2}, b_3^{-2}) > 0, \quad b_k = b_k(x, \dot{x}, \ddot{x}, \ldots) > 0, \quad k = 1, 2, 3. \tag{2.14}$$

In this case the isometric $\hat{\delta}$ has an arbitrary functional dependence on local co-ordinates and their derivatives, $\hat{\delta}(x, \dot{x}, \ddot{x}, \ldots)$. Yet the geometry is *isoflat*, that is, it verifies the axioms of flatness in isospace while its projection in the original space $E(x, \delta, \mathfrak{R})$ is evidently curved. An intriguing novel notion of the isoeuclidean geometry is the *isosphere of Class I* $\hat{x}^{\hat{2}} = (\hat{x}^t \hat{\delta} \hat{x})\hat{I} = \hat{I}$ which is a perfect sphere in isoeuclidean space. Nevertheless, its projection in the original Euclidean space is given by all infinitely possible ellipsoids $xb_1^2 x + y_2 b_y^2 + zb_3^2 z = 1$ (where, according to the convention assumed earlier, $\hat{x}$ is computed in $\hat{E}$ and x in E). In fact, in isospace the original sphere with radius 1 is subjected to the deformations of its axes $1_k \rightarrow b_k^2$ while the corresponding units are deformed in the *inverse* amounts, $1_k \rightarrow b_k^{-2}$, thus preserving the perfectly spherical character. The *isosphere of Class III* unifies all compact and noncompact curves $\pm xb_1^2 x \pm y_2 b^2 y \pm zb_3^2 z \neq 0$ in isospace. The *isosphere of Class IV* unifies all compact and non-compact surfaces plus all cones $\pm xb_1^2 x \pm y_2 b^2 y \pm zb_3^2 z = 0$. The *isosphere of Class V* is an additional novel notion of a sphere with arbitrary unit (e.g., a lattice).

(B) *Santilli's isominkowskian geometry of Class I* on isospace $\hat{M}(x, \hat{\eta}, \hat{\Re})$, $\hat{\eta} = \hat{T}\eta$, $\eta = \text{diag.}(1, 1, 1, -1)$ over the isoreals with 4×4-dimensional isounit reducible to the diagonal form

$$\hat{I} = \text{diag.}(b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^{-2}) > 0, \quad b_\mu = b_\mu(x, \dot{x}, \ddot{x}, \dots) > 0, \quad \mu = 1, 2, 3, 4, \tag{2.15}$$

which represents *locally varying speeds of light* $c = c_0 b_4 = c_0/n_4$ where c_0 is the speed of light in vacuum and n_4 is the local index of refraction. As such, the isominkowskian geometry is particularly suited for the representation of light propagating within inhomogeneous and anisotropic physical media such as our atmosphere. An important notion of the isominkowskian geometry is the *isolight cone of Class I* [which is a perfect cone in isominkowski space but, when projected in the conventional Minkowski space, represents all infinitely possible deformed light cones $xn_1^{-2} x + y_2 n^{-2} y + zn_3^{-2} z - tc_0 n_4^{-2} t = 0$. In fact, each axis of the original light cone is deformed $1_\mu \rightarrow n_\mu^{-2}$, while the corresponding units are deformed of the inverse amount, $1_\mu \rightarrow n_\mu^2$, thus preserving the original perfect cone. The axiom-preserving character of the isotopies is such that the maximal causal speeds of the Minkowski and isominkowski spaces coincide and are given by the speed of light in vacuum c_0.

(C) *Santilli's isoriemannian geometry* on isospaces $\hat{R}(\hat{x}, \hat{g}, \hat{\Re})$, $\hat{g} = \hat{T}g$ over isounit (2.14), which coincides with the conventional geometry at the abstract level. This implies that, unlike the isoeuclidean and isominkowskian geometries, the isoriemannian geometry is *isocurved*, that is, curved in isospace. As such, it permits the representation of interior gravitational problems with locally varying speeds of light, such as the bending of light within a physical medium with local speed $c = c_0/n_4 < c_0$, the contribution to cosmological redshift due to the decrease of the speed of light within astrophysical chromospheres, and other novel insights. An intriguing novel notion is that of *isogeodesics of Class I* which coincide in isospace with the geodesics in vacuum, but when projected in the original Riemannian space represents the actual non-geodesic trajectory of extended particles within physical media, such as that of a leaf in free fall in our atmosphere.

An isogeometry particularly important for the study of the Lie–Santilli theory is the *isosymplectic geometry* first presented in Ref. [57] (see also the more recent study [71]). Consider the conventional symplectic geometry (see, e.g. [34]) in canonical realization on the cotangent bundle $T^*E(x, \delta, \Re)$, $\delta = \text{diag.}(1, 1, 1)$, with local chart $a = (a^\mu) = \{x^k, p_k, k = 1, 2, 3, \mu = 1, 2, 3, 4, 5, 6$. As well-known, the above geometry is characterized by the canonical one-form

$$\theta = p_k \, \mathrm{d}x^k, \tag{2.16}$$

with nowhere-degenerate, canonical, symplectic two-form

$$\omega = dp_k \wedge dx^k = \tfrac{1}{2}\omega_{\mu\nu}\mathrm{d}a^\mu \wedge \mathrm{d}a^\nu, \quad (\omega_{\mu\nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.17}$$

which is exact, $\omega = \mathrm{d}\theta$, and therefore closed, $\mathrm{d}\omega = \mathrm{d}(\mathrm{d}\theta) \equiv 0$ (Poincaré Lemma [34]). Corresponding higher-order forms are constructed accordingly.

Santilli's isosymplectic geometry [71] is defined on the *isocotangent bundle* $T^*\hat{E}(\hat{x}, \hat{\delta}, \hat{\Re})$ with the local chart $\hat{a} = \{\hat{a}^\mu\} = \{\hat{x}^\mu, \hat{p}_k\}$, $\hat{x}^k \equiv x^k, \hat{p}_k \equiv p_k$, but now referred to the six-dimensional isounit given by the Kronecker product $\hat{I}_2 = \hat{I} \times \hat{T}$, resulting in the isodifferential forms $\hat{\mathrm{d}}\hat{x}^k = \hat{I}^k_i\,\mathrm{d}x^i$, $\hat{\mathrm{d}}\hat{p}_k = \hat{T}^i_k\,\mathrm{d}p_i$, $\hat{\partial}/\hat{\partial}\hat{x}^k = \hat{T}^i_k\,\partial/\partial x^i$, $\hat{\partial}/\hat{\partial}\hat{p}_k = \hat{I}^k_i\,\partial/\partial \hat{p}_i$, etc. We then have the *one isoform*

$$\hat{\theta} = \hat{p}_k\,\hat{\mathrm{d}}\hat{x}^k = p_k \hat{I}^k_i(x, p, \ldots)\,\mathrm{d}x^i, \tag{2.18}$$

and the *two-isoform*

$$\hat{\omega} = \hat{\mathrm{d}}\hat{p}_k \wedge \hat{\mathrm{d}}\hat{x}^k = \tfrac{1}{2}\,\omega_{\mu\nu}\hat{\mathrm{d}}\hat{a}^\mu \wedge \hat{\mathrm{d}}\hat{a}^\nu = \hat{T}^m_k(x, p, \ldots)\,\mathrm{d}p_m \wedge \hat{I}^k_n(x, p, \ldots)\,\mathrm{d}x^n \equiv \omega, \tag{2.19}$$

which is also nowhere degenerate as well as *isoexact*, $\hat{\omega} = \hat{\mathrm{d}}\hat{\theta}$, and therefore *isoclosed in the isocotangent bundle* (but not necessarily so in its projection on the original cotangent bundle), $\hat{\mathrm{d}}\omega = \hat{\mathrm{d}}(\hat{\mathrm{d}}\hat{\theta}) \equiv 0$ (this is the isotopic Poincaré Lemma [57, 71]). Isoform (2.19) is then called *isosymplectic*;. Higher-dimensional isoforms are then constructed accordingly.

An important geometric discovery which is permitted by the isosymplectic geometry is the following alternative of the Darboux theorem.

Theorem 2.1 (Santilli [61a, 71]). *Let $X(a)$ be a vector-field on a conventional tangent bundle and suppose that it is non-Hamiltonian in the point a, i.e., there exist no function $H(a)$ such that on a suitable neighborhood D of the chart a the following identities hold $\omega_{\mu\nu}X^\nu(a)\,\mathrm{d}a^\mu = \mathrm{d}H(a)$. Then, there always exists an isotopy within the fixed local chart a under which the same vector field becomes Hamiltonian, i.e., the following identities hold in a neighbourhood D of a*

$$\hat{\omega}_{\mu\nu}X^\nu(a)\,\hat{\mathrm{d}}a^\mu = \hat{\mathrm{d}}H(a)$$

in which case the vector-field is called isohamiltonian.

Recall from section 1.2 that a Darboux's transformation $a \to a' = a'(a) = (r'(r, p), p'(r, p))$ under which a vector-field becomes Hamiltonian *cannot* be generally used in physical applications because the transformed frame is highly nonlinear in the original co-ordinates, thus not realizable in actual experiment as well as highly non-inertial, thus incompatible with established relativities.

Santilli's motivation for the construction of the isosymplectic geometry is precisely to resolve these problematic aspects, by permitting a non-Hamiltonian vector-field to become Hamiltonian *under the preservation of the fixed a-frame of the experimenter and merely changing instead the basic unit (thus the basic differentials) of the geometry.*

Note that $\hat{\omega} \equiv \omega$ under the assumed conditions of $\hat{p}_k = p_k$ (see later on for differences) and this shows the 'hidden' character of the isotopies in the very structure of the conventional symplectic geometry. This also confirms that *the symplectic and isosymplectic geometries coincide at the abstract, realization-free level,* as established by the abstract identity of forms (2.15) and (2.18), or (2.17) and (2.19). Such an abstract identity is such that one can represent the isosymplectic geometry with the same symbols used for the co-ordinate free formulation of the symplectic geometry.

However, *the two geometries admit inequivalent realizations.* In fact, the symplectic geometry is strictly local-differential, does not admit nonlinearities in the velocities and it possesses a canonical structure. On the contrary, the isosymplectic geometry has an integro-differential structure (in the sense indicated earlier) and it is arbitrarily non-linear in the velocities.

All isogeometries indicated in this section admit intriguing isodual forms which can be easily identified by the reader via the rules of isodualities identified earlier. Regrettably, we have to refer the interested reader to monographs [61, 72] for details (see also paper [75] for topological aspects on isomanifolds).

At this writing, the isogeometries are minimally well-known for physical applications. Nevertheless, their mathematical study has yet to be initiated and a number of fundamental aspects remain open at this writing.

2.5. *Isotopies and isodualities of functional analysis*

As indicated earlier, the isotopies imply non-trivial generalizations of *all* mathematical structures of Lie's theory, inevitably leading to a generalization of functional analysis called by this author *functional isoanalysis* [22].

The generalized discipline begins with the isotopy of continuity (whose knowledge is assumed when dealing with the technical aspects of section 3), and includes the isotopies of conventional square-integrable, Banach and Hilbert spaces, as well as the isotopies of all operations on them.

In particular, functional isoanalysis includes a generalization of conventional special functions, distributions and transforms. For instance, the conventional Dirac delta distribution has no meaning under isotopy, mathematically, because of the loss of applicability of the conventional exponentiation and, physically, because particles are no longer point-like. The *isodirac distribution* is the reconstruction of the conventional distribution for an unrestricted unit permitting a direct treatment of the extended character of particles. The Fourier transform, Legendre polynomials, etc., also admit simple yet unique and unambiguous isotopies with important applications in various disciplines.

Regrettably, we are unable to review the above isotopies to prevent a prohibitive length of this paper, and refer the interested reader to [61a]. We shall merely identify in the next section only those isospecial functions which are necessary for an understanding of the Lie–Santilli isotheory. One should be aware that the elaboration of the Lie–Santilli isotheory via conventional functional analysis (e.g., the use of conventional trigonometry, logarithms, exponentiations, etc.) leads to inconsistencies which often remain undetected by the non-initiated reader.

2.6. *Isotopies and isodualities of classical mechanics*

As it is well-known (see, e.g., [13]), Lie's theory admits two fundamental realizations, one in classical and one in quantum mechanics, with interconnecting map given by the naive or symplectic quantization.

The preceding isotopies of fields, differential calculus, metric spaces, geometries, and functional analysis were used by Santilli for the construction of step-by-step isotopic generalizations of classical [72] and quantum [61] mechanics and their interconnecting maps. The new mechanics have been specifically conceived for the most general possible, non-linear, non-local and non-canonical, interior dynamical problems, as well as the fundamental classical and operator realizations of the Lie–Santilli isotheory. As a matter of fact, Santilli proposed the isotopies of Lie's theory precisely for quantitative treatment of the above generalized mechanics.

It is important to review at least the essential structural elements of the isotopic classical and operator mechanics because they provide the realizations of the Lie–Santilli isotheory most important for applications.

To conduct our outline, we shall keep using Santilli's notation of putting a 'hat' on all quantities belonging to isotopic formulations, while conventional symbols are used for quantities belonging to conventional formulations (see [71] for details). As it is well-known, conventional classical mechanics is formulated in the configuration space via the seven-dimensional space $E(t, \delta, \Re) \times E(x, \delta, \Re) \times E(v, \delta, \Re)$ where t is time, $x = \{x^k\}$ represents the space co-ordinates and $v = \{v^k\}$ represents the velocities, the latter being independent from the former.

The isotopies of classical mechanics in configuration space require their formulation in the isospace $\hat{S}(\hat{t}, \hat{x}, \hat{v}) = \hat{E}(\hat{t}, \hat{\mathrm{d}}, \hat{\Re})) \times \hat{E}(\hat{x}, \hat{\mathrm{d}}, \hat{\Re}) \times \hat{E}(\hat{v}, \hat{\mathrm{d}}, \hat{\Re})$ characterized by the total isounit $\hat{I}_{\text{tot}} = \hat{I}_t \times \hat{I} \times \hat{I}$, where: $\hat{I}_t = T_t^{-1}$ is the (one-dimensional) *isounit of time* and $\hat{I} = \hat{T}^{-1}$ is the (three-dimensional) *isounit of space*. By assuming that the isotime is contravariant we have $\hat{t} \equiv t$, while for the space components we have the general rules

$$\hat{x} = \{\hat{x}^k\} \equiv \{x^k\}, \quad \hat{x}_k = \hat{\mathrm{d}}_{ki}\hat{x}^i = \hat{T}^i_k \delta_{ij} x^j = \hat{T}^i_k x_i, \quad \hat{v} = \{\hat{v}^k\} \equiv \{v^k\} \equiv \{\mathrm{d}x^k/\mathrm{d}t\},$$
$$\hat{v}_k = \hat{\mathrm{d}}_{ki}\hat{v}^i = \hat{T}^i_k v_i.$$

The isodifferential calculus on $\hat{S}(\hat{t}, \hat{x}, \hat{v})$ is then based on the following *space and time isodifferentials* and *isoderivatives*,

$$\hat{\mathrm{d}}\hat{t} = \hat{I}_t \mathrm{d}t, \quad \hat{\mathrm{d}}\hat{x}^k = \hat{I}^k_i \mathrm{d}x^i, \quad \hat{\mathrm{d}}\hat{x}_k = \hat{T}^i_k \mathrm{d}\hat{x}_i, \quad \hat{\mathrm{d}}\hat{v}^k = \hat{I}^k_i \mathrm{d}v^i, \quad \hat{\mathrm{d}}\hat{v}_k = \hat{T}^i_k \mathrm{d}\hat{v}_k, \tag{2.20a}$$

$$\hat{\mathrm{d}}/\hat{\mathrm{d}}\hat{t} = \hat{T}_t \mathrm{d}/\mathrm{d}t, \quad \hat{\partial}/\hat{\partial}\hat{x}^k = \hat{T}^i_k \partial/\partial x^i, \quad \hat{\partial}/\hat{\partial}\hat{x}_k = \hat{I}^k_i \partial/\partial x_k,$$
$$\hat{\partial}/\hat{\partial}\hat{v}^k = \hat{T}^i_k \partial/\partial v^i, \hat{\partial}/\hat{\partial}\hat{v}_k = \hat{I}^k_i \partial/\partial \hat{v}_i, \tag{2.20b}$$

with basic properties

$$\hat{\partial}\hat{x}^i/\hat{\partial}\hat{x}^j = \delta^i_j, \quad \hat{\partial}\hat{x}_i/\hat{\partial}\hat{x}_j = \delta^j_i, \quad \hat{\partial}\hat{x}^i/\hat{\partial}\hat{x}_j = \hat{I}^i_j, \quad \hat{\partial}\hat{x}_i/\hat{\partial}\hat{x}^j = \hat{T}^j_i,$$
$$\hat{\partial}(\hat{v}^i \hat{\mathrm{d}}_{ij}\hat{v}^j)/\hat{\partial}\hat{v}^k = 2\hat{v}_k. \tag{2.21}$$

We then have the following isotopies of classical mechanics:

(1) *Isonewtonian mechanics.* Newton's equations of motion $m\,\mathrm{d}v_k/\mathrm{d}t + \partial V/\partial x^k - F_k^{\mathrm{NSA}} = 0$ on $S(t, x, v)$ over $\Re(n, +, \times)$ are lifted into the *Newton–Santilli equations* on the isospace $\hat{S}(\hat{t}, \hat{x}, \hat{v})$ first introduced in [71]

$$\hat{m}\frac{\hat{\mathrm{d}}\hat{v}_k}{\hat{\mathrm{d}}\hat{t}} + \frac{\hat{\partial}\hat{V}(\hat{x})}{\hat{\partial}\hat{x}^k} = 0, \tag{2.22}$$

which, when projected in the original space $S(t, x, v)$, assume the explicit form

$$\hat{m}\hat{I}_t\frac{\mathrm{d}[\hat{T}_k^i(t, x, v, \dots)v_i]}{\mathrm{d}t} + \hat{T}_k^i(t, x, v, \dots)\frac{\partial V(x)}{\partial x^i}$$

$$= \hat{T}_k^i[m\,\mathrm{d}v_i/\mathrm{d}t + \partial V(x)/\partial x^i + m\hat{I}_i^r(d\hat{T}_r^s/\mathrm{d}t)v_s] = 0, \tag{2.23}$$

where $\hat{m}\hat{I}_t = m$, and $\hat{m} = m\hat{T}_t$ is called the *isomass.*

As one can see, the Newton–Santilli equations permit the direct representation (i.e., representation in the fixed x-frame of the observer) of: (a) the actual, extended, non-spherical and deformable shape of the body considered; (b) non-local-integral interactions as permitted by the underlying integro-differential topology of $\hat{S}(\hat{t}, \hat{x}, \hat{v})$ [77]; and (c) the representation of all possible non-potential forces $F_i^{\mathrm{NSA}} = -m\hat{I}_i^r(d\hat{T}/\mathrm{d}t)v_r$ via the *isogeometry* itself, (i.e., via the covariant form $\hat{v}_k = \hat{T}_k^i v_i$) in such a way that all forces F^{NSA} 'disappear' in expression (2.22) in isospace.

As a specific example, consider an originally spherical body of mass m which moves along the x-axis within a resistive medium (say, gas or liquid) by acquiring the ellipsoidical shape σ with semiaxes (a^2, b^2, c^2). By ignoring potential forces for simplicity, suppose that the body experiences only a non-local-integral resistive force of the type $F_x^{\mathrm{NSA}} = -\gamma v_x^2 \int_\sigma \mathrm{d}\sigma\, \mathscr{F}(\sigma, \dots)$, where $\gamma > 0$ and $\mathscr{F}$ is a suitable kernel. The above systems can be directly represented in isoconfiguration space $\hat{S}(\hat{t}, \hat{x}, \hat{v})$ via the Newton–Santilli equation

$$\hat{m}\,\hat{\mathrm{d}}\hat{v}_x/\hat{\mathrm{d}}\hat{t} = 0, \text{ i.e., } m\,\mathrm{d}(\hat{T}_x^x v_x)/\mathrm{d}t = \hat{T}_x^x[m\,\mathrm{d}v_x/\mathrm{d}t + m\hat{I}_x^x(\mathrm{d}\hat{T}_x^x/\mathrm{d}t)v_x] = 0,$$

$$\hat{m} = m, \hat{I}_t = 1, \quad \hat{I}_x^x = \mathrm{diag.}(a^{-2}, b^{-2}, c^{-2})\exp\{-\gamma t v_x \int_\sigma \mathrm{d}\sigma\mathscr{F}(\sigma, \dots)\}. \tag{2.24}$$

The interested reader can then construct a virtually endless number of other examples. Note that, by comparison, the conventional Newton's equations can only represent *point-like particles under local-differential interactions.* By recalling that the terms 'Newtonian mechanics' are referred to point-particles under local-differential interactions, the emerging new mechanics for extended-deformable particles under integro-differential interactions shall be referred to as the *Newton–Santilli isomechanics.*

(2) *Isolagrangian mechanics.* A conventional first-order Lagrangian $L(x, v) = \frac{1}{2}mv^k v_k + V(x)$ on configuration space $S(t, x, v)$ acquires the form $\hat{L}(\hat{x}, \hat{v}) = \frac{1}{2}\hat{m}\hat{v}^k\hat{v}_k + \hat{V}(\hat{x})$ in isospace $\hat{S}(\hat{t}, \hat{x}, \hat{v})$. The isotopies of conventional variational principle of the *isoaction* $\hat{A} = \int_{\hat{t}_1}^{\hat{t}_2}\hat{L}(\hat{x}, \hat{v})\,\hat{\mathrm{d}}\hat{t}$ (see [71] for details) then lead to the

Lagrange–Santilli equations on isospace $\hat{S}(\hat{t}, \hat{x}, \hat{v})$

$$\frac{\hat{d}}{\hat{d}\hat{t}} \frac{\hat{\partial} \hat{L}(\hat{x}, \hat{v})}{\hat{\partial} \hat{v}^k} - \frac{\hat{\partial} \hat{L}(\hat{x}, \hat{v})}{\hat{\partial} \hat{x}^k} = 0, \tag{2.25}$$

which, under arbitrary but well behaved isolagrangians $\hat{L}(\hat{t}, \hat{x}, \hat{v})$, are *directly universal* for all possible isoequations (2.22). In fact, the above equations can be explicitly written on $S(t, x, v)$

$$\hat{I}_t \frac{d}{dt} \hat{T}_k^i \frac{\partial L(x, v)}{\partial v^i} - \frac{\partial L(x, v)}{\partial x^k} = 0 \tag{2.26}$$

by therefore coinciding with Equations (2.22). Note that the isolagrangian mechanics also permits the direct representation of extended, non-spherical and deformable bodies under conventional as well as non-local-integral nonpotential interactions with evident advances over the conventional formulation

(3) *Isohamiltonian mechanics.* The *isolegendre transform* is characterized by the invertible rules on a domain $\hat{D}$ of the local variables [61b, 71],

$$\operatorname{Det}\left(\frac{\hat{\partial}^{\hat{2}} \hat{L}}{\hat{\partial} \hat{v}^i \hat{\partial} \hat{v}^j}\right)(\hat{D}) \neq 0, \quad \hat{p}_k = \frac{\hat{\partial} \hat{L}}{\hat{\partial} \hat{v}^k} = m \hat{v}_k, \tag{2.27}$$

which are formulated on seven-dimensional *isophase space* $\hat{S}(\hat{t}, \hat{x}, \hat{p})$ with total isounit $\hat{I}_{\text{tot}} = \hat{I}_t \times \hat{I} \times \hat{I}$, yielding the *isocanonical action*

$$\hat{A} = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} (\hat{p}_k \hat{d} \hat{x}^k - \hat{H} \hat{d}t] = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} dt [p_k \hat{I}_i^k(t, x, p, \dots) dx^1 - H \hat{I}_t(t, x, p, \dots) dt], \tag{2.28}$$

with isohamiltonian $\hat{H} = \hat{p}_k \hat{p}^k / 2m + \hat{V}(x)$. The use again of the isovariations then yields the *Hamilton–Santilli equations* [61b, 71],

$$\frac{\hat{d} \hat{x}^k}{\hat{d} \hat{t}} = \frac{\hat{\partial} \hat{H}(\hat{x}, \hat{p})}{\hat{\partial} \hat{p}_k}, \quad \frac{\hat{d} \hat{p}_k}{\hat{d} \hat{t}} = -\frac{\hat{\partial} \hat{H}(\hat{x}, \hat{p})}{\hat{\partial} \hat{x}^k}, \tag{2.29}$$

The Hamilton–Jacobi equations are lifted into the *Hamilton–Jacobi–Santilli equations*

$$\hat{\partial}_t \hat{A} + \hat{H} = 0, \quad \hat{\partial} \hat{A} / \hat{\partial} \hat{x}^k - \hat{p}_k = 0, \quad \hat{\partial} \hat{A} / \hat{\partial} \hat{p}_k \equiv 0. \tag{2.30}$$

The conventional Poisson brackets, which are the realization in classical mechanics (CM) of the Lie product, are lifted into the *isopoisson brackets* first introduced in Ref. [47] (see also [71] for the explicit form below)

$$[A \hat{,} B]_{\text{CM}} = \frac{\hat{\partial} A}{\hat{\partial} \hat{x}^k} \frac{\hat{\partial} B}{\hat{\partial} \hat{p}_k} - \frac{\hat{\partial} B}{\hat{\partial} \hat{x}^k} \frac{\hat{\partial} A}{\hat{\partial} \hat{p}_k} = \frac{\partial A}{\partial x^i} T_j^i \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial x^i} T_j^i \frac{\partial B}{\partial p_j}, \tag{2.31}$$

which provide the desired *classical realization of the Lie–Santilli brackets.* In fact, it is easy to prove that the above brackets satisfy on isospace $\hat{S}(\hat{t}, \hat{x}, \hat{p})$ (but not in the original space) the Lie algebra axioms (see Refs. [61, 72] for a proof via the isotopies of the Poincaré Lemma of the symplectic geometry).

The exponentiated form of Hamilton's equation is a realization of a one-parameter Lie transformation group on $S(t, x, p)$. The exponentiated form of Eqs. (2.29) is

$$\hat{a}'^{\alpha} = \{e^{t\omega^{\mu\nu}\partial_{\mu}\hat{H}\partial_{\nu}}\}\hat{a}^{\alpha}, \quad \hat{a} = \{a^{\alpha}\} = \{\hat{x}^{k}, \hat{p}_{k}\}, \quad \alpha, \mu, \nu = 1, 2, \dots, 6, \tag{2.32}$$

which when properly written in an isotopic form (see next subsection), provide a realization of a one-dimensional Lie–Santilli transformation group on $\hat{S}(\hat{t}, \hat{x}, \hat{p})$.

The emerging new mechanics is called *Hamilton–Santilli isomechanics.* Some of the advantages over the conventional Hamiltonian mechanics are now evident. To begin, the Hamilton–Santilli equations preserve all essential features of the Newton–Santilli equations, thus permitting the representation, beginning at the classical level, of extended-deformable bodies with local-differential-potential as well as non-local-integral-nonpotential interactions. As we shall see in the next section, these features are mainly the classical foundations for corresponding operator formulations.

An important analytic discovery is given by the following.

Theorem 2.2 (Santilli [61b, 71]). *The Hamilton–Santilli equations (2.29) are 'direct universal' in the Newton–Santilli mechanics, that is, they can represent all infinitely possible, analytic and regular, integro-differential, variationally non-self-adjoint first-order systems in a star-shaped region of their variables (universality), directly in the frame of the experimenter (direct universality).*

The above property (which is the analytic counterpart of Theorem 2.1) can easily be proved by noting that a well-behaved action of *arbitrary order* always admit an identical *first-order* isotopic reformulation (2.28). The theorem can be equivalently established via the proof of the direct universality of equations (2.29) for all possible Hamiltonians $\hat{H}(\hat{t}, \hat{x}, \hat{p})$ and isounits $\hat{I}(\hat{t}, \hat{x}, \hat{p}, \dots)$. By comparison, the conventional Hamiltonian mechanics can directly represent only a rather small number of conservative Newtonian systems, and the more general Birkhoffian mechanics [49b] is directly universal only for (well-behaved) *local-differential* systems.

To understand this paper, the reader should keep in mind the above direct universality because it establishes the corresponding direct universality of the Lie–Santilli isotheory in classical mechanics with a corresponding direct universality for operator formulations indicated in the next subsection.

All formulations of this section admit isodual images on isospaces $\hat{S}^{d}(\hat{t}, \hat{x}, \hat{v})$ and $\hat{S}^{d}(\hat{t}, \hat{x}, \hat{p})$ over isodual isoreals $\hat{\mathfrak{R}}^{d}(\hat{n}^{d}, +, *^{d})$ which have produced the first *classical representation of antimatter* [71] known to this author. In particular, the representation occurs via particles with *negative-definite mass moving backward in time*, although defined with respect to *negative-definite units*, thus resulting to be equivalent (although antiautomorphic) to particles with positive-definite mass moving forward in time when defined with respect to positive-definite units. As an example, the *isodual Newton–Santilli equations* are given by [71]

$$\hat{m}^{d}\,\hat{d}\hat{v}_{k}^{d}/^{d}\,\hat{d}\hat{t}^{d} + \hat{\partial}\hat{V}^{d}(\hat{x})/^{d}\,\hat{\partial}\hat{x}^{dk} = 0$$

and characterize an antiparticle with mass $m^{d} = -m$ and time $\hat{t}^{d} = -t$. A similar situation occurs for the *isodual Lagrange–Santilli equations* as well as for the *isodual Hamilton–Santilli equations* and the *isodual Hamilton–Jacobi–Santilli equations* which all represent antiparticles in isodual isospaces on isodual isofields.

2.7. *Isotopies and isodualities of quantum mechanics*

We now outline the operator realization of the Lie–Santilli isotheory first identified in Ref. [48] and then studied in numerous subsequent papers (see Refs. [61b, 71] for the most recent accounts).

The isotopies of quantum mechanics were originally proposed by Santilli [48] under the name of *hadronic mechanics*, namely, a mechanics specifically built for strongly interacting particles called hadrons. Recall that quantum mechanics is strictly local and differential and has resulted to be *exactly valid* for electromagnetic and weak interactions, although there are historical doubts whether the same discipline can also be *exact* for the strong interactions, with the understanding that its *approximate validity* is unquestionable.

In fact, the charge radius of hadrons is of the same order of magnitude of the range of the strong interactions. Also, hadrons are some of the densest objects measured in laboratory until now. Therefore, the activation of the strong interactions requires the mutual penetration of these hyperdense particles, resulting in the historical expectation of non-linear, non-local-integral and non-Hamiltonian contributions whose quantitative treatment requires a suitable generalization of quantum mechanics.

Santilli [48] proposed the construction of the *isotopies of quantum mechanics* precisely for the treatment of the latter contributions in a form which preserves the original quantum mechanical axioms.

Let ξ be the enveloping associative operator algebra of quantum mechanics with elements $A, B, \ldots$, unit I and conventional associative product $A \times B = AB$, and let $\mathscr{H}$ be a conventional Hilbert space with states $|\psi\rangle, |\phi\rangle, \ldots$ and inner product $\langle\psi|\phi\rangle = \int d^3x\, \psi^\dagger(t,x)\phi(t,x)$ over the field complex numbers $C(c, +, \times)$.

By keeping the notation according to which quantities with a 'hat' are computed on isospaces over isofields while those without are computed on conventional spaces over conventional fields, hadronic mechanics is based on the following structures:

(1) the Class I lifting of the (space) unit $I \to \hat{I} = \hat{T}^{-1} > 0$ with consequential isofields of real $\hat{\Re}(\hat{n}, +, *)$ and complex isonumbers $\hat{C}(\hat{n}, +, *)$ (section 2.2);

(2) The corresponding lifting of the quantum mechanical representation spaces, such as the Euclidean $E(x, \delta, \Re)$ or Minkowskian spaces $M(x, \eta, \Re)$ into their isotopic form $\hat{E}(\hat{x}, \hat{\delta}, \hat{\Re})$ and $\hat{M}(\hat{x}, \hat{\eta}, \hat{\Re})$ (section 2.3)

(3) The lifting of the enveloping operator algebras ξ into the *enveloping isoassociative algebra* $\hat{\xi}$ with the same original elements $\hat{A} = A, \hat{B} = B, \ldots$ now equipped with the isounit $\hat{I}$ and the isoassociative product $\hat{A} * \hat{B} = \hat{A}\hat{T}\hat{B}$, as well as the lifting of the Hilbert space $\mathscr{H}$ into the *isohilbert space* $\hat{\mathscr{H}}$ with *isostates* $|\hat{\psi}\rangle, |\hat{\phi}\rangle, \ldots$ and *isoinner product*

$$\hat{\mathscr{H}}:\quad \langle\hat{\phi}\hat{|}\hat{\psi}\rangle = \langle\hat{\phi}|\hat{T}|\hat{\psi}\rangle\hat{I} \in \hat{C}(\hat{c}, +, *). \tag{2.33}$$

The fundamental dynamical equations of hadronic mechanics can be uniquely and unambiguously derived from the Hamilton–Santilli isomechanics via the isotopies of conventional or symplectic quantization. Recall that the *naive quantization* can be expressed via the mapping

$$A = \int_{t_1}^{t_2} (p\, dx^k - H\, dt) \to -i\hbar\, \mathrm{Ln}\, \psi(t, x). \tag{2.34}$$

Such a mapping is now inapplicable to isoaction (2.28) because $\hat{A} \neq A$. But the basic unit of quantum mechanics $\hbar = 1$ is replaced under isotopies by the (space) isounit $\hat{I}$. The consistent application of the isotopies then yields the generalized mapping identified by Animalu and Santilli here presented for simplicity for the isounit independent from the local time and co-ordinates (but dependent on the velocities as essential for contact resistive forces, see [61b] for the general case and references)

$$\hat{A} = \int_{t_1}^{t_2} [\hat{p}_k \hat{\mathrm{d}} x^k - \hat{H} \hat{\mathrm{d}} \hat{t}] \rightarrow -\mathrm{i} \hat{I} \,\mathrm{Ln}\, \hat{\psi}(\hat{t}, \hat{x}), \tag{2.35}$$

The above mapping is the *naive isoquantization* of the Hamilton–Jacob–Santilli equations (2.30) into the following fundamental dynamical equations of hadronic mechanics (see also Ref. [61b] for all references and details): *the isoschrödinger equations for the linear momentum*

$$-\mathrm{i} \hat{\partial}_k \hat{\psi}(\hat{t}, \hat{x}) = -\mathrm{i} T^i_k \partial_i \psi(t, r) = \hat{p}_k * \psi(t, x) = \hat{p}_k \hat{T} \hat{\psi}(\hat{t}, \hat{x}), \tag{2.36}$$

with the related *fundamental isocommutation rules*

$$[\hat{p}_i \hat{,} \hat{x}^j] = \hat{p}_i * \hat{x}^j - \hat{x}^j - \hat{x}^j * \hat{p}_i = -\delta^j_i, \quad [\hat{p}_i \hat{,} \hat{p}_j] = [\hat{x}^i \hat{,} \hat{x}^j] \equiv 0 \tag{2.37}$$

(where we have used properties (2.21)), first identified by Santilli; the *isoscrödinger equation for the energy*

$$\mathrm{i} \hat{\partial}_t \hat{\psi}(\hat{t}, \hat{x}) = \mathrm{i} \hat{T}_t \partial_t \psi(t, x) = \hat{H} * \hat{\psi}(\hat{t}, \hat{x}) = \hat{H} \hat{T} \hat{\psi}(\hat{t}, \hat{x}) = \hat{E} * \hat{\psi}(\hat{t}, \hat{x}) = E \hat{\psi}(\hat{t}, \hat{x}),$$

$$\hat{H} = \hat{H}^\dagger, \quad \hat{E} = E \hat{I} \in \hat{\mathfrak{R}}(\hat{n}, +, *), \quad E \in \mathfrak{R}(n, +, \times), \tag{2.38}$$

first identified by Myung and Santilli and, independently, by Mignani, with the conventional differential calculus, and finalized by Santilli with the isoderivatives; and the *Heisenberg–Santilli equation*

$$\mathrm{i} \hat{\mathrm{d}} \hat{Q} / \hat{\mathrm{d}} t = [\hat{Q} \hat{,} \hat{H}] = \hat{Q} * \hat{H} - \hat{H} * \hat{Q} = \hat{Q} \hat{T} \hat{H} - \hat{H} \hat{T} \hat{Q} \tag{2.39}$$

with integrated form

$$\hat{Q}(t) = \mathrm{e}^{\mathrm{i} \hat{H} \hat{T} t} \hat{Q}(0) \mathrm{e}^{-\mathrm{i} t \hat{T} \hat{H}}, \tag{2.40}$$

first identified by Santilli in the original proposal to build hadronic mechanics [48].

It should be recalled for subsequent need that *the condition of isohermiticity on an isohilbert space coincides with the conventional Hermiticity*, $\hat{H}^\dagger = H^\dagger$. As a consequence, *all operators which are Hermitean-observable in quantum mechanics remain so in hadronic mechanics*. Also, unitary transforms on $\mathscr{H}$, $UU^\dagger = U^\dagger = I$, are lifted under isotopies into the *isounitary transformations*

$$\hat{U} * \hat{U}^\dagger = \hat{U}^\dagger * U = \hat{I}, \tag{2.41}$$

As a matter of fact, any conventionally non-unitary operator U, $UU^\dagger = \hat{I} \neq I$, on $\mathscr{H}$ always admits an identical isounitary form on $\hat{\mathscr{H}}$ via the simple rule $U = \hat{U} \hat{T}^{1/2}$.

For the isotopies of the quantum mechanical axioms, isotopic laws and all other aspects we refer for brevity the interested reader to monograph [61b]. We here merely indicate that, from the positive-definiteness of the basic isounit $\hat{I}$, all distinctions between quantum and hadronic mechanics cease to exist at the abstract, realization-free level for which $\mathfrak{R} \approx \hat{\mathfrak{R}}$, $C \approx \hat{C}$, $\xi \approx \hat{\xi}$, $E \approx \hat{E}$, $\mathscr{H} \approx \hat{\mathscr{H}}$, etc. This ultimate abstract unity assures the correct axiomatic structure of hadronic mechanics to such an extent that criticisms on its structure may eventually result to be criticisms on quantum mechanics.

The *fundamental operator realization of the Lie–Santilli isoproduct* is then given by [48]

$$[\hat{A} \hat{,} \hat{B}] = \hat{A} * \hat{B} - \hat{B} * \hat{A} = ATB - BTA, \tag{2.42}$$

which, as one can easily verify, satisfy the Lie axioms in both isospace and in conventional spaces. The *fundamental operator realization of the isogroups* is then given by equation (2.40) which, as we shall see in the next section, can be identically rewritten in terms of the isounitary transforms.

Note that the naive (or symplectic) isoquantization apply for all possible isoaction (2.28). By recalling the direct universality of the Hamilton–Santilli isomechanics, we can therefore see that hadronic mechanics is also directly universal for all possible (well-behaved), integro-differential, operator systems which are non-linear in the wave function and its derivatives [61b]. This property is remarkable inasmuch as it establishes the direct universality of the Lie–Santilli isotheory in its operator realization.

The advantages of hadronic over quantum mechanics are similar to those of the Hamilton–Santilli over the Hamiltonian mechanics. In fact, quantum mechanics can only represent (in first quantization) point-like particles under action-at-a-distance interactions. By comparison, hadronic mechanics can represent (in first isoquantization) the actual non-spherical shape of hadrons, their deformations as well as non-local-integral interactions due to mutual penetrations of the hadrons. The possibilities for broader applications in various disciplines are then evident.

The isodual Hamilton–Santilli isomechanics is mapped via naive isoquantization into the *isodual hadronic mechanics* which is based on: (1) the isodual isofields of isoreals $\hat{\mathfrak{R}}^{\mathrm{d}}(\hat{n}^{\mathrm{d}}, +, *^{\mathrm{d}})$ or isocomplex numbers $\hat{C}^{\mathrm{d}}(\hat{c}^{\mathrm{d}}, +, *^{\mathrm{d}})$ (section 2.2); (2) the *isodual envelope* ξ^{d} with isodual isounit $\hat{I}^{\mathrm{d}} = -\hat{I}$, isodual elements $\hat{A}^{\mathrm{d}} = -A$, $\hat{B}^{\mathrm{d}} = -B$, etc., and isodual product $\hat{A}^{\mathrm{d}} *^{\mathrm{d}} \hat{B}^{\mathrm{d}} = -\hat{A}\hat{T}\hat{B}$; the *isodual isohilbert space* $\hat{\mathscr{H}}^{\mathrm{d}}$ with isodual isostates $|\hat{\psi}\rangle^{\mathrm{d}} = -\langle\hat{\psi}|$, etc, and isodual isoinner product $\langle\hat{\phi}|\hat{T}^{\mathrm{d}}|\hat{\psi}\rangle\hat{I}^{\mathrm{d}}$ over $\hat{C}^{\mathrm{d}}$.

In particular, at this operator level, the isodual map has resulted to be equivalent to charge conjugation (see [61b] for brevity), although with a number of differences. For instance, charge conjugation maps a particle into an antiparticle *in the same carrier space over the same field*, while isoduality maps a particles in a given carrier space over a given field *into a different carrier space over a different field* (the isodual ones); charge conjugation changes the sign of the charge but preserves the sign of energy and time, while isoduality changes the signs of all physical characteristics, although they are now defined over a field of negative-definite norm; etc.

As an example, the *isodual Heisenberg–Santilli equation* is given by

$$\mathrm{i}\,\hat{\partial}\hat{Q}^{\mathrm{d}}/\hat{\partial}\hat{t}^{\mathrm{d}} = \hat{Q}^{\mathrm{d}}\hat{T}^{\mathrm{d}}\hat{H}^{\mathrm{d}} - \hat{H}^{\mathrm{d}}\hat{T}^{\mathrm{d}}\hat{Q}^{\mathrm{d}},$$

where we have used the isoselfduality of the imaginary quantity i (section 2.2).

2.8. *Isolinearity, isolocality and isocanonicity*

In section 1 we pointed out that the primary limitations of the contemporary formulation of Lie's theory are those of being linear, local and canonical. The classical realizations identified earlier indicate rather clearly that the Lie–Santilli isotheory is non-linear, non-local and non-canonical, as desired.

It is important to understand that such non-linearity, non-locality and non-canonicity occur only when the theory is projected in the original space over the original fields because the theory reconstructs linearity, locality and canonicity in isospace (see [61] for all details and references).

Let $S(x, F)$ be a conventional vector space with local co-ordinates x over a field F, and let $x' = A(w)x$ be a linear, local and canonical transformation on $S(x, F)$, $w \in F$. The lifting $S(x, F) \to \hat{S}(\hat{x}, \hat{F})$ requires a corresponding necessary isotopy of the transformations [47]

$$\hat{x}' = \hat{A}(\hat{w}) * \hat{x} = \hat{A}(\hat{w})\, \hat{T} \hat{x}, \quad \hat{T} \text{ fixed}, \hat{x} \in \hat{S}(\hat{x}, \hat{F}), \quad \hat{w} = w\hat{I} \in \hat{\Re}, \hat{I} = \hat{T}^{-1}, \tag{2.43}$$

called *isotransforms*, with *isodual isotransforms* $\hat{x}' = \hat{A}^{d}(\hat{w}^{d} *{}^{d} \hat{x} = -\hat{A}(\hat{w}) * \hat{x}$.

It is easy to see that the above isotransforms satisfy the condition of linearity in isospaces, called *isolinearity*

$$\hat{A} * (\hat{a} * \hat{x} + \hat{b} * \hat{y}) = \hat{a} * (\hat{A} * \hat{x}) + \hat{b} * (\hat{A} * \hat{y}), \quad \forall \hat{x}, \hat{y} \in \hat{S}(\hat{x}, \hat{F}), \quad \hat{a}, \hat{b} \in \hat{F}, \tag{2.44}$$

although their projection in the original space $S(x, F)$ are non-linear because $x' = \hat{A}\, T(x, \dot{x}, \dots) x$.

Theorem 2.3 (Santilli [61a]). *All possible (well-behaved) non-linear, classical or-operator systems of equations or of transformations always admit an identical isolinear form.*

The above property illustrates the primary mechanisms according to which the Lie–Santilli isotheory applies to non-linear systems. In fact, as we shall see shortly, the latter theory is isolinear and, as such, it is capable of turning conventionally non-linear systems into identical forms which do verify the axioms of linearity in isospace, with evident advantages.

Isotransforms (2.39) are also *isolocal* in the sense that the theory formally deals with the local variables x while all non-local terms are embedded in the isounit, namely, all non-local-integral terms disappear at the abstract, realization-free level. Nevertheless, the theory is non-local when projected in the original space. Similarly, isotopic theories are *isocanonical* because they are derivable from the isoaction (2.28) which coincides at the abstract level with the canonical action.

3. Isotopies and isodualities of enveloping algebras, Lie algebras, Lie groups, symmetries, representation theory and their applications

As recalled in section 1, Lie's theory (see, e.g., [13, 15]) is centrally dependent on the basic n-dimensional unit $I = \text{diag.}\ (1, 1, \dots, 1)$ in all its major branches, such as enveloping algebras, Lie algebras, Lie groups, representation theory, etc. The main idea of the Lie–Santilli isotheory [47, 49, 61, 72] is the reformulation of the entire conventional theory with respect to the most general possible, integro-differential isounit $\hat{I}(x, \dot{x}, \ddot{x}, \dots)$.

One can therefore see from the outset the richness and novelty of the isotopic theory. In fact, it can be classified into five main classes as occurring for isofields, isospaces, etc., and admits novel realizations and applications, e.g., in the construction of the symmetries of deformed line elements of metric spaces.

In this section we shall continue to use the notation according to which quantities with a 'hat' are computed on isospaces over isofields while conventional quantities are computed on conventional spaces over conventional fields.

3.1. *Isotopies and isodualities of universal enveloping associative algebras*

Let ξ be a universal enveloping associative algebra [15] over a field F (of characteristic zero) with generic elements $A, B, C, \dots$, trivial associative product AB and unit I. Their isotopes $\hat{\xi}$ were first introduced in [47] under the name of *universal isoassociative enveloping algebras*. They coincide with ξ as vector spaces (i.e., $\hat{A} \equiv A$, $\hat{B} \equiv B$, etc.) but are equipped with the isoproduct so as to admit $\hat{I}$ as the correct (right and left) unit

$$\hat{\xi}\colon\quad \hat{A} * \hat{B} = \hat{A}\hat{T}\hat{B}, \quad \hat{T}\text{ fixed}, \quad \hat{I} * \hat{A} = \hat{A} * \hat{I} \equiv \hat{A} \equiv A \quad \forall \hat{A} \in \hat{\xi}, \quad \hat{I} = \hat{T}^{-1}. \tag{3.1}$$

Let $\xi = \xi(L)$ be the universal enveloping algebra of an N-dimensional Lie algebra L with ordered basis $\{X_k\}$, $k = 1, 2, \dots, N$, $[\xi(L)]^- \approx L$ over F, and let the infinite-dimensional basis of $\xi(L)$ be given by the Poincaré–Birkhoff–Witt theorem [15]. An important result achieved by Santilli in the original proposal [47] (see also [59, Vol. II, pp. 154–163]) is the following.

Theorem 3.1. *The cosets of* $\hat{I}$ *and the standard, isotopically mapped monomials*

$$\hat{I}, \hat{X}_k \quad \hat{X}_i * \hat{X}_j\, (i \leqslant j), \quad \hat{X}_i * \hat{X}_j * \hat{X}_k \quad (i \leqslant j \leqslant k), \dots \tag{3.2}$$

form a basis of the universal enveloping isoassociative algebra $\hat{\xi}(L)$ *of a Lie algebra* L.

The above theorem is fundamental for the entire analysis of this paper. A first consequence is given by the following isotopies of the conventional exponentiation, called *isoexponentiation*, here expressed for $\hat{w} = w\hat{I} \in \hat{F}$, $\hat{X} \equiv X$,

$$\begin{aligned} \mathrm{e}_{\hat{\xi}}^{\mathrm{i}wX} &= \hat{I} + (\mathrm{i}\hat{w} * \hat{X})/1! + (\mathrm{i}\hat{w} * \hat{X}) * (\mathrm{i}\hat{w} * \hat{X})/2! + \cdots = \hat{I}\{\mathrm{e}^{\mathrm{i}wTX}\} \\ &= \{\mathrm{e}^{\mathrm{i}XTw}\}\hat{I}. \end{aligned} \tag{3.3}$$

In turn, the notion of isoexponentiation permits the correct formulation of the isotransformations via expressions of the type $a' = \{\exp_{\hat{\xi}}(\mathrm{i}\hat{w} * \hat{X})\} * a = \{\exp_{\xi}(\mathrm{i}wTX)\}\hat{I}\hat{T}a = \{\exp_{\xi}(\mathrm{i}wTX)\}a$. The quantity $\hat{X}$ can first be a vector-field on an isomanifold with local chart a, thus providing a classical realization of the isotheory. The quantity $\hat{X}$ can also be a Hermitean operator on an isohilbert space, thus providing an operator realization of the isotheory. In fact, it is easy to prove that, for $\hat{X} = X^\dagger$, the quantity $\hat{U} = \exp_{\xi}(\mathrm{i}\hat{w} * \hat{X})$ is an isounitary operator satisfying (2.4).

The implications of Theorem 3.1 also emerge at the level of functional isoanalysis because all structures defined via the conventional exponentiation must be suitably lifted into a form compatible with Theorem 3.1. As an example, Fourier transforms are structurally dependent on the conventional exponentiation. As a result, they must be lifted under isotopies into the expressions [23]

$$f(x) = (1/2\pi)\int_{-\infty}^{+\infty} g(k) * \mathrm{e}_{\hat{\xi}}^{ikx}\,\mathrm{d}k, \quad g(k) = (1/2\pi)\int_{-\infty}^{+\infty} f(x) * \mathrm{e}_{\hat{\xi}}^{-ikx}\,\mathrm{d}x, \tag{3.4}$$

with similar liftings for Laplace transforms, Dirac-delta distribution, etc., not reviewed here for brevity.

On physical grounds, Theorem 3.1 implies that the isotransform of a Gaussian in functional isoanalysis is given by [23]

$$f(x) = N * \mathrm{e}_{\hat{\xi}}^{-x/2a^2} = N\,\mathrm{e}^{-x^2\hat{T}/2a^2} \to g(k) = N * \mathrm{e}_{\hat{\xi}}^{-k^2a^2/2} = N\mathrm{e}^{-k^2\hat{T}a^2/2}. \tag{3.5}$$

As a result, the widths are of the type $\Delta x \approx a\hat{T}^{-1/2}$, $\Delta k \approx a^{-1}\hat{T}^{-1/2}$. It then follows that the isotopies imply the loss of the conventional uncertainties $\Delta x\,\Delta k \approx 1$ in favor of the local-interior *isouncertainties* [61b]

$$\Delta x\,\Delta k \approx \hat{I}, \tag{3.6}$$

although the isoexpectation values recover the conventional value, $\langle \hat{I}\rangle = \langle|\hat{T}\hat{I}\hat{T}|\rangle/\langle|\hat{T}|\rangle = 1$ which allows to recover in full conventional uncertainties for the exterior, centre-of-mass behaviour of hadrons [61b].

The *isodual isoenvelopes* $\hat{\xi}^{\mathrm{d}}$ are characterized by the isodual basis $X_k^{\mathrm{d}} = -X_k$ defined with respect to the isodual isounits $\hat{I}^{\mathrm{d}} = -\hat{I}$ and isodual isotopic element $\hat{T}^{\mathrm{d}} = -\hat{T}$ over the isodual isofields $\hat{F}^{\mathrm{d}}$. The *isodual isoexponentiation* is then given by

$$\mathrm{e}_{\hat{\xi}^{\mathrm{d}}}^{i^{\mathrm{d}}w^{\mathrm{d}}\times^{\mathrm{d}}\hat{X}^{\mathrm{d}}} = \hat{I}^{\mathrm{d}}\{\mathrm{e}^{iwTX}\} = -\,\mathrm{e}_{\hat{\xi}}^{iwX} \tag{3.7}$$

and plays an important role for the characterization of antiparticles via isodual isosymmetries, with negative-definite energy and moving backward in time.

It is easy to see that Theorem 3.1 holds, as originally formulated [47], for Hermitean isounit of undefined signature now called of Class III, thus unifying isoenvelopes $\hat{\xi}$ and their isoduals $\hat{\xi}^{\mathrm{d}}$. In fact, the theorem was conceived to unify with one single envelope simple compact and non-compact algebras of the same dimension N. A first illustration was provided in [47] for the case of the Lie algebra $so(3)$ of the rotational group SO(3) with generators X_k, $k = 1, 2, 3$, in their fundamental three-dimensional representation, according to which $[\hat{\xi}(\mathrm{so}(3))]^- \approx so(3)$ for $\hat{I} = I = \mathrm{diag.}\,(1, 1, 1)$, $[\hat{\xi}(so(3))]^- \approx so(2.1)$ for $\hat{I} = \mathrm{diag.}\,(1, 1, -1)$ with more general realizations for more general forms of the isounits (see section 3.5 for more details). In the subsequent paper [51] Santilli illustrated how the isoenvelope $\hat{\xi}(so(4))$ unifies all possible simple, compact and noncompact six-dimensional Lie algebras, $so(4)$, $so(3.1)$, $so(2.2)$, as well as all their infinitely possible isotopes (see section 3.6 for more details). The possibility whether the preceding unifications holds for *all possible* simple Lie algebras of the same dimension was formulated by Santilli as a *conjecture* [61b], Appendix 8.A) which has remained unexplored until now to our knowledge.

Note that the isotopy $\xi \to \hat{\xi}$ is not a conventional map because the local coordinates x, the infinitesimal generators X_k and the parameters w_k are not changed by assumption. Only the underlying unit and related associative product are changed.

The non-triviality of the isotheory is first illustrated by the emergence of the non-linear-non-local isotopic element $\hat{T}$ directly in the exponent of isoexponentiation (3.3), thus ensuring the desired generalization. Also, in their operator realizations, the general transformation of the Lie into the Lie–Santilli isotheory is given by a non-unitary transformations for which

$$I \to \hat{I} = U I U^{\dagger},\ AB \to UABU^{\dagger} = A'\hat{T}B', \quad U(AB - BA)U^{\dagger} = A'\hat{T}B' - B'\hat{T}A', \tag{3.8a}$$

$$UU^{\dagger} = \hat{I} \neq I, \quad \hat{T} = (UU^{\dagger})^{-1}, \quad \hat{I} = \hat{T}^{-1}, \quad \hat{I} = \hat{I}^{\dagger}, \quad \hat{T} = \hat{T}^{\dagger},$$
$$A' = UAU^{\dagger}, \quad B' = UBU^{\dagger}, \tag{3.8b}$$

where one should note not only the emergence of the correct isotopic structure, but even that with the correct Hermiticity of $\hat{I}$ and $\hat{T}$. Once an isotopic structure is reached via non-unitary transforms, it remains form-invariant under the isounitary realization of non-unitary transforms [61b]. In fact, under a further non-unitary-isounitary transforms we have the invariance rules

$$\hat{U} * \hat{I} * \hat{U}^{\dagger} = \hat{I}, \quad \hat{U} * \hat{A} * \hat{B} * \hat{U}^{\dagger} = \hat{A}' * \hat{B}',$$
$$\hat{U} * (\hat{A} * \hat{B} - \hat{B} * \hat{A}) * \hat{U}^{\dagger} = A' * \hat{B}' - \hat{B}' * \hat{A}',$$

which establishes the form-invariance, first, of the fundamental isounit, and then of the isotheory.

The lack of equivalence of the two theories is further illustrated by the inequivalence between conventional eigenvalue equations, $H|b\rangle = E|b\rangle$, $H = H^{\dagger}$, $E \in \Re(n, +, \times)$, and their isotopic form in the same Hamiltonian

$$\hat{H} * |\hat{b}\rangle = \hat{H}\hat{T}|\hat{b}\rangle = \hat{E} * |\hat{b}\rangle \equiv E'|\hat{b}\rangle, \quad \hat{H} = H = H^{\dagger}, \quad E' \neq E,$$

with consequential *different eigenvalues for the same operator H* (see section 3.5 for an example). From the above occurrences it is easy to see that *the weights of the Lie and Lie–Santilli theories are different*, thus confirming the inequivalence of the two theories.

3.2. *Isotopies and isodualities of Lie algebras*

A (finite-dimensional) isospace $\hat{L}$ over the isofield $\hat{F}$ of isoreal $\hat{\Re}(\hat{n}, +, *)$ or isocomplex numbers $\hat{C}(\hat{c}, +, *)$ with isotopic element $\hat{T}$ and isounit $\hat{I} = \hat{T}^{-1}$ is called a *Lie–Santilli isoalgebra* over $\hat{F}$ (see [47, 49, 61, 72] for original studies and monographs [3, 24, 31, 76] with quoted papers for independent studies), when there is a composition $[\hat{A} \hat{,} \hat{B}]$ in $\hat{L}$, called *isocommutator*, which is isolinear (i.e., satisfies condition (2.44)) and such that for all $\hat{A}$, $\hat{B}$, $\hat{C} \in \hat{L}$

$$[\hat{A} \hat{,} \hat{B}] = -[\hat{B} \hat{,} \hat{A}], [\hat{A} \hat{,} [\hat{B} \hat{,} \hat{C}]] + [\hat{B} \hat{,} [\hat{C} \hat{,} \hat{A}]] + [\hat{C} \hat{,} [\hat{A} \hat{,} \hat{B}]] = 0, \tag{3.9a}$$

$$[\hat{A} * \hat{B} \hat{,} \hat{C}] = \hat{A} * [\hat{B} \hat{,} \hat{C}] + [\hat{A} \hat{,} \hat{C}] * \hat{B}. \tag{3.9b}$$

The isoalgebras are said to be: *isoreal* (*isocomplex*) when $\hat{F} = \hat{\Re}(\hat{F} = \hat{C})$, and *isoabelian* when $[\hat{A} \hat{,} \hat{B}] \equiv 0, \forall \hat{A}, \hat{B} \in \hat{L}$. A subset $\hat{L}_0$ of $\hat{L}$ is said to be an *isosubalgebra* of $\hat{L}$ when $[\hat{L}_0 \hat{,} L_0] \subset \hat{L}_0$ and an *isoideal* when $[\hat{L} \hat{,} \hat{L}_0] \subset \hat{L}_0$. A maximal isoideal which verifies the property $[\hat{L} \hat{,} \hat{L}_0] = 0$ is called the *isocenter* of $\hat{L}$. For the isotopies of conventional notions, theorems and properties of Lie algebras, one may see monograph [76].

We recall the *isotopic generalizations of the celebrated Lie's First, Second and Third Theorems* introduced in the original proposal [47], but which we do not review here for brevity (see [49b, 61b, 76]). For instance, the *Lie–Santilli second theorem* reads

$$[\hat{X}_i \hat{,} \hat{X}_j] = \hat{X}_i * \hat{X}_j - \hat{X}_j * \hat{X}_i = \hat{X}_i \hat{T}(\hat{x}, \ldots) \hat{X}_j - \hat{X}_j \hat{T}(\hat{x}, \ldots) \hat{X}_i$$
$$= \hat{C}^k_{ij}(\hat{x}, \ldots) * \hat{X}_k, \tag{3.10}$$

where the $\hat{X}$'s are vector-fields on an isomanifold with local chart $\hat{x}$, or operators on a isohilbert spaces, and the $\hat{C}$'s are called the *structure functions* because they generally have an explicit dependence on the local co-ordinates (see the example of section 3.5) restricted by certain conditions of the *Lie–Santilli* Third Theorem.

Let L be an N-dimensional Lie algebra with conventional commutation rules and structure constants C^k_{ij} on a space $S(x, F)$ with local co-ordinates x over a field F, and let L be (homomorphic to) the antisymmetric algebra $[\xi(L)]^-$ attached to the associative envelope $\xi(L)$. Then $\hat{L}$ can be equivalently defined as (homomorphic to) the antisymmetric algebra $[\hat{\xi}(L)]^-$ attached to the isoassociative envelope $\hat{\xi}(L)$ [47, 49, 76]. In this way, an infinite number of isoalgebras $\hat{L}$, depending on all possible isounits $\hat{I}$, can be constructed via the isotopies of one single Lie algebra L. It is easy to prove the following result.

Theorem 3.2 [61a]. *The isotopies $L \to \hat{L}$ of an N-dimensional Lie algebra L preserve the original dimensionality.*

In fact, the basis e_k, $k = 1, 2, \ldots, N$ of a Lie algebra L is not changed under isotopy, except for renormalization factors denoted $\hat{e}_k$. Let the commutation rules of L be given by $[e_i, e_j] = C^k_{ij}\, e_k$. The isocommutation rules of the isotopes $\hat{L}$ are then given by

$$[\hat{e}_i \hat{,} \hat{e}_j] = \hat{e}_i \hat{T} \hat{e}_j - \hat{e}_j \hat{T} \hat{e}_i = \tilde{C}^k_{ij}(x, \ldots) \hat{e}_k, \quad \tilde{C} = \hat{C} \hat{T}. \tag{3.11}$$

One can then see in this way the necessity of lifting the structure ⟨constants⟩ into structure ⟨functions⟩, as correctly predicted by the Lie–Santilli Second Theorem [47].

The structure theory of the above isoalgebras is still unexplored to a considerable extent. In the following we shall show that the main lines of the conventional structure of Lie theory do indeed admit a consistent isotopic lifting. To begin, we here introduce the *general isolinear and isocomplex Lie–Santilli algebras* denoted $G\hat{L}(n, \hat{C})$ as the vector isospaces of all $n \times n$ complex matrices over $\hat{C}$. It is easy to see that they are closed under isocommutators as in the conventional case. The *isocenter* of $G\hat{L}(n, \hat{C})$ is then given by $\hat{a} * \hat{I}$, $\forall \hat{a} \in \hat{\Re}$. The subset of all complex $n \times n$ matrices with null trace is also closed under isocommutators. We shall call it the *special, complex, isolinear isoalgebra* and denote it with $S\hat{L}(n, \hat{C})$. The subset of all antisymmetric $n \times n$ real matrices X, $X^t = -X$,, is also closed under isocommutators, it is called the *isoorthogonal algebra*, and it is denoted with $\hat{O}(n)$.

By proceeding along similar lines, we classify all classical, non-exceptional, Lie–Santilli algebras over an isofield of characteristic zero into the isotopes of the conventional forms, denoted with $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$ and $\hat{D}_n$ each one admitting realizations of Classes I–V (of which only Classes I–III are studied herein). In fact, $\hat{A}_{n-1} = S\hat{L}(n, \hat{C})$; $\hat{B}_n = \hat{O}(2n + 1, \hat{C})$; $\hat{C}_n = S\hat{P}(n, \hat{C})$; and $\hat{D}_n = \hat{O}(2n, \hat{C})$. One can begin to see in this way the richness of the isotopic theory as compared to the conventional theory.

The notions of *homomorphism, automorphism and isomorphism* of two isoalgebras $\hat{L}$ and $\hat{L}'$, as well as of *simplicity* and *semisimplicity* are the conventional ones.

Similarly, all properties of Lie algebras based on the addition, such as the *direct and semidirect sums*, carry over to the isotopic context unchanged (because of the preservation of the additive unit 0).

An *isoderivation* $\hat{D}$ of an isoalgebra $\hat{L}$ is an isolinear mapping of $\hat{L}$ into itself satisfying the property

$$\hat{D}([\hat{A}, \hat{B}]) = [\hat{D}(\hat{A})\hat{,}\hat{B}] + [\hat{A}\hat{,}\hat{D}(\hat{B})] \quad \forall \hat{A}, \hat{B} \in \hat{L}. \tag{3.12}$$

If two maps $\hat{D}_1$ and $\hat{D}_2$ are isoderivations, then $\hat{a} * \hat{D}_1 + \hat{b} * \hat{D}_2$ is also an isoderivation, and the isocommutators of $\hat{D}_1$ and $\hat{D}_2$ is also an isoderivation. Thus, the set of all isoderivations forms a Lie–Santilli isoalgebra as in the conventional case.

The isolinear map $\hat{ad}(\hat{L})$ of $\hat{L}$ into itself defined by

$$\hat{ad}\,\hat{A}(\hat{B}) = [\hat{A}\hat{,}\hat{B}], \quad \forall \hat{A}, \hat{B} \in \hat{L} \tag{3.13}$$

is called the *isoadjoint map*. It is an isoderivation, as one can prove via the iso-Jacobi identity. The set of all $\hat{ad}(\hat{A})$ is therefore an isolinear isoalgebra, called *isoadjoint algebra* and denoted $\hat{L}_a$. It also results to be an isoideal of the algebra of all isoderivations as in the conventional case.

Let $\hat{L}^{(0)} = \hat{L}$. Then $\hat{L}^{(1)} = [\hat{L}^{(0)}\hat{,}\hat{L}^{(0)}]$, $\hat{L}^{(2)} = [\hat{L}^{(1)}, \hat{L}^{(1)}]$, etc., are also isoideals of $\hat{L}$. $\hat{L}$ is then called *isosolvable* if, for some positive integer n, $\hat{L}^{(n)} \equiv 0$. Consider also the sequence

$$\hat{L}_{(0)} = L, \quad \hat{L}_{(1)} = [\hat{L}_{(0)}\hat{,}\hat{L}], \quad \hat{L}_{(2)} = [\hat{L}_{(1)}\hat{,}\hat{L}], \text{ etc.,}$$

Then $\hat{L}$ is said to be *isonilpotent* if, for some positive integer n, $\hat{L}_{(n)} \equiv 0$. One can then see that, as in the conventional case, an isonilpotent algebra is also isosolvable, but the converse is not necessarily true.

Let the *isotrace* of a matrix be given by the element of the isofield [61]

$$T\hat{r}\hat{A} = (\mathrm{Tr}\,A)\hat{I} \in \hat{F}, \tag{3.14}$$

where $\mathrm{Tr}\,A$ is the conventional trace. Then

$$\mathrm{T\hat{r}}(\hat{A} * \hat{B}) = (\mathrm{T\hat{r}}\,\hat{A}) * (\mathrm{T\hat{r}}\,\hat{B}), \quad \mathrm{T\hat{r}}(\hat{B}\,\hat{A}\,\hat{B}^{-1}) = \mathrm{T\hat{r}}\,\hat{A}.$$

Thus, the $\mathrm{T\hat{r}}\,\hat{A}$ preserves the axioms of $\mathrm{Tr}\,A$, by therefore being a correct isotopy. Then the isoscalar product

$$(\hat{A}\hat{,}\hat{B}) = \mathrm{T\hat{r}}[(\hat{ad}\,\hat{A}) * (\hat{ad}\,\hat{B})] \tag{3.15}$$

is here called the *isokilling form*. It is easy to see that $(\hat{A}\hat{,}\hat{B})$ is symmetric, bilinear, and verifies the property $(\hat{ad}\,\hat{X}(\hat{Y})\hat{,}Z) + (\hat{Y}\hat{,}Ad\,\hat{X}(\hat{Z})) = 0$, thus being a correct, axiom-preserving isotopy of the conventional Killing form.

Let e_k, $k = 1, 2, \ldots, N$, be the basis of L with one-to-one invertible map $e_k \to \hat{e}_k$ to the basis of $\hat{L}$. Generic elements in $\hat{L}$ can then be written in terms of local co-ordinates $\hat{x}, \hat{y}, \hat{z}$, $\hat{A} = \hat{x}^i \hat{e}_i$ and $\hat{B} = \hat{y}^j \hat{e}_j$, and $\hat{C} = \hat{z}^k \hat{e}_k = [\hat{A}\hat{,}\hat{B}] = \hat{x}^i \hat{y}^j [\hat{e}_i\hat{,}\hat{e}_j] = \hat{x}^i \hat{x}^j \tilde{C}^k_{ij} \hat{e}_k$. Thus,

$$[\hat{ad}\,\hat{A}(\hat{B})]^k = [\hat{A}\hat{,}\hat{B}]^k = \tilde{C}^k_{ij} \hat{x}^i \hat{x}^j. \tag{3.16}$$

We now introduce the *isocartan tensor* $\tilde{g}_{ij}$ of an isoalgebra $\hat{L}$ via the definition $(\hat{A}\hat{,}\hat{B}) = \tilde{g}_{ij} \hat{x}^i \hat{y}^j$ yielding

$$\tilde{g}_{ij}(\hat{x}, \ldots) = \tilde{C}^k_{ip} \tilde{C}^p_{jk}. \tag{3.17}$$

Note that the isocartan tensor has the general dependence of the isometric tensor of section 3, thus confirming the inner consistency among the various branches of the isotopic theory. In particular, the isocartan tensor is generally *non-linear, non-local and non-canonical* in all local variables as well as their derivatives with respect to an independent variable. This clarifies that the isotopic generalization of the Riemannian spaces $R(x, g, \mathfrak{R}) \rightarrow \hat{R}(\hat{x}, \hat{g}, \hat{\mathfrak{R}})$, $\hat{g} = \hat{g}(x, v, \ldots)$ [61a, 71], has its origin in the very structure of the Lie-isotopic theory.

The isocartan tensor also clarifies another fundamental point of section 1, that the isotopies naturally lead to an arbitrary dependence in the velocities and accelerations, exactly as needed for realistic treatments of the problems studied in this paper, and that their restriction to the non-linear dependence on the co-ordinates x only, as generally needed for the exterior (e.g., gravitational) problem, would be manifestly unnecessary.

The isotopies of the remaining aspects of the structure theory of Lie algebras can be completed by the interested reader. Here we limit ourselves to recall that when the isocartan form is positive- (or negative-)definite, $\hat{L}$ is compact, otherwise it is non-compact. Then it is easy to prove the following.

Theorem 3.3. *The Class* III *liftings* $\hat{L}$ *of a compact (noncompact) Lie algebra L are not necessarily compact (noncompact).*

The identification of the remaining properties which are not preserved under liftings of Class III is an instructive task for the interested reader. For instance, if the original structure is irreducible, its isotopic image is not necessarily so even for Class I, trivially, because the isounit itself can be reducible, thus yielding a reducible isotopic structure.

Let $\hat{L}$ be an isoalgebra with generators $\hat{X}_k$ and isounit $\hat{I} = \hat{T}^{-1} > 0$. From equations (3.7) we then see that the *isodual Lie–Santilli algebras* $\hat{L}^{\mathrm{d}}$ of $\hat{L}$ is characterized by the isocommutators

$$[\hat{X}_i \hat{,} \hat{X}_j]^{\mathrm{d}} = -[\hat{X}_i \hat{,} \hat{X}_j] = \tilde{C}_{ij}^{k(\mathrm{d})} \hat{X}_k^{\mathrm{d}}, \quad \tilde{C}_{ij}^{k(\mathrm{d})} = -\tilde{C}_{ij}^{k}. \tag{3.18}$$

$\hat{L}$ and $\hat{L}^{\mathrm{d}}$ are then (anti) isomorphic. Note that the isoalgebras of Class III contain all Class I isoalgebras $\hat{L}$ and all their isoduals $\hat{L}^{\mathrm{d}}$. The above remarks therefore show that the Lie–Santilli isotheory can be naturally formulated for Class III, as implicitly done in the original proposal [47]. The formulation of the same theory for Class IV or V is however considerably involved on technical grounds thus requiring specific studies.

The notion of isoduality applies also to conventional Lie algebras L, by permitting the identification of the *isodual Lie algebras* L^{d} via the rule [52, 53],

$$[\hat{X}_i \hat{,} \hat{X}_j]^{\mathrm{d}} = \hat{X}_i^{\mathrm{d}} I^{\mathrm{d}} \hat{X}_j^{\mathrm{d}} - \hat{X}_j^{\mathrm{d}} I^{\mathrm{d}} \hat{X}_i^{\mathrm{d}} = -[\hat{X}_i, \hat{X}_j] = C_{ij}^{k(\mathrm{d})} \hat{X}_k^{\mathrm{d}}, \quad C_{ij}^{k(\mathrm{d})} = -C_{ij}^{k}.$$

Note the necessity of the isotopies for the very construction of the isodual of conventional Lie algebras. In fact, they require the non-trivial lift of the unit $I \rightarrow I^{\mathrm{d}} = (-I)$, with consequential necessary generalization of the Lie product $AB - BA$ into the isotopic form $ATB - BTA$.

The following property is mathematically trivial, yet carries important physical applications.

Theorem 3.4. *All infinitely possible, Class* I *isotopes $\hat{L}$ of a (finite-dimensional) Lie algebra L are locally isomorphic to L, and all infinitely possible, Class* II *isodual isotopes $\hat{L}^d$ of L are anti-isomorphic to L.*

As indicated in section 2.6, the classical realization of the formulation of this section is provided by functions (or vector-fields) $\hat{X}_i, \hat{X}_j, \ldots$ on the isotangent bundle $T * \hat{E}(x, \hat{\delta}, \hat{\mathfrak{R}})$, $\hat{\delta} = \hat{T}\delta$, with local chart $\hat{a} = \{\hat{x}^k, \hat{p}_k\}$ and isoalgebra

$$[X_i \hat{,} X_j] = \frac{\hat{\partial} \hat{X}_i}{\hat{\partial} \hat{a}^\mu} \hat{\omega}^{\mu\nu} \frac{\hat{\partial} \hat{X}_j}{\hat{\partial} \hat{a}^\nu} = \frac{\partial X_i}{\partial x^m} T^n_m(x, p, \ldots) \frac{\partial X_j}{\partial p_n} = \hat{C}^k_{ij}\, T^m_k\, X_m. \tag{3.19}$$

where $\hat{\omega}^{\mu\nu} = \omega^{\mu\nu}$ is the conventional, canonical-Lie tensor. As outlined in section 2.7, the operator realization is given by operators X_k on an isohilbert space with a given isounit $\hat{I} = \hat{T}^{-1}$ and isoalgebra

$$[\hat{X}_i \hat{,} \hat{X}_j] = \hat{X}_i \hat{T} \hat{X}_j - \hat{X}_j \hat{T} \hat{X}_i = \hat{C}^k_{ij} \hat{T} \hat{X}_k. \tag{3.20}$$

The unique and unambiguous map interconnection realizations (3.19) and (3.20) is the *isoquantization* of section 2.7.

3.3. *Isotopies and isodualities of Lie groups*

A *right Lie–Santilli transformation isogroup* $\hat{G}$ (see [47, 49, 61, 72] for original studies and monographs [3, 24, 31, 76] with quoted papers for independent studies) on an isospace $\hat{S}(\hat{x}, \hat{F})$ over an isofield $\hat{F}, \hat{I} = T^{-1}$ (of isoreal $\hat{\mathfrak{R}}$ or isocomplex numbers $\hat{C}$), is a group which maps each element $\hat{x} \in \hat{S}(\hat{x}, \hat{F})$ into a new element $\hat{x}' \in \hat{S}(\hat{x}, \hat{F})$ via the isotransformations $\hat{x}' = \hat{U} * \hat{x} = \hat{U}\hat{T}\hat{x}$, $\hat{T}$ fixed, such that: (1) The map $(\hat{U}, \hat{x}) \rightarrow \hat{U} * \hat{x}$ of $\hat{G} \times \hat{S}(\hat{x}, \hat{F})$ onto $\hat{S}(\hat{x}, \hat{F})$ is isodifferentiable; (2) $\hat{I} * \hat{U} = \hat{U} * \hat{I} = \hat{U}$, $\forall \hat{U} \in \hat{G}$; and (3) $\hat{U}_1 * (\hat{U}_2 * \hat{x}) = (\hat{U}_1 * \hat{U}_2) * \hat{x}$, $\forall \hat{x} \in \hat{S}(\hat{x}, \hat{F})$ and $\hat{U}_1, \hat{U}_2 \in \hat{G}$. A *left transformation isogroup* is defined accordingly.

The notions of *connected or simply connected transformation groups* carry over to the isogroups in their entirety. We consider hereon the connected isotransformation groups. Right or left isogroups are characterized by the following laws [47]:

$$\hat{U}(0) = \hat{I}, \quad \hat{U}(\hat{w}) * \hat{U}(\hat{w}') = \hat{U}(\hat{w}') * \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}'),$$

$$\hat{U}(\hat{w}) * \hat{U}(-\hat{w}) = \hat{I}, \quad \hat{w} \in \hat{F}. \tag{3.21}$$

The most direct realization of the transformation isogroups is that via isoexponentiation (3.3),

$$\hat{U}(\hat{w}) = \prod_k \mathrm{e}_{\xi}^{\mathrm{i}\hat{w}_k * \hat{X}_k} = \prod_k \mathrm{e}_{\xi}^{\mathrm{i}\hat{X}_k * \hat{w}_k} = \hat{I}\left\{\prod_k \mathrm{e}^{\mathrm{i}\, w_k T X_k}\right\} = \left\{\prod_k \mathrm{e}_{\xi}^{\mathrm{i}\, X_k T w_k}\right\}\hat{I}, \tag{3.22}$$

where the X's and w's are the infinitesimal generator and parameters, respectively, of the original algebra L. Equations (3.22) hold for some open neighbourhood of N of the isoorigin of $\hat{L}$ and, in this way, characterize some open neighbourhood of the isounit of $\hat{G}$. Then the isotransformations can be reduced to an ordinary transform for computational convenience,

$$\hat{x}' = \hat{U} * \hat{x} = \left\{\prod_k \mathrm{e}_\xi^{\mathrm{i}X_k * w_k}\right\} * \hat{x} = \left\{\prod_k \mathrm{e}^{\mathrm{i}X_k T w_k}\right\} \hat{x}, \tag{3.23}$$

with the understanding that, on rigorous mathematical grounds, only the isotransform is correct.

Still another important result obtained in [47] is the proof that conventional group composition laws admit a consistent isotopic lifting, resulting in the following *isotopy of the Baker–Campbell–Hausdorff Theorem*

$$\{\mathrm{e}_\xi^{\hat{X}}\} * \{\mathrm{e}_\xi^{\hat{X}}\} = \mathrm{e}_\xi^{\hat{X}_3}, X_3 = \hat{X}_1 + \hat{X}_2 + [\hat{X}_1 \hat{,} \hat{X}_2]/2 + [(\hat{X}_1 - \hat{X}_2)\hat{,}[\hat{X}_1 \hat{,} \hat{X}_2]]/12 + \cdots \tag{3.24}$$

Note the crucial appearance of the isotopic element $\hat{T}(\hat{x}, \hat{v}, \dots)$ in the exponent of the isogroup. This ensures a structural generalization of Lie's theory of the desired non-linear, non-local and non-canonical form. For details see [49, 74].

The structure theory of isogroups is also vastly unexplored at this writing. In the following we shall point out that the conventional structure theory of Lie groups does indeed admit a consistent isotopic lifting. The isotopies of the notions of weak and strong continuity of [22] are a necessary pre-requisite. Let $\hat{L}$ be a (finite-dimensional) Lie–Santilli isoalgebra with (ordered) basis $\{\hat{X}_k\}$, $k = 1, 2, \dots, N$. For a sufficiently small neighborhood N of the isoorigin of $\hat{L}$, a generic element of $\hat{G}$ can be written

$$\hat{U}(\hat{w}) = \prod_{k=1,2,\dots,N}^{*} \mathrm{e}_\xi^{\mathrm{i}\hat{X}_k * \hat{w}_k}, \tag{3.25}$$

which characterizes some open neighbourhood M of the isounit $\hat{I}$ of $\hat{G}$. The map

$$\hat{\Phi}_{\hat{U}_1}(\hat{U}_2) = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-\hat{I}}, \tag{3.26}$$

for a fixed $\hat{U}_1 \in \hat{G}$, characterizes an *inner isoautomorphism* of $\hat{G}$ onto $\hat{G}$. The corresponding isoautomorphism of the algebra $\hat{L}$ can be readily computed by considering the above expression in the neighbourhood of the isounit $\hat{I}$. In fact, we have

$$\hat{U}_2' = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-\hat{I}} \cong \hat{U}_2 + \hat{w}_1 * \hat{w}_2 * [\hat{X}_2 \hat{,} \hat{X}_1] + O^{(2)}. \tag{3.27}$$

The reduction of the isogroups to isoalgebras requires the isodifferentials $\hat{\mathrm{d}}\hat{w} = \hat{I}\mathrm{d}\hat{w}$ and isoderivatives $\hat{\mathrm{d}}/\hat{\mathrm{d}}\hat{w} = \hat{T}\mathrm{d}/\mathrm{d}\hat{w}$, under which we have the following expression in one dimension:

$$i^{-1}\frac{\hat{\mathrm{d}}}{\hat{\mathrm{d}}\hat{w}}\hat{U}_{|w=0} = \hat{X} * \mathrm{e}_\xi^{\mathrm{i}\hat{w} * \hat{X}|_{\hat{w}=0|}} = \hat{X}. \tag{3.28}$$

Thus, to every inner isoautomorphism of $\hat{G}$, there corresponds an inner isoautomorphism of $\hat{L}$ which can be expressed in the form:

$$(\hat{L})_i^j = \hat{C}_{ki}^j * \hat{w}^k. \tag{3.29}$$

The isogroup $\hat{G}_a$ of all inner isoautomorphism of $\hat{G}$ is called the *isoadjoint group*. It is possible to prove that the Lie–Santilli algebra of $\hat{G}_a$ is the isoadjoint algebra $\hat{L}_a$ of $\hat{L}$. This establishes that the connections between algebras and groups carry over in their entirety under isotopies.

We mentioned before that the direct sum of isoalgebras is the conventional operation because the addition is not lifted under isotopies (otherwise there will be the loss of distributivity, see [59]). The corresponding operation for groups is the semidirect product which, as such, demands care in its formulation.

Let $\hat{G}$ be an isogroup and $\hat{G}_a$ the group of all its inner isoautomorphisms. Let $\hat{G}_a^0$ be a subgroup of $\hat{G}_a$, and let $\hat{\Lambda}(\hat{g})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}_a^0$. The *semidirect isoproduct* $\hat{G} \hat{\times} \hat{G}_a^0$ of $\hat{G}$ and $\hat{G}_a^0$ is the isogroup of all ordered pairs

$$(\hat{g}, \hat{\Lambda}) * (g', \hat{\Lambda}) = (\hat{g} * \hat{\Lambda}(\hat{g}'), \hat{\Lambda} * \hat{\Lambda}'), \tag{3.30}$$

with total isounit given by $(\hat{I}, \hat{I}_{\hat{\Lambda}})$ and inverse $(\hat{g}, \hat{\Lambda})^{-\hat{I}} = (\hat{\Lambda}^{-\hat{I}}(\hat{g}^{-\hat{I}}), \hat{\Lambda}^{-\hat{I}})$. The above notion plays an important role in the isotopies of the inhomogeneous space–time symmetries outlined later on.

Let $\hat{G}_1$ and $\hat{G}_2$ be two isogroups with respective isounits $\hat{I}_1$ and $\hat{I}_2$. The *direct isoproduct* $\hat{G}_1 \hat{\odot} \hat{G}_2$ of $\hat{G}_1$ and $\hat{G}_2$ is the isogroup of all ordered pairs $(\hat{g}_1, \hat{g}_2)$, $\hat{g}_1 \in \hat{G}_1, \hat{g}_2 \in \hat{G}_2$, with isomultiplication

$$(\hat{g}_1, \hat{g}_2) * (\hat{g}'_1, \hat{g}'_2) = (\hat{g}_1 * \hat{g}'_1, \hat{g}_2 * \hat{g}'_2). \tag{3.31}$$

total isounit $(\hat{I}_1, \hat{I}_2)$ and inverse $(\hat{g}_1^{-\hat{I}}, \hat{g}_2^{-\hat{I}})$. The isotopies of the remaining aspects of the structure theory of Lie groups can then be investigated by the interested reader.

Let $\hat{G}$ be an N-dimensional isotransformation group of Class I with infinitesimal generators $\hat{X}_k$, $k = 1, 2, \ldots, N$. The *isodual Lie–Santilli group* $\hat{G}^d$ of $\hat{G}$ [52, 53] is the N-dimensional isogroup with generators $\hat{X}_k^d = -\hat{X}_k$ constructed with respect to the isodual isounit $\hat{I}^d = -\hat{I}$ over the isodual isofield $\hat{F}^d$. By recalling that $\hat{w} \in \hat{F} \to \hat{w}^d \in \hat{F}^d$, $\hat{w}^d = -\hat{w}$, a generic element of $\hat{G}^d$ in a suitable neighbourhood of $\hat{I}^d$ is therefore given by

$$\hat{U}^d(\hat{w}^d) = e_{\xi}^{i^d \hat{w}^d * \hat{X}^d} = -e_{\xi}^{i\hat{w} * \hat{X}} = -\hat{U}(\hat{w}). \tag{3.32}$$

The above antiautomorphic conjugation can also be defined for conventional Lie group, yielding the *isodual Lie group* G^d of G with generic elements $U^d(w^d) = e_{\xi^d}^{i w^d X} = -e_{\xi}^{iwX}$.

The symmetries significant for this paper are the following ones: the conventional form G, its isodual G^d, the isotopic form $\hat{G}$ and the isodual isotopic form $\hat{G}^d$. These different forms are useful for the respective characterization of particles and antiparticles in vacuum (exterior problem) or within physical media (interior problem).

It is hoped that the reader can see from the above elements that the conventional Lie's theory does indeed admit a consistent and non-trivial lifting into the covering Lie–Santilli formulation. Particularly important are the isotopies of the conventional representation theory, known as the *isorepresentation theory*, which naturally yields the most general known, non-linear, non-local and non-canonical representations of Lie groups. Studies along these latter lines were initiated by Santilli with the isorepresentations of $S\hat{U}(2)$ and of $S\hat{U}(3)$ [61], by Klimyk and Santilli Klimyk [27], and others.

As received in section 2.6, a classical realization of the formulation of this section is formulated on the isotangent bundle $T * \hat{E}(\hat{x}, \hat{\delta}, \hat{\Re})$, $\hat{\delta} = \hat{T}\delta$, with local chart $\hat{a} = \{\hat{a}^\mu\} = \{\hat{x}^k, \hat{p}_k\}$, $\mu = 1, 2, \ldots, 6$, $k = 1, 2, 3$, and isounit $\hat{I}_2 = \hat{I} \times \hat{I}$ in terms of a vector-field $\hat{X}(\hat{a})$ (where $\hat{t} = t$)

$$\hat{a}(\hat{t}) = \{e_{\xi}^{\hat{X}\hat{t}}\} * \hat{a}(0) = \{e^{XTt}\}\, \hat{a}(0) \tag{3.33}$$

As recalled in section 2.7, an operator realization of the Lie–Santilli isogroups is given by *isounitary transforms* $\hat{x}' = \hat{U} * \hat{x}$ on an isohilbert space $\hat{\mathscr{H}}$, equation (2.41), with realization in terms of an *isohermitean operator* $\hat{H}$

$$\hat{a}(\hat{t}) = \hat{U} * \hat{a}(0) = \{e_{\xi}^{i\hat{H}\hat{t}}\} * \hat{a}(0) = \{e^{iHTt}\}\,\hat{a}(0). \tag{3.34}$$

The above classical and operator realizations are also interconnected in a unique and unambiguous way by the isoquantization (section 2.7).

3.4. *Santilli's fundamental theorem on isosymmetries*

We are now equipped to review without proof the following important result first formulated in [52] and then studied in detail in [61–72]:

Theorem 3.5. *Let G be an N-dimensional Lie group of isometries of an m-dimensional metric or pseudo-metric space $S(x, g, F)$ over a field F*

$$G:\quad x' = A(w)\,x, (x' - y')^{\dagger} A^{\dagger} g\, A(x - y) \equiv (x - y)^{\dagger} g (x - y),$$
$$A^{\dagger} g\, A = A\, g\, A^{\dagger} = g. \tag{3.35}$$

Then the infinitely possible isotopies $\hat{G}$ of G of Class III *characterized by the same generators and parameters of G and new isounits $\hat{I}$ (isotopic elements $\hat{T}$), automatically leave invariant the isocomposition on the isospaces $\hat{S}(\hat{x}, \hat{g}, \hat{F}), \hat{g} = Tg, \hat{I} = \hat{T}^{-1}$,*

$$\hat{G}:\quad \hat{x}' = \hat{A}(\hat{w}) * \hat{x}, (\hat{x}' - \hat{y}')^{\dagger} * \hat{A}^{\dagger}\, \hat{g}\, \hat{A} * (\hat{x} - \hat{y})$$
$$= (\hat{x} - \hat{y})^{\dagger}\, \hat{g}\, (\hat{x} - \hat{y}), \hat{A}^{\dagger}\, \hat{g}\, A = \hat{A}\, \hat{g}\, \hat{A}^{\dagger} = \hat{I}\, \hat{g}\, \hat{I}. \tag{3.36}$$

The 'direct universal' of the resulting isosymmetries for all infinitely possible Class III isotopies $g \to \hat{g}$ is then evident owing to the completely unrestricted functional dependence of the isotopic element $\hat{T}$ in the isometric $\hat{g} = \hat{T}g$. One should also note the insufficiency of the so-called *trivial isotopy*

$$\hat{X}_k \to \hat{X}'_k = \hat{X}_k \hat{I}, \tag{3.37}$$

for the achievement of the desired form-invariance. In fact, under the above mapping the isoexponentiation becomes

$$e_{\xi}^{i\hat{X}'_k * w_k} = \{e^{i\hat{X}'_k T w_k}\}\,\hat{I} = \{e^{iX_k w_k}\}\,\hat{I}, \tag{3.38}$$

namely, we have the disappearance precisely of the isotopic element T in the exponent which provides the invariance of the isoseparation.

3.5. *Isotopies and isodualities of the rotational symmetry*

We now illustrate the Lie–Santilli isotheory with the first mathematically and physically significant case, the *isotopies of the rotational symmetry*, also called *isorotational symmetry*. They were first studied in [53] and then treated in detail in [61b, 72b], including the isotopies of $SU(2)$, their isorepresentations, the iso-Clebsh–Gordon coefficients, etc.

Consider the lifting of the perfect sphere in Euclidean space $E(r, \delta, \mathfrak{R})$ with local co-ordinates $r = (x, y, z)$, and metric $\delta = \text{diag.}\,(1, 1, 1)$ over the reals $\mathfrak{R}$,

$$r^2 = r^t \delta r = x\,x + y\,y + z\,z \tag{3.39}$$

into the most general possible ellipsoid of Class III on isospace $\hat{E}^{\mathrm{III}}(\hat{r}, \hat{\delta}, \hat{\mathfrak{R}}), \hat{\delta} = \hat{T}\delta, \hat{T} = \text{diag.}(g_{11}, g_{22}, g_{33}), \hat{I} = \hat{T}^{-1}$.

$$r^{\hat{2}} = r^t \hat{\delta} r = x g_{11} y + y g_{22} y + z g_{33} z, \delta^\dagger = \hat{\delta}, \; g_{kk} = g_{kk}(t, r, \dot{r}, \ddot{r}, \ldots) \neq 0, \quad (3.40)$$

The invariance of the original separation r^2 is the conventional rotational symmetry $O(3)$. The Lie–Santilli isotheory then permit the construction, in the needed explicit and finite form, of the isosymmetries $\hat{O}(3)$ of all infinitely possible generalized invariants $r^{\hat{2}}$ via the following steps: (1) Identification of the basic isotopic element $\hat{T}$ in the lifting $\delta \to \hat{\delta} = \hat{T}\delta$ which, in this particular case, is given by the new metric $\hat{\delta}$ itself, $\hat{T} \equiv \hat{\delta}$, and identification of the fundamental unit of the theory, $\hat{I} = \hat{T}^{-1}$; (2) Consequently lifting of the basic field $\mathfrak{R}(n, +, \times) \to \hat{\mathfrak{R}}(\hat{n}, +, *)$; (3) Identification of the isospace in which the generalized metric $\hat{\delta}$ is defined, which is given by the three-dimensional isoeuclidean spaces $\hat{E}(\hat{r}, \hat{\delta}, \hat{\mathfrak{R}}), \hat{\delta} = T\delta, \hat{I} = \hat{T}^{-1}$; (4) Construction of the $\hat{O}(3)$ symmetry via the use of the original parameters of $O(3)$ (the Euler's angles θ_k, $k = 1, 2, 3$), although in their isotopic form $\hat{\theta}_k = \theta_k \hat{I}$, the original generators (the angular momentum components although computed in isospace $\hat{M}_k = \epsilon_{kij} \hat{r}^i \hat{p}_j$), and the new metric $\hat{\delta}$; and (5) Classification, interpretation and application of the results.

The explicit construction of $\hat{O}(3)$ is straightforward. According to the Lie–Santilli isotheory, the connected component $S\hat{O}(3)$ of $\hat{O}(3)$ is given by [53]

$$S\hat{O}(3): \quad \hat{r}' = \hat{R}(\hat{\theta}) * \hat{r}, \hat{R}(\hat{\theta}) = \prod^{*}_{k=1,2,3} \mathrm{e}_{\xi}^{\mathrm{i}\hat{M}_k * \hat{\theta}_k} = \left\{ \prod_{k=1,2,3} \mathrm{e}^{\mathrm{i} M_k T \theta_k} \right\} \hat{I}, \quad (3.41)$$

while the discrete component is given by the *isoinversions* [53] $\hat{r}' = \hat{\pi} * \hat{r} = \pi r = -r$, where π is the conventional inversion.

Under the assumed conditions on the isotopic element $\hat{T}$, the convergence of isoexponentiations is ensured by the original convergence, thus permitting the explicit construction of the isorotations for the case of the adjoint representation of $\hat{M}_k$. An example around the third axis, $z' = z$, is given by [53]

$$x' = x \cos[\theta_3 (g_{11} g_{22})^{1/2}] + y g_{22} (g_{11} g_{22})^{-1/2} \sin[\theta_3 (g_{11} g_{22})^{1/2}], \quad (3.42a)$$

$$y' = -x g_{11} (g_{11} g_{22})^{-1/2} \sin[\theta_3 (g_{11} g_{22})^{1/2}] + y \cos[g_{11} g_{22})^{1/2}], \quad (3.42b)$$

(see [61b] for general isorotations). One should note that the argument of the trigonometric functions as derived via the above isoexponentiation coincides with the isoangle of the isotrigonometry in $\hat{E}(\hat{r}, \hat{\delta}, \hat{\mathfrak{R}})$ (see paper [60]) thus confirming the remarkable compatibility and interconnections of the various branches of the isotopic theory:

The computation of the operator isoalgebras $\hat{o}(3)$ of $\hat{O}(3)$ is then straightforward [53]. The linear momentum operator has the isotopic form (2.36). The fundamental isocommutation rules are then given by (2.37). The operator isoalgebra $\hat{o}(3)$ with generators $\hat{M}_k = \epsilon_{kij} \hat{r}^i \hat{p}_j$ is then given by

$$\hat{o}(3): \quad [\hat{M}_i \hat{,} \hat{M}_j] = \hat{M}_i \hat{T} \hat{M}_j - \hat{M}_j \hat{T} \hat{M}_i = i\, \hat{\epsilon}_{ij}^k * \hat{M}_k, \; \hat{\epsilon}_{ij}^k = \epsilon_{ijk} \hat{I}, \quad (3.43)$$

namely the *product of the algebra is generalized, but the structure constants are the conventional ones.* The above results illustrates again the abstract identity of quantum and hadronic mechanics, this time in one of its most fundamental symmetries. The classical realization of $so(3)$ is formally identical to the above operator one with a unique interconnecting map (see [61b] for details).

The isocenter of $s\hat{o}(3)$ is characterized by the *isocasimir invariants*

$$C^{(0)} = \hat{I}, \quad C^{(2)} = \hat{M}^{\hat{2}} = \hat{M} * \hat{M} = \sum_{k=1,2,3} \hat{M}_k \hat{T} \hat{M}_k. \tag{3.44}$$

Note the non-linear-non-local-non-canonical character of isotransformations (3.42) owing to the unrestricted functional dependence of the diagonal elements g_{kk}. Note also the simplicity of the final results. In fact, the explicit symmetry transformations of separation (3.40) are provided by just plotting the given g_{kk} values into transformations (3.42) without any need of any additional computation.

Despite this simplicity, the implications of the above results are nontrivial. First, the Lie-isotopic theory permits the identification for the first time on record to our knowledge of the universal symmetry for the space-component of all infinitely possible Riemannian metrics, such as the space component of the Schwarzschild line element

$$\mathrm{d}r^2 = (1 - M/r)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\phi, \tag{3.45}$$

which results to be isomorphic to the rotational symmetry and which will be extended in the next section to the full space–time metrics.

Moreover, the isounit $\hat{I} > 0$ permits a direct representation of the non-spherical shapes, as well as all their deformations. By recalling that $O(3)$ is a *theory of rigid bodies*, $\hat{O}(3)$ results to be a *theory of deformable bodies* [53] with fundamentally novel physical applications in the classical mechanics, nuclear physics, particle physics, crystallography, and other fields [61, 72].

On mathematical grounds, we have equally intriguing novel insights. First, the Lie–Santilli isotheory disproves the rather popular belief that the rotational symmetry is broken for the ellipsoidical deformations of the sphere. In fact, the symmetry of the latter merely results to be a more general realization $\hat{O}(3)$ of the rotational symmetry $O(3)$, $\hat{O}(3) \approx O(3)$.

In addition, the rotational symmetry also results to be the symmetry of paraboloids. To see this occurrence, one must first understand the background isogeometry $\hat{E}_{\mathrm{III}}(\hat{r}, \hat{\delta}, \hat{\mathfrak{R}})$ which unifies all possible conics in $E(\hat{r}, \delta, \mathfrak{R})$ [61a], as mentioned in section 2.3. In fact, the geometric differences between (oblate or prolate) ellipsoids and (elliptic or hyperbolic) paraboloids have mathematical sense when projected in the conventional Euclidean space $E(r, \delta, \mathfrak{R})$. However, all these surfaces are geometrically unified with the perfect isosphere in $\hat{E}(\hat{r}, \hat{\delta}, \hat{\mathfrak{R}})$.

These occurrences permit the geometric and algebraic unification of $O(3)$ and $O(2.1)$, as well as of all their infinitely possible isotopes. In fact, the classification of all possible isosymmetries $\hat{O}(3)$, achieved in the original derivation [53], includes:

(1) The compact $O(3)$ symmetry evidently for $\hat{\delta} = \delta = \mathrm{diag.}\ (1, 1, 1)$;
(2) The noncompact $O(2.1)$ symmetry for $\hat{\delta} = \mathrm{diag.}\ (1, 1, -1)$;
(3) The isodual $O^{\mathrm{d}}(3)$ of $O(3)$ holding for $\hat{\delta} = \mathrm{diag.}\ (-1, -1, -1)$;
(4) The isodual $O^{\mathrm{d}}(2.1)$ of $O(2.1)$ holding for $\hat{\delta} = \mathrm{diag.}\ (-1, -1, 1)$;
(5) The infinite family of compact isotopes $\hat{O}(3) \approx O(3)$ for $\delta = \mathrm{diag.}\ (b_1^2, b_2^2, b_3^2)$, $b_k > 0$;
(6) The infinite family of non-compact isotopes $\hat{O}(2.1) \approx O(2.1)$ for $\delta = \mathrm{diag.}$ $(b_1^2, b_2^2, -b_3^2)$;

(7) The infinite family of compact isodual isotopes $\hat{O}^d(3) \approx O^d(3)$ for $\hat{\delta} = \text{diag.}(-b_1^2, -b_2^2, -b_3^2)$;
(8) The infinite family of isodual isotopes $\hat{O}^d(2.1) \approx O^d(2.1)$ for $\hat{\delta} = \text{diag.}(-b_1^2, -b_2^2, b_3^2)$.

Even greater differences between the Lie and Lie–Santilli theories occur in their representations because of the change in the eigenvalue equations due to the non-unitarity of the map indicated in section 3.2, from the familiar form $H\psi = E^0\psi$, to the isotopic form $H * \hat{\psi} = \hat{E} * \hat{\psi} \equiv E\hat{\psi}, E^0 \neq E$), thus implying generalized weights, generalized Cartan tensors and other structures studied earlier. The first differences emerge in the spectrum of eigenvalues of $\hat{O}(2)$ and $O(2)$. In fact, the $o(2)$ algebra on a conventional Hilbert space *solely* admits the spectrum $M = 0, 1, 2, 3$ (as a necessary condition of unitarity). For the covering $\hat{o}(2)$ isoalgebra on an isohilbert space with isotopic element $\hat{T} = \text{Diag.}(g_{11}, g_{22})$, the spectrum is instead given by $\hat{M} = g_{11}^{-1/2}\, g_{22}^{-1/2}\, M$ and, as such, it can acquire *continuous* values in a way fully consistent with the condition, this time, of isounitarity. For the general $\hat{O}(3)$ case see also the detailed studies of refs [61b].

Similar structural differences exist between the spectrum of eigenvalues of $su(2)$ and $s\hat{u}(2)$. In fact, the unitary irreducible representations of $su(2)$ characterize the familiar discrete spectrum

$$J_3\hat{\psi} = M\psi, \quad J^2\psi = J(J+1)\psi, \quad M = J, J-1, \dots, -J, \quad J = 0, \tfrac{1}{2}, 1, \dots \tag{3.46}$$

Three classes of irreducible isorepresentation of $s\hat{u}(2)$ were identified in [62] which, for the adjoint case, are given by the following generalizations of Pauli's matrices:

(1) *Regular isopauli matrices*

$$\hat{\sigma}_1 = \Delta^{-1/2}\begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix} \hat{\sigma}_2 = \Delta^{-1/2}\begin{pmatrix} 0 & -\mathrm{i}g_{11} \\ +\mathrm{i}g_{22} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \Delta^{-1/2}\begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix}, \tag{3.47a}$$

$$\hat{T} = \text{diag.}(g_{11}, g_{22}) > 0, \quad [\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = \hat{\sigma}_i \hat{T} \hat{\sigma}_j - \hat{\sigma}_j \hat{T} \hat{\sigma}_i = \mathrm{i}\, 2\Delta^{1/2}\, \epsilon_{ijk}\, \hat{\sigma}_k. \tag{3.47b}$$

$$\hat{\sigma}_3 * |\hat{b}\rangle = \pm\Delta^{1/2}|\hat{b}\rangle, \quad \hat{\sigma}^2 * |\hat{b}\rangle = 3\Delta|\hat{b}\rangle, \quad \Delta = \det T = g_{11}g_{22} > 0. \tag{3.47c}$$

(2) *Irregular isopauli matrices*

$$\hat{\sigma}'_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \hat{\sigma}'_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix} = \sigma_2, \quad \hat{\sigma}'_3 = \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} = \Delta\hat{I}\sigma_3, \tag{3.48a}$$

$$[\hat{\sigma}'_1 \hat{,} \hat{\sigma}'_2] = 2\mathrm{i}\hat{\sigma}'_3, \quad [\hat{\sigma}'_2 \hat{,} \hat{\sigma}'_3] = 2\mathrm{i}\Delta\hat{\sigma}'_1, \quad [\hat{\sigma}'_3 \hat{,} \hat{\sigma}'_1] = 2\mathrm{i}\Delta\hat{\sigma}'_2, \tag{3.48b}$$

$$\hat{T} = \text{diag.}(g_{11}, g_{22}) > 0, \quad \hat{\sigma}'_3 * |\hat{b}\rangle = \pm\Delta|\hat{b}\rangle, \quad \hat{\sigma}^2 * |\hat{b}\rangle = \Delta(\Delta + 2)|\hat{b}\rangle. \tag{3.48c}$$

(3) *Standard isopauli matrices*

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i}\lambda \\ \mathrm{i}\lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \tag{3.49a}$$

$$\hat{T} = \mathrm{diag.}\,(\lambda, \lambda^{-1}), \quad \lambda \neq 0, \quad \Delta = \det T = 1, \quad [\hat{\sigma}''_i \hat{,} \hat{\sigma}''_j] = \mathrm{i}\,\epsilon_{ijk}\,\hat{\sigma}''_k, \tag{3.49b}$$

$$\hat{\sigma}''_3 * |\hat{b}\rangle = \pm\,|\hat{b}\rangle, \qquad \hat{\sigma}''^{\hat{2}} * |\hat{b}\rangle = 3\,|\hat{b}\rangle. \tag{3.49c}$$

The primary differences in the above isorepresentations are the following. For the case of the regular isorepresentations, the isotopic contributions can be factorized with respect to the conventional Lie spectrum, as expected from the nonunitary character of the map $su(2) \to s\hat{u}(2)$. For the irregular case this is no longer possible. Finally, for the standard case we have conventional spectra of eigenvalues under a generalized structure of the matrix representations, as indicated by the appearance of a completely unrestricted, integro-differential function λ.

The regular and irregular representations of $\hat{o}(3)$ and $s\hat{u}(2)$ are applied to the angular momentum and spin of particles under extreme physical conditions, such as an electron in the core of a collapsing star. The standard isorepresentations are applied to conventional particles evidently because of the preservation of conventional quantum numbers. The appearance of the isotopic degrees of freedom then permit novel physical applications, that is, applications beyond the capacity of Lie's theory even for the simpler case of preservation of conventional spectra (see Section 3.7).

The spectrum-preserving map from the conventional representations J_g of a Lie-algebra L with metric tensor g to the covering isorepresentations $\hat{J}_{\hat{g}}$ of the Lie–Santilli isoalgebra $\hat{L}$ with isometric $\hat{g} = \hat{T}g$ and isounit $\hat{I} = \hat{T}^{-1}$ is important for physical application. It is called the *Klimyk rule* [27] and it is given by

$$\hat{J}_{\hat{g}} = J_g P, \quad P = N\hat{I}, \quad N \in \hat{F}, \tag{3.50}$$

under which Lie algebras are turned into Lie–Santilli isoalgebras

$$J_i J_j - J_j J_i = C^k_{ij} J_k \equiv (\hat{J}_i * \hat{J}_j - \hat{J}_j * \hat{J}_i) N^{-1} \hat{T} = C_{ijk} \hat{J}_k N^{-1} \hat{T}, \tag{3.51}$$

that is,

$$\hat{J}_i * \hat{J}_j - \hat{J}_j * \hat{J}_i = C^k_{ij} \hat{J}_k. \tag{3.52}$$

thus showing the preservation of the original structure constants.

However, by no means, the Klimyk rule can produce *all* Lie–Santilli isoalgebras, because the latter are generally characterized by *non-unitary* transforms of conventional algebras, with a general variation of the structure constants.

Nevertheless, the Klimyk rule is sufficient for a number of physical applications whenever the preservation of conventional quantum numbers is important, because the rule permits the identification of one specific and explicit form of standard isorepresentations with 'hidden' degrees of freedom represented by the isotopic element $\hat{T}$ available for basically novel applications. For instance, the standard isopauli matrices permit; the reconstruction of the exact isospin symmetry in nuclear physics under electromagnetic and weak interactions [62]; the construction of the isoquark theory with all conventional quantum numbers, yet an *exact confinement* (with an identically null probability of tunnel effects for free quarks because of the incoherence between the interior and exterior Hilbert spaces) [68]; the proof that the Bell's inequality and von neumann Theorem do not hold for the isotopic representation of $su(2)$, thus permitting a 'completion' of quantum mechanics precisely of isotopic character [61b]; and other novel applications.

3.6. *Isotopies and isodualities of the Lorentz and Poincaré symmetries*

Consider the line element $x^2 = x^\mu \eta_{\mu\nu} x^\nu$, $\mu, \nu = 1, 2, 3, 4$, in Minkowski space $M(x, \eta, \Re)$ with local co-ordinates $x = \{x^\mu\} = \{x^1, x^2, x^3, x^4\}$, $x^4 = c_0 t$ (c_0 being the speed of light in vacuum), and metric $\eta = \text{diag.}\,(1, 1, 1, -1)$ over the reals $\Re(n, +, \times)$. Its simple invariance group of *linear* transformations is the six-dimensional group $L(3.1)$ first identified by Lorentz in 1905 [34], which is characterized by the (ordered sets of) parameters given by the Euler's angles and speed parameter, $w = \{w_k\} = \{\theta, v\}$, $k = 1, 2, \ldots, 6$, and of generators $X = \{X_k\} = \{M_{\mu\nu} = x_\mu p_\nu - x_\nu P_\mu\}$, in their known fundamental representation (see, e.g., [31, 32]). The most general possible symmetry group of *linear* transformations leaving invariant the separation

$$(x - y)^\mu \eta_{\mu\nu} (x - y)^\nu, \quad \eta = (\eta_{\mu\nu}) = \text{diag.}\,(1, 1, 1, -1), \quad x, y, \in M(x, \eta, \Re) \tag{3.53}$$

is the ten-dimensional group $P(3.1) = L(3.1) \times T(3.1)$ first identified by Poincaré [45] in 1905, where $T(3.1)$ is the group of translations with parameters $a = \{a^\mu\}$.

In three of his most important papers, Santilli [51, 64, 67] introduced a step-by-step generalization of the historical papers by Lorentz and Poincaré, by achieving the universal symmetry of the most general possible, integro-differential, space–time separation with an arbitrarily non-linear dependence on the co-ordinates x, velocities $\dot{x}$, accelerations $\ddot{x}$ and any other needed interior quantity, such as density μ, temperature τ, index of refraction n, etc.,

$$x^{\hat{2}} = x^\mu \hat{\eta}_{\mu\nu}(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots) x^\nu, \quad \det \hat{\eta} \neq 0, \quad \hat{\eta} = \hat{\eta}^\dagger, \tag{3.54}$$

which, for the case $\text{sig}\,\hat{\eta} = \text{sig}\,\eta$, resulted to be locally isomorphic to the conventional Poincaré symmetry.

More specifically, in the first paper [51] Santilli introduced for the first time the isotopies $\hat{L}(3.1)$ of the Lorentz symmetry $L(3.1$, today called the *Lorentz–Santilli isosymmetry*, and identified the consequential isotopies of the special relativity for interior dynamical problems, today called *Santilli's isospecial relativity*. In the second paper [64] Santilli provided a detailed study of the isotopies $\hat{P}(3.1)$ of the Poincaré-symmetry $P(3.1)$, today called the *Poincaré–Santilli isosymmetry*, and presented a number of preliminary experimental verifications. In the third paper [67] Santilli constructed the isotopies $\hat{\mathscr{P}}(3.1) = S\hat{L}(2.\hat{C}) \hat{\times} \hat{T}(3.1)$ of the spinorial covering $\mathscr{P}(3.1) = SL(2.C) \times T(3.1)$ of the Poincaré symmetry $P(3.1) = L(3.1) X T(3.1)$ and their isoduals, applied the new theory to a novel understanding of the synthesis of neutrons from protons and electrons as occurring in the core of stars, and predicted a new form of *subnuclear* energy he called *hadronic energy*. Comprehensive operator studies were then presented in monographs [61] and their classical counterpart in monographs [62]. It should be finally indicated that Santilli is the *sole* author of published contributions at this writing, to our best knowledge, on the isotopies of the Minkowski space, the rotational, Lorentz and Poincaré symmetry and the special relativity. All other contributions which have appeared in the published literature deal with *applications* of the preceding basic isotopies.

Needless to say, to avoid a prohibitive length, in this section we can only indicate the most salient aspects of Santilli's space–time isosymmetries, and merely list in the following subsection some of the important references on applications.

Suppose that a well-behaved and diagonal but otherwise arbitrary deformation $\hat{\eta}(x, \dot{x}, \ldots)$ of the Minkowski metric η is assigned. The explicit form of the simple, six-dimensional *non-linear* invariance of generalized line element $x^\mu \hat{\eta}_{\mu\nu}(x, \dot{x}, \ldots) x^\nu$ can be constructed by following the space–time version of Steps 1–5 of the preceding subsection.

Step 1 is the identification of the fundamental isotopic element $\hat{T}$ via the factorization of the Minkowski in the deformed metric, $\hat{\eta} = \hat{T}\eta$ and the assumption of $\hat{I} = \hat{T}^{-1}$ as the fundamental new unit of the theory.

Step 2 is the lifting of the conventional numbers into the isonumbers via the isofields $\hat{\Re}(\hat{n}, +, *)$, $\hat{n} = n\hat{I}$, where $\hat{I}$ is the same generalized unit of Step 1 (which is different than that of $\hat{O}(3)$ because of the different dimension of the isounit).

Step 3 is the construction of the isospaces in which the isometric $\hat{\eta}$ is properly defined, which are given by the isominkowski spaces of Class III (section 2.3)

$$\hat{M}_{\mathrm{III}}(\hat{x}, \hat{\eta}, \hat{\Re}): \hat{\eta} = \hat{T}(x, \dot{x}, \ddot{x}, \ldots)\eta, \quad \hat{T} = \mathrm{diag.}(g_{11}, g_{22}, g_{33}, g_{44}), \hat{T} = \hat{T}^\dagger,$$
$$\det \hat{T} \neq 0. \tag{3.55a}$$

$$\hat{x}^{\hat{2}} = (\hat{x}^\mu \hat{\eta}(x, \dot{x}, \ddot{x}, \ldots) x^\nu)\hat{I} \in \hat{\Re}, \quad \hat{x} = \{\hat{x}^\mu\} = \{x^\mu\},$$
$$\hat{x}_\mu = \hat{\eta}_{\mu\nu}\hat{x}^\nu = \hat{T}_\mu^{\ \nu} x_\nu. \tag{3.55b}$$

Note that all possible (3 + 1)-dimensional Riemannian *metrics* $g(x)$ are *particular cases* of the much more general isominkowskian metric $\hat{\eta}(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots)$. However, the Riemannian *spaces* $R(x, g(x), \Re)$ are not particular cases of the isominkowskian ones $\hat{M}(\hat{x}, \hat{\eta}, \hat{\Re})$ because the units of the two spaces are different (and also because the former is curved) with respect to the trivial unit I while the latter is isoflat with respect to the isounit $\hat{I}$).

Step 4 is the construction of the *Lorentz–Santilli isosymmetry* $\hat{L}(3.1)$ [51], which can be characterized by the isotransformations

$$\hat{O}(3.1): \quad \hat{x}' = \hat{\Lambda}(\hat{w}) * \hat{x} = \tilde{\Lambda}(w)\hat{x}, \quad \hat{\Lambda} = \tilde{\Lambda}\hat{I} \tag{3.56}$$

verifying the properties

$$\hat{\Lambda}^\dagger \hat{\eta} \hat{\Lambda} = \hat{\Lambda}\hat{\eta}\hat{\Lambda}^\dagger = \hat{I}\hat{\eta}\hat{I}, \text{ or } \tilde{\Lambda}^\dagger \hat{\eta} \tilde{\Lambda} = \tilde{\Lambda}\hat{\eta}\tilde{\Lambda}^\dagger = \hat{\eta}, \tag{3.57a}$$

$$\mathrm{D\hat{e}t}\,\hat{\Lambda} = [\mathrm{Det}(\hat{\Lambda}\hat{T})]\hat{I} = \pm\hat{I}. \tag{3.57b}$$

It is easy to see that $\hat{L}(3.1)$ preserves the original connectivity properties of $L(3.1)$ (see [61b] for a detailed study). The connected component $S\hat{O}(3.1)$ of $\hat{L}(3.1)$ is characterized by $\mathrm{D\hat{e}t}\,\hat{\Lambda} = +\hat{I}$ and has the isogroup structure [51]

$$S\hat{O}(3.1): \quad \hat{\Lambda}(\hat{w}) = \prod_{k=1,2,\ldots,6}^{*} e_{\xi}^{i\hat{X}_k * \hat{w}_k} = \left\{ \prod_{k=1,2,\ldots,6} e^{iX_k T w_k} \right\} \hat{I}, \tag{3.58}$$

where $\hat{w} = w_k\hat{I}$ and the w's are the conventional parameters, $\hat{X}_k = X_k$ are the conventional generators in their fundamental representation and the isotopic element $\hat{T}$ is that identified in Step 1. The discrete part of $\hat{L}(3.1)$ is characterized by Dêt $\hat{\Lambda} = -\hat{1}$, and it is given by the *space–time isoinversions* [51]

$$\hat{\pi} * \hat{x} = \pi x = (-r, x^4), \quad \hat{\tau} * \hat{x} = \tau x = (r, -x^4). \tag{3.59}$$

Again, under the assumed conditions for $\hat{T}$, the convergence of infinite series (3.58) is ensured by the original convergence, thus permitting the explicit calculation of the symmetry transformations in the needed explicit, finite form. Their space components have been given in the preceding section 3.5. The additional *Lorentz–Santilli isoboosts* can also be explicitly computed, yielding the expression for all possible isometrics $\hat{\eta}$ [51]

$$x'^1 = x^1, \quad x'^2 = x^2, \tag{3.60a}$$

$$x'^3 = x^3 \cosh[v(g_{33}\, g_{44})^{1/2}] - x^4 g_{44}(g_{33}\, g_{44})^{1/2} \sinh[v(g_{33}\, g_{44})^{1/2}]$$
$$= \hat{\gamma}(x^3 - g_{33}^{-1/2} g_{44}^{1/2} \hat{\beta} x^4), \tag{3.60b}$$

$$x'^4 = -x^3 g_{33}(g_{33}\, g_{44})^{-1/2} \sinh[v(g_{33}\, g_{44})^{1/2}] + x^4 \cosh[v(g_{33}\, g_{44})^{1/2}]$$
$$= \hat{\gamma}(x^4 - g_{33}^{1/2} g_{44}^{-1/2} \hat{\beta} x^3), \tag{3.60c}$$

where

$$x^4 = c_0 t, \quad \beta = v/c_0, \quad \beta^2 = v^k g_{kk} v^k / c_0 g_{44} c_0, \tag{3.61a}$$

$$\cosh[v(g_{33}\, g_{44})^{1/2}] = \hat{\gamma} = (1 - \hat{\beta}^2)^{-1/2}, \quad \sinh[v(g_{33}\, g_{44})^{1/2}] = \hat{\beta}\hat{\gamma}. \tag{3.61b}$$

Again, one should note: (A) the unrestricted character of the functional dependence of the isometric $\hat{\eta}$; (B) the remarkable simplicity of the final results whereby the explicit symmetry transformations are merely given by plotting the values $g_{\mu\mu}$ in equations (3.60); and (C) the generally non-linear-non-local-non-canonical character of the isotransforms originating from the unrestricted functional dependence of the quantities $g_{\mu\mu}$.

The isocommutation rules of the *Lorentz–Santilli isoalgebra* $\hat{so}(3.1)$ when the generators $\hat{M}_{\mu\nu}$ are in their regular representation can also be readily computed and are given by [51]

$$\hat{o}(3.1)\colon [\hat{M}_{\mu\nu} \hat{,} \hat{M}_{\alpha\beta}] = \hat{\eta}_{\nu\alpha} \hat{M}_{\beta\mu} - \hat{\eta}_{\mu\alpha} \hat{M}_{\beta\nu} - \hat{\eta}_{\nu\beta} \hat{M}_{\alpha\mu} + \hat{\eta}_{\mu\beta} \hat{M}_{\alpha\nu}, \tag{3.62}$$

with isocasimirs

$$\hat{C}^{(0)} = \hat{I}, \quad \hat{C}^{(1)} = \tfrac{1}{2} \hat{M}_{\mu\nu} \hat{T} \hat{M}^{\mu\nu} = \hat{M} * \hat{M} - \hat{N} * \hat{N}, \tag{3.63a}$$

$$\hat{C}^{(3)} = \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} \hat{M}_{\mu\nu} \hat{T} \hat{M}_{\rho\sigma} = -\hat{M} * \hat{N}, \quad \hat{M} = \{\hat{M}_{12}, \hat{M}_{23}, \hat{M}_{31}\}, \tag{3.63b}$$

$$\hat{N} = \{\hat{M}_{01}, \hat{M}_{02}, \hat{M}_{03}\}. \tag{3.63c}$$

The classification of all possible isotopes $S\hat{O}(3.1)$ was also done in the original construction [51] via the realizations of the isotopic element

$$\hat{T} = \text{diag.}(\pm b_1^2, \pm b_2^2, \pm b_3^2, \pm b_4^2), \quad b_\mu > 0, \quad \mu = 1, 2, 3, 4, \tag{3.64}$$

where the b's are the characteristic functions of the interior medium, resulting in:

(1) The conventional orthogonal symmetry $SO(4)$ for $T = \text{diag.}(1, 1, 1, -1)$;
(2) The conventional Lorentz symmetry $SO(3.1)$ for $T = \text{diag.}(1, 1, 1, 1)$;

(3) the conventional de Sitter symmetry $SO(2.2)$ for $T = \text{diag.}(1, 1, -1, 1)$;
(4) the isodual $SO^d(4)$ for $T = \text{diag.}(-1, -1, -1, 1)$;
(5) the isodual $O^d(3.1)$ for $T = -\text{diag.}(1, 1, 1, 1)$;
(6) the isodual $SO^d(2.2)$ for $T = \text{diag.}(-1, -1, 1, -1)$;
(7) the infinite family of isotopes $S\hat{O}(4) \approx SO(4)$ for $\hat{T} = \text{diag.}(b_1^2, b_2^2, b_3^2, -b_4^2)$;
(8) the infinite family of isotopes $S\hat{O}(3.1) \approx SO(3.1)$ for $\hat{T} = \text{diag.}(b_1^2, b_2^2, b_3^2, b_4^2)$;
(9) the infinite family of isotopes $S\hat{O}(2.2) \approx SO(2.2)$ for $\hat{T} = \text{diag.}(-b_1^2, b_2^2, b_3^2, b_4^2)$;
(10) the infinite family of isoduals $S\hat{O}^d(4) \approx SO^d(4)$ for $\hat{T} = \text{diag.}(-b_1^2, b_2^2, b_3^2, b_4^2)$;
(11) the infinite family of isoduals $S\hat{O}^d(3.1) \approx SO(3.1)$ for $\hat{T} = -\text{diag.}(b_1^2, b_2^2, b_3^2, b_4^2)$;
(12) the infinite family of isoduals $S\hat{O}^d(2.2) \approx SO^d(2.2)$ for $T = \text{diag.}(-b_1^2, -b_2^2, -b_3^2, -b_4^2)$.

On the basis of the above results, Santilli [61] submitted the *conjecture* indicated in section 3.1 according to which *all simple Lie algebra of the same dimension over a field of characteristic zero in Cartan classification can be unified into one single abstract isotopic algebra of the same dimension.* This conjecture has remained unexplored beyond the cases $N = 3$ and 6.

In the above presentation we have shown that the lifting of the Lorentz symmetry can be naturally formulated for Class III. In fact, Santilli constructed in [51] the isotopies $S\hat{O}(4)$ of the *orthogonal* group $SO(4)$ of which $S\hat{O}(3.1)$ is a particular case. Nevertheless, whenever dealing with physical applications, the isotopic element is restricted to be positive-definite, $\hat{T} = +\text{diag.}(b_1^2, b_2^2, b_3^2, b_4^2) > 0$, for the description of matter and negative-definite, $\hat{T} = -\text{diag.}(b_1^2, b_2^2, b_3^2, b_4^2) < 0$, for the description of antimatter, thus restricting the isotopies to $S\hat{O}(3.1) \approx SO(3.1)$ and $S\hat{O}^d(3.1) \approx SO^d(3.1)$, respectively.

The operator realization of the Lorentz–Santilli isoalgebra is the following. The linear four-momentum admits the isotopic realization (section 2.7)

$$\hat{p}_\mu * |\hat{\psi}\rangle = -\mathrm{i}\hat{\partial}_\mu |\hat{\psi}\rangle = -\mathrm{i}\,\hat{T}_\mu^{\nu}\partial_\nu |\hat{\psi}\rangle, \quad \hat{\partial}_\mu = \partial/\partial\hat{x}^\mu, \tag{3.65}$$

Also, one can show that $\hat{\partial}_\mu \hat{x}^\nu = \eta_{\mu\nu}$. The fundamental relativistic isocommutation rules are then given by [61, 63]

$$[\hat{x}_\mu \hat{,} \hat{p}_\nu] = \mathrm{i}\hat{\eta}_{\mu\nu}, \quad [\hat{x}_\mu \hat{,} \hat{x}_\nu] = [\hat{p}_\mu \hat{,} \hat{p}_\nu] = 0, \tag{3.66}$$

The isocommutation rules for the generators $\hat{x}^\mu \hat{p}_\nu - \hat{x}^\nu \hat{p}_\mu$ are then given by

$$\hat{o}(3.1)\colon [\hat{M}_{\mu\nu} \hat{,} \hat{M}_{\alpha\beta} = \mathrm{i}(\hat{\eta}_{\nu\alpha}\hat{M}_{\beta\mu} - \hat{\eta}_{\mu\alpha}\hat{M}_{\beta\nu} - \hat{\eta}_{\nu\beta}\hat{M}_{\alpha\mu} + \hat{\eta}_{\mu\beta}\hat{M}_{\alpha\nu}), \tag{3.67}$$

(where $\hat{\eta}_{\mu\nu}$ is the isoMinkowski metric), thus confirming not only the local isomorphism $S\hat{O}(3.1) \approx SO(3.1)$ for all positive-definite $\hat{T}$, but also their identity $S\hat{O}(3.1) \equiv SO(3.1)$ at the abstract, realization-free level.

One should note that the use of the generators $\hat{M}_\nu^\mu = \hat{x}^\mu \hat{p}_\mu - \hat{x}^\nu \hat{p}_\mu$ with rules $[\hat{x}^\mu \hat{,} \hat{p}_\nu] = \mathrm{i}\delta_\nu^\mu$ would imply the *conventional structure constants in the isocommutation rules.*

The *Poincaré–Santilli isosymmetry* $\hat{P}(3.1) = \hat{L}(3.1) \times \hat{T}(3.1)$ and its isodual $\hat{P}^d(3.1) = \hat{L}^d(3.1) \times^d \hat{T}(3.1)$ have been constructed in their classical [72] and operator [61, 64] forms as well as in their isospinorial forms $\mathscr{P}(3.1) = S\hat{L}(2.\hat{C}) \times \hat{T}(3.1)$ [67]. We here limit ourselves to a brief outline of the nonspinorial case mainly to illustrate the advances in the structure of isoalgebras and isogroups studied in this paper.

A generic element of $\hat{P}(3.1)$ can be written $\hat{A} = (\hat{\Lambda}, \hat{a})$, $\hat{\Lambda} \in \hat{O}(3.1)$, $\hat{a} \in \hat{T}(3.1)$ with isocomposition

$$\hat{A}' * \hat{A} = (\hat{\Lambda}', \hat{a}') * (\hat{\Lambda}, \hat{a}) = (\hat{\Lambda} * \hat{\Lambda}', \hat{a} + \hat{\Lambda}' * \hat{a}'), \tag{3.68}$$

The realization important for physical applications is that via conventional generators in their adjoint representation for a system of n particles of non-null mass m_a

$$X = [X_k\} = \{\hat{M}_{\mu\nu} = \sum_a (\hat{x}_{a\mu} \hat{p}_{a\nu} - \hat{x}_{a\nu} \hat{p}_{a\mu}\}, \quad \hat{P}_\mu = \sum_a \hat{p}_{a\mu}\}, \quad k = 1, 2, \ldots, 10, \tag{3.69}$$

and conventional parameters $w = \{w_k\} = \{\theta, v, a\}$, where θ represents the Euler's angles, v represents the Lorentz parameters, and a characterizes conventional space–time translations.

The connected component of the Poincaré–Santilli isogroup is given by

$$\hat{P}(3.1): \quad \hat{x}' = \hat{A} * \hat{x}, \hat{A} = \prod_k^{*} \mathrm{e}_{\xi}^{\mathrm{i}X_k w_k} = \left\{\prod_k \mathrm{e}^{\mathrm{i}X_k T w_k}\right\}\hat{I}, \tag{3.70}$$

where the isotopic element $\hat{T}$ and the Lorentz generators $\hat{M}_{\mu\nu}$ have the same realization as for $\hat{O}(3.1)$. The primary different with isosymmetries $\hat{O}$q((3.1) is the appearance of the isotranslations

$$\hat{T}(3.1) \in \hat{x} = \{\mathrm{e}_{\xi}^{\mathrm{i}P\eta a}\} * \hat{x} = \mathrm{e}_{\xi}^{\mathrm{i}P\hat{g}a}\} * \hat{x} = \hat{x} + a\hat{A}, \quad \hat{T}(3.1) * \hat{p} \equiv 0, \tag{3.71}$$

where the quantities $\hat{A}_\mu$ are given by

$$\hat{A}_\mu = g_{\mu\mu} + a^\alpha [g_{\mu\mu} \hat{,} \hat{P}_\alpha]/1! + a^\alpha a^\beta [[g_{\mu\mu} \hat{,} \hat{P}_\alpha] \hat{,} \hat{P}_\beta]/2! + \cdots \tag{3.72}$$

The *general Poincaré–Santilli isotransformations are then given by* [61, 72, 64],

$$\hat{x}' = \hat{\Lambda} * \hat{x} \qquad \text{Lorentz–Santilli isotransforms,} \tag{3.73a}$$

$$\hat{x}' = \hat{x} + a\hat{A}(x, \dot{x}, \ddot{x}, \ldots), \qquad \text{isotranslations,} \tag{3.73b}$$

$$\hat{x}' = \hat{\pi}_r * \hat{x} = (-r, \hat{x}^4), \qquad \text{space isoinversions,} \tag{3.73c}$$

$$\hat{x}' = \hat{\pi}_t * \hat{x} = (\hat{r}, -\hat{x}^4), \qquad \text{time isoinversions.} \tag{3.73d}$$

The isocommutation rules of $\hat{P}(3.1)$ in the operator realizations indicated earlier are

$$[\hat{M}_{\mu\nu} \hat{,} \hat{M}_{\alpha\beta}] = \mathrm{i}(\hat{\eta}_{\nu\alpha} \hat{M}_{\beta\mu} - \hat{\eta}_{\mu\alpha} \hat{M}_{\beta\nu} - \hat{\eta}_{\nu\beta} \hat{M}_{\alpha\mu} + \hat{\eta}_{\mu\beta} \hat{M}_{\alpha\nu}), \tag{3.74a}$$

$$[\hat{M}_{\mu\nu} \hat{,} \hat{P}_\alpha] = \mathrm{i}(\hat{\eta}_{\mu\alpha} \hat{P}_\nu - \hat{\eta}_{\nu\alpha} \hat{P}_\mu), \quad [\hat{P}_\mu \hat{,} \hat{P}_\nu] = 0, \mu, \nu, \alpha, \beta = 1, 2, 3, 4, \tag{3.74b}$$

where, again, the use of the generators $\sum_a (\hat{x}_a^\mu \hat{p}_\nu - \hat{x}_{aa}^\nu \hat{P}_{a\mu})$ would yield isocommutation rules in terms of the conventional structure constants.

The isocenter is characterized by the isocasimirs

$$\hat{C}^{(0)} = \hat{I}, \quad \hat{C}^{(1)} = \hat{P}^{\hat{2}} = \hat{P}\hat{T}\hat{P} = \hat{\eta}^{\mu\nu} \hat{P}_\mu * \hat{P}_\nu, \tag{3.75a}$$

$$\hat{C}^{(2)} = \hat{W}^{\hat{2}} = \hat{W}_\mu \hat{\eta}^{\mu\nu} \hat{W}_\nu, \quad \hat{W}_\mu = \epsilon_{\mu\alpha\beta\rho} \hat{J}^{\alpha\beta} * \hat{P}^0. \tag{3.75b}$$

The *restricted Poincaré–Santilli isotransformations* occur when the isotopic element $\hat{T}$ is a diagonal matrix with positive constant elements.

The *isodual Poincaré–Santilli isosymmetry* $\hat{P}^\mathrm{d}(3.1)$ is characterized by the isodual generators $\hat{X}_k^\mathrm{d} = -\hat{X}_k$, the isodual parameters $\hat{w}_k^\mathrm{d} = -\hat{w}_k$, and the isodual isotopic element $\hat{T}^{d} = -\hat{T}$, resulting in the change of sign of isotransforms. This implies a novel *law of universal invariant under isoduality* which essentially state that any system which is invariant under a given symmetry is automatically invariant under its isodual. In turn, this law apparently permits novel advances in the study of antiparticles [61].

The main result outlined in this section can therefore be summarized via the following.

Theorem 3.6 (Santilli [51, 64]): *The Poincaré-Santilli isosymmetry $\hat{P}(3.1)$ constructed with respect to the Class III isounit $\hat{I} = \hat{T}^{-1}$ is the universal symmetry for all possible, non-linear, non-local and non-canical separations*

$$(x-y)^{\mu}\hat{\eta}_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)(x-y)^{\nu}, \quad \hat{\eta} = \hat{T}\eta, \tag{3.76}$$

As a first application, $\hat{P}(3.1)$ is the universal symmetry for *all possible exterior (3 + 1)-dimensional gravitational models for matter* with isounit $\hat{I} = [\hat{T}_{gr}(x)]^{-1}$, where $\hat{T}_{gr}$ originates from the factorization of the Riemannian metric $g(x) = \hat{T}_{gr}(x)\eta$, and for *antimatter* with the isounit $\hat{I}^{d}_{gr} = -\hat{I}_{gr}$. In fact, as indicated earlier, all possible Riemannian metrics $g(x)$ for matter are a *particular case* of the isominkowskian metric $\hat{\eta}(x, \dot{x}, \ddot{x}, \ldots)$, while the isodual Riemannian metrics for antimatter $g^{d}(x) = -g(x)$ yield the isodual isounit $\hat{I}^{d}_{gr}$ [61b]. Note that the above results *cannot* be obtained on *curved* Riemannian spaces and require their necessary reformulation as isominkowskian spaces which are *isoflat* with respect to $\hat{I}_{gr}$.

Theorem 3.6 is however much broader than that inasmuch as it implies that $\hat{P}(3.1)$ is the universal symmetry for all possible *interior* gravitational models of matter and antimatter with Class III isometrics $\hat{g}(x, \dot{x}, \ddot{x}, \ldots)$ now computed with respect to the isounit $\hat{I} = [\hat{T}_{gr}(x, \dot{x}, \ddot{x}, \ldots)]^{-1}$, where $\hat{T}_{gr}$ originates from the broader factorization $\hat{g}(x, \dot{x}, \ddot{x}, \ldots) = \hat{T}_{gr}(x, \dot{x}, \ddot{x}, \ldots)\eta$.

The simplicity of this universal invariance should be kept in mind and compared with the known complexity of other approaches to nonlinear (let alone integro-differential) symmetries. In fact, the symmetry is merely given by *plotting* the $\hat{T}_{\mu\mu}$ elements in isotransforms (3.60) or (3.70) without any need to verify anything, because the invariance of general separation (3.76) is ensured by Theorem 3.6. For numerous examples, see [61, 72].

The verification of total conservation laws (for a system assumed as isolated from the result of the universe), is intrinsic in the very structure of the isosymmetry $\hat{P}(3.1)$. In fact, the generators are the conventional ones and, since they are invariant under the action of the group, they characterize conventional total conservation laws. The simplicity of reading off the total conservation laws from the generators of the isosymmetry should be compared with other rather complex proofs, e.g., those for conventional gravitational theories.

The attentive reader has certainly noted that we have reviewed in this section the Poincaré–Santilli isosymmetry for gravitation *in its operator form.* As a result, the Lie–Santilli isotheory has permitted a novel (not necessarily unique) resolution of the historical problem of quantization of gravity via the *unification of gravitation and relativistic quantum mechanics.* As Santilli puts it [61b, Section 9.5.5, 'a consistent quantum gravity already exists. It did creep in unnoticed because it is embedded in the unit of conventional relativistic quantum mechanics'.

In conclusion, the Lie–Santilli isotheory permits the remarkable unification in one, single isosymmetry $\hat{P}$(3.1 of all possible linear or non-linear, local or non-local, Hamiltonian or non-Hamiltonian, relativistic or gravitational, exterior or interior and classical or operator systems [664].

3.7. *Mathematical and physical applications*

Lie's theory is known to be at the foundation of virtually all branches of mathematics. The emergence of intriguing and novel mathematical profiles from the Lie–Santilli theory is then unquestionable.

With the understanding that mathematical studies are at their first infancy, the isotopies have already identified new branches of mathematics, such as: the new branch of number theory dealing with isonumbers; the new branch of functional isoanalysis dealing with $\hat{T}$-operator special isofunctions, isotransforms and isodistribution; the new branch of topology dealing with the peculiar integro-differential topology of the Newton–Santilli mechanics; the new branch of the theory of manifold dealing with isomanifolds and their intriguing properties; the new branch of algebras, groups and their representations dealing with the Lie–Santilli isotheory; and so on. It is hoped that interested mathematicians will contribute to these novel mathematical advances which have been mainly identified and developed until now by physicists.

Lie's theory in its traditional linear-local-canonical formulation is also known to be at the foundation of all branches of contemporary physics. Profound physical implications due to the covering, non-linear-non-local-non-canonical Lie–Santilli isotheory cannot therefore be dismissed in a credible way.

With the understanding that these latter applications too are at the beginning and so much remains to be done, we here recall the following applications of the Poincaré = – Santilli isosymmetry $\hat{P}(3.1)$ (see [61, 72] for details):

(1) The universal invariance of all possible *conventional* gravitation outlined in the preceding subsection.
(2) The geometric unification of the special and general relativities.
(3) The universal invariance for all possible interior extensions of relativistic and gravitational theories.
(4) The reconstruction at the isotopic level of the *exact* $SU(2)$-isospin symmetry *under electromagnetic and weak interactions* via the use of the standard isopauli matrices (3.52) with $\lambda^2 = m_p/m_n$;
(5) An exact-numerical representation of Rauch's interferometric measures on the 4π-spinorial symmetry via the isotopies of Dirac's equation invariant under $\hat{\mathscr{P}}(3.1)$;
(6) The first numerical representation of the total magnetic moment of few-body nuclei via the $S\hat{O}(3)$ symmetry and its representation of the deformation of the charge distribution of nucleons with consequential alteration of their intrinsic magnetic moments [69];
(7) Nonlocal representation of the Bose–Einstein correlation from first isotopic principles via the isominkowskian geometrization of the p-$\bar{p}$ fireball in full agreement with the data from the UA1 experiments, while permitting a causal description of nonlocal interactions and the reconstruction of their exact Poincaré symmetry at the isotopic level [58, 8];
(8) A quantitative representation of the attraction among the identical electrons of the electron pairing in superconductivity as originating from non-local–non-potential effects in outstanding agreement with experimental data [1];

(9) An exact-numerical representation of the behaviour of the meanlives of unstable hadrons with speed (which, as well known, are anomalous between 30 and 100 GeV and conventional between 100 and 350 GeV for the K^0-system) via the isominkowskian geometrization of the physical medium in their interior [7, 8];

(10) Application to quarks theories via Klimyk rule on the standard isorepresentations of $S\hat{U}(3)$ with conventional quantum numbers, exact confinement of quarks (permitted by the incoherence of the interior isohilbert and exterior Hilbert spaces), and other intriguing possibilities, such as the regaining of *convergent* perturbative series for *strong* interactions (which is possible whenever $|T| \ll 1$) [68];

(11) A numerical representation of Arp's measures on quasars redshift as being partly due to the decrease of speed of light in their chromospheres under isominkowskian geometrization [37];

(12) A numerical representation of the joint redshift and blueshifts of pairs of quasars, particularly when proved via gamma spectroscopy to be physically connected to the associated galaxies, and prediction of a measurable isominkowskian redshift for sunlight at sunset [70];

(13) Application to local realism via the proof that Bell's inequality, von Neumann's theorem and all that are inapplicable (rather than 'violated') under isotopies (evidently because of the non-unitary structure of the lifting), thus permitting an isotopic completion of quantum mechanics much along the celebrated E–P–R argument [65];

(14) Application to q-deformations, discrete time theories and other ongoing studies via their axiomatization into a form invariant under their own time evolution [33];

(15) Novel possibilities in theoretical biology, such as a quantitative representation of the growth of sea shells which, according to computer simulations, crack during their growth if subjected to the conventional Minkowskian geometry, while admitting a normal growth under the covering isominkowskian geometry of Class III (the latter one being needed to represent bifurcations which require inversions of time) [60].

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