POWER-ASSOCIATIVE RINGS

BY

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INTRODUCTION

We use the term *ring* for any additive abelian group closed with respect to a product operation such that the two-sided distributive law holds. When the associative law for products also holds we call the ring an *associative* ring. Every element x of any ring $\mathfrak{A}$ generates a subring $\mathfrak{A}(x)$ of $\mathfrak{A}$ consisting of all finite sums of terms each of which is a finite product whose factors are all equal to x. We call $\mathfrak{A}$ a *power-associative* ring when every $\mathfrak{A}(x)$ is an associative subring of $\mathfrak{A}$.

We have shown elsewhere[1] that a ring $\mathfrak{A}$ whose characteristic is zero is power-associative if and only if $xx^2 = x^2x$ and $x^2x^2 = (x^2x)x$ for every x of $\mathfrak{A}$. This result is also true for all commutative rings having characteristic prime to 30, and the stated restrictions on the characteristic are actually necessary.

Our present investigation begins with a derivation of results on the decompositions of a power-associative ring relative to its idempotents. When e is an idempotent of a commutative power-associative ring $\mathfrak{A}$, the corresponding (right) multiplication R_e is an endomorphism of A having simple elementary divisors and roots $0, 1/2, 1$. There is a resulting decomposition of $\mathfrak{A}$ as the supplementary sum $\mathfrak{A}_e(1) + \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0)$ of submodules $\mathfrak{A}_e(\lambda)$ such that $xe = \lambda x$. Moreover the multiplication relations for these submodules are nearly those holding for the case[2] where $\mathfrak{A}$ is a Jordan ring. However, the situation becomes much more complicated when $\mathfrak{A}$ is not commutative since then the elementary divisors of R_e need not be simple and the characteristic roots are quite arbitrary.

It is true, nevertheless, that a decomposition theory may be obtained for all power-associative rings $\mathfrak{A}$ in which the equation $2x = a$ has a unique solution x in $\mathfrak{A}$ for every a of $\mathfrak{A}$. In this case we may always attach to $\mathfrak{A}$ a commutative ring $\mathfrak{A}^{(+)}$ which is the same additive group as $\mathfrak{A}$ and which has a product $x \cdot y$ defined in terms of the product xy of $\mathfrak{A}$ by $2(x \cdot y) = xy + yx$. The ring $\mathfrak{A}^{(+)}$ is power-associative when $\mathfrak{A}$ is, and every idempotent of $\mathfrak{A}$ is also an idempotent of $\mathfrak{A}^{(+)}$. This yields a decomposition of $\mathfrak{A} = \mathfrak{A}_e(1) + \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0)$ where $\mathfrak{A}_e(\lambda)$ is the set of all x such that $xe + ex = 2\lambda x$, and the submodules always have some of the multiplicative properties of the Jordan

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[1] *On the power-associativity of rings*, Summa Braziliensis Mathematicae vol. 2 (1948) pp. 21–33.

[2] Cf. §10 of the author's *A structure theory for Jordan algebras*, Annals of Mathematics vol. 48 (1947) pp. 546–567.

case. The list of properties grows when we assume that $\mathfrak{A}$ is a *flexible* ring, that is, $x(yx) = (xy)x$ for every x and y of $\mathfrak{A}$. It becomes essentially complete when $\mathfrak{A}^{(+)}$ is assumed to have one further property of Jordan rings.

While the theory of the decomposition relative to an idempotent is a basic part of a general structure theory for power-associative rings and algebras, one can hardly hope to derive a complete structure theory even for commutative power-associative algebras. It then becomes desirable to restrict the study by a proper selection of additional hypotheses. One possible line of investigation lies in the study of what are known as *shrinkable* algebras. We shall discuss all commutative shrinkable algebras of *shrinkability level* two here, and shall show that those algebras belonging to the classes of algebras containing algebras with a unity element are either Jordan algebras or are defined by the identity $x^2y^2 + (xy)^2 = (x^2y)y + (y^2x)x$. We shall also give a structure theory for algebras of the latter type.

A second line of investigation consists of an attempt at generalizing the Jordan algebra so as to delete the commutative law. We shall give a two postulate definition of a class of algebras including both Jordan and associative algebras and shall give a complete structure theory for these "standard" algebras. The simple standard algebras turn out to be merely associative or Jordan algebras and so this investigation does not yield any new types of simple algebras.

The final line of investigation we shall present here is a complete determination of those algebras $\mathfrak{A}$ such that $\mathfrak{A}^{(+)}$ is a simple Jordan algebra. We are first led to attach to any algebra $\mathfrak{B}$ over a field $\mathfrak{F}$ an algebra $\mathfrak{B}(\lambda)$ defined for every λ of $\mathfrak{F}$. This algebra is the same vector space over $\mathfrak{F}$ as $\mathfrak{B}$ but the product $x \cdot y$ in $\mathfrak{B}(\lambda)$ is defined in terms of the product xy of $\mathfrak{B}$ by $x \cdot y = \lambda xy + (1-\lambda)yx$. We then call an algebra $\mathfrak{A}$ over $\mathfrak{F}$ a *quasiassociative* algebra if there exists a scalar extension $\mathfrak{K}$ of $\mathfrak{F}$ (necessarily of degree $n = 1, 2$ over $\mathfrak{F}$), a quantity λ in $\mathfrak{K}$, and an associative algebra $\mathfrak{B}$ over $\mathfrak{K}$, such that $\mathfrak{A}_\mathfrak{K} = \mathfrak{B}(\lambda)$. The structure of quasiassociative algebras is readily determined and we shall conclude our work by showing that if $\mathfrak{A}^{(+)}$ is a simple Jordan algebra then $\mathfrak{A}$ is either $\mathfrak{A}^{(+)}$ or is a simple quasiassociative algebra.

CHAPTER I. NILRINGS AND IDEMPOTENTS

1. Power-associativity. While our results on power-associativity have been published elsewhere([2]) they may not be accessible readily and so will be summarized here.

If $\mathfrak{A}$ is any ring we shall say that the *characteristic* of $\mathfrak{A}$ is *prime* to n if the sum $nx = 0$ only if $x = 0$. We shall also say that $\mathfrak{A}$ has *characteristic zero* providing that the characteristic is prime to n for all integers $n > 1$.

The *right powers* of the elements x of a ring $\mathfrak{A}$ are defined by the formula $x^{k+1} = x^k x$, $k \geq 1$. Then $\mathfrak{A}$ is power-associative if and only if $x^\alpha x^\beta = x^{\alpha+\beta}$ for all positive integers α, β. Assume first that the characteristic of $\mathfrak{A}$ is prime to

two and define

$$[x, y] = xy - yx.$$

Then the hypothesis $x^3 = xx^2$ may be written as $[x, x^2] = 0$ and a linearization process implies that

(1) $$[xy + yx, z] + [yz + zy, x] + [zx + xz, y] = 0.$$

Conversely (1) implies that $xx^2 = x^2x$ if the characteristic of $\mathfrak{A}$ is prime to six. Formula (1) may then be used to obtain

LEMMA 1. *Let the characteristic of $\mathfrak{A}$ be prime to two, $n \geq 4$, and $x^\lambda x^\mu = x^{\lambda+\mu}$ for all positive integers λ, μ such that $\lambda + \mu < n$. Then*

(2) $$n[x^{n-1}, x] = 0, \quad [x^{n-\alpha}, x^\alpha] = \alpha[x^{n-1}, x] \qquad (\alpha = 1, \cdots, n - 1).$$

The hypothesis $x^2x^2 = x^3x$ yields, as the consequence of a linearization process, the relation

(3) $$\sum_6 (xy + yx)(zw + wz) = \sum_4 \left[\sum_3 (zw + wz)y \right] x$$

for any ring $\mathfrak{A}$ whose characteristic is prime to two. Here the sums are taken over all possible selections of the symbols involved and $\sum_k$ is a sum of k terms. Conversely (3) implies that $x^2x^2 = x^3x$ providing that the characteristic of $\mathfrak{A}$ is prime to six.

If the characteristic of $\mathfrak{A}$ is prime to two, the symbol $2^{-1}[x^{n-1}, x]$ has meaning, since either $[x^{n-1}, x] = 0$ or $[x^{n-1}, x] \neq 0$, $m[x^{n-1}, x] = 0$, and $2^{-1}[x^{n-1}, x] = \eta[x^{n-1}, x]$ where $m\xi + 2\eta = 1$ and m is an odd divisor of n. We use this concept in the statement of

LEMMA 2. *Let $\mathfrak{A}$ be a ring whose characteristic is prime to 30, $n \geq 5$, $x^\lambda x^\mu = x^{\lambda+\mu}$ for $\lambda + \mu < n$. Then*

(4) $$x^{n-\alpha}x^\alpha = x^{n-1}x + \frac{\alpha - 1}{2}[x^{n-1}, x] \qquad (\alpha = 1, \cdots, n - 1),$$

from which we may derive $[x^{n-\alpha}, x^\alpha] = \alpha[x^{n-1}, x]$, $n[x^{n-1}, x] = 0$ and thus $x^{n-\alpha}x^\alpha = x^n$ if n is prime to the characteristic of $\mathfrak{A}$.

The proof of the lemma above is made by replacing x by x^α, y by x^β, z by x^γ, w by x^δ in (3) where $\alpha + \beta + \gamma + \delta = n$. It has the following consequences.

LEMMA 3. *Let $\mathfrak{A}$ be a ring of characteristic zero and $x^2x = xx^2$, $x^2x^2 = (x^2x)x$ for every x of $\mathfrak{A}$. Then $\mathfrak{A}$ is power-associative.*

LEMMA 4. *Let $\mathfrak{A}$ be a commutative ring whose characteristic is prime to 30 and let $x^2x^2 = (x^2x)x$. Then $\mathfrak{A}$ is power-associative.*

The conditions on the characteristic given in these results are actually necessary as has been shown([1]) by counterexamples.

2. **The ring** $\mathfrak{A}^{(+)}$. We shall restrict our attention to power-associative rings whose characteristic is prime to six. We shall also assume that the equation $2x=a$ has a unique solution x in $\mathfrak{A}$ for every a of $\mathfrak{A}$. Then we may define an attached ring $\mathfrak{A}^{(+)}$ which is the same additive group as $\mathfrak{A}$ but which has a product operation $x \cdot y$ defined in terms of the product operation xy of $\mathfrak{A}$ by $2(x \cdot y) = xy + yx$. The ring $\mathfrak{A}^{(+)}$ is a commutative ring and powers in $\mathfrak{A}^{(+)}$ coincide with powers in $\mathfrak{A}$. Thus $\mathfrak{A}^{(+)}$ is power-associative. Note that the construction of $\mathfrak{A}^{(+)}$ could yield a power-associative ring even when $\mathfrak{A}$ itself is not power-associative. Indeed consider the algebra $\mathfrak{A}$ over the field $\mathfrak{F}$ with a basis a, a^2, aa^2, a^2a defined so that all products $a \cdots a$ with $n \geq 4$ factors are zero. In $\mathfrak{A}^{(+)}$ the identity $x \cdot x^2 = x^2 \cdot x$ is a trivial consequence of the commutative law, and $x^\alpha \cdot x^\beta = 0$ if $\alpha + \beta \geq 4$, $x^\alpha \cdot x^\beta = x^{\alpha+\beta}$ for all positive integers α, β. Then $\mathfrak{A}^{(+)}$ is power-associative but $aa^2 \neq a^2a$ in $\mathfrak{A}$.

In any ring $\mathfrak{A}$ the mapping $a \rightarrow ax$ is an endomorphism R_x of the additive group $\mathfrak{A}$ and is called a *right multiplication* of $\mathfrak{A}$. Similarly the endomorphism L_x defined by $a \rightarrow xa = aL_x$ is called a *left multiplication* of $\mathfrak{A}$. Under the assumption we have made, every endomorphism S of $\mathfrak{A}$ determines a unique endomorphism $S/2$ and the mapping

$$(5) \qquad T_x = 2^{-1}(R_x + L_x)$$

is the unique endomorphism of $\mathfrak{A}$ defining the generic multiplication $a \cdot x = aT_x$ of $\mathfrak{A}^{(+)}$. We shall use this property when we apply the theory of the decomposition of a commutative ring relative to an idempotent to non-commutative rings.

3. **Multiplication identities.** If $\mathfrak{A}$ is a power-associative ring whose characteristic is prime to six we have in (1) a result which may be written as

$$(6) \qquad R_{xy+yx} - L_{xy+yx} = (R_x + L_x)(R_y - L_y) + (R_y + L_y)(R_x - L_x).$$

Also (3) holds and may be written as

$$
\begin{aligned}
L_{(xy+yx)z} &+ L_{(yz+zy)x} + L_{(zx+xz)y} \\
&= (R_x + L_x)(R_{yz+zy} + L_{yz+zy} - R_y R_z - R_z R_y) \\
&\quad + (R_y + L_y)(R_{xz+zx} + L_{xz+zx} - R_x R_z - R_z R_x) \\
&\quad + (R_z + L_z)(R_{xy+yx} + L_{xy+yx} - R_x R_y - R_y R_x) \\
&\quad - (L_{xy+yx}R_z + L_{xz+zx}R_y + L_{yz+zy}R_x).
\end{aligned}
$$

(7)

Take $x = y$ in (6) and obtain

$$(8) \qquad R_{xx} - L_{xx} = (R_x + L_x)(R_x - L_x).$$

Also take $y = z = x$ in (7) and obtain

(9) $$L_{(xx)x} = (R_x + L_x)(R_{xx} + L_{xx} - R_x^2) - L_{xx}R_x.$$

We may now prove

LEMMA 5. *Let $\mathfrak{A}_x$ be the set of all finite sums of products of finite numbers of factors equal to R_x, L_x or R_{xx} so that $\mathfrak{A}_x$ is either zero or an associative ring for every x of $\mathfrak{A}$. Then $\mathfrak{A}_x$ contains R_w and L_w for all positive integral powers $w = x^k$ of x.*

The result has already been shown true for $k = 1$, 2 by (8). Assume it true for $k \leq t+1$ and write $w = x^{t+2}$. Put $y = x^{t+1}$ in (6) and have, since $xy + yx = 2w$,

(10) $$2(R_w - L_w) = (R_x + L_x)(R_y - L_y) + (R_y + L_y)(R_x - L_x).$$

Next replace z by x^t and y by x in (7) so that $(xy+yx)z = (yz+zy)x = (zx+xz)y = 6w$. The right member of (7) is then in $\mathfrak{A}_x$ by the hypothesis of our induction, L_w is in $\mathfrak{A}_x$, R_w is in $\mathfrak{A}_x$ by (10).

In the particular case where $x = e = e^2$ formulas (8) and (9) take the form

(11) $$(R_e + L_e - I)(R_e - L_e) = 0,$$

(12) $$(R_e + L_e)^2 - (R_e + L_e)R_e^2 - L_e(R_e + I) = 0,$$

where I is the identity endomorphism of $\mathfrak{A}$. It is not possible to eliminate R_e or L_e between these two equations so as to obtain a fixed characteristic equation of finite degree for either R_e or L_e. For the characteristic roots of R_e, in the case of a noncommutative power-associative algebra, may be completely arbitrary. We shall give an example illustrating this property later.

When $\mathfrak{A}$ is commutative (7) becomes

(13) $$\begin{aligned} &R_{x(yz)} + R_{y(zx)} + R_{z(xy)} \\ &= 4(R_x R_{yz} + R_y R_{zx} + R_z R_{xy}) - (R_{yz}R_x + R_{zx}R_y + R_{xy}R_z) \\ &\quad - [R_x(R_y R_z + R_z R_y) + R_y(R_z R_x + R_x R_z) + R_z(R_x R_y + R_y R_x)], \end{aligned}$$

and (6) is vacuous. Also (8) is vacuous while (9) becomes

(14) $$R_{(xx)x} = 4R_x R_{xx} - R_{xx}R_x - 2R_x^3.$$

We may then use (13) with $y = x$, $z = x^2$ to obtain a relation which may be combined with (14) to yield

(15) $$R_{(xx)(xx)} = R_{xx}^2 + 10R_x^2 R_{xx} - 6R_x R_{xx}R_x - 4R_x^4.$$

We shall use these formulas later.

4. Nilrings and nilideals. In a power-associative ring $\mathfrak{A}$ all positive integral powers a^k of any element a of $\mathfrak{A}$ are uniquely determined and so the meaning of the nilpotency of the elements of $\mathfrak{A}$ should be clear. We shall say that $\mathfrak{A}$ is a *nilring* if all elements of $\mathfrak{A}$ are nilpotent and shall say that $\mathfrak{A}$ has

bounded index $t>1$ if $a^t=0$ for every a of $\mathfrak{A}$, $b^{t-1}\neq 0$ for some element b of $\mathfrak{A}$.

An ideal $\mathfrak{B}$ of $\mathfrak{A}$ will be called a *nilideal* if $\mathfrak{B}\neq 0$ is a nilring. The sum of two nilideals is a nilideal. For if $\mathfrak{B}$ and $\mathfrak{C}$ are nilideals their sum $(\mathfrak{B}, \mathfrak{C})$ consists of all sums $b+c$ of elements b of $\mathfrak{B}$ and c of $\mathfrak{C}$. Then $b^k=0$, $(b+c)^k = b^k+c_1=c_1$ is in $\mathfrak{C}$, $c_1^t=(b+c)^{kt}=0$ for some t. Thus $(\mathfrak{B}, \mathfrak{C})$ is a nilideal.

It follows trivially that the sum $(\mathfrak{B}_1, \cdots, \mathfrak{B}_r)$ of any finite number of nilideals is a nilideal. We define the *union* $\mathfrak{N}$ of all nilideals of $\mathfrak{A}$ to be the set of all finite sums $b=b_1+ \cdots +b_r$ where b_i is in a nilideal $\mathfrak{B}_i$ of $\mathfrak{A}$. Then b is in $(\mathfrak{B}_1, \cdots, \mathfrak{B}_r)$ and is nilpotent. Evidently $\mathfrak{N}$ is a nilideal of $\mathfrak{A}$. We shall call $\mathfrak{N}$ the *nilradical* of $\mathfrak{A}$ and see that $\mathfrak{A}-\mathfrak{N}$ must have no nilideals.

In the case of a Jordan algebra $\mathfrak{A}$ the maximal nilideal of $\mathfrak{A}$ actually coincides with the maximal solvable ideal of $\mathfrak{A}$ and is, indeed, the maximal nilpotent ideal of $\mathfrak{A}$ in the sense that there exists an integer k such that all products of k elements of $\mathfrak{N}$ are zero. In the general power-associative ring case no such result is to be expected and indeed every simple Lie algebra is a nilring. One can then hardly expect to be able to prove that a nilring is nilpotent but a limited result of this type is provable.

THEOREM 1. *Let $\mathfrak{A}$ be a commutative power-associative ring of nilindex $t\leq 4$. Then $\mathfrak{A}_x$ is zero or a nilpotent associative ring for every x of $\mathfrak{A}$.*

We first observe that if $\mathfrak{A}$ has nilindex two then $(a+b)^2=a^2+2ab+b^2 = 2ab=0$ and so $ab=0$, $\mathfrak{A}$ is a zero ring, $\mathfrak{A}_x$ consists of the zero endomorphism only. We next let $t=3$ so that there exists an element z in $\mathfrak{A}$ such that $z^2\neq 0$, $R_z\neq 0$. The relation $x^2x=0$ implies that

$$(16) \qquad (xy)z + (xz)y + (yz)x = 0.$$

Indeed the left member of (16) arises in the derivation of (1) from the term x^2x in $x^2x=xx^2$. But then

$$(17) \qquad R_{xy} + R_xR_y + R_yR_x = 0.$$

Then $R_{xx}= -2R_x^2$, R_{xx} is commutative with R_x and $\mathfrak{A}$ is actually a Jordan ring. Also $R_{(xx)x}= -2R_xR_{xx}=4R_x^3=0$ by (17) and so $\mathfrak{A}_x$ is actually generated by the nilpotent endomorphism R_x, $\mathfrak{A}_x$ is nilpotent of nilindex at most three. In the case where $\mathfrak{A}$ is a Jordan algebra it is actually known that the union of all of the algebras $\mathfrak{A}_x$ is a nilpotent algebra.

There remains the case $t=4$. The hypothesis $x^2x^2=(x^2x)x$ implies that

$$
\begin{aligned}
(18) \quad & 4[(xy)(zw) + (xz)(yw) + (xw)(yz)] \\
& = x[y(zw) + z(wy) + w(yz)] + y[x(zw) + z(wx) + w(xz)] \\
& \quad + z[x(yw) + y(wx) + w(xy)] + w[x(yz) + y(zx) + z(xy)]
\end{aligned}
$$

when $\mathfrak{A}$ is commutative, and the left member of (18) arises from the linearization of x^2x^2. Then $x^2x^2=0$ implies that the left member of (18) is zero, that is,

(19) $R_xR_{yz} + R_yR_{zx} + R_zR_{xy} = 0.$

It follows that $R_xR_{xx}=0$ and we may use (15) to obtain $R_{xx}^2=4R_x^4$. But then $R_xR_{xx}^2=4R_v^5=0$, R_x is nilpotent. To prove that $\mathfrak{A}_x$ is nilpotent we observe that the quantities of $\mathfrak{A}_x$ are finite sums S of products $S_1 \cdots S_\alpha$ where $\alpha \geqq 1$ and $S_i = R_x$ or $S_i = R_{xx}$. Then every product of six elements of $\mathfrak{A}_x$ is a finite sum of products $P = S_1 \cdots S_\beta$ where $\beta \geqq 6$. If $S_1 = R_x$ then $P = 0$ regardless of the values of the S_j. If $S_1 = R_{xx} = S_2$ then $S_1S_2 = 4R_x^4$ and $S_1S_2S_3 = 4R_x^5$ or $4R_x^4R_{xx}$ and in either case $P = 0$. Finally let $S_1 = R_{xx}$ and $S_2 = R_x$. Then $S_2 \cdots S_6 = 0$ in every case and $P = 0$, $\mathfrak{A}_x$ is nilpotent.

We shall leave open the question as to whether or not commutative nil-rings of index $t \geqq 5$ possess the property of the theorem and pass on now to the case of a noncommutative ring not possessing this property. Indeed, let $\mathfrak{A}$ be an algebra over a field $\mathfrak{F}$ with a basis u, v, uv over $\mathfrak{F}$ such that

$$u^2 = v^2 = (uv)^2 = 0, \qquad vu + uv = 0,$$

$$(uv)u = -u(uv) = v, \qquad (uv)v = -v(uv) = u.$$

Then if $x = \lambda u + \mu v + \nu uv$ the quantity $x^2 = \lambda^2 u^2 + \mu^2 v^2 + \nu^2(uv)^2 + \lambda\mu(uv+vu)$ $+\lambda\nu[u(uv)+(uv)u]+\mu\nu[v(uv)+(uv)v] = 0$. It follows that $\mathfrak{A}$ is a power-associative nilring of nilindex two. But evidently both of the transformations R_u and R_v are not nilpotent. Indeed, their matrices are

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and R_u^4 and R_v^2 are idempotent.

It is easily verified that the algebra defined above is a simple Lie algebra. Indeed, if $\mathfrak{A}$ is any simple Lie algebra then $\mathfrak{A}$ has nilindex two. However Engel's theorem states that the multiplications R_x are all nilpotent only when $\mathfrak{A}$ is nilpotent. Note that even in the case of the solvable Lie algebra with a basis u, v, uv such that $u^2 = v^2 = (uv)^2 = (uv)v = v(uv) = 0$, $vu = -uv$, $(uv)u = -u(uv) = v$ we have R_u^4 idempotent.

5. **Idempotents in a commutative ring.** In a commutative power-associative ring (12) becomes

(20) $2R_e^3 - 3R_e^2 + R_e = (2R_e - I)(R_e - I)R_e = 0$

for every idempotent e of $\mathfrak{A}$. Define $\mathfrak{A}_e(\lambda)$ to be the set of all elements $a_e(\lambda)$ of $\mathfrak{A}$ such that

(21) $ea_e(\lambda) = \lambda a_e(\lambda),$

where (20) implies that necessarily $\lambda = 0$, 1, 1/2. Then $\mathfrak{A}_e(\lambda)$ is a submodule of $\mathfrak{A}$ and it is actually true that $\mathfrak{A}$ is the supplementary sum

(22) $$\mathfrak{A} = \mathfrak{A}_e(1) + \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0).$$

In fact every a of $\mathfrak{A}$ is uniquely expressible in the form $a_e(1) + a_e(1/2) + a_e(0)$ where

(23)
$$a_e(1) = a(2R_e^2 - R_e),$$
$$a_e(1/2) = 4a(R_e - R_e^2),$$
$$a_e(0) = a(I - 3R_e + 2R_e^2).$$

That $a_e(\lambda)$ is in $\mathfrak{A}_e(\lambda)$ follows by forming the product $ea_e(\lambda)$ and using (20). The uniqueness is then a consequence of the fact that if $a_e(1) + a_e(1/2) + a_e(0) = 0$ then we may multiply by e to obtain $a_e(1) + 2^{-1}a_e(1/2) = 0$, $a_e(1) + 4^{-1}a_e(1/2) = 0$, $4^{-1}a_e(1/2) = a_e(1/2) = 0 = a_e(1) = a_e(0)$. We now prove

THEOREM 2. *The modules $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ in a commutative power-associative ring are zero or orthogonal subrings of $\mathfrak{A}$. They are related to $\mathfrak{A}_e(1/2)$ by the inclusion relations*

(24)
$$\mathfrak{A}_e(1/2)\mathfrak{A}_e(1/2) \leqq \mathfrak{A}_e(1) + \mathfrak{A}_e(0),$$
$$\mathfrak{A}_e(1)\mathfrak{A}_e(1/2) \leqq \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0),$$
$$\mathfrak{A}_e(0)\mathfrak{A}_e(1/2) \leqq \mathfrak{A}_e(1/2) + \mathfrak{A}_e(1).$$

To prove these relations we put $z = w = e$ in (18) and let $xe = \lambda x$, $ye = \mu y$. This yields

(25)
$$2[(xy)e](\lambda + \mu) + 2[(xy)e]e + (xy)[2\lambda^2 + \lambda + 2\mu^2 + \mu]$$
$$= 8\lambda\mu xy + 4(xy)e.$$

When $\lambda = \mu = 1$ we obtain $(xy)(R_e^2 - I) = 0$ whence $(xy)(R_e - I) = 0$, that is, xy is in $\mathfrak{A}_e(1)$, $\mathfrak{A}_e(1)$ is a subring of $\mathfrak{A}$. Similarly $\lambda = \mu = 0$ yields $(xy)(R_e^2 - 2R_e) = (xy)R_e(R_e - 2I) = 0$, $R_e - 2I$ is nonsingular, $(xy)R_e = (xy)e = 0$ and $\mathfrak{A}_e(0)$ is a subring of $\mathfrak{A}$. The values $\lambda = 1$, $\mu = 0$ yield $(xy)(2R_e^2 - 2R_e + 3I) = 0$ and so $xy(2R_e^3 - 2R_e^2 + 3R_e) = 0$. By (20), $xy(R_e^2 + 2R_e) = 0$, $xyR_e = 0$, $3xy = 0$, $xy = 0$. There remains the case of products $\mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2)$, that is, $\mu = 1/2$. If $\lambda = 1$ then $2\lambda^2 + \lambda + 2\mu^2 + \mu - 8\lambda\mu = 2 + 1 + 1 - 4 = 0$ and $(xy)(2R_e^2 - R_e) = 0$, xy is in $\mathfrak{A}_e(1/2) + \mathfrak{A}_e(0)$ by (23). If $\lambda = 0$ then $(xy)(2R_e^2 - 3R_e + I) = 0$ and xy is in $\mathfrak{A}_e(1/2) + \mathfrak{A}_e(1)$. The remaining value $\lambda = 1/2$ yields $(xy)(R_e^2 - R_e) = 0$, that is, xy is in $\mathfrak{A}_e(1) + \mathfrak{A}_e(0)$.

In the case of Jordan algebras the last two inclusion relations of (24) may be replaced by the sharper result stating that

(26) $$\mathfrak{A}_e(1)\mathfrak{A}_e(1/2) \leqq \mathfrak{A}_e(1/2), \qquad \mathfrak{A}_e(0)\mathfrak{A}_e(1/2) \leqq \mathfrak{A}_e(1/2).$$

However (26) does not hold for all commutative power-associative rings. Indeed, let $\mathfrak{A}$ be an algebra over a field $\mathfrak{F}$ of characteristic not 2, 3, or 5 and let

$\mathfrak{A}$ have a basis e, f, g, h with a multiplication table given by $e^2 = e$, $ef = f$, $eg = 2^{-1}g$, $gh = f$, $eh = f^2 = g^2 = h^2 = fg = fh = 0$ and the commutative law. Then $\mathfrak{A}_e(1) = e\mathfrak{F} + f\mathfrak{F}$, $\mathfrak{A}_e(1/2) = g\mathfrak{F}$, $\mathfrak{A}_e(0) = h\mathfrak{F}$, $\mathfrak{A}_e(0)\mathfrak{A}_e(1/2) = f\mathfrak{F} \leqq \mathfrak{A}_e(1)$, $f\mathfrak{F}$ is not in $\mathfrak{A}_e(1/2)$ and (20) does not hold. Moreover $\mathfrak{A}$ is power-associative since if $x = \alpha e + \beta f + \gamma g + \delta h$ for α, β, γ, δ in $\mathfrak{F}$ then $x^2 = \alpha^2 e + 2(\alpha\beta + \gamma\delta)f + \alpha\gamma g$, $x^2 x^2 = \alpha^4 e + 4\alpha^2(\alpha\beta + \gamma\delta)f + \alpha^3\gamma g$, $x^2 x = \alpha^3 e + 3\alpha(\alpha\beta + \gamma\delta)f + \alpha^2\gamma g$, $(x^2 x)x = \alpha^4 e + 3\alpha^2(\alpha\beta + \gamma\delta)f + \alpha^3\gamma g + \alpha^3\beta f + \alpha^2\gamma\delta f = x^2 x^2$.

6. Idempotents in a noncommutative ring. An idempotent of a power-associative ring $\mathfrak{A}$ is also an idempotent of $\mathfrak{A}^{(+)}$. Since the additive group $\mathfrak{A}^{(+)}$ is precisely the same additive group as $\mathfrak{A}$ we may write (22) where the quantities $a_e(\lambda)$ of $\mathfrak{A}_e(\lambda)$ are now defined by

$$(27) \qquad\qquad e a_e(\lambda) + a_e(\lambda)e = 2\lambda a_e(\lambda) \qquad\qquad (\lambda = 0, 1, 1/2).$$

We may then prove

THEOREM 3. *The submodules $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ are orthogonal and such that*

$$(28) \qquad \begin{aligned} e a_e(1) &= a_e(1)e = a_e(1), \\ e a_e(0) &= a_e(0)e = 0 \end{aligned}$$

for all elements $a_e(1)$ of $\mathfrak{A}_e(1)$, $a_e(0)$ of $\mathfrak{A}_e(0)$.

To prove this result we use the relation (1) with $y = z = e$ and $xe + ex = \nu x$ to obtain $2xe + 2\nu ex = 2ex + 2\nu xe$, $(1 - \nu)xe = (1 - \nu)ex$, so that $xe = ex$ when $\nu \neq 1$. Then $\nu = 2\lambda$, $2ex = 2\lambda x$ and $ex = xe = \lambda x$ for $\lambda = 0, 1$. This yields (28). We next put $z = e$ and write $ex + xe = 2\lambda x$, $ey + ye = 2\mu y$ in (1) to obtain $2\lambda xy + 2\mu yx + (xy + yx)e = 2\lambda yx + 2\mu xy + e(xy + yx)$. If $\lambda = 1$ and $\mu = 0$ then $xy + yx = 0$ by Theorem 2 and so $2xy = 2yx$, $xy = yx$, $2xy = 0$, $xy = yx = 0$ as desired.

The characteristic roots of R_e need not be limited to the values 0, $1/2$, 1 of the commutative case if $\mathfrak{A}$ is not a commutative ring. Indeed, consider the algebra $\mathfrak{A}$ with a basis e, e_1, $\cdots$, e_t such that $ee_i = \alpha_i e_i$, $e_i e = (1 - \alpha_i)e_i$, $e_i e_j = 0$ for $i, j = 1, \cdots, t$. Then the characteristic roots α_i are arbitrary and we need only prove that $\mathfrak{A}$ is power-associative. We note that if $y = \xi_1 e_1 + \cdots + \xi_t e_t$ then $y^2 = 0$ and so $yy^2 = y^2 y$, $y^2 y^2 = (y^2 y)y$. For all other quantities x of $\mathfrak{A}$ we have $x = \lambda e + y$ where $\lambda \neq 0$ and the homogeneity of our relations implies that we may take $x = e + y$. But $ey + ye = y$ so that $x^2 = e + y = x$ and the properties $xx^2 = x^2 x$, $x^2 x^2 = (x^2 x)x$ are trivial.

It is also not true that the elementary divisors of R_e are simple. For consider an algebra with a basis e, f, g such that $g = ef$, $e^2 = e$, $f^2 = g^2 = fg = gf = eg = 0$, $fe = f - g$, $ge = g$. The matrix of R_e is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and $R_e - I \neq 0$, $(R_e - I)^2 = 0$. To show that $\mathfrak{A}$ is power-associative we note that if $y = \beta f + \gamma g$ then $y^2 = 0$. It then suffices to prove that $xx^2 = x^2x$, $x^2x^2 = (x^2x)x$ for $x = e + y$, $y = \beta f + \gamma g$, $ey + ye = \beta g + \beta(f - g) + \gamma g = y$, $x^2 = x$ and our proof is complete.

Let us next observe the algebra with a basis e, u, v over a field $\mathfrak{F}$ of characteristic prime to 30 such that $e^2 = e$, $eu = ue = u/2$, $ev = ve = v/2$, $uv = u$, $vu = -u$, $u^2 = v^2 = 0$. In this noncommutative algebra $\mathfrak{A}_e(1/2)$ is actually a subalgebra of $\mathfrak{A}$ and so the first relation of (24) does not hold. However if $x = \alpha e + \beta u + \gamma v$ then $x^2 = \alpha^2 e + \alpha(\beta u + \gamma v) = \alpha x$, so that the relations implying that $\mathfrak{A}$ is power-associative are trivial.

We shall finally give an example showing that the subsets $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ need not be subrings of $\mathfrak{A}$. We verify this result only for $\mathfrak{A}_e(0)$ but an example for the remaining case would be easy to construct. Let $\mathfrak{A}$ be an algebra with a basis e, u, v, w where $eu = ue = u$, $e^2 = e$, $ev = ve = ew = we = u^2 = v^2 = w^2 = uv$ $= vu = uw = wu = 0$, $vw = u = -wv$. Then $\mathfrak{A}_e(0)\mathfrak{A}_e(0) = u\mathfrak{F} \leqq \mathfrak{A}_e(1)$. If $x = \alpha e$ $+ \beta u + \gamma v + \delta w$ then $x^2 = \alpha^2 e + 2\alpha\beta u$, and so $x^2 x = \alpha^3 e + 3\alpha^2\beta u = xx^2$, $(x^2 x)x$ $= \alpha^4 e + 4\alpha^3\beta u = x^2 x^2$, $\mathfrak{A}$ is power associative.

It should be noted that if $\mathfrak{A}$ is a vector space over a field $\mathfrak{F}$ and $ae = \lambda a$, $ea = \mu a$ for λ and μ in $\mathfrak{F}$ then either a is in $\mathfrak{A}_e(1)$, or in $\mathfrak{A}_e(0)$ or in $\mathfrak{A}_e(1/2)$. For $ae + ea = (\lambda + \mu)a = 2a_e(1) + a_e(1/2)$. But then the uniqueness of (22) implies that $(\lambda + \mu)a_e(0) = 0$, $(\lambda + \mu - 1)a_e(1/2) = 0$, $(\lambda + \mu - 2)a_e(1) = 0$, and only one of the quantities $a_e(\alpha)$ can be different from zero. When a is in $\mathfrak{A}_e(1)$ we have $\lambda = \mu = 1$, and when a is in $\mathfrak{A}_e(0)$ we have $\lambda = \mu = 0$. However when a is in $\mathfrak{A}_e(1/2)$ we know only that $\lambda + \mu = 1$.

The examples we have given demonstrate that the relatively simple properties of a commutative power associative ring do not hold in general for noncommutative rings and so it is desirable to adjoin additional hypotheses.

7. **Flexible rings.** A ring $\mathfrak{A}$ will be called a *flexible* ring if $(xy)x = x(yx)$ for every x and y of $\mathfrak{A}$. Then $[(x+z)y](x+z) = (x+z)[y(x+z)]$ and multiplication and the use of the flexible law yields

$$(29) \qquad (xy)z + (zy)x = x(yz) + z(yx),$$

a relation holding for every x, y, and z of $\mathfrak{A}$. Conversely, if $\mathfrak{A}$ satisfies (29) and the characteristic of $\mathfrak{A}$ is prime to two then $\mathfrak{A}$ is flexible.

The property $x^2 x = xx^2$ holds for all flexible rings. Indeed, we may prove

THEOREM 4. *Let $\mathfrak{A}$ be a flexible ring whose characteristic is prime to 30. Then $\mathfrak{A}$ is power associative if and only if $x^2 x^2 = (x^2 x)x$ for every x of $\mathfrak{A}$.*

For proof we assume that $x^\alpha x^\beta = x^{\alpha+\beta}$ for all $\alpha + \beta < n$, where $n \geqq 4$, and put $y = x^{\lambda-1}$, $z = x^{n-\lambda}$ in (29) where $\lambda = 2, \cdots, n-1$. Then $x^\lambda x^{n-\lambda} + x^{n-1}x = xx^{n-1}$ $+ x^{n-\lambda}x^\lambda$. Take $\lambda = n-1$ and obtain $2x^{n-1}x = 2xx^{n-1}$ and so $x^{n-1}x = xx^{n-1}$, $x^{n-\lambda}x^\lambda = x^\lambda x^{n-\lambda}$. We now apply Lemma 2 and see that $x^\alpha x^\beta = x^{\alpha+\beta}$, for $\alpha + \beta = n$,

that is, that $\mathfrak{A}$ is power associative.

The flexible law does not seem to yield the first of the relations of (24) for noncommutative rings. However it does yield the remaining relations in their noncommutative form and we shall state the results as

THEOREM 5. *Let e be an idempotent of a flexible power associative ring $\mathfrak{A}$. Then $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ are zero or subrings of $\mathfrak{A}$, ex and xe are in $\mathfrak{A}_e(1/2)$ for every x of $\mathfrak{A}_e(1/2)$,*

$$
(30) \quad
\begin{aligned}
\mathfrak{A}_e(1)\mathfrak{A}_e(1/2) &\leqq \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0), & \mathfrak{A}_e(1/2)\mathfrak{A}_e(1) &\leqq \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0), \\
\mathfrak{A}_e(0)\mathfrak{A}_e(1/2) &\leqq \mathfrak{A}_e(1/2) + \mathfrak{A}_e(1), & \mathfrak{A}_e(1/2)\mathfrak{A}_e(0) &\leqq \mathfrak{A}_e(1/2) + \mathfrak{A}_e(1).
\end{aligned}
$$

Put $ye=ey=\alpha y$ and take $z=e$ in (29). Then $(xy)e+\alpha yx=\alpha(xy)+e(yx)$. Hence $\alpha(xy+yx) - e(xy+yx) = 2\alpha xy - [(xy)e+e(xy)]$. It follows that $(xy+yx)(\alpha I-L_e)=2xy(\alpha I-T_e)$, where $T_e=(R_e+L_e)/2$ is the multiplication for e in the algebra $\mathfrak{A}^{(+)}$. We use (23) with R_e replaced by T_e and first put $\alpha=1$, that is, let y be in $\mathfrak{A}_e(1)$. If x is also in $\mathfrak{A}_e(1)$, $xy+yx$ is in $\mathfrak{A}_e(1)$ since $\mathfrak{A}_e(1)$ is a subring of $\mathfrak{A}^{(+)}$. But then $(xy+yx)(I-L_e)=0$, $(xy)(I-T_e)=0$, and xy is in $\mathfrak{A}_e(1)$ by (23). We next let x and y be in $\mathfrak{A}_e(0)$, $\alpha=0$ so that $xy+yx$ is in $\mathfrak{A}_e(0)$, $(xy+yx)L_e=0$, $(xy)T_e=0$, and xy is in $\mathfrak{A}_e(0)$ by (23). This proves that $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ are subrings of $\mathfrak{A}$. They have already been shown to be orthogonal.

The next result is obtained by putting $y=e$ in (29). Then $\alpha=1$. Take x in $\mathfrak{A}_e(1/2)$ so that $xe+ex=x$. Then $x(I-L_e)=x-ex=xe=2xe(I-T_e)$, $xe=2(xe)T_e$, $(xe)T_e=xe/2$, xe is in $\mathfrak{A}_e(1/2)$. However x is in $\mathfrak{A}_e(1/2)$ and so $ex=x-xe$ is in $\mathfrak{A}_e(1/2)$.

To obtain the relations (30) we let $y=e$ in (29) and let x be in $\mathfrak{A}_e(1/2)$. Then $(xe)z+(ze)x=x(ez)+z(ex)$. However $xe=x-ex$ so that $xz+(ze)x=x(ez)+z(ex)+(ex)z$. If z is in $\mathfrak{A}_e(1)$ then $ez=ze=z$ and $xz=xz+zx-z(ex)-(ex)z$. But ex has already been shown to be in $\mathfrak{A}_e(1/2)$ and we may use (24) to see that both $xz+zx$ and $z(ex)+(ex)z$ are in $\mathfrak{A}_e(0)+\mathfrak{A}_e(1/2)$. It follows that both xz and zx are in $\mathfrak{A}_e(0)+\mathfrak{A}_e(1/2)$. We next put $ez=ze=0$ and obtain $xz=z(ex)+(ex)z$ which is in $\mathfrak{A}_e(1)+\mathfrak{A}_e(1/2)$ by the last relation of (24), zx is also in this submodule, and we have proved (30).

We shall be particularly interested in those flexible rings having the property expressed by (26). We define a ring $\mathfrak{A}$ to be a *stable* ring if $\mathfrak{A}$ is a flexible power associative ring and

$$
(31) \quad \mathfrak{A}_e(\lambda)\mathfrak{A}_e(1/2) \leqq \mathfrak{A}_e(1/2), \quad \mathfrak{A}_e(1/2)\mathfrak{A}_e(\lambda) \leqq \mathfrak{A}_e(1/2) \qquad (\lambda = 0, 1),
$$

for every idempotent e of $\mathfrak{A}$. We may then prove

THEOREM 6. *A flexible power associative ring $\mathfrak{A}$ is a stable ring if and only if $\mathfrak{A}^{(+)}$ is stable.*

For if xy and yx are in $\mathfrak{A}_e(1/2)$ for every y of $\mathfrak{A}_e(1/2)$ and every x of

$\mathfrak{A}_e(1) + \mathfrak{A}_e(0)$ their sum $xy + yx$ is also in $\mathfrak{A}_e(1/2)$ and the commutative ring $\mathfrak{A}^{(+)}$ is stable. Conversely let $\mathfrak{A}^{(+)}$ be stable so that (26) holds. Put $y = e$ in (29) and obtain $(xe)z + (ze)x = x(ez) + z(ex)$ so that, if $xe = ex = \lambda x$, then $\lambda xz + (ze)x = x(ez) + \lambda zx$. Then

$$\lambda xz + (ze + ez)x = [x(ez) + (ez)x] + \lambda zx.$$

If z is in $\mathfrak{A}_e(1/2)$ then $ze + ez = z$, ez is in $\mathfrak{A}_e(1/2)$ by Theorem 5, $w = x(ez) + (ez)x$ is in $\mathfrak{A}_e(1/2)$ by the assumption that $\mathfrak{A}^{(+)}$ is stable. Then $\lambda xz + (1-\lambda)zx$ is in $\mathfrak{A}_e(1/2)$. The value $\lambda = 1$ yields xz in $\mathfrak{A}_e(1/2)$ and the value $\lambda = 0$ yields zx in $\mathfrak{A}_e(1/2)$. But $xz + zx$ is in $\mathfrak{A}_e(1/2)$ and so both xz and zx are in $\mathfrak{A}_e(1/2)$ in every case, $\mathfrak{A}$ is stable.

It should be observed that when $\mathfrak{A}$ is a stable ring every subring $\mathfrak{B}$ of $\mathfrak{A}$ is stable. For $\mathfrak{B}$ is clearly flexible. If e is an idempotent of $\mathfrak{B}$ we may write $\mathfrak{B} = \mathfrak{B}_e(1) + \mathfrak{B}_e(1/2) + \mathfrak{B}_e(0)$ and the definitions of the modules $\mathfrak{B}_e(\lambda)$ clearly imply that $\mathfrak{B}_e(\lambda) \leqq \mathfrak{A}_e(\lambda)$. But then the relations (31) imply corresponding relations for the modules $\mathfrak{B}_e(\lambda)$ and so $\mathfrak{B}$ is stable.

We note finally that *when $\mathfrak{B}$ is an ideal of a stable algebra $\mathfrak{A}$, then $\mathfrak{A}_0 = \mathfrak{A} - \mathfrak{B}$ is also stable.* Indeed, let e_0 be an idempotent element of $\mathfrak{A}_0$. Then e_0 is a class $x + \mathfrak{B}$ where x is necessarily a non-nilpotent element of $\mathfrak{A}$, $e_0^k = x^k + \mathfrak{B} = e_0$. The algebra $\mathfrak{C} = \mathfrak{F}[x]$ is an associative commutative non-nilpotent algebra and contains a principal idempotent e. We may write $\mathfrak{C} = \mathfrak{C}_e(1) + \mathfrak{C}_e(0)$ where the elements of $\mathfrak{C}_e(0)$ are nilpotent. If e were in $\mathfrak{B}$ the algebra $\mathfrak{C}_e(1) = e\mathfrak{C}e$ would be in $\mathfrak{B}$ and we would have $x + \mathfrak{B} = y + \mathfrak{B}$ where y is in $\mathfrak{C}_e(0)$. But then $e_0 = y + \mathfrak{B}$ could not be idempotent. Hence e is not in $\mathfrak{B}$, $e + \mathfrak{B} = e(x) + \mathfrak{B}$ $= \alpha x + \mathfrak{B}$ for $\alpha \neq 0$ in $\mathfrak{F}$, $(e + \mathfrak{B})^2 = e^2 + \mathfrak{B} = e + \mathfrak{B} = \alpha^2 x^2 + \mathfrak{B}$, $\alpha^2 x^2 + \mathfrak{B} = \alpha x + \mathfrak{B}$, $\alpha^2 = \alpha$, $\alpha = 1$, $e + \mathfrak{B} = e_0$. We conclude that the homomorphic mapping of $\mathfrak{A}$ on $\mathfrak{A} - \mathfrak{B}$ maps every element z of $\mathfrak{A}$ such that $ze + ez = \lambda z$ on an element z_0 of $\mathfrak{A} - \mathfrak{B}$ such that $z_0 e_0 + e_0 z_0 = \lambda z_0$. Then the relations (31) for $\mathfrak{A}_e(\lambda)$ go into relations (31) for $\mathfrak{A}_{0e_0}(\lambda)$ and so $\mathfrak{A}_0$ is also stable.

8. Principal idempotents of algebras. An idempotent e of a ring $\mathfrak{A}$ is called a *principal idempotent* of $\mathfrak{A}$ if there is no idempotent u of $\mathfrak{A}$ orthogonal to e. If $\mathfrak{A}$ is power-associative this means that $\mathfrak{A}_e(0)$ contains no idempotent. When we assume that $\mathfrak{A}$ is also a flexible algebra then $\mathfrak{A}_e(0)$ is a subalgebra of $\mathfrak{A}$ and so e is principal if and only if $\mathfrak{A}_e(0)$ is a nilalgebra. We now make an additional assumption and prove

THEOREM 7. *Let e be a principal idempotent of a stable algebra $\mathfrak{A}$. Then the elements of $\mathfrak{A}_e(1/2)$ are nilpotent.*

The result is clearly only required for $\mathfrak{A}^{(+)}$ since powers in $\mathfrak{A}$ and in $\mathfrak{A}^{(+)}$ coincide. Hence it is sufficient to take the case where $\mathfrak{A}$ is commutative. We then use (14) with x in $\mathfrak{A}_e(1/2)$ and operate on e to obtain $ex^3 = 4(ex)x^2 - (ex^2)x - 2[(ex)x]x$. But $ex = x/2$ and so $ex^3 = x^3 - (ex^2)x$. By Theorem 2 we may write

(32) $$w = x^2 = w_1 + w_0$$

where w_0 is in $\mathfrak{A}_e(0)$ and w_1 is in $\mathfrak{A}_e(1)$ and use the property that $\mathfrak{A}$ is stable to see that $x^3 = (w_1 + w_0)x$ is in $\mathfrak{A}_e(1/2)$. Then $x^3/2 = x^3 - w_1 x$, $x^2 x = 2w_1 x = (w_1 + w_0)x$,

(33) $$w_1 x = w_0 x.$$

We now let e be principal so that $\mathfrak{A}_e(0)$ is a nilalgebra, $w_0^k = 0$ for some positive integer k. Put $z = x^{2k+1}$ and have $z^2 = w^{2k+1} = w_1^{2k+1} + w_0^{2k+1}$ since w_1 and w_0 are orthogonal. But $x^{2k} = w_0^k + w_1^k$ is in $\mathfrak{A}_e(1) + \mathfrak{A}_e(0)$, $z = x^{2k+1} = x^{2k}x$ is in $\mathfrak{A}_e(1/2)$, $z^2 = v_1 + v_0$ where $v_1 = w_1^{2k+1}$, $v_0 = w_0^{2k+1} = 0$. Hence we may apply (33) in this case, $v_1 z = v_0 z = 0$, $z^3 = (v_1 + v_0)z = 0$, $x^{6k+3} = 0$, x is nilpotent. This concludes our study of the decomposition of a power-associative ring relative to an idempotent and we pass on now to the study of certain types of stable power-associative algebras.

Chapter II. Trace-admissible algebras

1. Admissible trace functions. We shall leave the theory of rings and turn to the theory of algebras considering only power-associative stable algebras over a field $\mathfrak{F}$. Let $\mathfrak{A}$ be such an algebra and call $\mathfrak{A}$ a *trace-admissible algebra* if there exists a function $\tau(x, y)$ with arguments x and y in $\mathfrak{A}$ and values in $\mathfrak{F}$ such that

    I. $\tau(x, y)$ is a bilinear function of x and y.
    II. $\tau(x, y) = \tau(y, x)$.
    III. $\tau(xy, z) = \tau(x, yz)$.
    IV. $\tau(x, y) = 0$ if xy is nilpotent or zero.
    V. $\tau(e, e) \neq 0$ if e is an idempotent of $\mathfrak{A}$.
It should be noted that II and III imply

(1) $\tau(xy, z) = \tau(yz, x) = \tau(zx, y) = \tau(x, yz) = \tau(y, zx) = \tau(z, xy)$.

We shall call any function $\tau(x, y)$ satisfying I–V an admissible trace function for $\mathfrak{A}$. We now prove

Theorem 1. *Let $\mathfrak{A}$ be a trace-admissible algebra and let $\mathfrak{N}_\tau$ be the set of all elements x of $\mathfrak{A}$ such that $\tau(x, y) = 0$ for every y of $\mathfrak{A}$. Then $\mathfrak{N}_\tau$ is the nilradical $\mathfrak{N}$ of $\mathfrak{A}$.*

    For by I the set $\mathfrak{N}_\tau$ is a linear subspace of $\mathfrak{A}$. If x is in $\mathfrak{N}_\tau$ and y is in $\mathfrak{A}$, then $\tau(xy, z) = \tau(x, yz)$ by (1), $\tau(x, yz) = 0$ for every z, xy is in $\mathfrak{N}_\tau$. Similarly $\tau(yx, z) = \tau(x, zy)$ by (1) and yx is in $\mathfrak{N}_\tau$, $\mathfrak{N}_\tau$ is an ideal of $\mathfrak{A}$. If $\mathfrak{N}_\tau$ were not a nilalgebra it would contain[3] an idempotent e whereas $\tau(e, e) \neq 0$. Hence $\mathfrak{N}_\tau \leqq \mathfrak{N}$. Conversely if x is in $\mathfrak{N}$ then xy is in $\mathfrak{N}$ for every x of $\mathfrak{A}$, xy is nilpotent, $\tau(x, y) = 0$, $\mathfrak{N} \leqq \mathfrak{N}_\tau$, $\mathfrak{N} = \mathfrak{N}_\tau$ as desired.

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    [3] Cf. §S of the paper of footnote 2.

We have now shown that $\mathfrak{N}_\tau$ is independent of τ. Note also that if $\mathfrak{B}$ is any subalgebra of $\mathfrak{A}$ and $\tau(x, y)$ is an admissible trace function for $\mathfrak{A}$ then $\tau(x, y)$ is also an admissible trace function for $\mathfrak{B}$. For clearly I–V hold for $\mathfrak{B}$ when they hold for $\mathfrak{A}$. Also it has already been remarked that $\mathfrak{B}$ is stable when $\mathfrak{A}$ is.

2. **The radical of an ideal.** The radical of an ideal is completely determined by

THEOREM 2. *Let $\mathfrak{B}$ be an ideal of a trace-admissible algebra $\mathfrak{A}$. Then the intersection of $\mathfrak{B}$ and the nilradical $\mathfrak{N}$ of $\mathfrak{A}$ is the nilradical $\mathfrak{N}_0$ of $\mathfrak{B}$.*

For $\mathfrak{B} \cap \mathfrak{N}$ is a nilideal of $\mathfrak{B}$, $\mathfrak{B} \cap \mathfrak{N} \leqq \mathfrak{N}_0$. Conversely let g be in $\mathfrak{N}_0$ so that either g is in $\mathfrak{N}$ and hence in $\mathfrak{B} \cap \mathfrak{N}$ or there exists an element x of $\mathfrak{A}$ such that $\tau(g, x) \neq 0$. Then $y = gx$ is not nilpotent and there exist elements $\alpha_1, \cdots, \alpha_t$ in $\mathfrak{F}$ such that $e = \alpha_1 y \cdots \alpha_t y^t$ is idempotent. Now $\tau(e, e) = \sum_{i,j=1}^{t} \alpha_i \alpha_j \tau(y^i, y^j)$ $\neq 0$ and it follows that $\tau(y^i, y^j) \neq 0$ for some integers i and j. However by (1) we have $\tau(y^i, y^j) = \tau(y^{i+j-1}, y) = \tau(y^{i+j-1}, gx) = \tau(g, xy^{i+j-1}) = 0$ since y is in $\mathfrak{B}$, xy^{i+j-1} is in $\mathfrak{B}$, g is in $\mathfrak{N}_0$, $g(xy^{i+j-1})$ is in $\mathfrak{N}_0$ and is nilpotent. This proves that $\mathfrak{N}_0 \leqq \mathfrak{B} \cap \mathfrak{N}$, $\mathfrak{N}_0 = \mathfrak{B} \cap \mathfrak{N}$.

We define a trace-admissible algebra $\mathfrak{A}$ to be *semi-simple* if its nilradical $\mathfrak{N} = 0$ and immediately have

THEOREM 3. *Every ideal of a semi-simple trace-admissible algebra is semi-simple.*

3. **The nilradical in the Pierce decomposition.** We now refer to the decomposition $\mathfrak{A} = \mathfrak{A}_e(1) + \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0)$ and prove

THEOREM 4. *The nilradical $\mathfrak{N}_\lambda$ of $\mathfrak{A}_e(\lambda)$ is the intersection of $\mathfrak{A}_e(\lambda)$ and the nilradical $\mathfrak{N}$ of $\mathfrak{A}$ for $\lambda = 0, 1$.*

For $\mathfrak{N} \cap \mathfrak{A}_e(\lambda)$ is evidently a nilideal of $\mathfrak{A}_e(\lambda)$ and is contained in $\mathfrak{N}$. Conversely if x is in $\mathfrak{N}$ then xa_λ is nilpotent for every a_λ of $\mathfrak{A}_e(\lambda)$ and so $\tau(x, a_\lambda) = 0$. Also $\tau(x, a_{1-\lambda}) = 0$ for every $a_{1-\lambda}$ of $\mathfrak{A}_e(1-\lambda)$ since $xa_{1-\lambda} = 0$. Finally if b is in $\mathfrak{A}_e(1/2)$ then $\tau(x, b) = \tau(x, eb+be) = \tau(x, eb) + \tau(x, be) = \tau(xe, b) + \tau(ex, b)$ $= 2\lambda\tau(x, b)$, $\tau(x, b) = 0$ if $\lambda = 0, 1$. Hence $\tau(x, a) = 0$ for every a of $\mathfrak{A}$, x is in $\mathfrak{N} \cap \mathfrak{A}_e(\lambda)$, $\mathfrak{N}_\lambda = \mathfrak{N} \cap \mathfrak{A}_e(\lambda)$.

In the case of a principal idempotent we may prove

THEOREM 5. *Let e be a principal idempotent of a stable algebra $\mathfrak{A}$. Then $\mathfrak{A}_e(1/2) + \mathfrak{A}_e(0)$ is contained in the nilradical of $\mathfrak{A}$.*

For $\mathfrak{A}_e(0) \leqq \mathfrak{N}$ by Theorem 4 and the property that the maximal nilideal of $\mathfrak{A}_e(0)$ is $\mathfrak{A}_e(0)$ itself. By Theorem 1.7, if b is in $\mathfrak{A}_e(1/2)$ then b is nilpotent. It follows that b^2, $(b+c)^2$, and c^2 are nilpotent for every b and c of $\mathfrak{A}_e(1/2)$. Then $\tau(b+c, b+c) = 0 = \tau(b, b) + \tau(c, c) + 2\tau(b, c)$, $\tau(b, c) = 0$. But by the proof of Theorem 4, $\tau(b, x) = 0$ for every x in $\mathfrak{A}_e(0) + \mathfrak{A}_e(1)$, b is in $\mathfrak{N}$, $\mathfrak{A}_e(1/2) \leqq \mathfrak{N}$.

An algebra $\mathfrak{A}$ which is not a nilalgebra has an idempotent and hence a principal idempotent. Applying Theorem 5 we have

THEOREM 6. *A semi-simple trace-admissible algebra has a unity element.*

As a consequence of Theorems 3 and 6 we now have

THEOREM 7. *Let $\mathfrak{B}$ be an ideal of a trace-admissible semi-simple algebra $\mathfrak{A}$. Then $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$, $\mathfrak{B}$ has a unity element e and $\mathfrak{C} = \mathfrak{A}_e(0)$.*

For $\mathfrak{B}$ has a unity element e by Theorems 3 and 6, and if x is in $\mathfrak{A}_e(1/2)$ we have $xe + ex = x$ is in $\mathfrak{B}$, $xe + ex = x$, $2x = x$, $x = 0$, $\mathfrak{A}_e(1/2) = 0$, $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{A}_e(0)$.

Theorem 7 implies that every semi-simple trace-admissible algebra $\mathfrak{A}$ is a direct sum of simple algebras which are not nilalgebras.

CHAPTER III. COMMUTATIVE SHRINKABLE ALGEBRAS

1. Algebras of level one. Let $\mathfrak{A}$ be any algebra over a field $\mathfrak{F}$ so that the multiplications of $\mathfrak{A}$ are linear transformations R_x and L_x over $\mathfrak{F}$ of $\mathfrak{A}$. Use the notation $T(x)$ to represent either multiplication and the notation $T(x_1 \cdots x_s)$ to represent a multiplication $T(x)$ defined for a product $x = x_1 \cdots x_s$ of the factors $x_1, \cdots, x_s$ with an unspecified association. Then we shall call $\mathfrak{A}$ a *shrinkable algebra of shrinkability level s* if every $T(x_1 \cdots x_{s+1})$ is identically equal to a finite sum of products $\lambda T_1 \cdots T_r$ with fixed coefficients λ in $\mathfrak{F}$, where the transformations T_i are multiplications $T_i(z_i)$ defined for products z_i of t_i factors x_k, and $t_1 + \cdots + t_r = s + 1$. We propose to investigate here the relations possible for commutative power-associative shrinkable algebras of levels one and two over $\mathfrak{F}$ of characteristic not two or three.

A commutative algebra $\mathfrak{A}$ of level one has the property that $R_{xy} = \lambda R_x R_y + \mu R_y R_x$ for every x and y of $\mathfrak{A}$ where λ and μ are fixed elements of $\mathfrak{F}$. Since $xy = yx$ we also have $R_{xy} = R_{yx} = \lambda R_y R_x + \mu R_x R_y$. Then $2R_{xy} = (\lambda + \mu)(R_x R_y + R_y R_x)$ and, since $\mathfrak{F}$ does not have characteristic two, $R_{xy} = \nu(R_x R_y + R_y R_x)$. It follows that $R_{xx} = 2\nu R_x^2$ is commutative with R_x and that $\mathfrak{A}$ is a Jordan algebra.

However we may analyze the algebras of level one more deeply. We have $a(xy) = \nu(ax)y + \nu(ay)x$ and so $\nu R_{ax} = R_x R_a - \nu R_a R_x = \nu^2(R_x R_a + R_a R_x)$. Then

$$(1) \qquad (\nu^2 + \nu)R_a R_x = (1 - \nu^2)R_x R_a.$$

If $\nu = 0$ then $R_x R_a = 0$ for every x and a of $\mathfrak{A}$ and $R_{xa} = 0 = R_x R_a$, $\mathfrak{A}$ is an associative nilpotent commutative algebra in which all products of three elements are zero. If $\nu = 1$ then $R_a R_x = 0$ and we have the same result as in the case $\nu = 0$. When $\nu \neq 0$ and $\nu^2 \neq 1$ then $R_a R_x = \nu^{-1}(1 - \nu)R_x R_a$, $\nu(R_y R_x + R_x R_y) = (1 - \nu + \nu)R_x R_y = R_x R_y = R_{xy}$ and $\mathfrak{A}$ is a commutative associative algebra.

There remains the case where $\nu = -1$ and so $R_{xy} = -(R_x R_y + R_y R_x)$, $R_{xx} = -2R_x^2$, $x^3 + 2x^3 = 3x^3 = 0$. Then the Jordan algebra is a nilalgebra and is known to be nilpotent. We have proved

THEOREM 1. *A commutative shrinkable algebra of level one is either associative or is a nilpotent Jordan algebra satisfying the relation $x(yz)+y(zx)+z(xy) = 0$.*

2. Algebras of level two. The general relation satisfied by a commutative shrinkable algebra of level two expresses $R_{(xy)z}$ as a sum of terms of the form $\lambda R_{xy}R_z$, $\mu R_z R_{xy}$ and $\nu R_x R_y R_z$ with λ, μ, ν in $\mathfrak{F}$. If we use the hypothesis $xy = yx$ to see that $R_{(xy)z} + R_{(yx)z} = 2R_{(xy)z}$ we shall obtain a relation expressing $R_{(xy)z}$ as a sum of products unaltered by the interchange of x and y. The relation must then have the form

$$(2) \qquad R_{(xy)z} = \lambda_1 R_{xy}R_z + \lambda_2(R_{xz}R_y + R_{yz}R_x) + G + H,$$

where

$$(3) \qquad G = \mu_1 R_z R_{xy} + \mu_2(R_y R_{xz} + R_x R_{yz})$$

and

$$(4) \qquad \begin{aligned} H &= \nu_1 R_z(R_x R_y + R_y R_x) + \nu_2(R_x R_y + R_y R_x)R_z \\ &\quad + \nu_3(R_x R_z R_y + R_y R_z R_x), \end{aligned}$$

for coefficients λ_1, λ_2, μ_1, μ_2, ν_1, ν_2, ν_3 in $\mathfrak{F}$.

If w is in $\mathfrak{A}$ we form $w[(xy)z]$ and use (2) to write

$$(5) \qquad \begin{aligned} w[(xy)z] &= \lambda_1[w(xy)]z + \lambda_2[w(xz)]y + \lambda_2[w(yz)]x + \mu_1(wz)(xy) \\ &\quad + \mu_2(wx)(yz) + \mu_2(wy)(xz) + \nu_1[(wz)x]y + \nu_1[(wz)y]x \\ &\quad + \nu_2[(wx)y + (wy)x]z + \nu_3[(wx)z]y + \nu_3[(wy)z]x. \end{aligned}$$

We may interchange the symbols w and z and write the result as a transformation on w to obtain the relation

$$(6) \qquad \begin{aligned} R_{xy}R_z &= \lambda_1 R_{(xy)z} + \lambda_2(R_x R_z R_y + R_y R_z R_x) + G + \nu_1 R_z(R_x R_y + R_y R_x) \\ &\quad + \nu_2[R_{(xz)y} + R_{(yz)x}] + \nu_3(R_{xz}R_y + R_{yz}R_x). \end{aligned}$$

The elimination of the common term $G + \nu_1 R_z(R_x R_y + R_y R_x)$ between (6) and (2) then yields

$$(7) \qquad \begin{aligned} (1+\lambda_1)&R_{(xy)z} + \nu_2 R_{(xz)y} + \nu_2 R_{(yz)x} \\ &= (1+\lambda_1)R_{xy}R_z + (\lambda_2 - \nu_3)(R_{xz}R_y + R_{yz}R_x) \\ &\quad - (\lambda_2 - \nu_3)(R_x R_z R_y + R_y R_z R_x) + \nu_2(R_x R_y + R_y R_x)R_z. \end{aligned}$$

We may finally interchange the symbols x and z as well as the symbols y and w in (5) and write the result as an equation in transformations on w as

$$(8) \qquad \begin{aligned} R_z R_x R_y &= \lambda_1 R_z R_y R_x + \lambda_2[R_{(xz)y} + R_x R_y R_z] + G + \nu_1[R_{(xy)z} + R_{xy}R_z] \\ &\quad + \nu_2(R_{yz}R_x + R_y R_z R_x) + \nu_3[R_{(yz)x} + R_y R_x R_z]. \end{aligned}$$

At this point we shall further restrict our study. It should be clear that there is nothing to say about algebras of level two which are also of level one and so we may assume that our algebras do not have level one. Moreover a study of those classes of commutative algebras containing no algebras with a unity quantity should be of only secondary interest in this first investigation of the types of shrinkable algebras. Thus we shall attempt to determine the values of our parameters only for classes of power-associative algebras containing algebras with a unity quantity e and not of level one. For such algebras R_e is the identity transformation and a relation of the form

$$(9) \qquad \alpha R_{xy} + \beta R_x R_y + \gamma R_y R_x = 0$$

is possible only when $\alpha = 0$.

Let us now apply this property. Substitute first $z = e$ and then $x = e$ in (2) to obtain

$$(10) \qquad \lambda_1 + \mu_1 = \lambda_2 + \mu_2 = 1.$$

Make the same substitution in (6) and (7) and obtain

$$2\nu_2 = 0, \qquad 1 + \lambda_1 = \lambda_2 - \nu_3,$$

$$(11) \qquad \lambda_2 + \mu_1 + 2\nu_1 + \nu_3 = 0,$$

$$\lambda_2 + \mu_2 + \nu_1 + \nu_3 = 0.$$

Then $\nu_2 = 0$,

$$(12) \qquad \nu_1 + \nu_3 = -1, \qquad \lambda_2 + \mu_1 + \nu_1 = 1.$$

If $\lambda_1 + 1 = 0$ we may use (7) and the fact that $\nu_2 = 0$ to see that the corresponding class of algebras is defined by a relation of the form (2) with $\mu_1 = \mu_2 = 0$. Then the study we have made yields (10) and (11) and thus $\lambda_1 = \lambda_2 = 1$, $\nu_3 = -1$, $\nu_1 = \nu_2 = 0$. The resulting identity is $R_{(xy)z} = R_{xy}R_z + R_{xz}R_y + R_{yz}R_x - (R_xR_zR_y + R_yR_zR_x)$ and our algebras are known to be Jordan algebras.

There remains the case $\lambda_1 = -1$. In this case $\lambda_2 = \nu_3$ and $\mu_1 = 2$. Then we have the relation

$$
\begin{aligned}
R_{(xy)z} = {}& 2R_zR_{xy} + R_yR_{xz} + R_xR_{yz} - R_{xy}R_z - R_z(R_xR_y + R_yR_x) \\
(13) \qquad & + \nu_3\big[(R_xR_zR_y + R_yR_zR_x) - R_z(R_xR_y + R_yR_x) \\
& + (R_{xz}R_y + R_{yz}R_x) - (R_yR_{xz} + R_xR_{yz})\big].
\end{aligned}
$$

Compute $R_{(yz)x}$ and $R_{(zx)y}$ using (13) and add the three formulas so obtained to give

$$(14) \quad P = 2\nu_3\big[(R_{xy}R_z + R_{yz}R_x + R_{zx}R_y) - (R_zR_{xy} + R_xR_{yz} + R_yR_{zx})\big]$$

where

$$P = R_{(xy)z} + R_{(zx)y} + R_{(yz)x} - 4(R_zR_{xy} + R_yR_{xz} + R_xR_{yz})$$
$$+ (R_{xy}R_z + R_{yz}R_x + R_{zx}R_y)$$
$$+ [R_x(R_yR_z + R_zR_y) + R_y(R_xR_z + R_zR_x) + R_z(R_xR_y + R_yR_x)].$$

We now apply the assumption that $\mathfrak{A}$ is a power-associative commutative algebra. By (1.13), this is precisely the condition $P=0$ and formula (14) for $\nu_3 \neq 0$ yields a standard identity for Jordan algebras. The only remaining case is thus the case $\nu_3 = 0$ in (13) and we have proved

THEOREM 2. *Let $\mathfrak{A}$ be a commutative power-associative algebra over a field of characteristic not two or three and let $\mathfrak{A}$ have shrinkability level two and belong to a class of such algebras containing algebras with a unity quantity. Then $\mathfrak{A}$ is either a Jordan algebra or an algebra satisfying the relation*

$$(15) \quad R_{(xy)z} = 2R_zR_{xy} + R_yR_{xz} + R_xR_{yz} - R_{xy}R_z - R_z(R_xR_y + R_yR_x).$$

We shall refer to algebras defined by (15) as *static algebras* and shall obtain a structure theory for such algebras in the remaining sections of this chapter.

3. **Static algebras.** The identity (15) is equivalent to

$$(16) \quad [(xy)z]w + [(xy)w]z + [(zw)x]y + [(zw)y]x$$
$$= 2(xy)(zw) + (xz)(yw) + (yz)(xw),$$

a relation invariant under permutations of a transitive group of order eight on the four letters x, y, z, w. Put $y=x$ and $w=z$ to obtain $2(x^2z)z+2(z^2x)x = 2x^2z^2+2(xz)^2$. Then we have proved one of the implications in the following

THEOREM 3. *Let $\mathfrak{A}$ be a commutative algebra over a field $\mathfrak{F}$ of characteristic not two. Then $\mathfrak{A}$ is a static algebra if and only if*

$$(17) \quad x^2y^2 + (xy)^2 = x(xy^2) + y(yx^2)$$

for every x and y of $\mathfrak{A}$.

The remaining implication, (17) implies (16), is derived by first replacing y by $z+\lambda w$ in (17) and then equating the coefficients of λ. This yields $2x^2(zw)+2(xz)(xw) = 2x[x(zw)]+z(wx^2)+w(zx^2)$. The replacement of x by $x+\mu y$ and the equality of the coefficients of μ will then yield (16).

The relation (17) evidently implies that $x^2x^2 = x(xx^2) = (x^2x)x$ and so the commutative algebras which we are calling static algebras are necessarily power-associative.

4. **Solvable static algebras.** If $\mathfrak{B}$ is any subalgebra of an algebra $\mathfrak{A}$ we shall designate by $\mathfrak{B}^*$ the associative algebra generated by the multiplications R_b, L_b of $\mathfrak{A}$ which are defined for all quantities b of $\mathfrak{B}$. Every solvable subalgebra $\mathfrak{B}$ contains a maximal proper subalgebra $\mathfrak{C}$, such that $\mathfrak{C}$ contains the product of any two elements of $\mathfrak{B}$, $\mathfrak{B} - \mathfrak{C}$ is a zero algebra of order one,

$\mathfrak{B}=\mathfrak{C}+w\mathfrak{F}$. Define $\mathfrak{H}=\mathfrak{B}^*\mathfrak{C}^*+\mathfrak{C}^*$ and let $\mathfrak{A}$ be a static algebra. Then we may prove

LEMMA 1. *Let x be in $\mathfrak{B}$ and c be in $\mathfrak{C}$. Then $R_zR_x^2$ is in $\mathfrak{H}$.*

For $R_{(xx)c}=2R_cR_{xx}+2R_xR_{cx}-R_{xx}R_c-2R_cR_x^2$. Then $(xx)c$, xx, cx, c are all in $\mathfrak{C}$, $-2R_cR_x^2$ is in $\mathfrak{H}$, and we use our basic assumption that $\mathfrak{F}$ does not have characteristic two.

We next prove

LEMMA 2. *Let x be in $\mathfrak{B}$ and c and d be in $\mathfrak{C}$. Then $R_cR_dR_x$ is in $\mathfrak{H}$.*

For $R_{(xd)c}=2R_cR_{xd}+R_dR_{xc}+R_xR_{cd}-R_{xd}R_c-R_c(R_xR_d+R_dR_x)$ and our result follows from the fact that $(xd)c$, xd, xc, cd, c, d are all in $\mathfrak{C}$.

As in all cases of power-associative commutative algebras we have the relation

$$(18) \qquad R_{(xx)x} = 4R_xR_{xx} - R_{xx}R_x - 2R_x^3.$$

Then

$$(19) \qquad R_xR_{(xx)x} = 4R_x^2R_{xx} - R_xR_{xx}R_x - 2R_x^4$$

and

$$(20) \qquad R_{(xx)x}R_x = 4R_xR_{xx}R_x - R_{xx}R_x^2 - 2R_x^4.$$

It follows that

$$(21) \qquad R_xR_{(xx)x} - R_{(xx)x}R_x = 4R_x^2R_{xx} + R_{xx}R_x^2 - 5R_xR_{xx}R_x.$$

Then

$$
\begin{aligned}
(22) \qquad R_{(x^2x)x} &= 2R_xR_{(xx)x} + R_{xx}^2 + R_xR_{(xx)x} - R_{(xx)x}R_x \\
&\quad - R_x(R_xR_{xx} + R_{xx}R_x) \\
&= 2R_xR_{xxx} + R_{xx}^2 + R_{xx}R_x^2 + 3R_x^2R_{xx} - 6R_xR_{xx}R_x.
\end{aligned}
$$

By Lemma 1, $R_{xx}R_x^2$ is in $\mathfrak{H}$. But then $6R_xR_{xx}R_x$ is in $\mathfrak{H}$ and so $R_xR_{xx}R_x$ is in $\mathfrak{H}$. By (19) we have

LEMMA 3. *If x is in $\mathfrak{B}$ then R_x^4 is in $\mathfrak{H}$.*

Since R_x^4 is in $\mathfrak{H}$ for every x of $\mathfrak{B}$ it will be true that $(R_{x+\lambda y})^4=R_x^4+\lambda S+\lambda^2T+\lambda^3V+\lambda^4R_y^4$ will be in $\mathfrak{H}$ for every x and y of $\mathfrak{B}$. Then $\lambda S+\lambda^2T+\lambda^3V$ will be in $\mathfrak{H}$ and, since we are assuming always that the field $\mathfrak{F}$ has characteristic prime to six, S is in $\mathfrak{H}$. But $S=R_x^2(R_xR_y+R_yR_x)+(R_yR_x+R_xR_y)R_x^2$. Assume now that y is in $\mathfrak{C}$ so that $R_x^2R_xR_y$ is in $\mathfrak{H}$. By Lemma 1, $R_yR_x^2$ is in $\mathfrak{H}$ and, since $\mathfrak{B}^*\mathfrak{H}\leqq\mathfrak{H}$, we see that $R_xR_yR_x^2$ is in $\mathfrak{H}$. Also $2R_x^3=4R_xR_{xx}-R_{xx}R_x-R_{(xx)x}$

and so

$$(23) \qquad 2R_y R_x^3 = 4R_y R_x R_{xx} - R_y R_{xx} R_x - R_y R_{(xx)x}.$$

This transformation is in $\mathfrak{H}$ since $R_y R_{xx} R_x$ is in $\mathfrak{H}$ by Lemma 2. Then $R_y R_x^3$ is in $\mathfrak{H}$ and we have proved that S, $R_x^3 R_y$, $R_y R_x^3$, $R_x R_y R_x^2$ are all in $\mathfrak{H}$. This yields

LEMMA 4. *Let x be in $\mathfrak{B}$ and c be in $\mathfrak{C}$. Then $R_x^2 R_c R_x$ and $R_c R_x^3$ are in $\mathfrak{H}$.*

We finally compute $R_{x(cx)} = 3R_x R_{cx} + R_c R_{xx} - R_{cx} R_x - R_x(R_c R_x + R_x R_c)$ and form the product

$$R_d R_{x(cx)} = 3R_d R_x R_{cx} + R_d R_c R_{xx} - R_d R_{cx} R_x - R_d R_x R_c R_x - R_d R_x^2 R_c.$$

By Lemma 2 the term $R_d R_{cx} R_x$ is in $\mathfrak{H}$. All other terms have a right-hand factor R_y with y in $\mathfrak{C}$ and we have proved

LEMMA 5. *Let x be in $\mathfrak{B}$ and c and d be in $\mathfrak{C}$. Then $R_d R_x R_c R_x$ is in $\mathfrak{H}$.*

We may now prove that $(\mathfrak{B}^*)^4 \leq \mathfrak{H}$. It is evidently sufficient to prove that $P = R_x R_y R_z R_u$ is in $\mathfrak{H}$ for all x, y, z, u of $\mathfrak{B}$ and since P is linear in x, y, z, u it is sufficient to take the symbols x, y, z, u to be either in $\mathfrak{C}$ or equal to w in $\mathfrak{B}$. If u is in $\mathfrak{C}$ then $\mathfrak{H}$ contains P so we may take $u = w$. If z and y are in $\mathfrak{C}$ then P is in $\mathfrak{H}$ by Lemma 2. If z is in $\mathfrak{C}$ but $y = w$ then either x is in $\mathfrak{C}$ and P is in $\mathfrak{H}$ by Lemma 5 or $x = w$, $P = R_w^2 R_y R_w$ is in $\mathfrak{H}$ by Lemma 4. There remains the case $z = u = w$, $P = R_x R_y R_w^2$. If y is in $\mathfrak{C}$ then P is in $\mathfrak{H}$ by Lemma 1. Otherwise $P = R_x R_w^3$ and either $x = w$ and $P = R_w^4$ is in $\mathfrak{H}$ by Lemma 3 or x is in $\mathfrak{C}$, P is in $\mathfrak{H}$ by Lemma 4. This completes our argument and implies that $(\mathfrak{B}^*)^5 \leq \mathfrak{B}^*(\mathfrak{B}^*\mathfrak{C}^* + \mathfrak{C}^*) \leq \mathfrak{B}^*\mathfrak{C}^*$. An immediate induction implies that $(\mathfrak{B}^*)^{4k+1} \leq \mathfrak{B}^*(\mathfrak{C}^*)^k$.

If $\mathfrak{B}$ is a solvable subalgebra of order one of a static algebra $\mathfrak{A}$ then $\mathfrak{B} = w\mathfrak{F}$, $w^2 = 0 = w^3$. By (18) we see that $R_w^3 = 0$. Since $\mathfrak{B}^* = \mathfrak{F}[R_w]$ we see that $\mathfrak{B}^*$ is nilpotent. Let us assume then that $\mathfrak{B}$ has order m and that the property that $\mathfrak{B}$ is solvable implies that $\mathfrak{B}^*$ is nilpotent is valid for solvable subalgebras of order $m - 1$ of static algebras. Then the algebra $\mathfrak{C}$ of the argument above is solvable and has order $m - 1$, $\mathfrak{C}^*$ is nilpotent, $(\mathfrak{C}^*)^k = 0$ for some positive integer k, $(\mathfrak{B}^*)^{4k+1} = 0$, $\mathfrak{B}^*$ is nilpotent. We have proved

THEOREM 4. *Let $\mathfrak{B}$ be a solvable subalgebra of a static algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ of characteristic not two or three. Then $\mathfrak{B}^*$ is nilpotent.*

As in the theory of Jordan algebras we have the immediate

COROLLARY. *A solvable static algebra is nilpotent.*

5. **The radical of a static algebra.** We shall define the *radical* of a static algebra $\mathfrak{A}$ to be the maximal nilideal $\mathfrak{N}$ of $\mathfrak{A}$, and shall call $\mathfrak{A}$ semi-simple if $\mathfrak{N} = 0$. Then we may prove

THEOREM 5. *Let $\mathfrak{A}$ be a static algebra over a nonmodular field $\mathfrak{F}$. Then the radical of $\mathfrak{A}$ is the set $\mathfrak{N}_0$ of all quantities x of $\mathfrak{A}$ such that the trace $\tau(R_{xy}) = 0$ for every y of $\mathfrak{A}$.*

For $\mathfrak{N}_0$ is clearly a linear space over $\mathfrak{F}$. To prove that $\mathfrak{N}_0$ is an ideal, we need only show that whenever x is in $\mathfrak{N}_0$ and y is in $\mathfrak{A}$, then $R_{(xy)z} - R_{x(yz)}$ has trace zero for every z in $\mathfrak{A}$. This follows from (15) since

$$R_{(yz)x} = 2R_x R_{yz} + R_y R_{xz} + R_z R_{xy} - R_{yz} R_x - R_x(R_y R_z + R_z R_y),$$

and the trace of the transformation

$$(24) \quad \begin{aligned} R_{(xy)z} - R_{x(yz)} &= (R_{yz} R_x - R_x R_{yz}) + (R_z R_{xy} - R_{xy} R_z) \\ &\quad + [(R_x R_y) R_z - R_z(R_x R_y)] + [R_x(R_z R_y) - (R_z R_y)R_x] \end{aligned}$$

evidently vanishes. If $\mathfrak{N}_0$ were not a nilideal it would contain an idempotent $e = e^2$ and we would have $\tau(R_{ee}) = 0$ whereas $\mathfrak{A}_e(1)$ contains e and is not zero, the characteristic roots of R_e are 1, 1/2, 0 and are not all zero, $\tau(R_e) > 0$. Hence $\mathfrak{N}_0 \leqq \mathfrak{N}$. But if x is in $\mathfrak{N}$ then $z = xy$ is in $\mathfrak{N}$ and is a nilpotent quantity, $\mathfrak{B} = \mathfrak{F}[z]$ is an associative algebra defined by a nilpotent quantity and so is solvable, $\mathfrak{B}^*$ is nilpotent, R_z is nilpotent, $\tau(R_z) = 0$, x is in $\mathfrak{N}_0$. Hence $\mathfrak{N}_0 = \mathfrak{N}$.

6. Decomposition relative to an idempotent. Since a static algebra $\mathfrak{A}$ is a commutative algebra we see that

$$(25) \qquad\qquad \mathfrak{A} = \mathfrak{A}_e(1) + \mathfrak{A}_e(1/2) + \mathfrak{A}_e(0)$$

where $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$ are orthogonal algebras. Put $y = w = e$ in (16) and obtain

$$(26) \quad [(xe)z]e + [(xe)e]z + [x(ez)]e + [e(ez)]x = 3(xe)(ze) + (xz)e.$$

If x and z are in $\mathfrak{A}_e(1/2)$ then $xe = x/2$, $ze = z/2$ and (26) yields $(xz)e/2 + xz/4 + (xz)e/2 + xz/4 = 3xz/4 + (xz)e$, $xz/4 = 0$, $xz = 0$. Thus $\mathfrak{A}_e(1/2)$ is a zero algebra. Put $xe = x$, $ze = z/2$ and have $(xz)e + xz + (xz)e/2 + xz/4 = 3xz/2 + (xz)e$ and thus $(xz)(2R_e - I) = 0$, xz is in $\mathfrak{A}_e(1/2)$. Finally put $xe = 0$, $ze = z/2$ and obtain $(xz)e/2 + xz/4 = (xz)e$, $(xz)(2R_e - I) = 0$. We have proved

THEOREM 6. *Let e be an idempotent of a static algebra $\mathfrak{A}$. Then $\mathfrak{A}_e(1/2)$ is an ideal of $\mathfrak{A}$ and is a zero algebra.*

7. Structure of semi-simple static algebras. It follows from Theorem 6 that if a static algebra $\mathfrak{A}$ contains no solvable ideals and if e is an idempotent of $\mathfrak{A}$ then $\mathfrak{A}$ is the direct sum of $\mathfrak{A}_e(1)$ and $\mathfrak{A}_e(0)$. Let us assume now that $\mathfrak{N} = 0$. Then $\mathfrak{A}$ is not a nilalgebra and so must contain an idempotent. It follows that $\mathfrak{A}$ contains a principal idempotent e and that $\mathfrak{A} = \mathfrak{A}_e(1) \oplus \mathfrak{A}_e(0)$. But then $\mathfrak{A}_e(0)$ is a nilideal of $\mathfrak{A}$ and so is zero, e is the unity element of $\mathfrak{A}$. Write $e = u + v$ where u and v are pairwise orthogonal idempotents and have $\mathfrak{A} = \mathfrak{A}_u(1) \oplus \mathfrak{A}_u(0)$ where v is in $\mathfrak{A}_u(0)$. Then every a of $\mathfrak{A}$ has the form $a = b + c$ where b is in $\mathfrak{A}_u(1)$ and c is in $\mathfrak{A}_u(0)$, $ea = a = (u+v)(b+c) = b + vc = b + c$, $vc = c$. It is then

clear that $\mathfrak{A}_u(0) = \mathfrak{A}_v(1)$. We now decompose e as a sum $e = e_1 + \cdots + e_t$ of pairwise orthogonal primitive idempotents e_i and have a corresponding decomposition of $\mathfrak{A}$ as the direct sum $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_t$ where e_i is the unity element of $\mathfrak{A}_i$. Every nilideal of $\mathfrak{A}_i$ is a nilideal of $\mathfrak{A}$ and so $\mathfrak{A}_i$ must be semi-simple. Moreover a decomposition $\mathfrak{A}_i = \mathfrak{B}_i \oplus \mathfrak{C}_i$ of $\mathfrak{A}_i$ results in a decomposition $e_i = u_i + v_i$ of e_i contrary to our hypothesis that e_i is primitive. Hence $\mathfrak{A}_i$ must not have a decomposition as a direct sum. Actually $\mathfrak{A}_i$ must have no ideals other than zero and $\mathfrak{A}_i$ since it can have no nilideals and any other nonzero proper ideal of $\mathfrak{A}$ would contain an idempotent $u \neq e$. We may now write $\mathfrak{A}_i$ as a central simple algebra over its center $\mathfrak{Z}_i$. Extend $\mathfrak{Z}_i$ to be an algebraically closed field $\mathfrak{K}_i$ and designate the resulting simple algebra by $\mathfrak{B}_i$. Then $\mathfrak{B}_i$ is not a nilalgebra and if x is in $\mathfrak{B}_i$, then the only idempotent in $\mathfrak{F}[x]$ can be e_i since otherwise $\mathfrak{B}_i$ would be reducible. It follows that every element of $\mathfrak{B}_i$ has the form $\mathfrak{x} = \alpha e_i + f$ where f is nilpotent; α is in $\mathfrak{K}_i$. As in the case of Jordan algebras the trace criterion implies that the set of all nilpotent elements of $\mathfrak{B}_i$ is a nilideal. Then $x = \alpha e_i$, $\mathfrak{B}_i$ has order one over $\mathfrak{K}_i$, $\mathfrak{A}_i = \mathfrak{Z}_i$ and we have proved

THEOREM 7. *Every semi-simple static algebra is a commutative associative semi-simple algebra.*

CHAPTER IV. STANDARD ALGEBRAS

1. Flexible Jordan-admissible algebras. The theory of Jordan algebras is so much like the theory of associative algebras that it is natural to conjecture that both classes of algebras are members of a more general class with a similar theory. We shall present such a class of algebras here.

The class of algebras desired will satisfy the Jordan postulate $(xy)x^2 = x(yx^2)$ and the postulate of commutativity will be replaced by the postulate of flexibility. However, the formulation will be much more delicate since the structure of Jordan algebras is based upon formulas derived from the assumption $x(yx^2) = (xy)x^2$ and using the commutative law.

Let us first derive some consequences of the assumption of flexibility, that is, of the property that

$$(1) \qquad\qquad R_x L_x = L_x R_x$$

for every x of the algebra $\mathfrak{A}$. Then $R_{x+y}L_{x+y} - L_{x+y}R_{x+y} = (R_x L_x - L_x R_x) + (R_x L_y - L_y R_x) + (R_y L_x - L_x R_y) + (R_y L_y - L_y R_y) = 0$, and so

$$(2) \qquad\qquad R_x L_y - L_y R_x = L_x R_y - R_y L_x$$

for every x and y of $\mathfrak{A}$. However formula (2) is equivalent to the equation $(xy)z + (zy)x = x(yz) + z(yx)$ of (1.29) and this latter equation may be written as

$$(3) \qquad\qquad L_{xy} - L_y L_x = R_{yx} - R_y R_x.$$

In particular we have

$$(4) \qquad\qquad L_{xx} - L_x^2 = R_{xx} - R_x^2.$$

An algebra $\mathfrak{A}$ will be called a *Jordan-admissible algebra* if the algebra $\mathfrak{A}^{(+)}$ is a Jordan algebra. The multiplications S_x of $\mathfrak{A}^{(+)}$ are defined in terms of the multiplications of $\mathfrak{A}$ by $2S_x = R_x + L_x$ and the product x^2 in $\mathfrak{A}$ coincides with this product in $\mathfrak{A}^{(+)}$. It follows that $\mathfrak{A}$ is Jordan-admissible if and only if

$$(5) \qquad (R_x + L_x)(R_{xx} + L_{xx}) = (R_{xx} + L_{xx})(R_x + L_x)$$

for every x of $\mathfrak{A}$. Note that all Jordan algebras are Jordan-admissible since $\mathfrak{A}^{(+)}$ and $\mathfrak{A}$ coincide for commutative algebras. Also every associative algebra is Jordan-admissible since $R_x L_x = L_x R_x$ and $R_{xx} = R_x^2$, $L_{xx} = L_x^2$ in an associative algebra.

If x is any element of an algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ there is a corresponding polynomial algebra $\mathfrak{A}_x = \mathfrak{F}[R_x, L_x, R_{xx}]$. When $\mathfrak{A}$ is flexible (4) implies that $\mathfrak{A}_x = \mathfrak{F}[R_x, L_x, L_{xx}]$. We now prove

THEOREM 1. *Let $\mathfrak{A}$ be an algebra over a field $\mathfrak{F}$ of characteristic prime to* 30. *Then $\mathfrak{A}_x$ is a commutative algebra for every x of $\mathfrak{A}$ if and only if $\mathfrak{A}$ is flexible and one of the relations*

$$(6) \qquad \begin{aligned} x(yx^2) &= (xy)x^2, & (x^2y)x &= x^2(yx), \\ (yx)x^2 &= (yx^2)x, & x^2(xy) &= x(x^2y) \end{aligned}$$

holds for every x and y of $\mathfrak{A}$. Moreover if $\mathfrak{A}_x$ is commutative the algebra $\mathfrak{A}$ is Jordan-admissible and is power-associative, the algebra $\mathfrak{A}_x$ contains R_u and L_u for every power $u = x^k$ of x.

For the relations (6) are the commutativity relations

$$(7) \qquad \begin{aligned} L_x R_{xx} &= R_{xx} L_x, & R_x L_{xx} &= L_{xx} R_x, \\ R_x R_{xx} &= R_{xx} R_x, & L_x L_{xx} &= L_{xx} L_x. \end{aligned}$$

If $\mathfrak{A}_x$ is commutative all of the relations (7) hold and (5) holds trivially. Moreover (1) holds and $\mathfrak{A}$ is a flexible Jordan-admissible algebra. Also $x(xx) = (xx)x$, $x(xx^2) = x^2x^2 = (x^2x)x$ by (6) and the formula $x^\alpha x^\beta = x^{\alpha+\beta}$ holds for $\alpha + \beta = 3$, 4. Assume that this formula holds for $\alpha + \beta < n$ where $n > 4$ and write $y = x^{n-3}$ in (6) to obtain $xx^{n-1} = x^{n-2}x^2$, $x^{n-2}x^2 = x^{n-1}x$ so that $xx^{n-1} = x^{n-1}x$ and we may apply Lemma 1.2 to obtain the formula for $\alpha + \beta = n$. Hence $\mathfrak{A}$ is power-associative.

We now replace x by $x + \lambda z$ in the first relation of (6) and equate the coefficients of λ to obtain

$$(8) \qquad x[y(xz + zx)] + z(yx^2) = (xy)(xz + zx) + (zy)x^2.$$

This formula may be linearized and results in a relation which may be

written as

$$(9) \quad \begin{aligned} w[x(yz + zy)] &+ y[x(zw + wz)] + z[x(yw + wy)] \\ &= (wx)(yz + zy) + (yx)(zw + wz) + (zx)(yw + wy). \end{aligned}$$

This relation becomes

$$(10) \quad R_{x(yz+zy)} = R_x R_{yz+zy} + (R_z + L_z)(L_{yx} - L_x L_y) + (R_y + L_y)(L_{zx} - L_x L_z).$$

The assumption that $\mathfrak{A}_x$ contains R_u and L_u for $u = x^k$ and $k = 1, \cdots, t$ im-implies that R_u is in $\mathfrak{A}_x$ for $u = x^{t+1}$ when we replace y by x^{t-1} and put $z = x$ in (10). Then relation (3) implies that L_u is in $\mathfrak{A}_x$.

Conversely let $\mathfrak{A}$ be flexible and let one of the relations (6) hold. By (2) with $y = x^2$ we see that

$$(11) \quad R_x L_{xx} - L_{xx} R_x = L_x R_{xx} - R_{xx} L_x.$$

Hence the first two relations of (6) are equivalent when $\mathfrak{A}$ is flexible. The relation $R_x R_{xx} = R_{xx} R_x$ and (4) imply that L_{xx} is commutative with R_x while $L_x L_{xx} = L_{xx} L_x$ and (4) imply that $R_{xx} L_x = L_x R_{xx}$. Thus we see that the flexible law implies the equivalence of all of the relations (6) and all imply that $\mathfrak{A}_x$ is a commutative algebra.

It should be noted that the relations (6) may be linearized and imply

$$(12) \quad [R_x, L_{yz+zy}] + [R_y, L_{xz+zx}] + [R_z, L_{xy+yx}] = 0,$$

$$(13) \quad [R_x, R_{yz+zy}] + [R_y, R_{xz+zx}] + [R_z, R_{xy+yx}] = 0,$$

$$(14) \quad [L_x, L_{yz+zy}] + [L_y, L_{xz+zx}] + [L_z, L_{xy+yx}] = 0,$$

where, as usual, we mean $[S, T] = ST - TS$. The omitted relation is a trivial consequence of (12) and (2).

The relation (10) and the flexible law seem inadequate to yield a satis-factory theory and we shall strengthen both assumptions. It should be noted that all of the results above actually hold for rings.

2. **Lie-admissible rings.** Every ring $\mathfrak{A}$ determines an attached ring $\mathfrak{A}^{(-)}$ which is the same additive group as $\mathfrak{A}$ but has a product (x, y) defined in terms of the product xy of $\mathfrak{A}$ by $(x, y) = xy - yx$. The right multiplications T_x of $\mathfrak{A}^{(-)}$ will then be defined in terms of the multiplications of $\mathfrak{A}$ by $T_x = R_x - L_x$.

We call $\mathfrak{A}$ a *Lie-admissible ring* if $\mathfrak{A}^{(-)}$ is a Lie ring, that is, $T_{(x,y)} = T_x T_y - T_y T_x$. Then $\mathfrak{A}$ is Lie-admissible if and only if

$$(15) \quad R_{xy-yx} - L_{xy-yx} = (R_x - L_x)(R_y - L_y) - (R_y - L_y)(R_x - L_x).$$

Let us introduce the assumption that $\mathfrak{A}$ is also a flexible ring. We then use (3) to write $-R_{yx} + L_{xy} = -R_y R_x + L_y L_x$, $R_{xy} - L_{yx} = R_x R_y - L_x L_y$ and we add to obtain

$$(16) \quad R_{xy-yx} + L_{xy-yx} = (R_x R_y - R_y R_x) - (L_x L_y - L_y L_x).$$

The right member of (15) is $(R_xR_y-R_yR_x)+(L_xL_y-L_yL_x)+(L_yR_x-R_xL_y)$ $+(R_yL_x-L_xR_y)$ and $R_yL_x-L_xR_y=L_yR_x-R_xL_y$ by (2). Then we may add (15) and (16) and remove the factor 2 to obtain

$$(17) \qquad R_{xy-yx} = R_x(R_y - L_y) - (R_y - L_y)R_x.$$

We may also subtract (15) from (16) and obtain

$$(18) \qquad L_{xy-yx} = (L_y - R_y)L_x - L_x(L_y - R_y).$$

Conversely both (17) and (18) imply that $R_xL_x=L_xR_x$ so that $\mathfrak{A}$ is flexible and then (17) and (18) are equivalent to (15). We have proved

THEOREM 2. *A ring $\mathfrak{A}$ is a flexible Lie-admissible ring if and only if either* (17) *or* (18) *holds for every x and y of $\mathfrak{A}$.*

A Jordan ring is trivially flexible and Lie-admissible since (18) is satisfied when $xy=yx$, $R_y=L_y$. It is well known that an associative algebra is Lie admissible and is, of course, flexible. However, we may prove

THEOREM 3. *An alternative ring whose characteristic is prime to six is Lie-admissible if and only if it is associative.*

For if $\mathfrak{A}$ is alternative then $\mathfrak{A}$ is flexible and $x(xy)=(xx)y$, $x(zy)+z(xy)$ $=(xz+zx)y$,

$$(19) \qquad R_{xy} = R_xR_y + (L_xR_y - R_yL_x).$$

But by (2) $L_xR_y-R_yL_x=R_xL_y-L_yR_x$ and so from $R_{yx}=R_yR_x+(L_yR_x-R_xL_y)$ we obtain $R_{xy-yx}=(R_xR_y-R_yR_x)-2(L_yR_x-R_xL_y)$. This combined with (17) yields $3(L_yR_x-R_xL_y)=0$, $L_yR_x=R_xL_y$ since the characteristic of $\mathfrak{A}$ is prime to six. Thus $(yz)x=y(zx)$ for every x, y, z of $\mathfrak{A}$ and $\mathfrak{A}$ is associative.

3. **Standard algebras.** An algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ will be called a *standard algebra* if (17) holds and if

$$(20) \qquad R_{x(yz)} = R_xR_{yz} + R_y(R_{xz} - R_xR_z) + R_z(R_{xy} - R_xR_y)$$

for every x, y, z of $\mathfrak{A}$. (20) is known to be a consequence of the defining assumptions for Jordan algebras and is a trivial consequence of the property $R_{xy}=R_xR_y$ of associative algebras. We note the following simple result

THEOREM 4. *A Lie algebra $\mathfrak{A}$ is a standard algebra if and only if all products of four elements of $\mathfrak{A}$ are zero.*

For when $\mathfrak{A}$ is a Lie algebra $R_{x(yz)}=R_xR_{yz}-R_{yz}R_x$. Then (20) is equivalent to $-R_{yz}R_x=(R_zR_y-R_yR_z)R_x=R_yR_zR_x+R_zR_yR_x$, that is, to $2R_yR_zR_x=0$. Thus $[(wy)z]x=0$ for every x, y, z, w of $\mathfrak{A}$, $L_{(wy)z}=0=-R_{(wy)z}$, $R_{wy}R_z=R_zR_{wy}$, $R_{wy}R_x=0$, $R_zR_{wy}=0$, $(xz)(wy)=0$, all products of four elements of $\mathfrak{A}$ are zero. The converse clearly implies that $[(wy)z]x=0$, $R_yR_zR_x=0$ and so that (20) holds.

We shall now derive some identities which are a consequence of the relation (20) and the flexible law. We first write (20) as

$$(21) \quad w[x(yz)] + [(wy)x]z + [(wz)x]y = (wx)(yz) + (wy)(xz) + (wx)(xy)$$

and then have $R_{yz}L_w + L_{wy}R_z + L_{wz}R_y = L_w R_{yz} + R_z L_{wy} + R_y L_{wz}$. However $R_{yz}L_w - L_w R_{yz} = L_{yz}R_w - R_w L_{yz}$ by (2) and we replace w by x to obtain

$$(22) \qquad R_x L_{yz} + R_y L_{xz} + R_z L_{xy} = L_{yz}R_x + L_{xz}R_y + L_{xy}R_z.$$

Observe that if the characteristic of $\mathfrak{A}$ is prime to three then (22) implies that $R_x L_{xx} = L_{xx} R_x$ and so that every $\mathfrak{A}_x$ is commutative. We have proved

THEOREM 5. *Let $\mathfrak{A}$ be a standard algebra over a field $\mathfrak{F}$ whose characteristic is prime to six. Then $\mathfrak{A}$ is a power-associative Jordan-admissible algebra and $\mathfrak{A}_x$ is commutative for every x of $\mathfrak{A}$, that is, the transformations R_u, L_u, R_v, L_v commute for all powers u and v of x.*

The identity (21) may be rewritten with w and z interchanged and implies that $L_{(zy)x} + L_y L_x L_z + L_z R_x R_y = L_y L_{zx} + L_x L_{zy} + L_z R_{xy}$. However $R_{xy} - R_x R_y = L_{yx} - L_x L_y$ and so we have

$$(23) \qquad L_{(zy)x} = L_x L_{zy} + L_y(L_{zx} - L_x L_z) + L_z(L_{yx} - L_x L_y).$$

We note the consequences

$$(24) \qquad\qquad R_{x(xx)} = 3R_x R_{xx} - 2R_x^3$$

and

$$(25) \qquad\qquad L_{(xx)x} = 3L_x L_{xx} - 2L_x^3$$

of (20) and (23) respectively.

4. Solvable algebras. Our first major result will involve a subalgebra $\mathfrak{B}$ of a standard algebra $\mathfrak{A}$. Define $\mathfrak{B}_0$ to be the vector space of all mappings on $\mathfrak{A}$ of the form $R_x + L_y$ for x and y in $\mathfrak{B}$ and $\mathfrak{B}^*$ to be the associative algebra of all finite sums of products of elements of $\mathfrak{B}_0$. Then we may prove

THEOREM 6. *Let $\mathfrak{B}$ be a solvable subalgebra of a standard algebra $\mathfrak{A}$ over $\mathfrak{F}$ of characteristic prime to six. Then $\mathfrak{B}^*$ is nilpotent.*

The result is true for $\mathfrak{B}$ of order one since then $\mathfrak{B} = x\mathfrak{F}$, $x^2 = 0$, $R_x^3 = 0$ by (24), $L_x^3 = 0$ by (25), $\mathfrak{B}^*$ is the commutative algebra $\mathfrak{F}[R_x, L_x]$ and is clearly nilpotent. Assume the result true for algebras of order $m-1$ and let $\mathfrak{B}$ have order m. Every solvable algebra $\mathfrak{B}$ has the form $\mathfrak{B} = \mathfrak{C} + w\mathfrak{F}$ where $\mathfrak{B}\mathfrak{B} \leqq \mathfrak{C}$ and $\mathfrak{C}$ has order $m-1$ and so $\mathfrak{C}^*$ is nilpotent. We propose to show that $\mathfrak{B}_0^3 \leqq \mathfrak{H} = \mathfrak{B}^*\mathfrak{C}^* + \mathfrak{C}^*$.

Since $\mathfrak{B}_0$ is a vector space it is sufficient to prove that all products $S_x S_y S_z$ are in $\mathfrak{H}$ where S_x is the symbol for a multiplication and x, y, z are either in

$\mathfrak{C}$ or equal to w. Now (20) and $\mathfrak{BB} \leqq \mathfrak{C}$ imply that

$$(26) \qquad H_1(x,\ y,\ z)\ =\ R_x R_y R_z\ +\ R_z R_y R_x$$

is in $\mathfrak{H}$ for every x, y, z of $\mathfrak{B}$. Then $R_x R_y R_z$ is in $\mathfrak{H}$ if z is in $\mathfrak{C}$, $R_x R_y R_w$ $= H_1(x,\ y,\ z) - R_w R_y R_x$ is in $\mathfrak{H}$ if x is in $\mathfrak{C}$, $2R_w R_y R_w$ is in $\mathfrak{C}$ for every y. Thus $R_x R_y R_z$ is in $\mathfrak{H}$ for every x, y, z of $\mathfrak{B}$. Similarly (23) implies that every $L_x L_y L_z$ is in $\mathfrak{H}$. But $R_x L_y L_z = R_x(L_{zy} - R_{yx} + R_y R_z)$ is then in $\mathfrak{A}$ and so is $L_x R_y R_z = L_x(R_{yz} - L_{zy} - L_y L_z)$. It remains to consider products $R_x L_y R_z$, $L_x R_y L_z$, $R_x R_y L_z$, $L_x L_y R_z$.

We note next that the assumption (17) implies that $R_z R_{xy-yx} = R_z(R_x R_y - R_y R_x) + R_z(L_y R_x - R_x L_y)$ and so $H = R_z(L_y R_x - R_x L_y)$ is in $\mathfrak{H}$. If x is in $\mathfrak{C}$ then $R_z L_y R_x$ is in $\mathfrak{H}$. Hence let $x = w$. If y is in $\mathfrak{C}$ then $R_z L_y R_x = R_z R_x L_y + H$ is in $\mathfrak{H}$. It then remains to consider $R_z L_w R_w = (L_z R_w + L_w R_z - R_w L_z) R_w$ (by formula (21)) which is in $\mathfrak{H}$ if and only if $R_w L_z R_w$ is in $\mathfrak{H}$. We have already seen that $R_w L_z R_w$ is in $\mathfrak{H}$ if z is in $\mathfrak{C}$ so there remains the case of a product $R_w L_w R_w = L_w R_w R_w$ which is in $\mathfrak{H}$. It follows that all products $R_z L_y R_x$ and all products $R_z R_x L_y$ are in $\mathfrak{H}$. We also have $L_z R_{xy-yx} = L_z(R_x R_y - R_y R_x) + L_z(L_y R_x - R_x L_y)$ and so $L_z L_y R_x - L_z R_x L_y$ is in $\mathfrak{H}$. The remainder of the proof is exactly as above.

We have now shown that $\mathfrak{B}_0^3 \leqq \mathfrak{B}^*\mathfrak{C}^* + \mathfrak{C}^*$ so that $\mathfrak{B}^{*4} \leqq \mathfrak{B}^*\mathfrak{C}^*$, $\mathfrak{B}^{*3k+1} \leqq \mathfrak{B}^*\mathfrak{C}^{*k} = 0$ for some k, $\mathfrak{B}^*$ is nilpotent.

5. Nilalgebras. The procedure we shall follow in the study of standard nilalgebras is an extension to noncommutative algebras of that used for Jordan algebras. We observe first that if $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ then $x\mathfrak{B}^* \leqq \mathfrak{B}$ if and only if $\mathfrak{B}$ contains both xb and bx for every b of $\mathfrak{B}$. There always exists an idempotent linear transformation E on $\mathfrak{A}$ such that $\mathfrak{B} = \mathfrak{A}E$ and we define $\mathfrak{W}$ to be the set of all linear transformations T on $\mathfrak{A}$ such that $ET = ETE$. Then $\mathfrak{W}$ is an associative algebra and we have

LEMMA 1. *If x is in $\mathfrak{A}$ then $x\mathfrak{B}^* \leqq \mathfrak{B}$ if and only if R_x and L_x are in $\mathfrak{W}$.*

For the generic element of $\mathfrak{B}$ is $b = aE$ where a is the generic element of $\mathfrak{A}$. Then an element y is in $\mathfrak{A}E$ if and only if $y = yE$ and so $xb = x(aE) = a(EL_x)$ and $bx = (aE)x = a(ER_x)$ are in $\mathfrak{B}$ if and only if $EL_x = EL_x E$, $ER_x = ER_x E$, that is, R_x and L_x are in $\mathfrak{W}$.

Note that $b\mathfrak{B}^* \leqq \mathfrak{B}$ for every b of $\mathfrak{B}$ and so R_b and L_b are in $\mathfrak{W}$. We next prove

LEMMA 2. *Let $\mathfrak{B}$ be a subalgebra of a power-associative algebra $\mathfrak{A}$ and let R_x, L_x, R_{xx} be in $\mathfrak{W}$ for x in $\mathfrak{A}$. Then $\mathfrak{B}$ is an ideal of the algebra $\mathfrak{B}[x]$ of all polynomials in x with coefficients in either $\mathfrak{B}$ or $\mathfrak{F}$.*

For Lemma 1.5 implies that the algebra $\mathfrak{A}_x$ of all polynomials in R_x, L_x, R_{xx} contains R_u and L_u for all powers $u = x^k$ of x. Then $\mathfrak{A}_x \leqq \mathfrak{W}$ and every R_u is in $\mathfrak{W}$, $u\mathfrak{B}^* \leqq \mathfrak{B}$. It follows that $\mathfrak{B}[x] = \mathfrak{F}[x] + \mathfrak{B}$ and that $\mathfrak{B}$ is an ideal of $\mathfrak{B}[x]$.

The assumption that $\mathfrak{A}$ is a standard algebra will now be introduced and we shall prove

LEMMA 3. *Let R_x and L_x be in W and $y = bx^2$ where b is in $\mathfrak{B}$. Then R_y and L_y are in $\mathfrak{W}$.*

We first apply (20), (13), (23), (14), (4) to obtain the relations

$$(27) \qquad R_{x(bx)} = R_x R_{bx} + R_b(R_{xx} - R_x^2) + R_x(R_{xb} - R_x R_b),$$

$$(28) \qquad R_b R_{xx} + R_x R_{bx+xb} = R_{xx} R_b + R_{bx+xb} R_x,$$

$$(29) \qquad L_{(xb)x} = L_x L_{xb} + L_b(L_{xx} - L_x^2) + L_x(L_{bx} - L_x L_b),$$

$$(30) \qquad L_b L_{xx} + L_x L_{xb+bx} = L_{xb+bx} L_x + L_{xx} L_b,$$

$$(31) \qquad L_b L_{xx} = L_b(R_{xx} + L_x^2 - R_x^2), \qquad L_{xx} L_b = (R_{xx} + L_x^2 - R_x^2) L_b.$$

Since bx and xb are in $\mathfrak{B}$ we see from (27) that $R_b R_{xx}$ is in $\mathfrak{W}$, from (28) that $R_{xx} R_b$ is in $\mathfrak{W}$, from (29) that $L_b L_{xx}$ is in $\mathfrak{W}$, from (30) that $L_{xx} L_b$ is in $\mathfrak{W}$, and from (31) that $L_b R_{xx}$ and $R_{xx} L_b$ are in $\mathfrak{W}$. Now by (20) and (17) respectively,

$$(32) \qquad R_{b(xx)} = R_b R_{xx} + 2R_x(R_{bx} - R_b R_x),$$

$$(33) \qquad R_{(xx)b} = R_{b(xx)} + R_{xx}(R_b - L_b) - (R_b - L_b) R_{xx}$$

so that both R_y and $R_{(xx)b}$ are in $\mathfrak{W}$. But by (3),

$$(34) \qquad R_{(xx)b} - R_{xx} R_b = L_{b(xx)} - L_{xx} L_b$$

and so L_y is also in $\mathfrak{W}$ as desired.

We write $u = xx^2$ and use (24) together with Theorem 5 to write $R_b R_u = 3R_b R_{xx} R_x - 2R_b R_x^3$, $R_u R_b = 3R_x R_{xx} R_b - 2R_x^3 R_b$. Since $R_{xx} R_b$ and $R_b R_{xx}$ are in $\mathfrak{W}$, we see that $R_u R_b$ and $R_b R_u$ are in $\mathfrak{W}$ for every b of $\mathfrak{B}$. Similarly, $R_u L_b$ and $L_b R_u$ are in $\mathfrak{W}$. But by (20) we have

$$(35) \qquad R_{bu} = R_b R_u + R_{xx}(R_{bx} - R_b R_x) + R_x(R_y - R_b R_{xx})$$

which implies that R_{bu} is in $\mathfrak{W}$, and by (17),

$$(36) \qquad R_{ub} = R_{bu} + R_u(R_b - L_b) - (R_b - L_b) R_u$$

and then R_{ub} is in $\mathfrak{W}$. But

$$(37) \qquad R_{u(bx)} = R_u R_{bx} + R_b(R_v - R_u R_x) + R_x(R_{ub} - R_u R_b)$$

where $v = ux = x^4$, bx is in $\mathfrak{B}$ and so $R_{u(bx)}$ is in $\mathfrak{W}$. This shows that $R_b R_v$ is in $\mathfrak{W}$. However $R_v = R_{xu}$ and the same computation used in the commutative Jordan case gives

$$(38) \qquad R_v = R_{xx}^2 + 4R_{xx} R_x^2 - 4R_x^4$$

and so $R_b R_{xx}^2$ is in $\mathfrak{W}$. We now use (28) to form

(39) $R_{xx}R_bR_{xx} = - R_x(R_{xx}R_{bx+xb}) + R_{xx}^2R_b + R_{xx}R_{bx+xb}R_x$

by multiplication on the left and

(40) $R_{xx}R_bR_{xx} = R_bR_{xx}^2 + R_xR_{bx+xb}R_{xx} - (R_{bx+xb}R_{xx})R_x.$

Then we subtract to see that $R_{xx}^2R_b$ is in $\mathfrak{W}$.

We now write $y^2 = y(bx^2)$ and form

(41) $R_{yy} = R_y^2 + R_b(R_{y(xx)} - R_yR_{xx}) + R_{xx}(R_{yb} - R_yR_b).$

We then use $R_{y(xx)} - R_yR_{xx} = 2R_x(R_{yx} - R_yR_x)$ as well as

(42) $R_{xy} = R_xR_y + R_b(R_u - R_xR_{xx}) + R_{xx}(R_{xb} - R_xR_b),$

(43) $R_{yx} = R_{xy} + R_y(R_x - L_x) - (R_x - L_x)R_y$

to see that R_{xy} and R_{yx} are in $\mathfrak{W}$, $R_b(R_{y(xx)} - R_yR_{xx})$ is in $\mathfrak{W}$. The term $R_{xx}R_yR_b = R_{xx}[R_bR_{xx} + 2R_x(R_{bx} - R_bR_x)]R_b = (R_{xx}R_b)^2 + 2R_x(R_{xx}R_{bx}R_b - R_{xx}R_bR_xR_b)$ is in $\mathfrak{W}$ and we finally compute

(44) $R_{by} = R_bR_y + R_{xx}(R_{bb} - R_b^2) + R_b(R_y - R_bR_{x^2})$

and

(45) $R_{yb} = R_{by} + R_y(R_b - L_b) + (R_b - L_b)R_y.$

The product $[R_{xx}(R_b - L_b)]R_y$ is in $\mathfrak{W}$, the product $R_{xx}R_y(R_b - L_b) = R_{xx}R_bR_{xx}(R_b-L_b) + 2R_xR_{xx}(R_{bx} - R_bR_x)(R_b - L_b)$ is in $\mathfrak{W}$, and the product $R_{xx}R_{by} = (R_{xx}R_b)R_y + R_{xx}^2(R_{bb} - R_b^2) + (R_{xx}R_b)(R_y - R_bR_{xx})$ is in $\mathfrak{W}$. But then $R_{xx}R_{yb}$ is in $\mathfrak{W}$ and we have proved that R_{yy} is in $\mathfrak{W}$, which completes the proof of

LEMMA 4. *Let R_x and L_x be in $\mathfrak{W}$, $y = bx^2$ where b is in $\mathfrak{B}$. Then $\mathfrak{A}_y$ is in $\mathfrak{W}$.*

We are now ready to prove

THEOREM 7. *Let $\mathfrak{A}$ be a standard nilalgebra. Then A is a solvable algebra.*

The result is trivial for algebras of order one and so we assume it true for algebras of lower order than that of $\mathfrak{A}$. Also the result is true if $\mathfrak{A}$ consists of the associative algebra $\mathfrak{F}[x]$ generated by an element x of $\mathfrak{A}$. Hence assume that $\mathfrak{A} \neq \mathfrak{F}[x]$ and thus that $\mathfrak{A}$ contains a proper subalgebra. Let $\mathfrak{B}$ be a maximal proper subalgebra of $\mathfrak{A}$. Then by the hypothesis of the induction, $\mathfrak{B}$ is solvable, $\mathfrak{B}^*$ is nilpotent, $\mathfrak{A}\mathfrak{B}^{*k} = 0$. Let t be the least integer such that $\mathfrak{A}\mathfrak{B}^{*t} \leq \mathfrak{B}$. Then clearly $t \geq 1$ and there exists an element z of $\mathfrak{A}$ such that $z\mathfrak{B}^{*t} \leq \mathfrak{B}$, $z\mathfrak{B}^{*t-1}$ is not $\leq \mathfrak{B}$. Thus there is an element x in $z\mathfrak{B}^{*t-1}$ but not in $\mathfrak{B}$ and $x\mathfrak{B}^* \leq \mathfrak{B}$. By Lemmas 1 and 4, if b is in $\mathfrak{B}$, then $y\mathfrak{B}^* \leq \mathfrak{B}$, $y^2\mathfrak{B}^* \leq \mathfrak{B}$ for $y = bx^2$. Thus we have either some such y not in $\mathfrak{B}$ or $\mathfrak{B}x^2 \leq \mathfrak{B}$. In either case we

have shown the existence of an element w not in $\mathfrak{B}$ such that $w\mathfrak{B}^* \leqq \mathfrak{B}$, $\mathfrak{B}w^2 \leqq \mathfrak{B}$. This latter condition implies R_{w^2} in $\mathfrak{W}$ as the proof of Lemma 1 shows and so the hypotheses of Lemma 2 are satisfied. Since x is not in $\mathfrak{B}$, $\mathfrak{B}[x] = \mathfrak{A}$ by the maximality of $\mathfrak{A}$ and by Lemma 2, $\mathfrak{B}$ is a solvable ideal of $\mathfrak{A}$. The nilalgebra $\mathfrak{A} - \mathfrak{B}$ is solvable by the induction hypothesis and so $\mathfrak{A}$ is solvable.

6. **Trace admissibility of standard algebras.** Let $\mathfrak{A}$ be a standard algebra and define its trace function $\tau(x, y)$ to be the trace of the linear transformation R_{xy}. By (17) we have $\tau(x, y) = \tau(y, x)$. Also $R_{x(yz)} - R_{z(xy)} = R_x R_{yz} + R_y(R_{xz} - R_x R_z)$ $+ R_z(R_{xy} - R_x R_y) - R_z R_{xy} - R_y(R_{zx} - R_z R_x) - R_x(R_{zy} - R_z R_y) = R_x R_{yz-zy}$ $+ R_y R_{xz-zx} + R_y(R_z R_x) - (R_z R_x)R_y + (R_x R_z)R_y - R_y(R_x R_z)$. However $R_x R_{yz-z}$ $+ R_y R_{xz-zx} = R_x[R_y(R_z - L_z) - (R_z - L_z)R_y] + R_y[R_x(R_z - L_z) - (R_z - L_z)R_x]$ $= R_x[R_y(R_z - L_z)] - [R_y(R_z - L_z)]R_x + R_y[R_x(R_z - L_z)] - [R_x(R_z - L_z)]R_y$. Then $\tau(x, yz) - \tau(z, xy) = 0$, $\tau(x, yz) = \tau(xy, z)$. If xy is nilpotent it generates a solvable algebra, R_{xy} is nilpotent, $\tau(x, y) = 0$. If e is an idempotent $\tau(e, e)$ is the trace of R_e and is not zero. We have proved that $\tau(x, y)$ is an admissible trace function for $\mathfrak{A}$.

However, we may now show that $\tau(x, y)$ is an admissible trace function for $\mathfrak{A}^{(+)}$. For $\tau(x, y) = \tau(y, x)$, $\tau(x \cdot y, z) = \tau(xy + yx, z)/2 = [\tau(xy, z) + \tau(yx, z)]/2$ $= (\tau x, yz)/2 + \tau(y, x_z)/2$, $\tau(y, xz) = \tau(xz, y) = \tau(x, zy)$, $\tau(x \cdot y, z) = [\tau(x, yz)$ $+ \tau(x, zy)]/2 = \tau(x, y \cdot z)$. If $x \cdot y$ is nilpotent so is $xy + yx$, R_{xy+yx} has zero trace, $\tau(x, y) + \tau(y, x) = 2\tau(x, y) = 0$, $\tau(x, y) = 0$. If e is idempotent $\tau(e, e) \neq 0$ as was shown above.

It follows now that the set of all elements x such that $\tau(x, y) = 0$ for every y of $\mathfrak{A}$ is the radical of both $\mathfrak{A}$ and $\mathfrak{A}^{(+)}$. We have proved

THEOREM 8. *Let $\mathfrak{A}$ be a standard algebra over a nonmodular field. Then $\mathfrak{A}$ is trace-admissible, there is a trace function which is admissible both for $\mathfrak{A}$ and for $\mathfrak{A}^{(+)}$, the radical of A coincides with the radical of $\mathfrak{A}^{(+)}$, $\mathfrak{A}$ is semi-simple if and only if $\mathfrak{A}^{(+)}$ is semi-simple.*

CHAPTER V. QUASIASSOCIATIVE ALGEBRAS

1. **The algebra $\mathfrak{A}(\lambda)$.** If $\mathfrak{A}$ is any algebra over a field $\mathfrak{F}$ of characteristic not two and λ is in $\mathfrak{F}$ we define $\mathfrak{A}(\lambda)$ to be that algebra which is the same vector space over $\mathfrak{F}$ as $\mathfrak{A}$ but whose product $x \cdot y$ is defined in terms of the product xy of $\mathfrak{A}$ by $x \cdot y = \lambda xy + (1 - \lambda)yx$. Then $\mathfrak{A}(1) = \mathfrak{A}$, $\mathfrak{A}(0)$ is antiisomorphic to $\mathfrak{A}$, $\mathfrak{A}(1/2) = \mathfrak{A}^{(+)}$.

THEOREM 1. *If $\mathfrak{A}$ is power-associative so is $\mathfrak{A}(\lambda)$ for every λ of $\mathfrak{F}$ and indeed powers in $\mathfrak{A}(\lambda)$ coincide with powers in $\mathfrak{A}$. Also $\mathfrak{A}(\lambda)$ is flexible for every λ if and only if $\mathfrak{A}$ is flexible.*

For when $\mathfrak{A}$ is power-associative the power x^n of every x of $\mathfrak{A}$ is uniquely defined in $\mathfrak{A}$ for every positive integer n. Suppose that x^t is the same in $\mathfrak{A}(\lambda)$ as in $\mathfrak{A}$ for every $t < n$, a result true for $t = 1$. Then $x^r \cdot x^{n-r} = \lambda x^r x^{n-r}$ $+ (1 - \lambda)x^{n-r}x^r = \lambda x^n + (1 - \lambda)x^n = x^n$ for every $r = 1, \cdots, n - 1$. It follows

that nth powers of x in $\mathfrak{A}(\lambda)$ all coincide with nth powers in $\mathfrak{A}$ and $\mathfrak{A}(\lambda)$ is power-associative. If $\mathfrak{A}(\lambda)$ is flexible for every λ then in particular $\mathfrak{A}(1) = \mathfrak{A}$ is flexible. Conversely if $\mathfrak{A}$ is flexible then $(x \cdot y) \cdot x = \lambda [\lambda xy + (1 - \lambda)yx]$ $\cdot x + (1 - \lambda)x [\lambda xy + (1 - \lambda)yx] = \lambda^2(xy)x + (1 - \lambda)^2 x(yx) + \lambda(1 - \lambda)[(yx)x + x(xy)]$ and $x \cdot (y \cdot x) = \lambda x [\lambda yx + (1 - \lambda)xy] + (1 - \lambda)[\lambda yx + (1 - \lambda)xy]x = \lambda^2 x(yx) + (1 - \lambda)^2$ $\cdot (xy)x + \lambda(1 - \lambda)[x(xy) + (yx)x]$. The flexibility of $\mathfrak{A}$ implies that $\lambda^2(xy)x$ $+ (1 - \lambda)^2 x(yx) = \lambda^2 x(yx) + (1 - \lambda)^2 (xy)x$ and so $x \cdot (y \cdot x) = (x \cdot y) \cdot x$, $A(\lambda)$ is flexible.

THEOREM 2. *An algebra $\mathfrak{A}$ is Jordan-admissible if and only if $\mathfrak{A}(\lambda)$ is Jordan-admissible for every λ. Indeed $\mathfrak{A}^{(+)} = [\mathfrak{A}(\lambda)]^{(+)}$ for every λ.*

For $[\mathfrak{A}(\lambda)]^{(+)}$ is defined relative to the product $(x \cdot y + y \cdot x)/2 = [\lambda xy + (1 - \lambda)yx + \lambda yx + (1 - \lambda)xy]/2 = (xy + yx)/2$.

In a similar fashion we may study $[\mathfrak{A}(\lambda)]^{(-)}$. This algebra is defined relative to the product $(x, y) = x \cdot y - y \cdot x = \lambda xy + (1 - \lambda)yx - \lambda yx - (1 - \lambda)xy = (2\lambda - 1)(xy - yx)$. If $\lambda = 1/2$ then $\mathfrak{A}(\lambda)$ is the commutative algebra $\mathfrak{A}^{(+)}$, $[\mathfrak{A}(\lambda)]^{(-)}$ is a zero algebra and is always Lie-admissible. Otherwise $(x, y) = xT_y$ where $T_y = (2\lambda - 1)S_y$ and S_y is the generic right multiplication for the algebra $\mathfrak{A}^{(-)}$. The algebra $\mathfrak{A}$ is Lie-admissible if and only if $S_{[x,y]} = S_x S_y - S_y S_x$ where $[x, y] = xy - yx = xS_y$ and if $\alpha = 2\lambda - 1 \neq 0$ we see that $T_{(x,y)} = \alpha T_{[x,y]}$ $= \alpha^2 S_{[x,y]} = \alpha^2(S_x S_y - S_y S_x) = T_x T_y - T_y T_x$. Conversely if $T_{(x,y)} = T_x T_y - T_y T_x$ then $S_{[x,y]} = S_x S_y - S_y S_x$ and we have proved

THEOREM 3. *An algebra $\mathfrak{A}$ is Lie-admissible if and only if $\mathfrak{A}(\lambda)$ is Lie-admissible for every λ of $\mathfrak{F}$.*

Let us henceforth restrict our attention to the case of algebras $\mathfrak{A}(\lambda)$ defined for $\lambda \neq 1/2$. Then $x \cdot y - y \cdot x = (2\lambda - 1)(xy - yx)$, $x \cdot y + y \cdot x = xy + yx$ and so $2xy = (2\lambda - 1)^{-1}[x \cdot y - y \cdot x + (2\lambda - 1)(x \cdot y + y \cdot x)] = (2\lambda - 1)^{-1}2[\lambda x \cdot y + (\lambda - 1)y \cdot x]$. Since $1 - (2\lambda - 1)^{-1}(\lambda - 1) = (2\lambda - 1)^{-1}[2\lambda - 1 - (\lambda - 1)]$ $= \lambda(2\lambda - 1)^{-1}$, we have proved

THEOREM 4. *Let $\lambda \neq 1/2$ and $\mathfrak{H} = \mathfrak{A}(\lambda)$. Then $\mathfrak{A} = \mathfrak{H}(\mu)$ where $\mu = (2\lambda - 1)^{-1}\lambda$.*

Theorem 4 implies that the ideals of $\mathfrak{A}$ and of $\mathfrak{H} = \mathfrak{A}(\lambda)$ coincide for every $\lambda \neq 1/2$. Indeed if $\mathfrak{B}$ is a subspace of $\mathfrak{A}$ such that by and yb are in $\mathfrak{B}$ for every b of $\mathfrak{B}$ and y of $\mathfrak{A}$ it is true that $b \cdot y$ and $y \cdot b$ are in $\mathfrak{B}$ and conversely. When $\mathfrak{A}$ is a direct sum $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$ then $bc = cb = 0$ for every b of $\mathfrak{B}$ and c of $\mathfrak{C}$, $b \cdot c = c \cdot b = 0$, $\mathfrak{A}(\lambda) = \mathfrak{B}(\lambda) \oplus \mathfrak{C}(\lambda)$. When $\mathfrak{B}$ is solvable so is $\mathfrak{B}(\lambda)$ and when $\mathfrak{B}$ is strongly nilpotent so is $\mathfrak{B}(\lambda)$. Evidently the maximal solvable ideal of $\mathfrak{A}$ coincides with the maximal solvable ideal of $\mathfrak{A}(\lambda)$. Similarly the maximal nilideal of $\mathfrak{A}$ coincides with that of $\mathfrak{A}(\lambda)$ for every $\lambda \neq 1/2$. Finally if every nilideal of $\mathfrak{A}(\lambda)$ is solvable (strongly nilpotent) this is true for every nilideal of $\mathfrak{A}$.

The results just given imply that if $\mathfrak{A}(\lambda)$ has the property that when its

maximal solvable ideal is zero it is a direct sum of simple algebras this will also be true of $\mathfrak{A}$. Moreover $\mathfrak{A}(\lambda)$ is simple if and only if $\mathfrak{A}$ is simple. In particular every simple associative algebra $\mathfrak{A}$ determines a class of flexible, Jordan-admissible and Lie-admissible simple algebras $\mathfrak{A}(\lambda)$. These are not commutative when $\mathfrak{A}$ is not commutative since $x \cdot y - y \cdot x = (2\lambda - 1)(xy - yx)$. They are also not associative, in general, even when $\mathfrak{A}$ is associative. This result follows for total matric algebras by use of the identity we shall derive in the next section.

2. **Quasiassociative algebras.** Let $\mathfrak{A}$ be an algebra over a field $\mathfrak{F}$ of characteristic not two. Then $\mathfrak{A}$ will be called a *quasiassociative algebra* if there exists a scalar extension $\mathfrak{K}$ of $\mathfrak{F}$ and a quantity λ in $\mathfrak{K}$ such that $\mathfrak{A}_\mathfrak{K} = \mathfrak{B}(\lambda)$ where $\mathfrak{B}$ is an associative algebra over $\mathfrak{K}$. If $\lambda = 0, 1$ then $\mathfrak{A}_\mathfrak{K}$ is either isomorphic or antiisomorphic to $\mathfrak{B}$ and $\mathfrak{A}_\mathfrak{K}$ is associative, $\mathfrak{A}$ is associative. If $\lambda = 1/2$ then $\mathfrak{A}_\mathfrak{K}$ is a Jordan algebra, $xy = yx$ and $x(yx^2) = (xy)x^2$ for every x and y of $\mathfrak{A}_\mathfrak{K}$, the same relations hold for every x, y of $\mathfrak{A}$, $\mathfrak{A}$ is a Jordan algebra. Assume henceforth that $\lambda \neq 0, 1, 1/2$ whence $\mathfrak{B} = \mathfrak{A}_\mathfrak{K}(\mu)$ where $\mu = \lambda(2\lambda - 1)^{-1}$ and $\mathfrak{A}_\mathfrak{K}(\mu)$ is associative. Evidently

(1) $$\mu \neq 0, 1, 1/2.$$

If A is quasiassociative then $\mathfrak{B}(\lambda) = \mathfrak{A}_\mathfrak{K}$ is flexible, $x(yx) = (xy)x$ for every x, y of $\mathfrak{A}$, $\mathfrak{A}$ is flexible. We now prove

THEOREM 5. *Let $\mathfrak{A}$ be a quasiassociative algebra over a field $\mathfrak{F}$ of characteristic not two and let $\mathfrak{A}$ be neither associative nor a Jordan algebra. Then there exists a quantity $\alpha_0 \neq 0, -1/4$ in $\mathfrak{F}$ and a scalar extension $\mathfrak{K} = \mathfrak{F}(\mu)$ of $\mathfrak{F}$ such that $\mu^2 = \mu + \alpha_0$, $\mathfrak{A}_\mathfrak{K}(\mu)$ is an associative algebra.*

For the defining operation of $\mathfrak{A}_\mathfrak{K}(\mu)$ is $x \cdot y = \mu xy + (1 - \mu)yx$ where xy is the product in $\mathfrak{A}$ and in $\mathfrak{A}_\mathfrak{K}$. We compute

(2) $$x \cdot (y \cdot z) = \mu x[\mu(yz) + (1 - \mu)zy] + (1 - \mu)[\mu yz + (1 - \mu)zy]x$$

and

(3) $$(x \cdot y) \cdot z = \mu[\mu xy + (1 - \mu)yx]z + (1 - \mu)z[\mu xy + (1 - \mu)yx]$$

to see that $\mathfrak{A}_\mathfrak{K}(\mu)$ is associative if and only if $\mu^2[x(yz) - (xy)z] + (1 - \mu)^2 [(zy)x - z(yx)] + \mu(1 - \mu)[x(zy) + (yz)x - (yx)z - z(xy)] = 0$ for every x, y, z of $\mathfrak{A}_\mathfrak{K}$. However $\mathfrak{A}_\mathfrak{K}$ is flexible and so

(4) $$x(yz) - (xy)z = (zy)x - z(yx).$$

It follows that $\mathfrak{A}_\mathfrak{K}(\mu)$ is associative if and only if $\mathfrak{A}_\mathfrak{K}$ is flexible and

$$[\mu^2 + (1 - \mu)^2][x(yz) - (xy)z] = (\mu^2 - \mu)[x(zy) + (yz)x - (yx)z - z(xy)]$$

for every x, y, z of $\mathfrak{A}_\mathfrak{K}$. Since this relation is linear in x, y, z it holds for every x, y, z of $\mathfrak{A}_\mathfrak{K}$ if and only if it holds for every x, y, z of $\mathfrak{A}$. Define $\alpha_0 = \mu^2 - \mu$ and see that $\mathfrak{A}_\mathfrak{K}(\mu)$ is associative if and only if $\mathfrak{A}$ is flexible and

(5) $(2\alpha_0+1)[x(yz) - (xy)z] = \alpha_0[x(zy) + (yz)x - (yx)z - z(xy)]$

for every x, y, z of $\mathfrak{A}$. If α_0 were not in $\mathfrak{F}$ then a sum $a+b\alpha_0=0$ for a and b in $\mathfrak{A}$ only if $a=b=0$, (5) would imply that $x(yz) = (xy)z$, that is, $\mathfrak{A}$ would be associative. It follows that $\alpha_0=\mu^2-\mu$ is in $\mathfrak{F}$ and the hypothesis (1) implies that $\alpha_0\neq0$, $-1/4$. This proves our theorem.

As an immediate corollary of our proof we have

THEOREM 6. *Let $\mathfrak{A}$ be an algebra over a field $\mathfrak{F}$ of characteristic not two and let $\mathfrak{A}$ be neither associative nor a Jordan algebra. Then $\mathfrak{A}$ is quasiassociative if and only if $\mathfrak{A}$ is flexible and there exists an element $\alpha_0\neq0$, $-1/4$ in $\mathfrak{F}$ such that (5) holds for every x, y, z of $\mathfrak{A}$.*

Let us observe that if $\mathfrak{A}$ is associative and $\mu\neq0$, 1 we may use (2) and (3) to see that $\mathfrak{A}(\mu)$ is associative if and only if $x(zy)+(yz)x=(yx)z+z(xy)$, that is, $(xz-zx)y=y(xz-zx)$. Then $\mathfrak{A}(\mu)$ is associative if and only if $xz-zx$ is in the center of $\mathfrak{A}$ for every x and z of $\mathfrak{A}$. But if $\mathfrak{A}$ is a total matric algebra of degree $n>1$ over $\mathfrak{F}$ we may use the usual basis e_{ij} of $\mathfrak{A}$ with $x=e_{12}$, $z=e_{21}$ to see that $xz-zx=e_{11}-e_{22}$ which is not in the center of $\mathfrak{A}$ since $e_{12}(e_{11}-e_{22})$ $= -e_{12}$, $(e_{11}-e_{12})e_{12}=e_{12}$.

3. **Defining identities.** A quasiassociative algebra $\mathfrak{A}$ which is neither associative nor a Jordan algebra is defined by the existence of an element $\alpha_0\neq0$, $-1/4$ such that $x(yx)=(xy)x$ and (5) holds. Conversely if (5) and the relation $x(yx)=(xy)x$ hold the corresponding algebra is quassiassociative. It should be noted that associative and Jordan algebras do not, in general, satisfy (5).

We may write (5) in the form

(6) $(2\alpha_0 + 1)(R_{yz} - R_yR_z) = \alpha_0(R_{zy} + L_{yz} - L_yR_z - R_yL_z)$.

By the flexible law $L_{yz}=R_{zy}-R_zR_y+L_zL_y$ and so (6) becomes

(7) $2\alpha_0(R_{yz} - R_{zy}) + R_{yz} = (2\alpha_0 + 1)R_yR_z + \alpha_0(L_zL_y - R_zR_y - L_yR_z - R_yL_z)$.

Interchange y and z to obtain

(8) $2\alpha_0(R_{zy} - R_{yz}) + R_{zy} = (2\alpha_0 + 1)R_zR_y + \alpha_0(L_yL_z - R_yR_z - L_zR_y - R_zL_y)$.

Add and use $R_yL_z+R_zL_y=L_yR_z+L_zR_y$ (by (4.2)) to obtain

(9) $R_{yz+zy} = (\alpha_0 + 1)(R_yR_z + R_zR_y) + \alpha_0(L_yL_z + L_zL_y) - 2\alpha_0(R_yL_z + R_zL_y)$.

We may also subtract (8) from (7) to obtain

(10) $(4\alpha_0 + 1)R_{yz-zy} = (3\alpha_0 + 1)(R_yR_z - R_zR_y) + \alpha_0(L_zL_y - L_yL_z)$
                    $+ \alpha_0(L_zR_y + R_zL_y - L_yR_z - R_yL_z)$,

and from (5) with x and y interchanged we have

(11) $(2\alpha_0 + 1)(R_zL_y - L_yR_z) = \alpha_0(L_zL_y + R_zR_y - R_yR_z - L_yL_z)$.

Finally, by (4.2) we may write

$$(12) \qquad R_y L_z - L_z R_y = L_y R_z - R_z L_y.$$

If $\alpha_0 = -1/2$, then $L_z L_y - L_y L_z = R_y R_z - R_z R_y$, and so (7) reduces to

$$(13) \qquad R_{zy} = - (L_z L_y - R_z R_y - L_y R_z - R_y L_z)/2.$$

Otherwise we have $\alpha_0(L_z R_y + R_z L_y - L_y R_z - R_y L_z) = 2\alpha_0(R_z L_y - L_y R_z)$. Thus
$(4\alpha_0 + 1)R_{yz-zy} = (3\alpha_0 + 1)(R_y R_z - R_z R_y) + \alpha_0(L_z L_y - L_y L_z) + 2\alpha_0(R_z L_y - L_y R_z)$
$= (4\alpha_0 + 1)(R_y R_z - R_z R_y) + 2\alpha_0(R_z L_y - L_y R_z), \quad \alpha_0(L_z L_y - L_y L_z + R_z R_y - R_y R_z)$
and so

$$(14) \qquad (4\alpha_0 + 1)R_{yz-zy} = (4\alpha_0 + 1)(R_y R_z - R_z R_y) + (4\alpha_0 + 1)(R_z L_y - L_y R_z)$$

which implies

$$(15) \qquad R_{yz-zy} = (R_y R_z - R_z R_y) + (L_z R_y - R_y L_z).$$

Thus we have

$$(16) \qquad \begin{aligned} 2R_{yz} &= (\alpha_0 + 2)(R_y R_z - R_z R_y) + \alpha_0(L_y L_z + L_z L_y) \\ &\quad - 2\alpha_0(R_y L_z + R_z L_y) + L_z R_y - R_y L_z. \end{aligned}$$

This proves the following theorem.

THEOREM 7. *Let $\mathfrak{A}$ be a quasiassociative algebra. Then $\mathfrak{A}$ is shrinkable of level one.*

4. J-semi-simple algebras. An algebra $\mathfrak{A}$ will be called J-*semi-simple* if $\mathfrak{A}^{(+)}$ is a semi-simple Jordan algebra and will be called J-simple if $\mathfrak{A}^{(+)}$ is a simple Jordan algebra.

THEOREM 8. *Every J-semi-simple algebra is a direct sum $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ where the components $\mathfrak{A}_i$ are J-simple. Every J-simple algebra $\mathfrak{A}$ is simple and the center of $\mathfrak{A}$ is the center of $\mathfrak{A}^{(+)}$.*

For, if $\mathfrak{B}$ is an ideal of the semi-simple Jordan algebra $\mathfrak{A}^{(+)}$ then $\mathfrak{B}$ has a unity element e and $\mathfrak{A}^{(+)} = \mathfrak{B} \oplus \mathfrak{C}$, $\mathfrak{B} = \mathfrak{A}_e^{(+)}(1)$, $\mathfrak{C} = \mathfrak{A}_e^{(+)}(0)$. It follows that $\mathfrak{A} = \mathfrak{A}_e(1) \oplus \mathfrak{A}_e(0)$ and that $\mathfrak{B} = \mathfrak{A}_e(1)^{(+)}$, $\mathfrak{C} = \mathfrak{A}_e(0)^{(+)}$. If we decompose $\mathfrak{A}^{(+)}$ as a direct sum $\mathfrak{A}^{(+)} = \mathfrak{B}_1 \oplus \cdots \oplus \mathfrak{B}_t$ of simple Jordan algebras $\mathfrak{B}_i$ then each $\mathfrak{B}_i = \mathfrak{A}_{e_i}^{(+)}(1) = \mathfrak{A}_i^{(+)}$ where $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_t$, the algebras $\mathfrak{A}_i$ are J-simple. Assume now that $\mathfrak{A}$ is J-simple. Then every ideal $\mathfrak{M}$ of $\mathfrak{A}$ is also an ideal of $\mathfrak{A}^{(+)}$ and if $\mathfrak{M} \neq 0$ we have $\mathfrak{M} = \mathfrak{A}$. Hence $\mathfrak{A}$ is simple. If $\mathfrak{Z}$ is the center of $\mathfrak{A}$ we may express $\mathfrak{A}$ as an algebra over $\mathfrak{Z}$. But then $\mathfrak{A}^{(+)}$ will be an algebra over $\mathfrak{Z}$ and so $\mathfrak{Z}$ is contained in the center of $\mathfrak{A}^{(+)}$. However it is true for arbitrary simple algebras that $\mathfrak{A}$ is central simple over $\mathfrak{Z}$ if and only if every scalar extension over $\mathfrak{Z}$ of $\mathfrak{A}$ is a simple algebra. Thus if $\mathfrak{Z}$ is not the center of $\mathfrak{A}^{(+)}$ there will be a scalar extension $\mathfrak{K}$ of $\mathfrak{Z}$ such that $\mathfrak{A}_{\mathfrak{K}}^{(+)} = \mathfrak{B} \oplus \mathfrak{C}$. By the proof above $\mathfrak{A}_{\mathfrak{K}}$ is a direct sum of the corresponding components contrary to the

hypothesis that $\mathfrak{Z}$ is the center of $\mathfrak{A}$ and so $\mathfrak{A}_{\mathfrak{K}}$ is simple for every $\mathfrak{K}$ over $\mathfrak{Z}$.

5. **The structure of quasiassociative algebras.** The general theorems on the structure of quasiassociative algebras may be stated as follows:

THEOREM 9. *Let $\mathfrak{A}$ be a quasiassociative algebra over an infinite field $\mathfrak{F}$ of characteristic not two. Then if $\mathfrak{A}$ is solvable it is strongly nilpotent and if $\mathfrak{A}$ is a nilalgebra it is solvable.*

THEOREM 10. *Every quasiassociative algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ contains a maximal solvable ideal $\mathfrak{N}$ called the radical of $\mathfrak{A}$, such that every nilideal of $\mathfrak{A}$ is contained in $\mathfrak{N}$. If $\mathfrak{F}$ is nonmodular and $\mathfrak{W}$ is any scalar extension of $\mathfrak{F}$ the radical of $\mathfrak{A}_{\mathfrak{W}}$ is $\mathfrak{N}_{\mathfrak{W}}$. Also $\mathfrak{A} - \mathfrak{N}$ is semi-simple, that is, $\mathfrak{A} - \mathfrak{N}$ has no nonzero nilideal.*

THEOREM 11. *A semi-simple quasiassociative algebra is J-simi-simple and so is a direct sum $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ of J-simple algebras $\mathfrak{A}_i$. Each $\mathfrak{A}_i$ is quasi-associative and so is flexible. A quasiassociative simple algebra with center $\mathfrak{Z}$ is quasiassociative over $\mathfrak{Z}$.*

The results just stated clearly hold for Jordan algebras and for associative algebras. Theorems 9 and 10 have been seen to hold for those algebras $\mathfrak{A}$ for which there exists an element λ in $\mathfrak{F}$ such that $\lambda \neq 0$, 1, 1/2 and $\mathfrak{A}(\lambda)$ is associative. Evidently Theorem 11 also holds for such algebras. There remains the case $\mathfrak{A}_{\mathfrak{K}} = \mathfrak{B}(\mu)$ for a quadratic field $\mathfrak{K} = \mathfrak{F}(\mu)$ over $\mathfrak{F}$ and an associative algebra $\mathfrak{B}$ over $\mathfrak{K}$. Note that the properties above already hold for $\mathfrak{A}_{\mathfrak{K}}$.

Let $\mu^2 = \mu + \alpha_0$ for α_0 in $\mathfrak{F}$ and write $\theta = 2\mu - 1$ so that $\theta^2 = 4(\mu^2 - \mu) + 1 = 4\alpha_0 + 1$ is in $\mathfrak{F}$ and must not be the square of any element of $\mathfrak{F}$. Then every vector subspace $\mathfrak{M}$ over $\mathfrak{K}$ of $\mathfrak{A}_{\mathfrak{K}}$ has elements of the form $x = x_0 + x_1\theta$ where x_0 and x_1 are in $\mathfrak{A}$. If $y = y_0 + y_1\theta$ is in $\mathfrak{M}$ and ξ and η are in $\mathfrak{F}$ the element $\xi x + \eta y = (\xi x_0 + \eta y_0) + (\xi x_1 + \eta y_1)\theta$ is in $\mathfrak{M}$. Hence if $\mathfrak{M}_0$ is the set of all x_0 then $\mathfrak{M}_0$ is a vector subspace over $\mathfrak{F}$ of $\mathfrak{A}$. But $\theta^{-1}x = x_1 + (4\alpha_0 + 1)^{-1}x_0\theta$ is in $\mathfrak{M}$ and so x_1 is in $\mathfrak{M}_0$, $\mathfrak{M} \leq (\mathfrak{M}_0)_{\mathfrak{K}}$.

Suppose now that $\mathfrak{M}$ is an ideal of $\mathfrak{A}_{\mathfrak{K}}$. Then if y is in $\mathfrak{A}$ and $x = x_0 + x_1\theta$ is in $\mathfrak{M}$ the products $xy = x_0y + (x_1y)\theta$ and $yx = yx_0 + (yx_1)\theta$ are in $\mathfrak{M}$, x_0y and yx_0 are in $\mathfrak{M}_0$, $\mathfrak{M}_0$ is an ideal of $\mathfrak{A}$. Thus every ideal $\mathfrak{M}$ of $\mathfrak{A}_{\mathfrak{K}}$ determines an ideal $\mathfrak{M}_0$ of $\mathfrak{A}$ such that $\mathfrak{M} \leq (\mathfrak{M}_0)_{\mathfrak{K}}$. Conversely every ideal $\mathfrak{M}_0$ of $\mathfrak{A}$ determines an ideal $(\mathfrak{M}_0)_{\mathfrak{K}}$ of $\mathfrak{A}_{\mathfrak{K}}$.

Assume now that $\mathfrak{A}$ is solvable. Then $\mathfrak{A}_{\mathfrak{K}}$ is solvable. Indeed if we define $\mathfrak{A}^{(k+1)} = \mathfrak{A}^{(k)}\mathfrak{A}^{(k)}$ then $[\mathfrak{A}_{\mathfrak{W}}]^{(k+1)} = [\mathfrak{A}_{\mathfrak{W}}]^{(k)}[\mathfrak{A}_{\mathfrak{W}}]^{(k)}$ for every scalar extension $\mathfrak{W}$ of $\mathfrak{F}$ and $\mathfrak{A}^{(k)} = 0$ implies that $[\mathfrak{A}_{\mathfrak{W}}]^{(k)} = 0$. But then $\mathfrak{A}_{\mathfrak{K}}$ is strongly nilpotent, that is, every product of t elements of $\mathfrak{A}_{\mathfrak{K}}$ is zero for some positive integer t. Clearly then $\mathfrak{A}$ is strongly nilpotent. If $\mathfrak{A}$ is a nilalgebra and $u_1, \cdots, u_n$ form a basis of $\mathfrak{A}$ then $(\xi_1 u_1 + \cdots + \xi_n u_n)^n = 0$ for $\xi_1, \cdots, \xi_n$ indeterminates over $\mathfrak{F}$. Otherwise we could find values in $\mathfrak{F}$ of $\xi_1, \cdots, \xi_n$ such that the corresponding element of $\mathfrak{A}$ is not nilpotent. But then we may replace the ξ_i by any elements of $\mathfrak{K}$, $\mathfrak{A}_{\mathfrak{K}}$ must be a nilalgebra, $\mathfrak{A}_{\mathfrak{K}}$ is strongly nilpotent, so is $\mathfrak{A}$.

The first statement in Theorem 10 holds by virtue of Theorem 9. To prove the second part we first prove that if $\mathfrak{F}$ is nonmodular with $\mathfrak{N}$ and $\mathfrak{M}$ the respective radicals of $\mathfrak{A}$ and $\mathfrak{A}_\mathfrak{K}$ then $\mathfrak{N}_\mathfrak{K} = \mathfrak{M}$. Evidently $\mathfrak{N}_\mathfrak{K} \leqq \mathfrak{M}$. Conversely let x be in $\mathfrak{M}$, T be any transformation of the algebra of polynomials in the multiplications of the elements of $\mathfrak{A}$ and $y = xT$. Then y is in $\mathfrak{M}$ and so $aR_y^{k+1} = (ay)R_y^k$ is a product of $k+1$ elements of $\mathfrak{M}$ and is zero for every a of $\mathfrak{A}_\mathfrak{K}$ and a suitable k, $R_y^{k+1} = 0$. Similarly $aL_y^{k+1} = (ya)L_y^k = 0$, R_y and L_y are nilpotent and have zero trace. Write the general element x of $\mathfrak{M}$ as $x = x_0 + x_1\theta$ and see that $y = xR_{x_0}^k = x_0^{k+1} + (x_1R_{x_0}^k)\theta = u + v\theta$, $\tau(R_y) = \tau(R_u) + \theta\tau(R_v) = 0$ so that $\tau(R_u) = 0$. Similarly $\tau(L_u) = 0$ and so the linear transformation $S_u = (R_u + L_u)/2$ has zero trace for $u = x_0^{k+1}$ and every positive integer k. But x_0 is in the Jordan algebra $\mathfrak{A}_\mathfrak{K}^{(+)}$, S_u is the right multiplication for x_0^{k+1} and has zero trace. If $\mathfrak{F}[x_0^2]$ were to contain an idempotent e, we would have $\tau(S_e) \neq 0$ whereas e is a polynomial in x_0^2 and S_x is a linear function of x. Thus there is no such idempotent, x_0^2 is nilpotent, x_0 is nilpotent, the ideal $\mathfrak{M}_0$ defined above is a nilideal of $\mathfrak{A}$, $\mathfrak{M}_0 \leqq \mathfrak{N}$, $\mathfrak{M} \leqq (\mathfrak{M}_0)_\mathfrak{K} \leqq \mathfrak{N}_\mathfrak{K}$, $\mathfrak{M} = \mathfrak{N}_\mathfrak{K}$.

To prove Theorem 11 we note that if $\mathfrak{A}$ is a semi-simple quasiassociative algebra, then $\mathfrak{A}_\mathfrak{K}$ is semi-simple and so the related associative algebra $\mathfrak{B}$ is semi-simple. But then $\mathfrak{B}^{(+)}$ is semi-simple, $\mathfrak{B}^{(+)} = \mathfrak{A}_\mathfrak{K}^{(+)} = (\mathfrak{A}^{(+)})_\mathfrak{K}$ is a semi-simple Jordan algebra. It follows that $x(yx^2) = (xy)x^2$ for every x and y of the commutative algebra $\mathfrak{A}^{(+)}$ and so $\mathfrak{A}^{(+)}$ is a semi-simple Jordan algebra. Write $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ and see that $\mathfrak{A}_\mathfrak{K} = (\mathfrak{A}_1)_\mathfrak{K} \oplus \cdots \oplus (\mathfrak{A}_r)_\mathfrak{K}$. However if $xy = yx = 0$ then $x \cdot y = \lambda xy + (1-\lambda)yx = y \cdot x = 0$ and so $\mathfrak{A}_\mathfrak{K}(\lambda) = (\mathfrak{A}_1)_\mathfrak{K}(\lambda) \oplus \cdots \oplus (\mathfrak{A}_r)_\mathfrak{K}(\lambda)$ where $(\mathfrak{A}_i)_\mathfrak{K}(\lambda)$ must be associative. Assume, finally, that $\mathfrak{A}$ is a simple quasiassociative algebra with center $\mathfrak{Z}$ and that $\mathfrak{A}_\mathfrak{K}(\lambda)$ is associative where $\mathfrak{K} = \mathfrak{F}(\lambda)$, $\lambda^2 - \lambda = \alpha_0$ in $\mathfrak{F}$. If λ is in $\mathfrak{F}$ then $\mathfrak{A}$ is clearly quasiassociative over $\mathfrak{Z}$. Hence let $\mathfrak{K}$ have degree two over $\mathfrak{F}$. Then either the composite of $\mathfrak{Z}$ and $\mathfrak{K}$ is a field $\mathfrak{Z}(\lambda)$ of degree two over $\mathfrak{Z}$ and $\mathfrak{A}$ is quasiassociative over $\mathfrak{Z}$ or $\mathfrak{Z}$ has a quadratic subfield $\mathfrak{W}$ isomorphic over $\mathfrak{F}$ to $\mathfrak{K}$. Then $\mathfrak{W}_\mathfrak{K} = e_1\mathfrak{K} + e_2\mathfrak{K}$ where e_1 and e_2 are pairwise orthogonal idempotents, $\mathfrak{A}_\mathfrak{K} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$ where $\mathfrak{B}_1 = \mathfrak{A}_\mathfrak{K}e_1$ and $\mathfrak{B}_2 = \mathfrak{A}_\mathfrak{K}e_2$. Let $u_1, \cdots, u_n$ be a basis of $\mathfrak{A}$ over $\mathfrak{W}$ and let $u_iu_j = \sum \gamma_{ijk}u_k$ where the γ_{ijk} are in $\mathfrak{W}$. Then $(u_ie_1)(u_je_1) = (u_iu_j)e_1 = \sum(\gamma_{ijk}e_1)u_ke_1$ where the elements u_ie_1 are a basis of $\mathfrak{B}_1$. Moreover $\mathfrak{W} = \mathfrak{F}(x)$ where $x^2 = \beta_0$ in $\mathfrak{F}$, $\mathfrak{K} = \mathfrak{F}(\theta)$ where $\theta^2 = \beta_0$, $x = \theta e_1 - \theta e_2$, $xe_1 = \theta e_1$, $\gamma_{ijk}(x)e_1 = \gamma_{ijk}(\theta)e_1$. It follows that $\mathfrak{B}_1$ is an algebra of order $2n$ over $\mathfrak{F}$ isomorphic to $\mathfrak{A}$ over $\mathfrak{F}$ under a mapping which maps x on θe_1. Evidently $\mathfrak{A}_\mathfrak{K}(\lambda) = \mathfrak{B}_1(\lambda) \oplus \mathfrak{B}_2(\lambda)$ and so $\mathfrak{B}_1(\lambda)$ is associative, $\mathfrak{A}((x+1)/2)$ is associative. $\mathfrak{A}$ is quasiassociative over $\mathfrak{Z}$.

We have now reduced the study of quasiassociative algebras to the case of central simple quasiassociative algebras $\mathfrak{A}$ such that $\mathfrak{A}^{(+)}$ is a central simple Jordan algebra over $\mathfrak{F}$. Then $\mathfrak{A}_\mathfrak{K} = \mathfrak{B}(\lambda)$ where $\mathfrak{K} = \mathfrak{F}(\lambda)$ has degree one or two over $\mathfrak{F}$ and $\mathfrak{B}$ is a central simple associative algebra over $\mathfrak{K}$. It is not true, in general, that there always exists an element λ in $\mathfrak{F}$ such that $\mathfrak{A} = \mathfrak{B}(\lambda)$. For we shall later show exactly when this is true for algebras of degree two and

order four. However it might be true that there always exists an associative central simple algebra $\mathfrak{B}$ over $\mathfrak{F}$ such that $\mathfrak{A}_{\mathfrak{R}} = \mathfrak{B}_{\mathfrak{R}}(\lambda)$ and we leave this as an unsolved problem.

6. Algebras of degree two. Let $\mathfrak{A}$ be a flexible J-simple algebra of degree two over a field $\mathfrak{F}$ of characteristic not two. Then $\mathfrak{A}$ has a unity element e and a basis $u_1 = e, u_2, \cdots, u_n$ where $n \geq 3$, $u_i^2 = \alpha_i e$ for $\alpha_i \neq 0$ in $\mathfrak{F}$, $u_i u_j + u_j u_i = 0$ for $i \neq j$ and $i, j = 2, \cdots, n$. It follows that the general element $x = \xi_1 u_1 + \cdots + \xi_n u_n$ of $\mathfrak{A}$ satisfies the equation $(x - \xi_1 e)^2 = f(x)e$, $f(x) = \alpha_2 \xi_2^2 + \cdots + \alpha_n \xi_n^2$.

Assume the multiplication table $u_i u_j = \sum_{k=1}^{n} \gamma_{ijk} u_k$ for γ_{ijk} in $\mathfrak{F}$. It is evidently necessary only to study the elements γ_{ijk} for $i \neq j$ and $i, j = 2, \cdots, n$. The condition $u_i u_j = -u_j u_i$ is expressed by

$$(17) \qquad\qquad \gamma_{ijk} = -\gamma_{jik} \qquad\qquad (i \neq j; i, j = 2, \cdots, n).$$

The assumption that $\mathfrak{A}$ is flexible is now equivalent to the relation $x(yz) + z(yx) = (xy)z + (zy)x$ for x, y, z taken to be basal elements u_i for $i > 1$. In case two of the variables are equal we have as one case $u_i(u_i u_j) + u_j(u_i u_i) = (u_i u_i)u_j + (u_j u_i)u_i$, that is, $u_i(u_i u_j) + (u_i u_j)u_i = 0$. Then $2\gamma_{ij1} u_i + 2\gamma_{iji}\alpha_i e = 0$. Similarly $u_j(u_j u_i) + (u_j u_i)u_j = 0$, that is, $2\gamma_{ij1}u_j + 2\gamma_{ijj}\alpha_j e = 0$. This yields

$$(18) \qquad\qquad \gamma_{ij1} = \gamma_{iji} = \gamma_{ijj} = 0 \qquad\qquad (i \neq j; i, j = 2, \cdots, n),$$

a result stating that every product of two distinct basal elements u_i, u_j is a linear combination of the basal elements u_k distinct from e, u_i and u_j.

The flexibility relation used above may now be employed with $x = z = u_j$ and $y = u_i$ to yield $u_j(u_i u_i) = (u_j u_i)u_j$. This is equivalent to $u_j(u_i u_i) + (u_i u_j)u_j = 0$, a relation already satisfied by (18). Thus there remains only the case where $x = u_i$, $y = u_j$, $z = u_k$ with i, j, k all distinct. The relation becomes $u_i(u_j u_k) + u_k(u_j u_i) = (u_i u_j)u_k + (u_k u_j)u_i$ which is equivalent to $u_i(u_j u_k) + (u_j u_k)u_i = (u_i u_j)u_k + u_k(u_i u_j)$ by using skew-commutivity. Hence we have

$$(19) \qquad\qquad \alpha_i \gamma_{jki} = \alpha_k \gamma_{ijk} \qquad\qquad (i \neq j; i, j, k = 2, \cdots, n)$$

where (19) is satisfied by (18) in case $k = i$ or $k = j$.

We have now determined a set of conditions which are necessary and sufficient conditions that an algebra $\mathfrak{A}$ of degree two shall be both flexible and J-simple. A complete normalization of the multiplication table of the general algebra of order n of our type is manifestly impossible and is not even particularly significant for algebras of order $n > 4$. We shall therefore content ourselves with a discussion of algebras of order three and four.

When $n = 3$ the product $u_2 u_3 = 0$ by (18) and so $u_3 u_2 = 0$, $\mathfrak{A}$ is commutative and is a Jordan algebra.

Assume that $n = 4$ and use the notations $u = u_2$, $v = u_3$, $w = u_4$, $u^2 = \alpha e$, $v^2 = \beta e$, $w^2 = \gamma e$ where $\alpha = \alpha_2 \neq 0$, $\beta = \alpha_3 \neq 0$ and $\gamma = \alpha_4 \neq 0$ are in $\mathfrak{F}$. Then $uv = \rho w$ where $\rho = \gamma_{234}$ by (18). Apply (19) to see that $\alpha_4 \gamma_{231} = \alpha_2 \gamma_{342}$ and so

$\gamma_{342} = \alpha^{-1}\gamma\rho$, $vw = \alpha^{-1}\gamma\rho u$. Also $\alpha_3\gamma_{243} = -\alpha_3\gamma_{423} = -\alpha_4\gamma_{234}$, $\gamma_{243} = -\beta^{-1}\gamma\rho$, uw $= -\beta^{-1}\gamma\rho v$. When $\rho = 0$ the algebra so defined is the general simple Jordan algebra of degree two and order four over $\mathfrak{F}$. When $\rho \neq 0$ we may select a new basis with $w = uv$ and so take $\rho = 1$. Then we have the multiplication table

$$(20) \qquad u^2 = \alpha e, \ v^2 = \beta e, \ w^2 = \gamma e, \ uv = w, \ uw = -\beta^{-1}\gamma v, \ vw = \alpha^{-1}\gamma u.$$

When $\mathfrak{A}$ is associative $uw = u(uv) = u^2 v = \alpha v$, that is, $\gamma = -\alpha\beta$. It follows that the algebra defined by (20) is associative if and only if $\gamma = -\alpha\beta$.

It remains to study the quasiassociativity of $\mathfrak{A}$. Define an algebra $\mathfrak{B}$ with a basis e_0, u_0, v_0, w_0 and the associative multiplication table given by the assumption that e_0 is the unity quantity of $\mathfrak{B}$ and the relations $u_0^2 = \alpha e_0$, $v_0^2 = \beta e_0$, $w_0^2 = -\alpha\beta e_0$, $w_0 = u_0 \cdot v_0$. Form $\mathfrak{B}(\lambda)$ by defining a new product $xy = \lambda x \cdot y + (1-\lambda)y \cdot x$ and see that $u_0 v_0 = \lambda w_0 + (1-\lambda)(-w_0) = (2\lambda-1)w_0$, $(u_0 v_0)^2 = \gamma e_0$ if and only if $\gamma = -(2\lambda-1)^2\alpha\beta$. Since $\mathfrak{B}$ is associative the algebra $\mathfrak{B}(\lambda)$ is flexible and the relation $(u_0 v_0)^2 = \gamma e_0$ is sufficient to imply that $\mathfrak{B}(\lambda)$ is isomorphic to the given algebra $\mathfrak{A}$ of (20). It follows that $\mathfrak{A}$ is isomorphic to some $\mathfrak{B}(\lambda)$ if and only if $-\alpha\beta\gamma$ is the square of an element of $\mathfrak{F}$. When $-\alpha\beta\gamma$ is not such a square we have $\mathfrak{A}_{\mathfrak{K}}$ isomorphic to $\mathfrak{B}_{\mathfrak{K}}(\lambda)$ where $\mathfrak{K} = \mathfrak{F}(\lambda)$ and $(2\lambda-1)^2 = -(\alpha\beta)^{-1}\gamma$, so that $\mathfrak{K}$ is a quadratic field over $\mathfrak{F}$. We have proved that all flexible J-simple algebras of degree two and order four over their centers are quasiassociative. Since the order of a simple associative algebra of degree two over its center is necessarily four the flexible J-simple algebras of degree two and order $n > 4$ are necessarily not quasiassociative.

One of the major problems in the study of nonassociative algebras is the question as to the existence of real nonassociative division algebras of order 2^t with $t > 3$. A special case of this problem is the same question for flexible algebras of degree two. These algebras are necessarily J-simple and their study may be expedited by use of the following result:

THEOREM 12. *Let $\mathfrak{A}$ be a flexible J-simple algebra of degree two over $\mathfrak{F}$ of characteristic not two. Then $\mathfrak{A}$ is a division algebra if and only if $\mathfrak{A}(\lambda)$ is a division algebra for every $\lambda \neq 1/2$ in $\mathfrak{F}$.*

To demonstrate this result we first note that when $\mathfrak{A}$ is a division algebra its associative subalgebras $\mathfrak{F}[x]$ must all be fields. This occurs if and only if the norm form $N(x) = \xi_1^2 - (\alpha_2\xi_2^2 + \cdots + \alpha_n\xi_n^2)$ is not a null form. We next prove the

LEMMA. *Let $N(x)$ be a non-null form. Then $\mathfrak{A}$ is a division algebra if and only if there exist no elements x and y in $\mathfrak{A}$ such that $x^2 = N(x)e \neq 0$, $y^2 = N(y)e \neq 0$, $xy = -yx = 0$.*

For when $\mathfrak{A}$ is not a division algebra there exist nonzero quantities a and b in $\mathfrak{A}$ such that $ab = 0$. Since a and b are necessarily not in the same subfield of

$\mathfrak{A}$ we may always write $a = \xi e + x$, $b = \eta e + \lambda x + y$ where ξ, η, λ are in $\mathfrak{F}$ and $x^2 = N(x)e \neq 0$, $y^2 = N(y)e \neq 0$, $yx = -xy$. Then we may select a basis of $\mathfrak{A}$ with $u_2 = x$, $u_3 = y$ and so see that xy is linearly independent of e, x, y. But $ab = (\xi\eta + N(x))e + (\lambda\xi + \eta)x + \xi y + xy = 0$ so that $xy = 0$ as desired.

To complete our proof we observe that if $\mathfrak{A}(\lambda)$ is not a division algebra then there exist elements x and y in $\mathfrak{A}$ such that $x \cdot y = -y \cdot x = 0$. But $xy = \mu x \cdot y + (1 - \mu)y \cdot x$ where $\mu = \lambda(2\lambda - 1)^{-1}$, $xy = (2\mu - 1)x \cdot y = 0$, $\mathfrak{A}$ cannot be a division algebra.

7. Algebras of degree $t > 2$. Let $\mathfrak{C}$ be a central simple alternative algebra of degree two over a nonmodular field $\mathfrak{F}$. Then $\mathfrak{C}$ has order 1, 2, 4, or 8 over $\mathfrak{F}$ and has an involutorial antiautomorphism $x \rightarrow \bar{x}$ which is the identity automorphism only when $s = 1$. Define $\mathfrak{C}_t$ to be the set of all t-rowed square matrices $A = (a_{ij})$ with a_{ij} in $\mathfrak{C}$ and define $\mathfrak{J}_t(\mathfrak{C})$ to be the subspace of $\mathfrak{C}_t$ consisting of all matrices $A = (a_{ij})$ such that $a_{ij} = \bar{a}_{ji}$. Then $\mathfrak{J}_t(\mathfrak{C})$ is a subalgebra of $\mathfrak{C}_t^{(+)}$ and is a Jordan algebra except when $t > 3$ and $s = 8$. Conversely every central simple reduced Jordan algebra is an algebra $\mathfrak{J}_t(\mathfrak{C})$.

We now let $\mathfrak{A}$ be a flexible algebra over $\mathfrak{F}$ and assume that $\mathfrak{A}^{(+)}$ is a central simple Jordan algebra of degree $t > 2$ over $\mathfrak{F}$. Let us assume that $\mathfrak{F}$ has been extended, if necessary, so that $\mathfrak{A}^{(+)}$ is a split algebra. We propose to determine the resulting multiplication table of $\mathfrak{A}$. We first write the unity quantity e of $\mathfrak{A}^{(+)}$ as a sum $e = e_1 + \cdots + e_t$ of pairwise orthogonal idempotents e_i which may be taken to be absolutely primitive. Then $\mathfrak{A}_{e_i}(1) = \mathfrak{A}_{ii} = e_i\mathfrak{F}$. Decompose $\mathfrak{A}^{(+)}$ and so write $\mathfrak{A}$ as the supplementary sum $\mathfrak{A} = \sum_{i<j} \mathfrak{A}_{ij}$ of subspaces $\mathfrak{A}_{ij}$ where $\mathfrak{A}_{ii}$ has already been defined and $\mathfrak{A}_{ij}$ is the intersection of $\mathfrak{A}_{e_i}(1/2)$ and $\mathfrak{A}_{e_j}(1/2)$ for $i < j$. The general element x_{ij} of $\mathfrak{A}_{ij}$ may then be regarded as a t-rowed square matrix with the element x of an algebra $\mathfrak{C}$ in the ith row and jth column, with $\bar{x}$ in the jth row and ith column and with zeros elsewhere. Use the notation $x \cdot y$ to designate products in $\mathfrak{C}$ and translate the product in $\mathfrak{A}^{(+)}$ in terms of the product in $\mathfrak{A}$ to write

$$(21) \qquad x_{ij}y_{jk} + y_{jk}x_{ij} = (x \cdot y)_{ik} \qquad (i < j < k),$$

$$(22) \qquad x_{ij}y_{ik} + y_{ik}x_{ij} = (\bar{x} \cdot y)_{jk} \qquad (i < j < k),$$

$$(23) \qquad x_{ik}y_{jk} + y_{jk}x_{ik} = (x \cdot \bar{y})_{ik} \qquad (i < j < k).$$

We also have the results

$$(24) \qquad e_ix_{ij} + x_{ij}e_i = e_jx_{ij} + x_{ij}e_j = x_{ij} \qquad (i < j).$$

Then $(e_i - e_j)x_{ij} = x_{ij}(e_j - e_i)$ and $(e_i + e_j)x_{ij} = x_{ij}(e_i + e_j) = x_{ij}$ whence

$$(25) \qquad e_ix_{ij} = x_{ij}e_j, \qquad e_jx_{ij} = x_{ij}e_i \qquad (i < j).$$

The subset $\mathfrak{S}_{ij} = \mathfrak{A}_{ii} + \mathfrak{A}_{jj} + \mathfrak{A}_{ij} = \mathfrak{A}_{e_i + e_j}(1)$ by the known properties of $\mathfrak{A}^{(+)}$ and so $\mathfrak{S}_{ij}$ is a subalgebra of $\mathfrak{A}$. Then $\mathfrak{S}_{ij}^{(+)}$ is a central simple Jordan algebra of degree 2 over $\mathfrak{F}$, $\mathfrak{S}_{ij}$ is stable,

$$\mathfrak{A}_{ii}\mathfrak{A}_{ij} \leqq \mathfrak{A}_{ij}, \qquad \mathfrak{A}_{ij}\mathfrak{A}_{ii} \leqq \mathfrak{A}_{ij}, \qquad \mathfrak{A}_{jj}\mathfrak{A}_{ij} \leqq \mathfrak{A}_{ij}, \qquad \mathfrak{A}_{ij}\mathfrak{A}_{jj} \leqq \mathfrak{A}_{ij}$$

and

$$\mathfrak{A}_{ii}\mathfrak{A}_{jj} = \mathfrak{A}_{jj}\mathfrak{A}_{ii} = 0, \qquad \mathfrak{A}_{ij}^{2} \leqq \mathfrak{A}_{ii} + \mathfrak{A}_{jj} + \mathfrak{A}_{ij}.$$

It is evident from an examination of algebras of degree two that the property $\mathfrak{A}_{ij}^{2} \leqq \mathfrak{A}_{ii} + \mathfrak{A}_{jj}$ is not true in general.

It remains to compute products $x_{ij}y_{kp}$ where at least three of i, j, k, p are distinct. We note first that $\mathfrak{A}_{ij}\mathfrak{A}_{kp} = 0$ if neither i nor j is equal to k or p. Indeed $\mathfrak{A}_{ij} \leqq \mathfrak{A}_{e_i+e_j}(1)$ while $\mathfrak{A}_{kp} \leqq \mathfrak{A}_{e_i+e_j}(0)$. It then remains to compute the products appearing in (27), (28), (29), (30). We observe first that

(26) $$\mathfrak{A}_{ij}\mathfrak{A}_{jk} \leqq \mathfrak{A}_{ik}, \qquad \mathfrak{A}_{jk}\mathfrak{A}_{ij} \leqq \mathfrak{A}_{ik} \qquad\qquad (i < j < k).$$

Indeed $\mathfrak{A}_{ij} \leqq \mathfrak{A}_{e_i+e_j}(1)$, $\mathfrak{A}_{jk} \leqq \mathfrak{A}_{e_i+e_j}(1/2)$ and so $\mathfrak{A}_{ij}\mathfrak{A}_{jk}$ and $\mathfrak{A}_{jk}\mathfrak{A}_{ij}$ are both subspaces of $\mathfrak{A}_{e_i+e_j}(1/2)$. However they are subsets of the algebra $\mathfrak{A}_{e_i+e_j+e_k}(1)$ $= \mathfrak{A}_{ii} + \mathfrak{A}_{jj} + \mathfrak{A}_{kk} + \mathfrak{A}_{ij} + \mathfrak{A}_{ik} + \mathfrak{A}_{jk}$ and it follows that the products are contained in $\mathfrak{A}_{ik} + \mathfrak{A}_{jk}$. Similarly $\mathfrak{A}_{ij} \leqq \mathfrak{A}_{e_i+e_k}(1/2)$, $\mathfrak{A}_{jk} \leqq \mathfrak{A}_{e_j+e_k}(1)$ so that their products are subspaces of $\mathfrak{A}_{ik} + \mathfrak{A}_{ij}$. The intersection of these subspaces is $\mathfrak{A}_{ik}$ and we have (26).

We now use the flexibility property $z(xy) + y(xz) = (zx)y + (yx)z$. We take $z = e_k$ and compute $e_k(x_{ij}y_{jk}) + y_{jk}(x_{ij}e_k) = (e_kx_{ij})y_{jk} + (y_{jk}x_{ij})e_k$. But $x_{ij}e_k = e_kx_{ij}$ $= 0$ and so $e_k(x_{ij}y_{jk}) + (x_{ij}y_{jk})e_k = (x_{ij}y_{jk} + y_{jk}x_{ij})e_k$. Since $x_{ij}y_{jk}$ is in $\mathfrak{A}_{ik}$ we have $e_k(x_{ij}y_{jk}) + (x_{ij}y_{jk})e_k = x_{ij}y_{jk}$. We may then use (23) and have

(27) $$x_{ij}y_{jk} = e_i(x \cdot y)_{ik} = (x \cdot y)_{ik}e_k \qquad\qquad (i < j < k).$$

Since $(e_i+e_k)(x \cdot y)_{ik} = (x \cdot y)_{ik}$ we evidently have

(28) $$y_{jk}x_{ij} = e_k(x \cdot y)_{ik} = (x \cdot y)_{ik}e_i \qquad\qquad (i < j < k).$$

In exactly the same fashion we compute $e_k(x_{ij}y_{ik}) + y_{ik}(x_{ij}e_k) = (e_kx_{ij})y_{ik}$ $+ (y_{ik}x_{ij})e_k$ to obtain $x_{ij}y_{ik} = (x_{ij}y_{ik} + y_{ik}x_{ij})e_k$, that is,

(29) $$\begin{aligned} x_{ij}y_{ik} &= e_i(\bar{x} \cdot y)_{ik} = (\bar{x} \cdot y)_{ik}e_k, \\ y_{ik}x_{ij} &= e_k(\bar{x} \cdot y)_{ik} = (\bar{x} \cdot y)_{ik}e_i. \end{aligned} \qquad\qquad (i < j < k).$$

Finally we compute $e_j(x_{ik}y_{jk}) + y_{jk}(x_{ik}e_j) = (e_jx_{ik})y_{jk} + (y_{jk}x_{ik})e_j$ whence $x_{ik}y_{jk}$ $= (x_{ik}y_{jk} + y_{jk}x_{ik})e_j$ and so

(30) $$\begin{aligned} x_{ik}y_{jk} &= e_i(x \cdot \bar{y})_{ij} = (x \cdot \bar{y})_{ij}e_j, \\ y_{jk}x_{ik} &= e_j(x \cdot \bar{y})_{ij} = (x \cdot \bar{y})_{ij}e_i. \end{aligned} \qquad\qquad (i < j < k).$$

We have now computed all products required, but have not exhausted the possible implications of the flexible law. Use $z = e_j$ and have $e_j(x_{ij}y_{jk})$ $+ y_{jk}(x_{ij}e_j) = (e_jx_{ij})y_{jk} + (y_{jk}x_{ij})e_j$. Then $e_j(x_{ij}y_{jk}) = (y_{jk}x_{ij})e_j = 0$ since both $x_{ij}y_{jk}$ and $y_{jk}x_{ij}$ are in $\mathfrak{A}_{ik} \leqq \mathfrak{A}_{e_j}(0)$. Then $y_{jk}(x_{ij}e_j) = (e_jx_{ij})y_{jk}$, $x_{ij}y_{jk} = (e_ix_{ij}$ $+ x_{ij}e_j)y_{jk} = (x_{ij}e_j)y_{jk} + y_{jk}(x_{ij}e_j)$. Define

(31) $$e_i x_{ij} = (xS_{ij})_{ij}$$

where S_{ij} is then a uniquely determined linear transformation of the vector space $\mathfrak{C}$. Then we have proved that

(32) $$x_{ij} y_{jk} = (xS_{ij} \cdot y)_{ik} \qquad (i < j < k).$$

To obtain a similar formula with y, the transformed vector, we compute $e_j(y_{jk}x_{ij}) + x_{ij}(y_{jk}e_j) = (e_j y_{jk})x_{ij} + (x_{ij}y_{jk})e_j$. As before $e_j(y_{jk}x_{ij}) = (x_{ij}y_{jk})e_j = 0$ and so $x_{ij}(y_{jk}e_j + e_j y_{jk}) = x_{ij}y_{jk} = (e_j y_{jk})x_{ij} + x_{ij}(e_j y_{jk})$, that is,

(33) $$x_{ij} y_{jk} = (x \cdot yS_{jk})_{ik} \qquad (i < j < k).$$

However our first value of $x_{ij}y_{jk}$ may be written as $[(x \cdot y)S_{ik}]_{ik}$ and so we have proved that

(34) $$xS_{ij} \cdot y = x \cdot yS_{jk} = (x \cdot y)S_{ik} \qquad (i < j < k).$$

We now take $y = f$ to be the unity quantity of $\mathfrak{C}$ and so have $xS_{ij} = xS_{ik}$ for every x, that is, $S_{ij} = S_{ik}$ for $i < j < k$. Then $S_{12} = S_{13} = \cdots = S_{1t}$. Similarly take $x = f$ and have $yS_{jk} = yS_{ik}$, $S_{ik} = S_{jk}$, $S_{1k} = S_{jk}$ for $k > j > 1$ and so $S_{12} = S_{13} = S_{23}$, $S_{jk} = S_{12}$ for all $j < k$ and we have proved that every $S_{ij} = S = S_{12}$. Moreover we now have $xS \cdot y = x \cdot yS = (x \cdot y)S$, that is,

(35) $$SR_y = R_yS = R_{yS}$$

for every y in the algebra $\mathfrak{C}$. This implies that $S = R_{fS}$ is commutative with every element of $\mathfrak{C}$ and this is possible for algebras $\mathfrak{C}$ of order four and eight, cases where $\mathfrak{C}$ is central simple, only when S is a scalar transformation.

Let us now consider the algebra $\mathfrak{G} = \mathfrak{A}_{11} + \mathfrak{A}_{12} + \mathfrak{A}_{22}$, a flexible J-simple subalgebra of $\mathfrak{A}$. The order of $\mathfrak{G}$ is $g = 3$, 4, 6 or 8 according as the order of $\mathfrak{C}$ is 1, 2, 4 or 8. We may assume that we are working over an extension of the base field so that $\mathfrak{G}$ is a split algebra, that is, $\mathfrak{G}$ has a basis $e = u_1, u_2, \cdots, u_g$ with $u_i^2 = e$. Moreover we may define $e_1^{(t)} = (e - u_i)/2$, $e_2^{(t)} = (e + u_i)/2$ and then see that the subspace $\mathfrak{A}_{12}$ of $\mathfrak{G}$ is spanned by the $g - 2$ elements u_j for $j \neq 1$, i. When $g = 6$, 8 the mapping $S = S(i)$ is a scalar mapping for every selection of $e_1^{(t)}$ and so $e_1^{(t)} u_j = \lambda_{ij} u_j$. But then $u_i u_j = (e - 2e_1^{(t)})u_j = (1 - 2\lambda_{ij})u_j$ which is possible, by (18), only if $1 - 2\lambda_{ij} = 0$, that is, $\mathfrak{G}$ is a Jordan algebra. We have also proved that $\mathfrak{G}$ is a Jordan algebra when $g = 3$ in §5.5.

The multiplication table of $\mathfrak{A}$ has been shown to depend upon relations which can vary only as the algebra $\mathfrak{G}$ varies. If $\mathfrak{A}$ is taken to be a central simple (split) Jordan algebra we know that $\mathfrak{A} = \mathfrak{A}^{(+)}$ and that $\mathfrak{G}$ is a Jordan algebra. But then our study shows that when $\mathfrak{G}$ is a Jordan algebra so is $\mathfrak{A}$. If $\mathfrak{B}$ is a total matric algebra the algebra $\mathfrak{B}^{(+)}$ is a split Jordan algebra, $\mathfrak{B}$ has the subalgebra $\mathfrak{G}_0 = \mathfrak{B}_{11} + \mathfrak{B}_{12} + \mathfrak{B}_{22}$, $\mathfrak{A} = \mathfrak{B}(\lambda)$ has the subalgebra $\mathfrak{G} = \mathfrak{G}_0(\lambda) = \mathfrak{A}_{11} + \mathfrak{A}_{12} + \mathfrak{A}_{22}$. Then when $\mathfrak{G}$ is prescribed to be quasiassociative of degree two and order four the structure of $\mathfrak{A}$ is completely known and $\mathfrak{A} = \mathfrak{B}(\lambda)$ is quasiassociative. The extension of the base field which splits $\mathfrak{G}$ may now be

eliminated since clearly a central J-simple algebra $\mathfrak{A}$ is quasiassociative if and only if any scalar extension of $\mathfrak{A}$ is quasiassociative. We have proved

THEOREM 13. *Let $\mathfrak{A}$ be a flexible J-simple algebra. Then $\mathfrak{A}$ is either a Jordan algebra or is quasiassociative.*

We shall now apply our results to a determination of all simple standard algebras. Every simple standard algebra $\mathfrak{A}$ is flexible and J-simple and so is a quasiassociative algebra. Moreover, after a suitable extension of the center of $\mathfrak{A}$, if necessary, we may assume that $\mathfrak{A} = \mathfrak{B}(\lambda)$ where $\mathfrak{B}$ is a total matric algebra. Then $\mathfrak{A}$ has a subalgebra $\mathfrak{G} = \mathfrak{A}_{11} + \mathfrak{A}_{12} + \mathfrak{A}_{22} = \mathfrak{M}_2(\lambda)$ where $\mathfrak{M}_2$ is the total matric algebra of degree two over $\mathfrak{F}$. It follows that $\mathfrak{G}$ has a basis e, u, v, w over $\mathfrak{F}$ where e is the unity quantity of $\mathfrak{G}$ and $u^2 = v^2 = w^2 = e$, $uv = \rho w$, $uw = -\rho v$, $vw = \rho u$. Since $\mathfrak{G}$ is a standard algebra it satisfies the identity $w[x(yz)] + [(wy)x]z + [(wz)x]y = (wx)(yz) + (wy)(xz) + (wz)(xy)$. Take $y = z = u$ and $x = v$ to obtain $w[v(uu)] + 2[(wu)v]u = (wv)(uu) + 2(wu)(vu)$. But $w[v(uu)] = wv = (wv)(uu)$ and so $[(wu)v]u = (wu)(vu)$, $[\rho(vv)]u = (\rho v)(-\rho w)$, $(\rho + \rho^3)u = 0$, $\rho(1 + \rho^2) = 0$. If $\rho = 0$ then $\mathfrak{G}$ is a Jordan algebra, $\lambda = 1/2$, $\mathfrak{A}$ is a Jordan algebra. If $1 + \rho^2 = 0$ then $(uv)^2 = -e$ and $\mathfrak{G}$ is associative, $\lambda = 1$ and $\mathfrak{A}$ is associative. We have proved

THEOREM 14. *A standard simple algebra is either an associative algebra or a Jordan algebra.*

8. **Problems.** We shall close our discussion with a list of some unsolved problems of our theory. A major question is that of the existence of simple nilalgebras in the class of algebras we have called static algebras. An investigation of shrinkable non-commutative algebras of low level seems desirable and might lead to some interesting new types of algebras. A further study of commutative simple algebras also seems desirable. Indeed examples of simple power-associative algebras are needed and some may be furnished by a study which the author will make of the power-associativity of crossed products and some types of crossed extensions. It is desirable to investigate the structure of algebras like Lie algebras and, in particular, of all algebras $\mathfrak{A}$ which are flexible, power-associative, and such that $\mathfrak{A}^{(-)}$ is a semi-simple Lie algebra. The type of study which led to our definition of standard algebras may evidently be extended and other forms of Jordan-like identities may yield new classes of simple algebras. Finally, it would be desirable to extend the theory of standard algebras to rings with chain conditions, an extension not yet made even for Jordan algebras.

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