THE FOUNDATION OF THE GENERAL THEORY OF RELATIVITY

BY

A. EINSTEIN

Translated from "Die Grundlage der allgemeinen Relativitätstheorie," Annalen der Physik, 49, 1916.

THE FOUNDATION OF THE GENERAL THEORY OF RELATIVITY

By A. EINSTEIN

A. Fundamental Considerations on the Postulate of Relativity

§ 1. Observations on the Special Theory of Relativity

The special theory of relativity is based on the following postulate, which is also satisfied by the mechanics of Galileo and Newton.

If a system of co-ordinates K is chosen so that, in relation to it, physical laws hold good in their simplest form, the *same* laws also hold good in relation to any other system of co-ordinates K′ moving in uniform translation relatively to K. This postulate we call the "special principle of relativity." The word "special" is meant to intimate that the principle is restricted to the case when K′ has a motion of uniform translation relatively to K, but that the equivalence of K′ and K does not extend to the case of non-uniform motion of K′ relatively to K.

Thus the special theory of relativity does not depart from classical mechanics through the postulate of relativity, but through the postulate of the constancy of the velocity of light *in vacuo,* from which, in combination with the special principle of relativity, there follow, in the well-known way, the relativity of simultaneity, the Lorentzian transformation, and the related laws for the behaviour of moving bodies and clocks.

The modification to which the special theory of relativity has subjected the theory of space and time is indeed far-reaching, but one important point has remained unaffected.

For the laws of geometry, even according to the special theory of relativity, are to be interpreted directly as laws relating to the possible relative positions of solid bodies at rest; and, in a more general way, the laws of kinematics are to be interpreted as laws which describe the relations of measuring bodies and clocks. To two selected material points of a stationary rigid body there always corresponds a distance of quite definite length, which is independent of the locality and orientation of the body, and is also independent of the time. To two selected positions of the hands of a clock at rest relatively to the privileged system of reference there always corresponds an interval of time of a definite length, which is independent of place and time. We shall soon see that the general theory of relativity cannot adhere to this simple physical interpretation of space and time.

§ 2. The Need for an Extension of the Postulate of Relativity

In classical mechanics, and no less in the special theory of relativity, there is an inherent epistemological defect which was, perhaps for the first time, clearly pointed out by Ernst Mach. We will elucidate it by the following example:—Two fluid bodies of the same size and nature hover freely in space at so great a distance from each other and from all other masses that only those gravitational forces need be taken into account which arise from the interaction of different parts of the same body. Let the distance between the two bodies be invariable, and in neither of the bodies let there be any relative movements of the parts with respect to one another. But let either mass, as judged by an observer at rest relatively to the other mass, rotate with constant angular velocity about the line joining the masses. This is a verifiable relative motion of the two bodies. Now let us imagine that each of the bodies has been surveyed by means of measuring instruments at rest relatively to itself, and let the surface of S_1 prove to be a sphere, and that of S_2 an ellipsoid of revolution. Thereupon we put the question—What is the reason for this difference in the two bodies? No answer can be admitted as epistemologically satisfactory,* unless the reason given is an *observable fact of experience.* The law of causality has not the significance of a statement as to the world of experience, except when *observable facts* ultimately appear as causes and effects.

Newtonian mechanics does not give a satisfactory answer to this question. It pronounces as follows :—The laws of mechanics apply to the space R_1, in respect to which the body S_1 is at rest, but not to the space R_2, in respect to which the body S_2 is at rest. But the privileged space R_1 of Galileo, thus introduced, is a merely *factitious* cause, and not a thing that can be observed. It is therefore clear that Newton's mechanics does not really satisfy the requirement of causality in the case under consideration, but only apparently does so, since it makes the factitious cause R_1 responsible for the observable difference in the bodies S_1 and S_2.

The only satisfactory answer must be that the physical system consisting of S_1 and S_2 reveals within itself no imaginable cause to which the differing behaviour of S_1 and S_2 can be referred. The cause must therefore lie *outside* this system. We have to take it that the general laws of motion, which in particular determine the shapes of S_1 and S_2, must be such that the mechanical behaviour of S_1 and S_2 is partly conditioned, in quite essential respects, by distant masses which we have not included in the system under consideration. These distant masses and their motions relative to S_1 and S_2 must then be regarded as the seat of the causes (which must be susceptible to observation) of the different behaviour of our two bodies S_1 and S_2. They take over the rôle of the factitious cause R_1. Of all imaginable spaces R_1, R_2, etc., in any kind of motion relatively to one another, there is none which we may look upon as privileged *a priori* without reviving the above-mentioned epistemological objection. *The laws of physics must be of such a nature that they apply to systems of reference in any kind of motion.* Along this road we arrive at an extension of the postulate of relativity.

In addition to this weighty argument from the theory of

* Of course an answer may be satisfactory from the point of view of epistemology, and yet be unsound physically, if it is in conflict with other experi-

knowledge, there is a well-known physical fact which favours an extension of the theory of relativity. Let K be a Galilean system of reference, i.e. a system relatively to which (at least in the four-dimensional region under consideration) a mass, sufficiently distant from other masses, is moving with uniform motion in a straight line. Let K′ be a second system of reference which is moving relatively to K in *uniformly accelerated* translation. Then, relatively to K′, a mass sufficiently distant from other masses would have an accelerated motion such that its acceleration and direction of acceleration are independent of the material composition and physical state of the mass.

Does this permit an observer at rest relatively to K′ to infer that he is on a "really" accelerated system of reference? The answer is in the negative; for the above-mentioned relation of freely movable masses to K′ may be interpreted equally well in the following way. The system of reference K′ is unaccelerated, but the space-time territory in question is under the sway of a gravitational field, which generates the accelerated motion of the bodies relatively to K′.

This view is made possible for us by the teaching of experience as to the existence of a field of force, namely, the gravitational field, which possesses the remarkable property of imparting the same acceleration to all bodies.* The mechanical behaviour of bodies relatively to K′ is the same as presents itself to experience in the case of systems which we are wont to regard as "stationary" or as "privileged." Therefore, from the physical standpoint, the assumption readily suggests itself that the systems K and K′ may both with equal right be looked upon as "stationary," that is to say, they have an equal title as systems of reference for the physical description of phenomena.

It will be seen from these reflexions that in pursuing the general theory of relativity we shall be led to a theory of gravitation, since we are able to "produce" a gravitational field merely by changing the system of co-ordinates. It will also be obvious that the principle of the constancy of the velocity of light *in vacuo* must be modified, since we easily

* Eötvös has proved experimentally that the gravitational field has this property in great accuracy.

recognize that the path of a ray of light with respect to K′ must in general be curvilinear, if with respect to K light is propagated in a straight line with a definite constant velocity.

§ 3. The Space-Time Continuum. Requirement of General Co-Variance for the Equations Expressing General Laws of Nature

In classical mechanics, as well as in the special theory of relativity, the co-ordinates of space and time have a direct physical meaning. To say that a point-event has the X_1 co-ordinate x_1 means that the projection of the point-event on the axis of X_1, determined by rigid rods and in accordance with the rules of Euclidean geometry, is obtained by measuring off a given rod (the unit of length) x_1 times from the origin of co-ordinates along the axis of X_1. To say that a point-event has the X_4 co-ordinate $x_4 = t$, means that a standard clock, made to measure time in a definite unit period, and which is stationary relatively to the system of co-ordinates and practically coincident in space with the point-event,* will have measured off $x_4 = t$ periods at the occurrence of the event.

This view of space and time has always been in the minds of physicists, even if, as a rule, they have been unconscious of it. This is clear from the part which these concepts play in physical measurements; it must also have underlain the reader's reflexions on the preceding paragraph (§ 2) for him to connect any meaning with what he there read. But we shall now show that we must put it aside and replace it by a more general view, in order to be able to carry through the postulate of general relativity, if the special theory of relativity applies to the special case of the absence of a gravitational field.

In a space which is free of gravitational fields we introduce a Galilean system of reference K (x, y, z, t), and also a system of co-ordinates K′ (x', y', z', t') in uniform rotation relatively to K. Let the origins of both systems, as well as their axes

* We assume the possibility of verifying "simultaneity" for events immediately proximate in space, or—to speak more precisely—for immediate proximity or coincidence in space-time, without giving a definition of this fundamental concept.

of Z, permanently coincide. We shall show that for a space-time measurement in the system K′ the above definition of the physical meaning of lengths and times cannot be maintained. For reasons of symmetry it is clear that a circle around the origin in the X, Y plane of K may at the same time be regarded as a circle in the X′, Y′ plane of K′. We suppose that the circumference and diameter of this circle have been measured with a unit measure infinitely small compared with the radius, and that we have the quotient of the two results. If this experiment were performed with a measuring-rod at rest relatively to the Galilean system K, the quotient would be π. With a measuring-rod at rest relatively to K′, the quotient would be greater than π. This is readily understood if we envisage the whole process of measuring from the "stationary" system K, and take into consideration that the measuring-rod applied to the periphery undergoes a Lorentzian contraction, while the one applied along the radius does not. Hence Euclidean geometry does not apply to K′. The notion of co-ordinates defined above, which presupposes the validity of Euclidean geometry, therefore breaks down in relation to the system K′. So, too, we are unable to introduce a time corresponding to physical requirements in K′, indicated by clocks at rest relatively to K′. To convince ourselves of this impossibility, let us imagine two clocks of identical constitution placed, one at the origin of co-ordinates, and the other at the circumference of the circle, and both envisaged from the "stationary" system K. By a familiar result of the special theory of relativity, the clock at the circumference—judged from K—goes more slowly than the other, because the former is in motion and the latter at rest. An observer at the common origin of co-ordinates, capable of observing the clock at the circumference by means of light, would therefore see it lagging behind the clock beside him. As he will not make up his mind to let the velocity of light along the path in question depend explicitly on the time, he will interpret his observations as showing that the clock at the circumference "really" goes more slowly than the clock at the origin. So he will be obliged to define time in such a way that the rate of a clock depends upon where the clock may be.

We therefore reach this result:—In the general theory of relativity, space and time cannot be defined in such a way that differences of the spatial co-ordinates can be directly measured by the unit measuring-rod, or differences in the time co-ordinate by a standard clock.

The method hitherto employed for laying co-ordinates into the space-time continuum in a definite manner thus breaks down, and there seems to be no other way which would allow us to adapt systems of co-ordinates to the four-dimensional universe so that we might expect from their application a particularly simple formulation of the laws of nature. So there is nothing for it but to regard all imaginable systems of co-ordinates, on principle, as equally suitable for the description of nature. This comes to requiring that:—

The general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any substitutions whatever (generally co-variant).

It is clear that a physical theory which satisfies this postulate will also be suitable for the general postulate of relativity. For the sum of *all* substitutions in any case includes those which correspond to all relative motions of three-dimensional systems of co-ordinates. That this requirement of general co-variance, which takes away from space and time the last remnant of physical objectivity, is a natural one, will be seen from the following reflexion. All our space-time verifications invariably amount to a determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meetings of two or more of these points. Moreover, the results of our measurings are nothing but verifications of such meetings of the material points of our measuring instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point-events happening at the same place at the same time.

The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such coincidences. We allot to the universe four space-time variables x_1, x_2, x_3, x_4 in such a way that for every point-event there is a corresponding system of values of the variables $x_1 \ldots x_4$. To two coincident point-events there corresponds one system of values of the variables $x_1 \ldots x_4$, i.e. coincidence is characterized by the identity of the co-ordinates. If, in place of the variables $x_1 \ldots x_4$, we introduce functions of them, x'_1, x'_2, x'_3, x'_4, as a new system of co-ordinates, so that the systems of values are made to correspond to one another without ambiguity, the equality of all four co-ordinates in the new system will also serve as an expression for the space-time coincidence of the two point-events. As all our physical experience can be ultimately reduced to such coincidences, there is no immediate reason for preferring certain systems of co-ordinates to others, that is to say, we arrive at the requirement of general co-variance.

§ 4. The Relation of the Four Co-ordinates to Measurement in Space and Time

It is not my purpose in this discussion to represent the general theory of relativity as a system that is as simple and logical as possible, and with the minimum number of axioms; but my main object is to develop this theory in such a way that the reader will feel that the path we have entered upon is psychologically the natural one, and that the underlying assumptions will seem to have the highest possible degree of security. With this aim in view let it now be granted that:—

For infinitely small four-dimensional regions the theory of relativity in the restricted sense is appropriate, if the co-ordinates are suitably chosen.

For this purpose we must choose the acceleration of the infinitely small ("local") system of co-ordinates so that no gravitational field occurs; this is possible for an infinitely small region. Let X_1, X_2, X_3, be the co-ordinates of space, and X_4 the appertaining co-ordinate of time measured in the appropriate unit.* If a rigid rod is imagined to be given as the unit measure, the co-ordinates, with a given orientation of the system of co-ordinates, have a direct physical meaning

* The unit of time is to be chosen so that the velocity of light *in vacuo* as measured in the "local" system of co-ordinates is to be equal to unity.

in the sense of the special theory of relativity. By the special theory of relativity the expression

$$ds^2 = -dX_1^2 - dX_2^2 - dX_3^2 + dX_4^2 \quad . \quad . \quad (1)$$

then has a value which is independent of the orientation of the local system of co-ordinates, and is ascertainable by measurements of space and time. The magnitude of the linear element pertaining to points of the four-dimensional continuum in infinite proximity, we call ds. If the ds belonging to the element $dX_1 \ldots dX_4$ is positive, we follow Minkowski in calling it time-like ; if it is negative, we call it space-like.

To the " linear element " in question, or to the two infinitely proximate point-events, there will also correspond definite differentials $dx_1 \ldots dx_4$ of the four-dimensional co-ordinates of any chosen system of reference. If this system, as well as the " local " system, is given for the region under consideration, the dX_ν will allow themselves to be represented here by definite linear homogeneous expressions of the dx_σ:—

$$dX_\nu = \sum_\sigma a_{\nu\sigma} dx_\sigma \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Inserting these expressions in (1), we obtain

$$ds^2 = \sum_{\tau\sigma} g_{\sigma\tau} dx_\sigma dx_\tau, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where the $g_{\sigma\tau}$ will be functions of the x_σ. These can no longer be dependent on the orientation and the state of motion of the "local" system of co-ordinates, for ds^2 is a quantity ascertainable by rod-clock measurement of point-events infinitely proximate in space-time, and defined independently of any particular choice of co-ordinates. The $g_{\sigma\tau}$ are to be chosen here so that $g_{\sigma\tau} = g_{\tau\sigma}$; the summation is to extend over all values of σ and τ, so that the sum consists of 4×4 terms, of which twelve are equal in pairs.

The case of the ordinary theory of relativity arises out of the case here considered, if it is possible, by reason of the particular relations of the $g_{\sigma\tau}$ in a finite region, to choose the system of reference in the finite region in such a way that the $g_{\sigma\tau}$ assume the constant values

$$\left.\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{array}\right\} \quad . \quad . \quad . \quad (4)$$

We shall find hereafter that the choice of such co-ordinates is, in general, not possible for a finite region.

From the considerations of § 2 and § 3 it follows that the quantities $g_{\tau\sigma}$ are to be regarded from the physical standpoint as the quantities which describe the gravitational field in relation to the chosen system of reference. For, if we now assume the special theory of relativity to apply to a certain four-dimensional region with the co-ordinates properly chosen, then the $g_{\sigma\tau}$ have the values given in (4). A free material point then moves, relatively to this system, with uniform motion in a straight line. Then if we introduce new space-time co-ordinates x_1, x_2, x_3, x_4, by means of any substitution we choose, the $g^{\sigma\tau}$ in this new system will no longer be constants, but functions of space and time. At the same time the motion of the free material point will present itself in the new co-ordinates as a curvilinear non-uniform motion, and the law of this motion will be independent of the nature of the moving particle. We shall therefore interpret this motion as a motion under the influence of a gravitational field. We thus find the occurrence of a gravitational field connected with a space-time variability of the g_{σ}. So, too, in the general case, when we are no longer able by a suitable choice of co-ordinates to apply the special theory of relativity to a finite region, we shall hold fast to the view that the $g_{\sigma\tau}$ describe the gravitational field.

Thus, according to the general theory of relativity, gravitation occupies an exceptional position with regard to other forces, particularly the electromagnetic forces, since the ten functions representing the gravitational field at the same time define the metrical properties of the space measured.

B. Mathematical Aids to the Formulation of Generally Covariant Equations

Having seen in the foregoing that the general postulate of relativity leads to the requirement that the equations of physics shall be covariant in the face of any substitution of the co-ordinates $x_1 \ldots x_4$, we have to consider how such generally covariant equations can be found. We now turn to this purely mathematical task, and we shall find that in its solution a fundamental rôle is played by the invariant ds given in equation (3), which, borrowing from Gauss's theory of surfaces, we have called the "linear element."

The fundamental idea of this general theory of covariants is the following:—Let certain things ("tensors") be defined with respect to any system of co-ordinates by a number of functions of the co-ordinates, called the "components" of the tensor. There are then certain rules by which these components can be calculated for a new system of co-ordinates, if they are known for the original system of co-ordinates, and if the transformation connecting the two systems is known. The things hereafter called tensors are further characterized by the fact that the equations of transformation for their components are linear and homogeneous. Accordingly, all the components in the new system vanish, if they all vanish in the original system. If, therefore, a law of nature is expressed by equating all the components of a tensor to zero, it is generally covariant. By examining the laws of the formation of tensors, we acquire the means of formulating generally covariant laws.

§ 5. Contravariant and Covariant Four-vectors

Contravariant Four-vectors.—The linear element is defined by the four "components" dx_ν, for which the law of transformation is expressed by the equation

$$dx'_\sigma = \sum_\nu \frac{\partial x'_\sigma}{\partial x_\nu} dx_\nu \quad . \quad . \quad . \quad . \quad (5)$$

The dx'_σ are expressed as linear and homogeneous functions of the dx_ν. Hence we may look upon these co-ordinate differentials as the components of a "tensor" of the particular kind which we call a contravariant four-vector. Any thing which is defined relatively to the system of co-ordinates by four quantities A^ν, and which is transformed by the same law

$$A'^\sigma = \sum_\nu \frac{\partial x'_\sigma}{\partial x_\nu} A^\nu, \quad . \quad . \quad . \quad . \quad (5a)$$

we also call a contravariant four-vector. From (5a) it follows at once that the sums $A^\sigma \pm B^\sigma$ are also components of a four-vector, if A^σ and B^σ are such. Corresponding relations hold for all "tensors" subsequently to be introduced. (Rule for the addition and subtraction of tensors.)

Covariant Four-vectors.—We call four quantities A_ν the components of a covariant four-vector, if for any arbitrary choice of the contravariant four-vector B^ν

$$\sum_\nu A_\nu B^\nu = \text{Invariant} \quad . \quad . \quad . \quad (6)$$

The law of transformation of a covariant four-vector follows from this definition. For if we replace B^ν on the right-hand side of the equation

$$\sum_\sigma A'_\sigma B'^\sigma = \sum_\nu A_\nu B^\nu$$

by the expression resulting from the inversion of (5a),

$$\sum_\sigma \frac{\partial x_\nu}{\partial x'_\sigma} B'^\sigma,$$

we obtain

$$\sum_\sigma B'^\sigma \sum_\nu \frac{\partial x_\nu}{\partial x'_\sigma} A_\nu = \sum_\sigma B'^\sigma A' \, .$$

Since this equation is true for arbitrary values of the B'^σ, it follows that the law of transformation is

$$A'_\sigma = \sum_\nu \frac{\partial x_\nu}{\partial x'_\sigma} A_\nu \quad . \quad . \quad . \quad . \quad (7)$$

Note on a Simplified Way of Writing the Expressions.— A glance at the equations of this paragraph shows that there is always a summation with respect to the indices which occur twice under a sign of summation (e.g. the index ν in (5)), and only with respect to indices which occur twice. It is therefore possible, without loss of clearness, to omit the sign of summation. In its place we introduce the convention:— If an index occurs twice in one term of an expression, it is always to be summed unless the contrary is expressly stated.

The difference between covariant and contravariant four-vectors lies in the law of transformation ((7) or (5) respectively). Both forms are tensors in the sense of the general remark above. Therein lies their importance. Following Ricci and

Levi-Civita, we denote the contravariant character by placing the index above, the covariant by placing it below.

§ 6. Tensors of the Second and Higher Ranks

Contravariant Tensors.—If we form all the sixteen products $A^{\mu\nu}$ of the components A^{μ} and B^{ν} of two contravariant four-vectors

$$A^{\mu\nu} = A^{\mu}B^{\nu} \quad . \quad . \quad . \quad . \quad (8)$$

then by (8) and (5a) $A^{\mu\nu}$ satisfies the law of transformation

$$A'^{\sigma\tau} = \frac{\partial x'_{\sigma}}{\partial x_{\mu}}\frac{\partial x'_{\tau}}{\partial x_{\nu}}A^{\mu\nu} \quad . \quad . \quad . \quad (9)$$

We call a thing which is described relatively to any system of reference by sixteen quantities, satisfying the law of transformation (9), a contravariant tensor of the second rank. Not every such tensor allows itself to be formed in accordance with (8) from two four-vectors, but it is easily shown that any given sixteen $A^{\mu\nu}$ can be represented as the sums of the $A^{\mu}B^{\nu}$ of four appropriately selected pairs of four-vectors. Hence we can prove nearly all the laws which apply to the tensor of the second rank defined by (9) in the simplest manner by demonstrating them for the special tensors of the type (8).

Contravariant Tensors of Any Rank.—It is clear that, on the lines of (8) and (9), contravariant tensors of the third and higher ranks may also be defined with 4^3 components, and so on. In the same way it follows from (8) and (9) that the contravariant four-vector may be taken in this sense as a contravariant tensor of the first rank.

Covariant Tensors.—On the other hand, if we take the sixteen products $A_{\mu\nu}$ of two covariant four-vectors A_{μ} and B_{ν},

$$A_{\mu\nu} = A_{\mu}B_{\nu}, \quad . \quad . \quad . \quad . \quad (10)$$

the law of transformation for these is

$$A'_{\sigma\tau} = \frac{\partial x_{\mu}}{\partial x'_{\sigma}}\frac{\partial x_{\nu}}{\partial x'_{\tau}}A_{\mu\nu} \quad . \quad . \quad . \quad (11)$$

This law of transformation defines the covariant tensor of the second rank. All our previous remarks on contravariant tensors apply equally to covariant tensors.

NOTE.—It is convenient to treat the scalar (or invariant) both as a contravariant and a covariant tensor of zero rank.

Mixed Tensors.—We may also define a tensor of the second rank of the type

$$A_\mu^\nu = A_\mu B^\nu \quad . \quad . \quad . \quad . \quad (12)$$

which is covariant with respect to the index μ, and contravariant with respect to the index ν. Its law of transformation is

$$A'^\tau_\sigma = \frac{\partial x'_\tau}{\partial x_\nu} \frac{\partial x_\mu}{\partial x'_\sigma} A_\mu^\nu \quad . \quad . \quad . \quad (13)$$

Naturally there are mixed tensors with any number of indices of covariant character, and any number of indices of contravariant character. Covariant and contravariant tensors may be looked upon as special cases of mixed tensors.

Symmetrical Tensors.—A contravariant, or a covariant tensor, of the second or higher rank is said to be symmetrical if two components, which are obtained the one from the other by the interchange of two indices, are equal. The tensor $A^{\mu\nu}$, or the tensor $A_{\mu\nu}$, is thus symmetrical if for any combination of the indices μ, ν,

$$A^{\mu\nu} = A^{\nu\mu}, \quad . \quad . \quad . \quad . \quad (14)$$

or respectively,

$$A_{\mu\nu} = A_{\nu\mu}. \quad . \quad . \quad . \quad . \quad (14a)$$

It has to be proved that the symmetry thus defined is a property which is independent of the system of reference. It follows in fact from (9), when (14) is taken into consideration, that

$$A'^{\sigma\tau} = \frac{\partial x'_\sigma}{\partial x_\mu} \frac{\partial x'_\tau}{\partial x_\nu} A^{\mu\nu} = \frac{\partial x'_\sigma}{\partial x_\mu} \frac{\partial x'_\tau}{\partial x_\nu} A^{\nu\mu} = \frac{\partial x'_\sigma}{\partial x_\nu} \frac{\partial x'_\tau}{\partial x_\mu} A^{\mu\nu} = A'^{\tau\sigma}.$$

The last equation but one depends upon the interchange of the summation indices μ and ν, i.e. merely on a change of notation.

Antisymmetrical Tensors.—A contravariant or a covariant tensor of the second, third, or fourth rank is said to be antisymmetrical if two components, which are obtained the one from the other by the interchange of two indices, are equal and of opposite sign. The tensor $A^{\mu\nu}$, or the tensor $A_{\mu\nu}$, is therefore antisymmetrical, if always

$$A^{\mu\nu} = -A^{\nu\mu}, \quad . \quad . \quad . \quad . \quad (15)$$

or respectively,

$$A_{\mu\nu} = -A_{\nu\mu} \quad . \quad . \quad . \quad . \quad (15a)$$

Of the sixteen components $A^{\mu\nu}$, the four components $A^{\mu\mu}$ vanish; the rest are equal and of opposite sign in pairs, so that there are only six components numerically different (a six-vector). Similarly we see that the antisymmetrical tensor of the third rank $A^{\mu\nu\sigma}$ has only four numerically different components, while the antisymmetrical tensor $A^{\mu\nu\sigma\tau}$ has only one. There are no antisymmetrical tensors of higher rank than the fourth in a continuum of four dimensions.

§ 7. Multiplication of Tensors

Outer Multiplication of Tensors.—We obtain from the components of a tensor of rank n and of a tensor of rank m the components of a tensor of rank $n + m$ by multiplying each component of the one tensor by each component of the other. Thus, for example, the tensors T arise out of the tensors A and B of different kinds,

$$\begin{aligned} T_{\mu\nu\sigma} &= A_{\mu\nu}B_{\sigma}, \\ T^{\mu\nu\sigma\tau} &= A^{\mu\nu}B^{\sigma\tau}, \\ T^{\sigma\tau}_{\mu\nu} &= A_{\mu\nu}B^{\sigma\tau}. \end{aligned}$$

The proof of the tensor character of T is given directly by the representations (8), (10), (12), or by the laws of transformation (9), (11), (13). The equations (8), (10), (12) are themselves examples of outer multiplication of tensors of the first rank.

"*Contraction*" *of a Mixed Tensor.*—From any mixed tensor we may form a tensor whose rank is less by two, by equating an index of covariant with one of contravariant character, and summing with respect to this index ("contraction"). Thus, for example, from the mixed tensor of the fourth rank $A^{\sigma\tau}_{\mu\nu}$, we obtain the mixed tensor of the second rank,

$$A^{\tau}_{\nu} = A^{\mu\tau}_{\mu\nu} \quad \left(= \sum_{\mu} A^{\mu\tau}_{\mu\nu} \right),$$

and from this, by a second contraction, the tensor of zero rank,

$$A = A^{\nu}_{\nu} = A^{\mu\nu}_{\mu\nu}.$$

The proof that the result of contraction really possesses the tensor character is given either by the representation of a tensor according to the generalization of (12) in combination with (6), or by the generalization of (13).

Inner and Mixed Multiplication of Tensors.—These consist in a combination of outer multiplication with contraction.

Examples.—From the covariant tensor of the second rank $A_{\mu\nu}$ and the contravariant tensor of the first rank B^{σ} we form by outer multiplication the mixed tensor

$$D_{\mu\nu}^{\sigma} = A_{\mu\nu}B^{\sigma}.$$

On contraction with respect to the indices ν and σ, we obtain the covariant four-vector

$$D_{\mu} = D_{\mu\nu}^{\nu} = A_{\mu\nu}B^{\nu}.$$

This we call the inner product of the tensors $A_{\mu\nu}$ and B^{σ}. Analogously we form from the tensors $A_{\mu\nu}$ and $B^{\sigma\tau}$, by outer multiplication and double contraction, the inner product $A_{\mu\nu}B^{\mu\nu}$. By outer multiplication and one contraction, we obtain from $A_{\mu\nu}$ and $B^{\sigma\tau}$ the mixed tensor of the second rank $D_{\mu}^{\tau} = A_{\mu\nu}B^{\nu\tau}$. This operation may be aptly characterized as a mixed one, being "outer" with respect to the indices μ and τ, and "inner" with respect to the indices ν and σ.

We now prove a proposition which is often useful as evidence of tensor character. From what has just been explained, $A_{\mu\nu}B^{\mu\nu}$ is a scalar if $A_{\mu\nu}$ and $B^{\sigma\tau}$ are tensors. But we may also make the following assertion: If $A_{\mu\nu}B^{\mu\nu}$ is a scalar *for any choice of the tensor* $B^{\mu\nu}$, then $A_{\mu\nu}$ has tensor character. For, by hypothesis, for any substitution,

$$A'_{\sigma\tau}B'^{\sigma\tau} = A_{\mu\nu}B^{\mu\nu}.$$

But by an inversion of (9)

$$B^{\mu\nu} = \frac{\partial x_{\mu}}{\partial x'_{\sigma}}\frac{\partial x_{\nu}}{\partial x'_{\tau}}B'^{\sigma\tau}.$$

This, inserted in the above equation, gives

$$\left(A'_{\sigma\tau} - \frac{\partial x_{\mu}}{\partial x'_{\sigma}}\frac{\partial x_{\nu}}{\partial x'_{\tau}}A_{\mu\nu}\right)B'^{\sigma\tau} = 0.$$

This can only be satisfied for arbitrary values of $B'^{\sigma\tau}$ if the bracket vanishes. The result then follows by equation (11). This rule applies correspondingly to tensors of any rank and character, and the proof is analogous in all cases.

The rule may also be demonstrated in this form: If B^{μ} and C^{ν} are any vectors, and if, for all values of these, the inner product $A_{\mu\nu}B^{\mu}C^{\nu}$ is a scalar, then $A_{\mu\nu}$ is a covariant tensor. This latter proposition also holds good even if only the more special assertion is correct, that with any choice of the four-vector B^{μ} the inner product $A_{\mu\nu}B^{\mu}B^{\nu}$ is a scalar, if in addition it is known that $A_{\mu\nu}$ satisfies the condition of symmetry $A_{\mu\nu} = A_{\nu\mu}$. For by the method given above we prove the tensor character of $(A_{\mu\nu} + A_{\nu\mu})$, and from this the tensor character of $A_{\mu\nu}$ follows on account of symmetry. This also can be easily generalized to the case of covariant and contravariant tensors of any rank.

Finally, there follows from what has been proved, this law, which may also be generalized for any tensors: If for any choice of the four-vector B^{ν} the quantities $A_{\mu\nu}B^{\nu}$ form a tensor of the first rank, then $A_{\mu\nu}$ is a tensor of the second rank. For, if C^{μ} is any four-vector, then on account of the tensor character of $A_{\mu\nu}B^{\nu}$, the inner product $A_{\mu\nu}B^{\nu}C^{\mu}$ is a scalar for any choice of the two four-vectors B^{ν} and C^{μ}. From which the proposition follows.

§ 8. Some Aspects of the Fundamental Tensor $g_{\mu\nu}$

The Covariant Fundamental Tensor.—In the invariant expression for the square of the linear element,

$$ds^2 = g_{\mu\nu}dx_{\mu}dx_{\nu},$$

the part played by the dx_{μ} is that of a contravariant vector which may be chosen at will. Since further, $g_{\mu\nu} = g_{\nu\mu}$, it follows from the considerations of the preceding paragraph that $g_{\mu\nu}$ is a covariant tensor of the second rank. We call it the "fundamental tensor." In what follows we deduce some properties of this tensor which, it is true, apply to any tensor of the second rank. But as the fundamental tensor plays a special part in our theory, which has its physical basis in the peculiar effects of gravitation, it so happens that the relations to be developed are of importance to us only in the case of the fundamental tensor.

The Contravariant Fundamental Tensor.—If in the determinant formed by the elements $g_{\mu\nu}$, we take the co-factor of each of the $g_{\mu\nu}$ and divide it by the determinant $g = | g_{\mu\nu} |$, we obtain certain quantities $g^{\mu\nu} (= g^{\nu\mu})$ which, as we shall demonstrate, form a contravariant tensor.

By a known property of determinants

$$g_{\mu\sigma} g^{\nu\sigma} = \delta_\mu^\nu \quad . \quad . \quad . \quad . \quad (16)$$

where the symbol δ_μ^ν denotes 1 or 0, according as $\mu = \nu$ or $\mu \neq \nu$.

Instead of the above expression for ds^2 we may thus write

$$g_{\mu\sigma} \delta_\nu^\sigma dx_\mu dx_\nu$$

or, by (16)

$$g_{\mu\sigma} g_{\nu\tau} g^{\sigma\tau} dx_\mu dx_\nu.$$

But, by the multiplication rules of the preceding paragraphs, the quantities

$$d\xi_\sigma = g_{\mu\sigma} dx_\mu$$

form a covariant four-vector, and in fact an arbitrary vector, since the dx_μ are arbitrary. By introducing this into our expression we obtain

$$ds^2 = g^{\sigma\tau} d\xi_\sigma d\xi_\tau.$$

Since this, with the arbitrary choice of the vector $d\xi_\sigma$, is a scalar, and $g^{\sigma\tau}$ by its definition is symmetrical in the indices σ and τ, it follows from the results of the preceding paragraph that $g^{\sigma\tau}$ is a contravariant tensor.

It further follows from (16) that δ_μ is also a tensor, which we may call the mixed fundamental tensor.

The Determinant of the Fundamental Tensor.—By the rule for the multiplication of determinants

$$| g_{\mu\alpha} g^{\alpha\nu} | = | g_{\mu\alpha} | \times | g^{\alpha\nu} | .$$

On the other hand

$$| g_{\mu\alpha} g^{\alpha\nu} | = | \delta_\mu^\nu | = 1.$$

It therefore follows that

$$| g_{\mu\nu} | \times | g^{\mu\nu} | = 1 \quad . \quad . \quad . \quad (17)$$

The Volume Scalar.—We seek first the law of transformation of the determinant $g = | g_{\mu\nu} |$. In accordance with (11)

$$g' = \left| \frac{\partial x_\mu}{\partial x'_\sigma} \frac{\partial x}{\partial x'_\tau} g_{\mu\nu} \right|.$$

Hence, by a double application of the rule for the multiplication of determinants, it follows that

$$g' = \left| \frac{\partial x_\mu}{\partial x'_\sigma} \right| \cdot \left| \frac{\partial x_\nu}{\partial x'_\tau} \right| \cdot | g_{\mu\nu} | = \left| \frac{\partial x_\mu}{\partial x'_\sigma} \right|^2 g,$$

or

$$\sqrt{g'} = \left| \frac{\partial x_\mu}{\partial x'_\sigma} \right| \sqrt{g}.$$

On the other hand, the law of transformation of the element of volume

$$d\tau = \int dx_1 dx_2 dx_3 dx_4$$

is, in accordance with the theorem of Jacobi,

$$d\tau' = \left| \frac{\partial x'_\sigma}{\partial x_\mu} \right| d\tau.$$

By multiplication of the last two equations, we obtain

$$\sqrt{g'} d\tau' = \sqrt{g} d\tau \quad . \quad . \quad . \quad (18).$$

Instead of $\sqrt{g}$, we introduce in what follows the quantity $\sqrt{-g}$, which is always real on account of the hyperbolic character of the space-time continuum. The invariant $\sqrt{-g}d\tau$ is equal to the magnitude of the four-dimensional element of volume in the "local" system of reference, as measured with rigid rods and clocks in the sense of the special theory of relativity.

Note on the Character of the Space-time Continuum.—Our assumption that the special theory of relativity can always be applied to an infinitely small region, implies that ds^2 can always be expressed in accordance with (1) by means of real quantities $dX_1 \ . \ . \ . \ dX_4$. If we denote by $d\tau_0$ the "natural" element of volume dX_1, dX_2, dX_3, dX_4, then

$$d\tau_0 = \sqrt{-g} d\tau \quad . \quad . \quad . \quad (18a)$$

If $\sqrt{-g}$ were to vanish at a point of the four-dimensional continuum, it would mean that at this point an infinitely small "natural" volume would correspond to a finite volume in the co-ordinates. Let us assume that this is never the case. Then g cannot change sign. We will assume that, in the sense of the special theory of relativity, g always has a finite negative value. This is a hypothesis as to the physical nature of the continuum under consideration, and at the same time a convention as to the choice of co-ordinates.

But if $-g$ is always finite and positive, it is natural to settle the choice of co-ordinates *a posteriori* in such a way that this quantity is always equal to unity. We shall see later that by such a restriction of the choice of co-ordinates it is possible to achieve an important simplification of the laws of nature.

In place of (18), we then have simply $d\tau' = d\tau$, from which, in view of Jacobi's theorem, it follows that

$$\left| \frac{\partial x'_\sigma}{\partial x_\mu} \right| = 1 \quad . \quad . \quad . \quad . \quad (19)$$

Thus, with this choice of co-ordinates, only substitutions for which the determinant is unity are permissible.

But it would be erroneous to believe that this step indicates a partial abandonment of the general postulate of relativity. We do not ask "What are the laws of nature which are co-variant in face of all substitutions for which the determinant is unity?" but our question is "What are the generally co-variant laws of nature?" It is not until we have formulated these that we simplify their expression by a particular choice of the system of reference.

The Formation of New Tensors by Means of the Fundamental Tensor.—Inner, outer, and mixed multiplication of a tensor by the fundamental tensor give tensors of different character and rank. For example,

$$A^\mu = g^{\mu\sigma} A_\sigma,$$
$$A = g_{\mu\nu} A^{\mu\nu}.$$

The following forms may be specially noted:—

$$A^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} A_{\alpha\beta},$$
$$A_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} A^{\alpha\beta}$$

(the " complements " of covariant and contravariant tensors respectively), and

$$B_{\mu\nu} = g_{\mu\nu}g^{\alpha\beta}A_{\alpha\beta}.$$

We call $B_{\mu\nu}$ the reduced tensor associated with $A_{\mu\nu}$. Similarly,

$$B^{\mu\nu} = g^{\mu\nu}g_{\alpha\beta}A^{\alpha\beta}.$$

It may be noted that $g^{\mu\nu}$ is nothing more than the complement of $g_{\mu\nu}$, since

$$g^{\mu\alpha}g^{\nu\beta}g_{\alpha\beta} = g^{\mu\alpha}\delta_{\alpha}^{\nu} = g^{\mu\nu}.$$

§ 9. The Equation of the Geodetic Line. The Motion of a Particle

As the linear element ds is defined independently of the system of co-ordinates, the line drawn between two points P and P′ of the four-dimensional continuum in such a way that $\int ds$ is stationary—a geodetic line—has a meaning which also is independent of the choice of co-ordinates. Its equation is

$$\delta \int_{P}^{P'} ds = 0 \quad . \quad . \quad . \quad . \quad (20)$$

Carrying out the variation in the usual way, we obtain from this equation four differential equations which define the geodetic line; this operation will be inserted here for the sake of completeness. Let λ be a function of the co-ordinates x_ν, and let this define a family of surfaces which intersect the required geodetic line as well as all the lines in immediate proximity to it which are drawn through the points P and P′. Any such line may then be supposed to be given by expressing its co-ordinates x_ν as functions of λ. Let the symbol δ indicate the transition from a point of the required geodetic to the point corresponding to the same λ on a neighbouring line. Then for (20) we may substitute

$$\left.\begin{aligned} \int_{\lambda_1}^{\lambda_2} \delta w d\lambda &= 0 \\ w^2 &= g_{\mu\nu}\frac{dx_\mu}{d\lambda}\frac{dx_\nu}{d\lambda} \end{aligned}\right\} \quad . \quad . \quad . \quad (20a)$$

But since

$$\delta w = \frac{1}{w}\left\{\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\frac{dx_\mu}{d\lambda}\frac{dx_\nu}{d\lambda}\delta x_\sigma + g_{\mu\nu}\frac{dx_\mu}{d\lambda}\delta\left(\frac{dx_\nu}{d\lambda}\right)\right\},$$

and

$$\delta\left(\frac{dx_\nu}{d\lambda}\right) = \frac{d}{d\lambda}(\delta x_\nu),$$

we obtain from (20a), after a partial integration,

$$\int_{\lambda_1}^{\lambda_2} \kappa_\sigma \delta x_\sigma d\lambda = 0,$$

where

$$\kappa_\sigma = \frac{d}{d\lambda}\left\{\frac{g_{\mu\nu}}{w}\frac{dx_\mu}{d\lambda}\right\} - \frac{1}{2w}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\frac{dx_\mu}{d\lambda}\frac{dx_\nu}{d\lambda} \quad . \quad (20b)$$

Since the values of δx_σ are arbitrary, it follows from this that

$$\kappa_\sigma = 0 \quad . \quad . \quad . \quad . \quad (20c)$$

are the equations of the geodetic line.

If ds does not vanish along the geodetic line we may choose the "length of the arc" s, measured along the geodetic line, for the parameter λ. Then $w = 1$, and in place of (20c) we obtain

$$g_{\mu\nu}\frac{d^2x_\mu}{ds^2} + \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\frac{dx_\sigma}{ds}\frac{dx_\mu}{ds} - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = 0$$

or, by a mere change of notation,

$$g_{\alpha\sigma}\frac{d^2x_\alpha}{ds^2} + [\mu\nu, \sigma]\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = 0 \quad . \quad . \quad (20d)$$

where, following Christoffel, we have written

$$[\mu\nu, \sigma] = \frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right) \quad . \quad . \quad (21)$$

Finally, if we multiply (20d) by $g^{\sigma\tau}$ (outer multiplication with respect to τ, inner with respect to σ), we obtain the equations of the geodetic line in the form

$$\frac{d^2x_\tau}{ds^2} + \{\mu\nu, \tau\}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = 0 \quad . \quad . \quad . \quad (22)$$

where, following Christoffel, we have set

$$\{\mu\nu, \tau\} = g^{\tau\alpha}[\mu\nu, \alpha] \quad . \quad . \quad . \quad (23)$$

§ 10. The Formation of Tensors by Differentiation

With the help of the equation of the geodetic line we can now easily deduce the laws by which new tensors can be formed from old by differentiation. By this means we are able for the first time to formulate generally covariant differential equations. We reach this goal by repeated application of the following simple law :—

If in our continuum a curve is given, the points of which are specified by the arcual distance s measured from a fixed point on the curve, and if, further, ϕ is an invariant function of space, then $d\phi/ds$ is also an invariant. The proof lies in this, that ds is an invariant as well as $d\phi$.

As

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x_\mu}\frac{dx_\mu}{ds}$$

therefore

$$\psi = \frac{\partial\phi}{dx_\mu}\frac{dx_\mu}{ds}$$

is also an invariant, and an invariant for all curves starting from a point of the continuum, that is, for any choice of the vector dx_μ. Hence it immediately follows that

$$\mathrm{A}_\mu = \frac{\partial\phi}{\partial x_\mu} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

is a covariant four-vector—the "gradient" of ϕ.

According to our rule, the differential quotient

$$\chi = \frac{d\psi}{ds}$$

taken on a curve, is similarly an invariant. Inserting the value of ψ, we obtain in the first place

$$\chi = \frac{\partial^2\phi}{\partial x_\mu \partial x_\nu}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds} + \frac{\partial\phi}{\partial x_\mu}\frac{d^2x_\mu}{ds^2}.$$

The existence of a tensor cannot be deduced from this forthwith. But if we may take the curve along which we have differentiated to be a geodetic, we obtain on substitution for d^2x_ν/ds^2 from (22),

$$\chi = \left(\frac{\partial^2\phi}{\partial x_\mu \partial x_\nu} - \{\mu\nu, \tau\}\frac{\partial\phi}{\partial x_\tau}\right)\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}.$$

Since we may interchange the order of the differentiations, and since by (23) and (21) $\{\mu\nu, \tau\}$ is symmetrical in μ and ν, it follows that the expression in brackets is symmetrical in μ and ν. Since a geodetic line can be drawn in any direction from a point of the continuum, and therefore dx_μ/ds is a four-vector with the ratio of its components arbitrary, it follows from the results of § 7 that

$$A_{\mu\nu} = \frac{\partial^2\phi}{\partial x_\mu \partial x_\nu} - \{\mu\nu, \tau\}\frac{\partial\phi}{\partial x_\tau} \quad . \quad . \quad . \quad (25)$$

is a covariant tensor of the second rank. We have therefore come to this result: from the covariant tensor of the first rank

$$A_\mu = \frac{\partial\phi}{\partial x_\mu}$$

we can, by differentiation, form a covariant tensor of the second rank

$$A_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \{\mu\nu, \tau\}A_\tau \quad . \quad . \quad . \quad (26)$$

We call the tensor $A_{\mu\nu}$ the " extension " (covariant derivative) of the tensor A_μ. In the first place we can readily show that the operation leads to a tensor, even if the vector A_μ cannot be represented as a gradient. To see this, we first observe that

$$\psi\frac{\partial\phi}{\partial x_\mu}$$

is a covariant vector, if ψ and ϕ are scalars. The sum of four such terms

$$S_\mu = \psi^{(1)}\frac{\phi\partial^{(1)}}{\partial x_\mu} + . + . + \psi^{(4)}\frac{\partial\phi^{(4)}}{\partial x_\mu},$$

is also a covariant vector, if $\psi^{(1)}, \phi^{(1)} \ . \ . \ . \ \psi^{(4)}, \phi^{(4)}$ are scalars. But it is clear that any covariant vector can be represented in the form S_μ. For, if A_μ is a vector whose components are any given functions of the x_ν, we have only to put (in terms of the selected system of co-ordinates)

$$\begin{aligned}
\psi^{(1)} &= A_1, & \phi^{(1)} &= x_1, \\
\psi^{(2)} &= A_2, & \phi^{(2)} &= x_2, \\
\psi^{(3)} &= A_3, & \phi^{(3)} &= x_3, \\
\psi^{(4)} &= A_4, & \phi^{(4)} &= x_4,
\end{aligned}$$

in order to ensure that S_μ shall be equal to A_μ.

Therefore, in order to demonstrate that $A_{\mu\nu}$ is a tensor if *any* covariant vector is inserted on the right-hand side for A_μ, we only need show that this is so for the vector S_μ. But for this latter purpose it is sufficient, as a glance at the right-hand side of (26) teaches us, to furnish the proof for the case

$$A_\mu = \psi \frac{\partial\phi}{\partial x_\mu}.$$

Now the right-hand side of (25) multiplied by ψ,

$$\psi \frac{\partial^2\phi}{\partial x_\mu \partial x_\nu} - \{\mu\nu, \tau\}\psi\frac{\partial\phi}{\partial x_\tau}$$

is a tensor. Similarly

$$\frac{\partial\psi}{\partial x_\mu}\frac{\partial\phi}{\partial x_\nu}$$

being the outer product of two vectors, is a tensor. By addition, there follows the tensor character of

$$\frac{\partial}{\partial x_\nu}\left(\psi \frac{\partial\phi}{\partial x_\mu}\right) - \{\mu\nu, \tau\}\left(\psi \frac{\partial\phi}{\partial x_\tau}\right).$$

As a glance at (26) will show, this completes the demonstration for the vector

$$\psi \frac{\partial\phi}{\partial x_\mu}$$

and consequently, from what has already been proved, for any vector A_μ.

By means of the extension of the vector, we may easily define the "extension" of a covariant tensor of any rank. This operation is a generalization of the extension of a vector. We restrict ourselves to the case of a tensor of the second rank, since this suffices to give a clear idea of the law of formation.

As has already been observed, any covariant tensor of the second rank can be represented * as the sum of tensors of the

* By outer multiplication of the vector with arbitrary components A_{11}, A_{12}, A_{13}, A_{14} by the vector with components 1, 0, 0, 0, we produce a tensor with components

A_{11}	A_{12}	A_{13}	A_{14}
0	0	0	0
0	0	0	0
0	0	0	0.

By the addition of four tensors of this type, we obtain the tensor $A_{\mu\nu}$ with any ssigned components.

type $A_\mu B_\nu$. It will therefore be sufficient to deduce the expression for the extension of a tensor of this special type. By (26) the expressions

$$\frac{\partial A_\mu}{\partial x_\sigma} - \{\sigma\mu, \tau\}A_\tau,$$

$$\frac{\partial B_\nu}{\partial x_\sigma} - \{\sigma\nu, \tau\}B_\tau,$$

are tensors. On outer multiplication of the first by B_ν, and of the second by A_μ, we obtain in each case a tensor of the third rank. By adding these, we have the tensor of the third rank

$$A_{\mu\nu\sigma} = \frac{\partial A_{\mu\nu}}{\partial x_\sigma} - \{\sigma\mu, \tau\}A_{\tau\nu} - \{\sigma\nu, \tau\}A_{\mu\tau} \quad . \quad . \quad (27)$$

where we have put $A_{\mu\nu} = A_\mu B_\nu$. As the right-hand side of (27) is linear and homogeneous in the $A_{\mu\nu}$ and their first derivatives, this law of formation leads to a tensor, not only in the case of a tensor of the type $A_\mu B_\nu$, but also in the case of a sum of such tensors, i.e. in the case of any covariant tensor of the second rank. We call $A_{\mu\nu\sigma}$ the extension of the tensor $A_{\mu\nu}$.

It is clear that (26) and (24) concern only special cases of extension (the extension of the tensors of rank one and zero respectively).

In general, all special laws of formation of tensors are included in (27) in combination with the multiplication of tensors.

§ 11. Some Cases of Special Importance

The Fundamental Tensor.—We will first prove some lemmas which will be useful hereafter. By the rule for the differentiation of determinants

$$dg = g^{\mu\nu}g\,dg_{\mu\nu} = -g_{\mu\nu}g\,dg^{\mu\nu} \quad . \quad . \quad (28)$$

The last member is obtained from the last but one, if we bear in mind that $g_{\mu\nu}g^{\mu'\nu} = \delta_\mu^{\mu'}$, so that $g_{\mu\nu}g^{\mu\nu} = 4$, and consequently

$$g_{\mu\nu}dg^{\mu\nu} + g^{\mu\nu}dg_{\mu\nu} = 0.$$

From (28), it follows that

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x_\sigma} = \frac{1}{2}\frac{\partial \log(-g)}{\partial x_\sigma} = \frac{1}{2}g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x_\sigma} = -\frac{1}{2}g_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x_\sigma}. \quad (29)$$

Further, from $g_{\mu\sigma}g^{\nu\sigma} = \delta^\nu_\mu$, it follows on differentiation that

$$\left.\begin{aligned} g_{\mu\sigma}dg^{\nu\sigma} &= -g^{\nu\sigma}dg_{\mu\sigma} \\ g_{\mu\sigma}\frac{\partial g^{\nu\sigma}}{\partial x_\lambda} &= -g^{\nu\sigma}\frac{\partial g_{\mu\sigma}}{\partial x_\lambda} \end{aligned}\right\} \quad . \quad . \quad . \quad (30)$$

From these, by mixed multiplication by $g^{\sigma\tau}$ and $g_{\nu\lambda}$ respectively, and a change of notation for the indices, we have

$$\left.\begin{aligned} dg^{\mu\nu} &= -g^{\mu\alpha}g^{\nu\beta}\,dg_{\alpha\beta} \\ \frac{\partial g^{\mu\nu}}{\partial x_\sigma} &= -g^{\mu\alpha}g^{\nu\beta}\frac{\partial g_{\alpha\beta}}{\partial x_\sigma} \end{aligned}\right\} \quad . \quad . \quad . \quad (31)$$

and

$$\left.\begin{aligned} dg_{\mu\nu} &= -g_{\mu\alpha}g_{\nu\beta}\,dg^{\alpha\beta} \\ \frac{\partial g_{\mu\nu}}{\partial x_\sigma} &= -g_{\mu\alpha}g_{\nu\beta}\frac{\partial g^{\alpha\beta}}{\partial x_\sigma} \end{aligned}\right\} \quad . \quad . \quad . \quad (32)$$

The relation (31) admits of a transformation, of which we also have frequently to make use. From (21)

$$\frac{\partial g_{\alpha\beta}}{\partial x_\sigma} = [\alpha\sigma, \beta] + [\beta\sigma, \alpha] \quad . \quad . \quad . \quad (33)$$

Inserting this in the second formula of (31), we obtain, in view of (23)

$$\frac{\partial g^{\mu\nu}}{\partial x_\sigma} = -g^{\mu\tau}\{\tau\sigma, \nu\} - g^{\nu\tau}\{\tau\sigma, \mu\} \quad . \quad . \quad (34)$$

Substituting the right-hand side of (34) in (29), we have

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x_\sigma} = \{\mu\sigma, \mu\} \quad . \quad . \quad (29a)$$

The "Divergence" of a Contravariant Vector.—If we take the inner product of (26) by the contravariant fundamental tensor $g^{\mu\nu}$, the right-hand side, after a transformation of the first term, assumes the form

$$\frac{\partial}{\partial x_\nu}(g^{\mu\nu}A_\mu) - A_\mu\frac{\partial g^{\mu\nu}}{\partial x_\nu} - \frac{1}{2}g^{\tau\alpha}\left(\frac{\partial g_{\mu\alpha}}{\partial x_\nu} + \frac{\partial g_{\nu\alpha}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right)g^{\mu\nu}A_\tau.$$

In accordance with (31) and (29), the last term of this expression may be written

$$\frac{1}{2}\frac{\partial g^{\tau\nu}}{\partial x_\nu}A_\tau + \frac{1}{2}\frac{\partial g^{\tau\mu}}{\partial x_\mu}A_\tau + \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x_\alpha}g^{\mu\nu}A_\tau.$$

As the symbols of the indices of summation are immaterial, the first two terms of this expression cancel the second of the one above. If we then write $g^{\mu\nu}A_\mu = A^\nu$, so that A^ν like A_μ is an arbitrary vector, we finally obtain

$$\Phi = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x_\nu}(\sqrt{-g}A^\nu). \quad . \quad . \quad . \quad (35)$$

This scalar is the *divergence* of the contravariant vector A^ν.

The "Curl" of a Covariant Vector.—The second term in (26) is symmetrical in the indices μ and ν. Therefore $A_{\mu\nu} - A_{\nu\mu}$ is a particularly simply constructed antisymmetrical tensor. We obtain

$$B_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu} \quad . \quad . \quad . \quad (36)$$

Antisymmetrical Extension of a Six-vector.—Applying (27) to an antisymmetrical tensor of the second rank $A_{\mu\nu}$, forming in addition the two equations which arise through cyclic permutations of the indices, and adding these three equations, we obtain the tensor of the third rank

$$B_{\mu\nu\sigma} = A_{\mu\nu\sigma} + A_{\nu\sigma\mu} + A_{\sigma\mu\nu} = \frac{\partial A_{\mu\nu}}{\partial x_\sigma} + \frac{\partial A_{\nu\sigma}}{\partial x_\mu} + \frac{\partial A_{\sigma\mu}}{\partial x_\nu} \quad (37)$$

which it is easy to prove is antisymmetrical.

The Divergence of a Six-vector.—Taking the mixed product of (27) by $g^{\mu\alpha}g^{\nu\beta}$, we also obtain a tensor. The first term on the right-hand side of (27) may be written in the form

$$\frac{\partial}{\partial x_\sigma}(g^{\mu\alpha}g^{\nu\beta}A_{\mu\nu}) - g^{\mu\alpha}\frac{\partial g^{\nu\beta}}{\partial x_\sigma}A_{\mu\nu} - g^{\nu\beta}\frac{\partial g^{\mu\alpha}}{\partial x_\sigma}A_{\mu\nu}.$$

If we write $A_\sigma^{\alpha\beta}$ for $g^{\mu\alpha}g^{\nu\beta}A_{\mu\nu\sigma}$ and $A^{\alpha\beta}$ for $g^{\mu\alpha}g^{\nu\beta}A_{\mu\nu}$, and in the transformed first term replace

$$\frac{\partial g^{\nu\beta}}{\partial x_\sigma} \text{ and } \frac{\partial g^{\mu\alpha}}{\partial x_\sigma}$$

by their values as given by (34), there results from the right-hand side of (27) an expression consisting of seven terms, of which four cancel, and there remains

$$A^{\alpha\beta}_{\sigma} = \frac{\partial A^{\alpha\beta}}{\partial x_\sigma} + \{\sigma\gamma, \alpha\}A^{\gamma\beta} + \{\sigma\gamma, \beta\}A^{\alpha\gamma}. \quad . \quad (38)$$

This is the expression for the extension of a contravariant tensor of the second rank, and corresponding expressions for the extension of contravariant tensors of higher and lower rank may also be formed.

We note that in an analogous way we may also form the extension of a mixed tensor :—

$$A^{\alpha}_{\mu\sigma} = \frac{\partial A^{\alpha}_{\mu}}{\partial x_\sigma} - \{\sigma\mu, \tau\}A^{\alpha}_{\tau} + \{\sigma\tau, \alpha\}A^{\tau}_{\mu}. \quad . \quad (39)$$

On contracting (38) with respect to the indices β and σ (inner multiplication by δ^{σ}_{β}), we obtain the vector

$$A^{\alpha} = \frac{\partial A^{\alpha\beta}}{\partial x_\beta} + \{\beta\gamma, \beta\}A^{\alpha\gamma} + \{\beta\gamma, \alpha\}A^{\gamma\beta}.$$

On account of the symmetry of $\{\beta\gamma, \alpha\}$ with respect to the indices β and γ, the third term on the right-hand side vanishes, if $A^{\alpha\beta}$ is, as we will assume, an antisymmetrical tensor. The second term allows itself to be transformed in accordance with (29a). Thus we obtain

$$A^{\alpha} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}A^{\alpha\beta})}{\partial x_\beta}. \quad . \quad . \quad (40)$$

This is the expression for the divergence of a contravariant six-vector.

The Divergence of a Mixed Tensor of the Second Rank.— Contracting (39) with respect to the indices α and σ, and taking (29a) into consideration, we obtain

$$\sqrt{-g}A_{\mu} = \frac{\partial(\sqrt{-g}A^{\sigma}_{\mu})}{\partial x_\sigma} - \{\sigma\mu, \tau\}\sqrt{-g}A^{\sigma}_{\tau}. \quad (41)$$

If we introduce the contravariant tensor $A^{\rho\sigma} = g^{\rho\tau}A^{\sigma}_{\tau}$ in the last term, it assumes the form

$$-[\sigma\mu, \rho]\sqrt{-g}A^{\rho\sigma}.$$

If, further, the tensor $A^{\rho\sigma}$ is symmetrical, this reduces to

$$-\tfrac{1}{2}\sqrt{-g}\frac{\partial g_{\rho\sigma}}{\partial x_\mu}A^{\rho\sigma}.$$

Had we introduced, instead of $A^{\rho\sigma}$, the covariant tensor $A_{\rho\sigma} = g_{\rho\alpha}g_{\sigma\beta}A^{\alpha\beta}$, which is also symmetrical, the last term, by virtue of (31), would assume the form

$$\tfrac{1}{2}\sqrt{-g}\frac{\partial g^{\rho\sigma}}{\partial x_\mu}A_{\rho\sigma}.$$

In the case of symmetry in question, (41) may therefore be replaced by the two forms

$$\sqrt{-g}A_\mu = \frac{\partial(\sqrt{-g}A_\mu^\sigma)}{\partial x_\sigma} - \tfrac{1}{2}\frac{\partial g_{\rho\sigma}}{\partial x_\mu}\sqrt{-g}A^{\rho\sigma} \quad . \quad (41a)$$

$$\sqrt{-g}A_\mu = \frac{\partial(\sqrt{-g}A_\mu^\sigma)}{\partial x_\sigma} + \tfrac{1}{2}\frac{\partial g^{\rho\sigma}}{\partial x_\mu}\sqrt{-g}A_{\rho\sigma} \quad . \quad (41b)$$

which we have to employ later on.

§ 12. The Riemann-Christoffel Tensor

We now seek the tensor which can be obtained from the fundamental tensor *alone*, by differentiation. At first sight the solution seems obvious. We place the fundamental tensor of the $g_{\mu\nu}$ in (27) instead of any given tensor $A_{\mu\nu}$, and thus have a new tensor, namely, the extension of the fundamental tensor. But we easily convince ourselves that this extension vanishes identically. We reach our goal, however, in the following way. In (27) place

$$A_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \{\mu\nu, \rho\}A_\rho,$$

i.e. the extension of the four-vector A_μ. Then (with a somewhat different naming of the indices) we get the tensor of the third rank

$$A_{\mu\sigma\tau} = \frac{\partial^2 A_\mu}{\partial x_\sigma \partial x_\tau} - \{\mu\sigma, \rho\}\frac{\partial A_\rho}{\partial x_\tau} - \{\mu\tau, \rho\}\frac{\partial A_\rho}{\partial x_\sigma} - \{\sigma\tau, \rho\}\frac{\partial A_\mu}{\partial x_\rho}$$
$$+ \left[-\frac{\partial}{\partial x_\tau}\{\mu\sigma, \rho\} + \{\mu\tau, \alpha\}\{\alpha\sigma, \rho\} + \{\sigma\tau, \alpha\}\{\alpha\mu, \rho\}\right]A_\rho.$$

This expression suggests forming the tensor $A_{\mu\sigma\tau} - A_{\mu\tau\sigma}$. For, if we do so, the following terms of the expression for $A_{\mu\sigma\tau}$ cancel those of $A_{\mu\tau\sigma}$, the first, the fourth, and the member corresponding to the last term in square brackets; because all these are symmetrical in σ and τ. The same holds good for the sum of the second and third terms. Thus we obtain

$$A_{\mu\sigma\tau} - A_{\mu\tau\sigma} = B^{\rho}_{\mu\sigma\tau}A_{\rho} \quad . \quad . \quad . \quad (42)$$

where

$$B^{\rho}_{\mu\sigma\tau} = -\frac{\partial}{\partial x_{\tau}}\{\mu\sigma, \rho\} + \frac{\partial}{\partial x_{\sigma}}\{\mu\tau, \rho\} - \{\mu\sigma, a\}\{a\tau, \rho\}$$

$$+ \{\mu\tau, a\}\{a\sigma, \rho\} \quad (43)$$

The essential feature of the result is that on the right side of (42) the A_{ρ} occur alone, without their derivatives. From the tensor character of $A_{\mu\sigma\tau} - A_{\mu\tau\sigma}$ in conjunction with the fact that A_{ρ} is an arbitrary vector, it follows, by reason of § 7, that $B^{\rho}_{\mu\sigma\tau}$ is a tensor (the Riemann-Christoffel tensor).

The mathematical importance of this tensor is as follows: If the continuum is of such a nature that there is a co-ordinate system with reference to which the $g_{\mu\nu}$ are constants, then all the $B^{\rho}_{\mu\sigma\tau}$ vanish. If we choose any new system of co-ordinates in place of the original ones, the $g_{\mu\nu}$ referred thereto will not be constants, but in consequence of its tensor nature, the transformed components of $B^{\rho}_{\mu\sigma\tau}$ will still vanish in the new system. Thus the vanishing of the Riemann tensor is a necessary condition that, by an appropriate choice of the system of reference, the $g_{\mu\nu}$ may be constants. In our problem this corresponds to the case in which,* with a suitable choice of the system of reference, the special theory of relativity holds good for a *finite* region of the continuum.

Contracting (43) with respect to the indices τ and ρ we obtain the covariant tensor of second rank

* The mathematicians have proved that this is also a *sufficient* condition.

$$\left.\begin{aligned} G_{\mu\nu} &= B^{\rho}_{\mu\nu\rho} = R_{\mu\nu} + S_{\mu\nu} \\ \text{where} \qquad R_{\mu\nu} &= -\frac{\partial}{\partial x_{\alpha}}\{\mu\nu, \alpha\} + \{\mu\alpha, \beta\}\{\nu\beta, \alpha\} \\ S_{\mu\nu} &= \frac{\partial^2 \log\sqrt{-g}}{\partial x_{\mu}\partial x_{\nu}} - \{\mu\nu, \alpha\}\frac{\partial \log\sqrt{-g}}{\partial x_{\alpha}} \end{aligned}\right\} \quad (44)$$

Note on the Choice of Co-ordinates.—It has already been observed in § 8, in connexion with equation (18a), that the choice of co-ordinates may with advantage be made so that $\sqrt{-g} = 1$. A glance at the equations obtained in the last two sections shows that by such a choice the laws of formation of tensors undergo an important simplification. This applies particularly to $G_{\mu\nu}$, the tensor just developed, which plays a fundamental part in the theory to be set forth. For this specialization of the choice of co-ordinates brings about the vanishing of $S_{\mu\nu}$, so that the tensor $G_{\mu\nu}$ reduces to $R_{\mu\nu}$.

On this account I shall hereafter give all relations in the simplified form which this specialization of the choice of co-ordinates brings with it. It will then be an easy matter to revert to the *generally* covariant equations, if this seems desirable in a special case.

C. Theory of the Gravitational Field

§ 13. Equations of Motion of a Material Point in the Gravitational Field. Expression for the Field-components of Gravitation

A freely movable body not subjected to external forces moves, according to the special theory of relativity, in a straight line and uniformly. This is also the case, according to the general theory of relativity, for a part of four-dimensional space in which the system of co-ordinates K_0, may be, and is, so chosen that they have the special constant values given in (4).

If we consider precisely this movement from any chosen system of co-ordinates K_1, the body, observed from K_1, moves, according to the considerations in § 2, in a gravitational field. The law of motion with respect to K_1 results without difficulty from the following consideration. With respect to K_0 the law of motion corresponds to a four-dimensional straight line, i.e. to a geodetic line. Now since the geodetic line is defined independently of the system of reference, its equations will also be the equation of motion of the material point with respect to K_1. If we set

$$\Gamma^{\tau}_{\mu\nu} = -\{\mu\nu, \tau\} \quad . \quad . \quad . \quad (45)$$

the equation of the motion of the point with respect to K_1, becomes

$$\frac{d^2 x_\tau}{ds^2} = \Gamma^{\tau}_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} \quad . \quad . \quad . \quad (46)$$

We now make the assumption, which readily suggests itself, that this covariant system of equations also defines the motion of the point in the gravitational field in the case when there is no system of reference K_0, with respect to which the special theory of relativity holds good in a finite region. We have all the more justification for this assumption as (46) contains only *first* derivatives of the $g_{\mu\nu}$, between which even in the special case of the existence of K_0, no relations subsist.*

If the $\Gamma^{\tau}_{\mu\nu}$ vanish, then the point moves uniformly in a straight line. These quantities therefore condition the deviation of the motion from uniformity. They are the components of the gravitational field.

§ 14. The Field Equations of Gravitation in the Absence of Matter

We make a distinction hereafter between "gravitational field" and "matter" in this way, that we denote everything but the gravitational field as "matter." Our use of the word therefore includes not only matter in the ordinary sense, but the electromagnetic field as well.

Our next task is to find the field equations of gravitation in the absence of matter. Here we again apply the method

* It is only between the second (and first) derivatives that, by § 12, the relations $B^{\rho}_{\mu\sigma\tau} = 0$ subsist.

employed in the preceding paragraph in formulating the equations of motion of the material point. A special case in which the required equations must in any case be satisfied is that of the special theory of relativity, in which the $g_{\mu\nu}$ have certain constant values. Let this be the case in a certain finite space in relation to a definite system of co-ordinates K_0. Relatively to this system all the components of the Riemann tensor $B^{\rho}_{\mu\sigma\tau}$, defined in (43), vanish. For the space under consideration they then vanish, also in any other system of co-ordinates.

Thus the required equations of the matter-free gravitational field must in any case be satisfied if all $B^{\rho}_{\mu\sigma\tau}$ vanish. But this condition goes too far. For it is clear that, e.g., the gravitational field generated by a material point in its environment certainly cannot be "transformed away" by any choice of the system of co-ordinates, i.e. it cannot be transformed to the case of constant $g_{\mu\nu}$.

This prompts us to require for the matter-free gravitational field that the symmetrical tensor $G_{\mu\nu}$, derived from the tensor $B^{\rho}_{\mu\nu\tau}$, shall vanish. Thus we obtain ten equations for the ten quantities $g_{\mu\nu}$, which are satisfied in the special case of the vanishing of all $B^{\rho}_{\mu\nu\tau}$. With the choice which we have made of a system of co-ordinates, and taking (44) into consideration, the equations for the matter-free field are

$$\left.\begin{aligned} \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} &= 0 \\ \sqrt{-g} &= 1 \end{aligned}\right\} \quad . \quad . \quad . \quad (47)$$

It must be pointed out that there is only a minimum of arbitrariness in the choice of these equations. For besides $G_{\mu\nu}$ there is no tensor of second rank which is formed from the $g_{\mu\nu}$ and its derivatives, contains no derivations higher than second, and is linear in these derivatives.*

These equations, which proceed, by the method of pure

* Properly speaking, this can be affirmed only of the tensor

$$G_{\mu\nu} + \lambda g_{\mu\nu} g^{\alpha\beta} G_{\alpha\beta},$$

where λ is a constant. If, however, we set this tensor = 0, we come back again to the equations $G_{\mu\nu} = 0$.

mathematics, from the requirement of the general theory of relativity, give us, in combination with the equations of motion (46), to a first approximation Newton's law of attraction, and to a second approximation the explanation of the motion of the perihelion of the planet Mercury discovered by Leverrier (as it remains after corrections for perturbation have been made). These facts must, in my opinion, be taken as a convincing proof of the correctness of the theory.

§ 15. The Hamiltonian Function for the Gravitational Field. Laws of Momentum and Energy

To show that the field equations correspond to the laws of momentum and energy, it is most convenient to write them in the following Hamiltonian form :—

$$\left.\begin{aligned} &\delta\int \mathrm{H}\,d\tau = 0 \\ &\mathrm{H} = g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \\ &\sqrt{-g} = 1 \end{aligned}\right\} \quad . \quad . \quad . \quad (47a)$$

where, on the boundary of the finite four-dimensional region of integration which we have in view, the variations vanish.

We first have to show that the form (47a) is equivalent to the equations (47). For this purpose we regard H as a function of the $g^{\mu\nu}$ and the $g^{\mu\nu}_{\sigma}$ $(= \partial g^{\mu\nu}/\partial x_{\sigma})$.

Then in the first place

$$\begin{aligned} \delta\mathrm{H} &= \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\,\delta g^{\mu\nu} + 2g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\,\delta\Gamma^{\beta}_{\nu\alpha} \\ &= -\,\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\,\delta g^{\mu\nu} + 2\Gamma^{\alpha}_{\mu\beta}\,\delta(g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha}). \end{aligned}$$

But

$$\delta\left(g^{\mu\nu}\Gamma^{\beta}_{\nu\alpha}\right) = -\,\tfrac{1}{2}\delta\left[g^{\mu\nu}g^{\beta\lambda}\left(\frac{\partial g_{\nu\lambda}}{\partial x_{\alpha}} + \frac{\partial g_{\alpha\lambda}}{\partial x_{\nu}} - \frac{\partial g_{\alpha\nu}}{\partial x_{\lambda}}\right)\right].$$

The terms arising from the last two terms in round brackets are of different sign, and result from each other (since the denomination of the summation indices is immaterial) through interchange of the indices μ and β. They cancel each other in the expression for $\delta\mathrm{H}$, because they are multiplied by the quantity $\Gamma^{\alpha}_{\mu\beta}$, which is symmetrical with respect to the indices μ and β. Thus there remains only the first term in round brackets to be considered, so that, taking (31) into account, we obtain

$$\delta\mathrm{H} = -\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\delta g^{\mu\nu} + \Gamma^{\alpha}_{\mu\beta}\delta g^{\mu\beta}_{\alpha}.$$

Thus

$$\left.\begin{aligned} \frac{\partial\mathrm{H}}{\partial g^{\mu\nu}} &= -\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \\ \frac{\partial\mathrm{H}}{\partial g^{\mu\nu}_{\sigma}} &= \Gamma^{\sigma}_{\mu\nu} \end{aligned}\right\} \quad . \quad . \quad . \quad (48)$$

Carrying out the variation in (47a), we get in the first place

$$\frac{\partial}{\partial x_{\alpha}}\left(\frac{\partial\mathrm{H}}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial\mathrm{H}}{\partial g^{\mu\nu}} = 0, \quad . \quad . \quad . \quad (47\mathrm{b})$$

which, on account of (48), agrees with (47), as was to be proved.

If we multiply (47b) by $g^{\mu\nu}_{\sigma}$, then because

$$\frac{\partial g^{\mu\nu}_{\sigma}}{\partial x_{\alpha}} = \frac{\partial g^{\mu\nu}_{\alpha}}{\partial x_{\sigma}}$$

and, consequently,

$$g^{\mu\nu}_{\sigma}\frac{\partial}{\partial x_{\alpha}}\left(\frac{\partial\mathrm{H}}{\partial g^{\mu\nu}_{\alpha}}\right) = \frac{\partial}{\partial x_{\alpha}}\left(g^{\mu\nu}_{\sigma}\frac{\partial\mathrm{H}}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial\mathrm{H}}{\partial g^{\mu\nu}_{\alpha}}\frac{\partial g^{\mu\nu}_{\alpha}}{\partial x_{\sigma}},$$

we obtain the equation

$$\frac{\partial}{\partial x_{\alpha}}\left(g^{\mu\nu}_{\sigma}\frac{\partial\mathrm{H}}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial\mathrm{H}}{\partial x_{\sigma}} = 0$$

or *

$$\left.\begin{aligned} \frac{\partial t^{\alpha}_{\sigma}}{\partial x_{\alpha}} &= 0 \\ -2\kappa t^{\alpha}_{\sigma} &= g^{\mu\nu}_{\sigma}\frac{\partial\mathrm{H}}{\partial g^{\mu\nu}_{\alpha}} - \delta^{\alpha}_{\sigma}\mathrm{H} \end{aligned}\right\} \quad . \quad . \quad . \quad (49)$$

where, on account of (48), the second equation of (47), and (34)

$$\kappa t^{\alpha}_{\sigma} = \tfrac{1}{2}\delta^{\alpha}_{\sigma}g^{\mu\nu}\Gamma^{\lambda}_{\mu\beta}\Gamma^{\beta}_{\nu\lambda} - g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\sigma} \quad . \quad . \quad (50)$$

* The reason for the introduction of the factor -2κ will be apparent later.

It is to be noticed that t^α_σ is not a tensor; on the other hand (49) applies to all systems of co-ordinates for which $\sqrt{-g} = 1$. This equation expresses the law of conservation of momentum and of energy for the gravitational field. Actually the integration of this equation over a three-dimensional volume V yields the four equations

$$\frac{d}{dx_4}\int t^4_\sigma dV = \int (lt^1_\sigma + mt^2_\sigma + nt^3_\sigma)dS \quad . \quad (49a)$$

where l, m, n denote the direction-cosines of direction of the inward drawn normal at the element dS of the bounding surface (in the sense of Euclidean geometry). We recognize in this the expression of the laws of conservation in their usual form. The quantities t^α_σ we call the "energy components" of the gravitational field.

I will now give equations (47) in a third form, which is particularly useful for a vivid grasp of our subject. By multiplication of the field equations (47) by $g^{\nu\sigma}$ these are obtained in the "mixed" form. Note that

$$g^{\nu\sigma}\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\left(g^{\nu\sigma}\Gamma^\alpha_{\mu\nu}\right) - \frac{\partial g^{\nu\sigma}}{\partial x_\alpha}\Gamma^\alpha_{\mu\nu},$$

which quantity, by reason of (34), is equal to

$$\frac{\partial}{\partial x_\alpha}\left(g^{\nu\sigma}\Gamma^\alpha_{\mu\nu}\right) - g^{\nu\beta}\Gamma^\sigma_{\alpha\beta}\Gamma^\alpha_{\mu\nu} - g^{\sigma\beta}\Gamma^\nu_{\beta\alpha}\Gamma^\alpha_{\mu\nu},$$

or (with different symbols for the summation indices)

$$\frac{\partial}{\partial x_\alpha}\left(g^{\sigma\beta}\Gamma^\alpha_{\mu\beta}\right) - g^{\gamma\delta}\Gamma^\sigma_{\gamma\beta}\Gamma^\beta_{\delta\mu} - g^{\nu\sigma}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}.$$

The third term of this expression cancels with the one arising from the second term of the field equations (47); using relation (50), the second term may be written

$$\kappa(t^\sigma_\mu - \tfrac{1}{2}\delta^\sigma_\mu t),$$

where $t = t^\alpha_\alpha$. Thus instead of equations (47) we obtain

$$\left.\begin{aligned}\frac{\partial}{\partial x_\alpha}\left(g^{\sigma\beta}\Gamma^\alpha_{\mu\beta}\right) &= -\kappa(t^\sigma_\mu - \tfrac{1}{2}\delta^\sigma_\mu t)\\ \sqrt{-g} &= 1\end{aligned}\right\} \quad . \quad . \quad (51)$$

§ 16. The General Form of the Field Equations of Gravitation

The field equations for matter-free space formulated in § 15 are to be compared with the field equation

$$\nabla^2\phi = 0$$

of Newton's theory. We require the equation corresponding to Poisson's equation

$$\nabla^2\phi = 4\pi\kappa\rho,$$

where ρ denotes the density of matter.

The special theory of relativity has led to the conclusion that inert mass is nothing more or less than energy, which finds its complete mathematical expression in a symmetrical tensor of second rank, the energy-tensor. Thus in the general theory of relativity we must introduce a corresponding energy-tensor of matter T^α_σ, which, like the energy-components t_σ [equations (49) and (50)] of the gravitational field, will have mixed character, but will pertain to a symmetrical covariant tensor.*

The system of equation (51) shows how this energy-tensor (corresponding to the density ρ in Poisson's equation) is to be introduced into the field equations of gravitation. For if we consider a complete system (e.g. the solar system), the total mass of the system, and therefore its total gravitating action as well, will depend on the total energy of the system, and therefore on the ponderable energy together with the gravitational energy. This will allow itself to be expressed by introducing into (51), in place of the energy-components of the gravitational field alone, the sums $t^\sigma_\mu + T^\sigma_\mu$ of the energy-components of matter and of gravitational field. Thus instead of (51) we obtain the tensor equation

$$\left.\begin{aligned} \frac{\partial}{\partial x_\alpha}(g^{\sigma\beta}\Gamma^\alpha_{\mu\beta}) &= -\kappa[(t^\sigma_\mu + T^\sigma_\mu) - \tfrac{1}{2}\delta^\sigma_\mu(t + T)], \\ \sqrt{-g} &= 1 \end{aligned}\right\} \quad . \quad (52)$$

where we have set $T = T^\mu_\mu$ (Laue's scalar). These are the

* $g_{\alpha\tau}T^\alpha_\sigma = T_{\sigma\tau}$ and $g^{\sigma\beta}T^\alpha_\sigma = T^{\alpha\beta}$ are to be symmetrical tensors.

required general field equations of gravitation in mixed form. Working back from these, we have in place of (47)

$$\left.\begin{aligned}\frac{\partial}{\partial x_\alpha}\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} &= -\kappa(\mathrm{T}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathrm{T}),\\ \sqrt{-g} &= 1\end{aligned}\right\} \quad . \quad (53)$$

It must be admitted that this introduction of the energy-tensor of matter is not justified by the relativity postulate alone. For this reason we have here deduced it from the requirement that the energy of the gravitational field shall act gravitatively in the same way as any other kind of energy. But the strongest reason for the choice of these equations lies in their consequence, that the equations of conservation of momentum and energy, corresponding exactly to equations (49) and (49a), hold good for the components of the total energy. This will be shown in § 17.

§ 17. The Laws of Conservation in the General Case

Equation (52) may readily be transformed so that the second term on the right-hand side vanishes. Contract (52) with respect to the indices μ and σ, and after multiplying the resulting equation by $\frac{1}{2}\delta^\sigma_\mu$, subtract it from equation (52). This gives

$$\frac{\partial}{\partial x_\alpha}(g^{\sigma\beta}\Gamma^\alpha_{\mu\beta} - \tfrac{1}{2}\delta^\sigma_\mu g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta}) = -\kappa(t^\sigma_\mu + \mathrm{T}^\sigma_\mu). \quad . \quad (52\mathrm{a})$$

On this equation we perform the operation $\partial/\partial x_\sigma$. We have

$$\frac{\partial^2}{\partial x_\alpha \partial x_\sigma}\left(g^{\sigma}\Gamma^{\ \alpha}_{\beta\mu}\right) = -\frac{1}{2}\frac{\partial^2}{\partial x_\alpha \partial x_\sigma}\left[g^{\sigma\beta}g^{\alpha\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x_\beta} + \frac{\partial g_{\beta\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\beta}}{\partial x_\lambda}\right)\right].$$

The first and third terms of the round brackets yield contributions which cancel one another, as may be seen by interchanging, in the contribution of the third term, the summation indices α and σ on the one hand, and β and λ on the other. The second term may be re-modelled by (31), so that we have

$$\frac{\partial^2}{\partial x_\alpha \partial x_\sigma}\left(g^{\sigma\beta}\Gamma^\alpha_{\mu\beta}\right) = \frac{1}{2}\frac{\partial^3 g^{\alpha\beta}}{\partial x_\alpha \partial x_\beta \partial x_\mu} \quad . \quad . \quad (54)$$

The second term on the left-hand side of (52a) yields in the first place

$$-\frac{1}{2}\frac{\partial^2}{\partial x_\alpha \partial x_\mu}\left(g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta}\right)$$

or

$$\frac{1}{4}\frac{\partial^2}{\partial x_\alpha \partial x_\mu}\left[g^{\lambda\beta}g^{\alpha\delta}\left(\frac{\partial g_{\delta\lambda}}{\partial x_\beta} + \frac{\partial g_{\delta\beta}}{\partial x_\lambda} - \frac{\partial g_{\lambda\beta}}{\partial x_\delta}\right)\right].$$

With the choice of co-ordinates which we have made, the term deriving from the last term in round brackets disappears by reason of (29). The other two may be combined, and together, by (31), they give

$$-\frac{1}{2}\frac{\partial^3 g^{\alpha\beta}}{\partial x_\alpha \partial x_\beta \partial x_\mu},$$

so that in consideration of (54), we have the identity

$$\frac{\partial^2}{\partial x_\alpha \partial x_\sigma}\left(g^{\rho\beta}\Gamma_{\mu\beta} - \frac{1}{2}\delta^\sigma_\mu g^{\lambda\beta}\Gamma^\alpha_{\lambda\beta}\right) \equiv 0 \quad . \quad . \quad (55)$$

From (55) and (52a), it follows that

$$\frac{\partial(t^\sigma_\mu + T^\sigma_\mu)}{\partial x_\sigma} = 0. \quad . \quad . \quad . \quad (56)$$

Thus it results from our field equations of gravitation that the laws of conservation of momentum and energy are satisfied. This may be seen most easily from the consideration which leads to equation (49a); except that here, instead of the energy components t^σ of the gravitational field, we have to introduce the totality of the energy components of matter and gravitational field.

§ 18. The Laws of Momentum and Energy for Matter, as a Consequence of the Field Equations

Multiplying (53) by $\partial g^{\mu\nu}/\partial x_\sigma$, we obtain, by the method adopted in § 15, in view of the vanishing of

$$g_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x_\sigma},$$

the equation

$$\frac{\partial t^\alpha_\sigma}{\partial x_\alpha} + \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x_\sigma}T_{\mu\nu} = 0,$$

or, in view of (56),

$$\frac{\partial T^{\alpha}_{\sigma}}{\partial x_{\alpha}} + \tfrac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x_{\sigma}}T_{\mu\nu} = 0 \quad . \quad . \quad . \quad (57)$$

Comparison with (41b) shows that with the choice of system of co-ordinates which we have made, this equation predicates nothing more or less than the vanishing of divergence of the material energy-tensor. Physically, the occurrence of the second term on the left-hand side shows that laws of conservation of momentum and energy do not apply in the strict sense for matter alone, or else that they apply only when the $g^{\mu\nu}$ are constant, i.e. when the field intensities of gravitation vanish. This second term is an expression for momentum, and for energy, as transferred per unit of volume and time from the gravitational field to matter. This is brought out still more clearly by re-writing (57) in the sense of (41) as

$$\frac{\partial T^{\alpha}_{\sigma}}{\partial x_{\alpha}} = -\Gamma^{\beta}_{\alpha\sigma}T^{\alpha}_{\beta} \quad . \quad . \quad . \quad (57a)$$

The right side expresses the energetic effect of the gravitational field on matter.

Thus the field equations of gravitation contain four conditions which govern the course of material phenomena. They give the equations of material phenomena completely, if the latter is capable of being characterized by four differential equations independent of one another.*

D. Material Phenomena

The mathematical aids developed in part B enable us forthwith to generalize the physical laws of matter (hydrodynamics, Maxwell's electrodynamics), as they are formulated in the special theory of relativity, so that they will fit in with the general theory of relativity. When this is done, the general principle of relativity does not indeed afford us a further limitation of possibilities ; but it makes us acquainted with the influence of the gravitational field on all processes,

* On this question cf. H. Hilbert, Nachr. d. K. Gesellsch. d. Wiss. zu Gottingen, Math.-phys. Klasse, 1915, p. 3.

without our having to introduce any new hypothesis whatever.

Hence it comes about that it is not necessary to introduce definite assumptions as to the physical nature of matter (in the narrower sense). In particular it may remain an open question whether the theory of the electromagnetic field in conjunction with that of the gravitational field furnishes a sufficient basis for the theory of matter or not. The general postulate of relativity is unable on principle to tell us anything about this. It must remain to be seen, during the working out of the theory, whether electromagnetics and the doctrine of gravitation are able in collaboration to perform what the former by itself is unable to do.

§ 19. Euler's Equations for a Frictionless Adiabatic Fluid

Let p and ρ be two scalars, the former of which we call the "pressure," the latter the "density" of a fluid; and let an equation subsist between them. Let the contravariant symmetrical tensor

$$T^{\alpha\beta} = -g^{\alpha\beta}p + \rho\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} \quad . \quad . \quad . \quad (58)$$

be the contravariant energy-tensor of the fluid. To it belongs the covariant tensor

$$T_{\mu\nu} = -g_{\mu\nu}p + g_{\mu\alpha}g_{\mu\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}\rho, \quad . \quad . \quad (58a)$$

as well as the mixed tensor *

$$T_\sigma^\alpha = -\delta_\sigma^\alpha p + g_{\sigma\beta}\frac{dx_\beta}{ds}\frac{dx_\alpha}{ds}\rho \quad . \quad . \quad (58b)$$

Inserting the right-hand side of (58b) in (57a), we obtain the Eulerian hydrodynamical equations of the general theory of relativity. They give, in theory, a complete solution of the problem of motion, since the four equations (57a), together

* For an observer using a system of reference in the sense of the special theory of relativity for an infinitely small region, and moving with it, the density of energy T_4^4 equals $\rho - p$. This gives the definition of ρ. Thus ρ is not constant for an incompressible fluid.

with the given equation between p and ρ, and the equation

$$g_{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 1,$$

are sufficient, $g_{\alpha\beta}$ being given, to define the six unknowns

$$p, \rho, \frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_3}{ds}, \frac{dx_4}{ds}.$$

If the $g_{\mu\nu}$ are also unknown, the equations (53) are brought in. These are eleven equations for defining the ten functions $g_{\mu\nu}$, so that these functions appear over-defined. We must remember, however, that the equations (57a) are already contained in the equations (53), so that the latter represent only seven independent equations. There is good reason for this lack of definition, in that the wide freedom of the choice of co-ordinates causes the problem to remain mathematically undefined to such a degree that three of the functions of space may be chosen at will.*

§ 20. Maxwell's Electromagnetic Field Equations for Free Space

Let ϕ_ν be the components of a covariant vector—the electromagnetic potential vector. From them we form, in accordance with (36), the components $F_{\rho\sigma}$ of the covariant six-vector of the electromagnetic field, in accordance with the system of equations

$$F_{\rho\sigma} = \frac{\partial\phi_\rho}{\partial x_\sigma} - \frac{\partial\phi_\sigma}{\partial x_\rho} \quad . \quad . \quad . \quad (59)$$

It follows from (59) that the system of equations

$$\frac{\partial F_{\rho\sigma}}{\partial x_\tau} + \frac{\partial F_{\sigma\tau}}{\partial x_\rho} + \frac{\partial F_{\tau\rho}}{\partial x_\sigma} = 0 \quad . \quad . \quad . \quad (60)$$

is satisfied, its left side being, by (37), an antisymmetrical tensor of the third rank. System (60) thus contains essentially four equations which are written out as follows:—

* On the abandonment of the choice of co-ordinates with $g = -1$, there remain *four* functions of space with liberty of choice, corresponding to the four arbitrary functions at our disposal in the choice of co-ordinates.

$$\left.\begin{aligned}\frac{\partial F_{23}}{\partial x_4}+\frac{\partial F_{34}}{\partial x_2}+\frac{\partial F_{42}}{\partial x_3}&=0\\\frac{\partial F_{34}}{\partial x_1}+\frac{\partial F_{41}}{\partial x_3}+\frac{\partial F_{13}}{\partial x_4}&=0\\\frac{\partial F_{41}}{\partial x_2}+\frac{\partial F_{12}}{\partial x_4}+\frac{\partial F_{24}}{\partial x_1}&=0\\\frac{\partial F_{12}}{\partial x_3}+\frac{\partial F_{23}}{\partial x_1}+\frac{\partial F_{31}}{\partial x_2}&=0\end{aligned}\right\} \quad (60a)$$

This system corresponds to the second of Maxwell's systems of equations. We recognize this at once by setting

$$\left.\begin{aligned}F_{23}=H_x,\ F_{14}=E_x\\F_{31}=H_y,\ F_{24}=E_y\\F_{12}=H_z,\ F_{34}=E_z\end{aligned}\right\} \quad . \quad . \quad . \quad (61)$$

Then in place of (60a) we may set, in the usual notation of three-dimensional vector analysis,

$$\left.\begin{aligned}-\frac{\partial H}{\partial t}&=\text{curl } E\\\text{div } H&=0\end{aligned}\right\} \quad . \quad . \quad . \quad (60b)$$

We obtain Maxwell's first system by generalizing the form given by Minkowski. We introduce the contravariant six-vector associated with $F^{\alpha\beta}$

$$F^{\mu\nu}=g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta} \quad . \quad . \quad . \quad (62)$$

and also the contravariant vector J^μ of the density of the electric current. Then, taking (40) into consideration, the following equations will be invariant for any substitution whose invariant is unity (in agreement with the chosen co-ordinates) :—

$$\frac{\partial}{\partial x_\nu}F^{\mu\nu}=J^\mu \quad . \quad . \quad . \quad . \quad (63)$$

Let

$$\left.\begin{aligned}F^{23}=H'_x,\ F^{14}=-E'_x\\F^{31}=H'_y,\ F^{24}=-E'_y\\F^{12}=H'_z,\ F^{34}=-E'_z\end{aligned}\right\} \quad . \quad . \quad (64)$$

which quantities are equal to the quantities $H_x \ldots E^z$ in the special case of the restricted theory of relativity ; and in addition

$$J^1 = j_x,\ J^2 = j_y,\ J^3 = j_z,\ J^4 = \rho,$$

we obtain in place of (63)

$$\left.\begin{aligned}\frac{\partial E'}{\partial t} + j &= \operatorname{curl} H' \\ \operatorname{div} E' &= \rho\end{aligned}\right\} \quad . \quad . \quad . \quad (63a)$$

The equations (60), (62), and (63) thus form the generalization of Maxwell's field equations for free space, with the convention which we have established with respect to the choice of co-ordinates.

The Energy-components of the Electromagnetic Field.—We form the inner product

$$\kappa_\sigma = F_{\sigma\mu}J^\mu \quad . \quad . \quad . \quad . \quad . \quad (65)$$

By (61) its components, written in the three-dimensional manner, are

$$\left.\begin{aligned}\kappa_1 &= \rho E_x + [j \,.\, H]^x \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ \kappa_4 &= -(jE)\end{aligned}\right\} \quad . \quad . \quad . \quad (65a)$$

κ_σ is a covariant vector the components of which are equal to the negative momentum, or, respectively, the energy, which is transferred from the electric masses to the electromagnetic field per unit of time and volume. If the electric masses are free, that is, under the sole influence of the electromagnetic field, the covariant vector κ_σ will vanish.

To obtain the energy-components T_σ^ν of the electromagnetic field, we need only give to equation $\kappa_\sigma = 0$ the form of equation (57). From (63) and (65) we have in the first place

$$\kappa_\sigma = F_{\sigma\mu}\frac{\partial F^{\mu\nu}}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}(F_{\sigma\mu}F^{\mu\nu}) - F^{\mu\nu}\frac{\partial F_{\sigma\mu}}{\partial x_\nu}.$$

The second term of the right-hand side, by reason of (60), permits the transformation

$$F^{\mu\nu}\frac{\partial F_{\sigma\mu}}{\partial x_\nu} = -\tfrac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x_\sigma} = -\tfrac{1}{2}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}\frac{\partial F_{\mu\nu}}{\partial x_\sigma},$$

which latter expression may, for reasons of symmetry, also be written in the form

$$-\tfrac{1}{4}\left[g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}\frac{\partial F_{\mu\nu}}{\partial x_\sigma} + g^{\mu\alpha}g^{\nu\beta}\frac{\partial F_{\alpha\beta}}{\partial x_\sigma}F_{\mu\nu}\right].$$

But for this we may set

$$-\tfrac{1}{4}\frac{\partial}{\partial x_\sigma}(g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu}) + \tfrac{1}{4}F_{\alpha\beta}F_{\mu\nu}\frac{\partial}{\partial x_\sigma}(g^{\mu\alpha}g^{\nu\beta}).$$

The first of these terms is written more briefly

$$-\tfrac{1}{4}\frac{\partial}{\partial x_\sigma}(F^{\mu\nu}F_{\mu\nu});$$

the second, after the differentiation is carried out, and after some reduction, results in

$$-\tfrac{1}{2}F^{\mu\tau}F_{\mu\nu}g^{\nu\rho}\frac{\partial g_{\sigma\tau}}{\partial x_\sigma}.$$

Taking all three terms together we obtain the relation

$$\kappa_\sigma = \frac{\partial T_\sigma^\nu}{\partial x_\nu} - \tfrac{1}{2}g^{\tau\mu}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}T_\tau^\nu \quad . \quad . \quad . \quad (66)$$

where

$$T_\sigma^\nu = -F_{\sigma\alpha}F^{\nu\alpha} + \tfrac{1}{4}\delta_\sigma^\nu F_{\alpha\beta}F^{\alpha\beta}.$$

Equation (66), if κ_σ vanishes, is, on account of (30), equivalent to (57) or (57*a*) respectively. Therefore the T_σ^ν are the energy-components of the electromagnetic field. With the help of (61) and (64), it is easy to show that these energy-components of the electromagnetic field in the case of the special theory of relativity give the well-known Maxwell-Poynting expressions.

We have now deduced the general laws which are satisfied by the gravitational field and matter, by consistently using a system of co-ordinates for which $\sqrt{-g} = 1$. We have thereby achieved a considerable simplification of formulæ and calculations, without failing to comply with the requirement of general covariance; for we have drawn our equations from generally covariant equations by specializing the system of co-ordinates.

Still the question is not without a formal interest, whether with a correspondingly generalized definition of the energy-components of gravitational field and matter, even without specializing the system of co-ordinates, it is possible to formulate laws of conservation in the form of equation (56), and field equations of gravitation of the same nature as (52) or (52a), in such a manner that on the left we have a divergence (in the ordinary sense), and on the right the sum of the energy-components of matter and gravitation. I have found that in both cases this is actually so. But I do not think that the communication of my somewhat extensive reflexions on this subject would be worth while, because after all they do not give us anything that is materially new.

E

§ 21. Newton's Theory as a First Approximation

As has already been mentioned more than once, the special theory of relativity as a special case of the general theory is characterized by the $g_{\mu\nu}$ having the constant values (4). From what has already been said, this means complete neglect of the effects of gravitation. We arrive at a closer approximation to reality by considering the case where the $g_{\mu\nu}$ differ from the values of (4) by quantities which are small compared with 1, and neglecting small quantities of second and higher order. (First point of view of approximation.)

It is further to be assumed that in the space-time territory under consideration the $g_{\mu\nu}$ at spatial infinity, with a suitable choice of co-ordinates, tend toward the values (4); i.e. we are considering gravitational fields which may be regarded as generated exclusively by matter in the finite region.

It might be thought that these approximations must lead us to Newton's theory. But to that end we still need to approximate the fundamental equations from a second point of view. We give our attention to the motion of a material point in accordance with the equations (16). In the case of the special theory of relativity the components

$$\frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_3}{ds}$$

may take on any values. This signifies that any velocity

$$v = \sqrt{\left(\frac{dx_1}{dx_4}\right)^2 + \left(\frac{dx_2}{dx_4}\right)^2 + \left(\frac{dx_3}{dx_4}\right)^2}$$

may occur, which is less than the velocity of light *in vacuo.* If we restrict ourselves to the case which almost exclusively offers itself to our experience, of v being small as compared with the velocity of light, this denotes that the components

$$\frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_3}{ds}$$

are to be treated as small quantities, while dx_4/ds, to the second order of small quantities, is equal to one. (Second point of view of approximation.)

Now we remark that from the first point of view of approximation the magnitudes $\Gamma^{\tau}_{\mu\nu}$ are all small magnitudes of at least the first order. A glance at (46) thus shows that in this equation, from the second point of view of approximation, we have to consider only terms for which $\mu = \nu = 4$. Restricting ourselves to terms of lowest order we first obtain in place of (46) the equations

$$\frac{d^2x_\tau}{dt^2} = \Gamma^{\tau}_{44}$$

where we have set $ds = dx_4 = dt$; or with restriction to terms which from the first point of view of approximation are of first order :—

$$\frac{d^2x_\tau}{dt^2} = [44, \tau] \quad (\tau = 1, 2, 3)$$

$$\frac{d^2x_4}{dt^2} = -[44, 4].$$

If in addition we suppose the gravitational field to be a quasi-static field, by confining ourselves to the case where the motion of the matter generating the gravitational field is but slow (in comparison with the velocity of the propagation of light), we may neglect on the right-hand side differentiations with respect to the time in comparison with those with respect to the space co-ordinates, so that we have

$$\frac{d^2x_\tau}{dt^2} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_\tau} \quad (\tau = 1, 2, 3) \quad . \quad . \quad (67)$$

This is the equation of motion of the material point according to Newton's theory, in which $\frac{1}{2}g_{44}$ plays the part of the gravitational potential. What is remarkable in this result is that the component g_{44} of the fundamental tensor alone defines, to a first approximation, the motion of the material point.

We now turn to the field equations (53). Here we have to take into consideration that the energy-tensor of "matter" is almost exclusively defined by the density of matter in the narrower sense, i.e. by the second term of the right-hand side of (58) [or, respectively, (58a) or (58b)]. If we form the approximation in question, all the components vanish with the one exception of $T_{44} = \rho = T$. On the left-hand side of (53) the second term is a small quantity of second order; the first yields, to the approximation in question,

$$\frac{\partial}{\partial x_1}[\mu\nu, 1] + \frac{\partial}{\partial x_2}[\mu\nu, 2] + \frac{\partial}{\partial x_3}[\mu\nu, 3] - \frac{\partial}{\partial x_4}[\mu\nu, 4].$$

For $\mu = \nu = 4$, this gives, with the omission of terms differentiated with respect to time,

$$-\frac{1}{2}\left(\frac{\partial^2 g_{44}}{\partial x_1^2} + \frac{\partial^2 g_{44}}{\partial x_2^2} + \frac{\partial^2 g_{44}}{\partial x_3^2}\right) = -\frac{1}{2}\nabla^2 g_{44}.$$

The last of equations (53) thus yields

$$\nabla^2 g_{44} = \kappa\rho \quad . \quad . \quad . \quad . \quad . \quad (68)$$

The equations (67) and (68) together are equivalent to Newton's law of gravitation.

By (67) and (68) the expression for the gravitational potential becomes

$$-\frac{\kappa}{8\pi}\int\frac{\rho d\tau}{r} \quad . \quad . \quad . \quad . \quad (68a)$$

while Newton's theory, with the unit of time which we have chosen, gives

$$-\frac{K}{c^2}\int\frac{\rho d\tau}{r}$$

in which K denotes the constant $6{\cdot}7 \times 10^{-8}$, usually called the constant of gravitation. By comparison we obtain

$$\kappa = \frac{8\pi K}{c^2} = 1{\cdot}87 \times 10^{-27} \quad . \quad . \quad (69)$$

§ 22. Behaviour of Rods and Clocks in the Static Gravitational Field. Bending of Light-rays. Motion of the Perihelion of a Planetary Orbit

To arrive at Newton's theory as a first approximation we had to calculate only one component, g_{44}, of the ten $g_{\mu\nu}$ of the gravitational field, since this component alone enters into the first approximation, (67), of the equation for the motion of the material point in the gravitational field. From this, however, it is already apparent that other components of the $g_{\mu\nu}$ must differ from the values given in (4) by small quantities of the first order. This is required by the condition $g = -1$.

For a field-producing point mass at the origin of co-ordinates, we obtain, to the first approximation, the radially symmetrical solution

$$\left.\begin{aligned} g_{\rho\sigma} &= -\delta_{\rho\sigma} - a\frac{x_\rho x_\sigma}{r^3} \ (\rho, \sigma = 1, 2, 3) \\ g_{\rho 4} &= g_{4\rho} = 0 \qquad (\rho = 1, 2, 3) \\ g_{44} &= 1 - \frac{a}{r} \end{aligned}\right\} \quad . \quad (70)$$

where $\delta_{\rho\sigma}$ is 1 or 0, respectively, accordingly as $\rho = \sigma$ or $\rho \neq \sigma$, and r is the quantity $+\sqrt{x_1^2 + x_2^2 + x_3^2}$. On account of (68a)

$$a = \frac{\kappa M}{4\pi}, \quad . \quad . \quad . \quad . \quad (70a)$$

if M denotes the field-producing mass. It is easy to verify that the field equations (outside the mass) are satisfied to the first order of small quantities.

We now examine the influence exerted by the field of the mass M upon the metrical properties of space. The relation

$$ds^2 = g_{\mu\nu}dx_\mu dx_\nu.$$

always holds between the "locally" (§ 4) measured lengths and times ds on the one hand, and the differences of co-ordinates dx_ν on the other hand.

For a unit-measure of length laid "parallel" to the axis of x, for example, we should have to set $ds^2 = -1$; $dx_2 = dx_3 = dx_4 = 0$. Therefore $-1 = g_{11}dx_1^2$. If, in addition, the unit-measure lies on the axis of x, the first of equations (70) gives

$$g_{11} = -\left(1 + \frac{a}{r}\right).$$

From these two relations it follows that, correct to a first order of small quantities,

$$dx = 1 - \frac{a}{2r} \quad . \quad . \quad . \quad . \quad (71)$$

The unit measuring-rod thus appears a little shortened in relation to the system of co-ordinates by the presence of the gravitational field, if the rod is laid along a radius.

In an analogous manner we obtain the length of co-ordinates in tangential direction if, for example, we set

$ds^2 = -1$; $dx_1 = dx_3 = dx_4 = 0$; $x_1 = r$, $x_2 = x_3 = 0$.
The result is

$$-1 = g_{22}dx_2^2 = -dx_2^2 \quad . \quad . \quad . \quad (71a)$$

With the tangential position, therefore, the gravitational field of the point of màss has no influence on the length of a rod.

Thus Euclidean geometry does not hold even to a first approximation in the gravitational field, if we wish to take one and the same rod, independently of its place and orientation, as a realization of the same interval; although, to be sure, a glance at (70a) and (69) shows that the deviations to be expected are much too slight to be noticeable in measurements of the earth's surface.

Further, let us examine the rate of a unit clock, which is arranged to be at rest in a static gravitational field. Here we have for a clock period $ds = 1$; $dx_1 = dx_2 = dx_3 = 0$
Therefore

$$1 = g_{44}dx_4^2;$$

$$dx_4 = \frac{1}{\sqrt{g_{44}}} = \frac{1}{\sqrt{(1 + (g_{44} - 1))}} = 1 - \tfrac{1}{2}(g_{44} - 1)$$

or

$$dx_4 = 1 + \frac{\kappa}{8\pi}\int \rho\frac{d\tau}{r} \quad . \quad . \quad . \quad (72)$$

Thus the clock goes more slowly if set up in the neighbourhood of ponderable masses. From this it follows that the spectral lines of light reaching us from the surface of large stars must appear displaced towards the red end of the spectrum.*

We now examine the course of light-rays in the static gravitational field. By the special theory of relativity the velocity of light is given by the equation

$$- dx_1^2 - dx_2 - dx_3^2 + dx_4^2 = 0$$

and therefore by the general theory of relativity by the equation

$$ds^2 = g_{\mu\nu}dx_\mu dx_\nu = 0 \quad . \quad . \quad . \quad (73)$$

If the direction, i.e. the ratio $dx_1 : dx_2 : dx_3$ is given, equation (73) gives the quantities

$$\frac{dx_1}{dx_4}, \frac{dx_2}{dx_4}, \frac{dx_3}{dx_4}$$

and accordingly the velocity

$$\sqrt{\left(\frac{dx_1}{dx_4}\right)^2 + \left(\frac{dx_2}{dx_4}\right)^2 + \left(\frac{dx_3}{dx_4}\right)^2} = \gamma$$

defined in the sense of Euclidean geometry. We easily recognize that the course of the light-rays must be bent with regard to the system of co-ordinates, if the $g_{\mu\nu}$ are not constant. If n is a direction perpendicular to the propagation of light, the Huyghens principle shows that the light-ray, envisaged in the plane (γ, n), has the curvature $-\partial\gamma/\partial n$.

We examine the curvature undergone by a ray of light passing by a mass M at the distance Δ. If we choose the system of co-ordinates in agreement with the accompanying diagram, the total bending of the ray (calculated positively if

* According to E. Freundlich, spectroscopical observations on fixed stars of certain types indicate the existence of an effect of this kind, but a crucial test of this consequence has not yet been made.

concave towards the origin) is given in sufficient approximation by

$$B = \int_{-\infty}^{+\infty} \frac{\partial \gamma}{\partial x_1} dx_2,$$

while (73) and (70) give

$$\gamma = \sqrt{\left(-\frac{g_{44}}{g_{22}}\right)} = 1 - \frac{a}{2r}\left(1 + \frac{x_2^2}{r^2}\right).$$

Carrying out the calculation, this gives

$$B = \frac{2a}{\Delta} = \frac{\kappa M}{2\pi\Delta}. \quad . \quad . \quad . \quad (74)$$

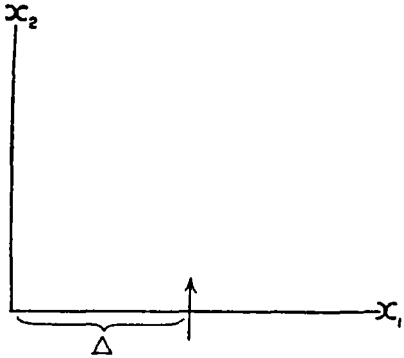

Fig. 8.

According to this, a ray of light going past the sun undergoes a deflexion of 1·7″; and a ray going past the planet Jupiter a deflexion of about ·02″.

If we calculate the gravitational field to a higher degree of approximation, and likewise with corresponding accuracy the orbital motion of a material point of relatively infinitely small mass, we find a deviation of the following kind from the Kepler-Newton laws of planetary motion. The orbital ellipse of a planet undergoes a slow rotation, in the direction of motion, of amount

$$\epsilon = 24\pi^3 \frac{a^2}{T^2 c^2 (1 - e^2)} \quad . \quad . \quad . \quad (75)$$

per revolution. In this formula a denotes the major semi-axis, c the velocity of light in the usual measurement, e the eccentricity, T the time of revolution in seconds.*

Calculation gives for the planet Mercury a rotation of the orbit of 43″ per century, corresponding exactly to astronomical observation (Leverrier); for the astronomers have discovered in the motion of the perihelion of this planet, after allowing for disturbances by other planets, an inexplicable remainder of this magnitude.

*For the calculation I refer to the original papers: A. Einstein, Sitzungsber. d. Preuss. Akad. d. Wiss., 1915, p. 831; K. Schwarzschild, *ibid*, 1916, p. 189.

HAMILTON'S PRINCIPLE AND THE GENERAL THEORY OF RELATIVITY

BY

A. EINSTEIN

Translated from "Hamiltonsches Princip und allgemeine Relativitätstheorie," Sitzungsberichte der Preussischen Akad. d. Wissenschaften, 1916.

HAMILTON'S PRINCIPLE AND THE GENERAL THEORY OF RELATIVITY

By A. EINSTEIN

THE general theory of relativity has recently been given in a particularly clear form by H. A. Lorentz and D. Hilbert,* who have deduced its equations from one single principle of variation. The same thing will be done in the present paper. But my purpose here is to present the fundamental connexions in as perspicuous a manner as possible, and in as general terms as is permissible from the point of view of the general theory of relativity. In particular we shall make as few specializing assumptions as possible, in marked contrast to Hilbert's treatment of the subject. On the other hand, in antithesis to my own most recent treatment of the subject, there is to be complete liberty in the choice of the system of co-ordinates.

§ 1. The Principle of Variation and the Field-equations of Gravitation and Matter

Let the gravitational field be described as usual by the tensor † of the $g_{\mu\nu}$ (or the $g^{\mu\nu}$); and matter, including the electromagnetic field, by any number of space-time functions $q_{(\rho)}$. How these functions may be characterized in the theory of invariants does not concern us. Further, let $\mathfrak{H}$ be a function of the

$$g^{\mu\nu}, g^{\mu\nu}_{\sigma}\left(=\frac{\partial g^{\mu\nu}}{\partial x_\sigma}\right) \text{ and } g^{\mu\nu}_{\sigma\tau}\left(=\frac{\partial^2 g^{\mu\nu}}{\partial x_\sigma \partial x_\tau}\right), \text{the } q_{(\rho)} \text{ and } q_{(\rho)\alpha}\left(=\frac{\partial q_{(\rho)}}{\partial x_\alpha}\right).$$

* Four papers by Lorentz in the Publications of the Koninkl. Akad. van Wetensch. te Amsterdam, 1915 and 1916; D. Hilbert, Gottinger Nachr., 1915, Part 3.

† No use is made for the present of the tensor character of the $g_{\mu\nu}$.

The principle of variation

$$\delta\int \mathfrak{H} d\tau = 0 \quad . \quad . \quad . \quad . \quad (1)$$

then gives us as many differential equations as there are functions $g_{\mu\nu}$ and $q_{(\rho)}$ to be defined, if the $g^{\mu\nu}$ and $q_{(\rho)}$ are varied independently of one another, and in such a way that at the limits of integration the $\delta q_{(\rho)}$, $\delta g^{\mu\nu}$, and $\frac{\partial}{\partial x_\sigma}(\delta g_{\mu\nu})$ all vanish.

We will now assume that $\mathfrak{H}$ is linear in the $g_{\sigma\tau}$, and that the coefficients of the $g^{\mu\nu}_{\sigma\tau}$ depend only on the $g^{\mu\nu}$. We may then replace the principle of variation (1) by one which is more convenient for us. For by appropriate partial integration we obtain

$$\int \mathfrak{H} d\tau = \int \mathfrak{H}^* d\tau + \mathrm{F} \quad . \quad . \quad . \quad (2)$$

where F denotes an integral over the boundary of the domain in question, and $\mathfrak{H}^*$ depends only on the $g^{\mu\nu}$, $g^{\mu\nu}_{\sigma}$, $q_{(\rho)}$, $q_{(\rho)\alpha}$, and no longer on the $g^{\mu\nu}_{\sigma\tau}$. From (2) we obtain, for such variations as are of interest to us,

$$\delta\int \mathfrak{H} d\tau = \delta\int \mathfrak{H}^* d\tau, \quad . \quad . \quad . \quad (3)$$

so that we may replace our principle of variation (1) by the more convenient form

$$\delta\int \mathfrak{H}^* d\tau = 0. \quad . \quad . \quad . \quad (1a)$$

By carrying out the variation of the $g^{\mu\nu}$ and the $q_{(\rho)}$ we obtain, as field-equations of gravitation and matter, the equations †

$$\frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathfrak{H}^*}{\partial g^{\mu\nu}_\alpha}\right) - \frac{\partial \mathfrak{H}^*}{\partial g^{\mu\nu}} = 0 \quad . \quad . \quad . \quad (4)$$

$$\frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathfrak{H}^*}{\partial q_{(\rho)\alpha}}\right) - \frac{\partial \mathfrak{H}^*}{\partial q_{(\rho)}} = 0 \quad . \quad . \quad . \quad (5)$$

† For brevity the summation symbols are omitted in the formulæ. Indices occurring twice in a term are always to be taken as summed. Thus in (4), for example, $\frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathfrak{H}^*}{\partial g^{\mu\nu}_\alpha}\right)$ denotes the term $\sum_\alpha \frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathfrak{H}^*}{\partial g^{\mu\nu}_\alpha}\right)$.

§ 2. Separate Existence of the Gravitational Field

If we make no restrictive assumption as to the manner in which $\mathfrak{H}$ depends on the $g^{\mu\nu}$, $g^{\mu\nu}_{\sigma}$, $g^{\mu\nu}_{\sigma\tau}$, $q_{(\rho)}$, $q_{(\rho)\alpha}$, the energy-components cannot be divided into two parts, one belonging to the gravitational field, the other to matter. To ensure this feature of the theory, we make the following assumption

$$\mathfrak{H} = \mathfrak{G} + \mathfrak{M} \quad . \quad . \quad . \quad . \quad (6)$$

where $\mathfrak{G}$ is to depend only on the $g^{\mu\nu}$, $g^{\mu\nu}_{\sigma}$, $g^{\mu\nu}_{\sigma\tau}$, and $\mathfrak{M}$ only on $g^{\mu\nu}$, $q_{(\rho)}$, $q_{(\rho)\alpha}$. Equations (4), (4a) then assume the form

$$\frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathfrak{G}^*}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial \mathfrak{G}^*}{\partial g^{\mu\nu}} = \frac{\partial \mathfrak{M}}{\partial g^{\mu\nu}} \quad . \quad . \quad . \quad (7)$$

$$\frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathfrak{M}}{\partial q_{(\rho)\alpha}}\right) - \frac{\partial \mathfrak{M}}{\partial q_{(\rho)}} = 0 \quad . \quad . \quad . \quad (8)$$

Here $\mathfrak{G}^*$ stands in the same relation to $\mathfrak{G}$ as $\mathfrak{H}^*$ to $\mathfrak{H}$.

It is to be noted carefully that equations (8) or (5) would have to give way to others, if we were to assume $\mathfrak{M}$ or $\mathfrak{H}$ to be also dependent on derivatives of the $q_{(\rho)}$ of order higher than the first. Likewise it might be imaginable that the $q_{(\rho)}$ would have to be taken, not as independent of one another, but as connected by conditional equations. All this is of no importance for the following developments, as these are based solely on the equations (7), which have been found by varying our integral with respect to the $g^{\mu\nu}$.

§ 3. Properties of the Field Equations of Gravitation Conditioned by the Theory of Invariants

We now introduce the assumption that

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \quad . \quad . \quad . \quad . \quad (9)$$

is an invariant. This determines the transformational character of the $g_{\mu\nu}$. As to the transformational character of the $q_{(\rho)}$, which describe matter, we make no supposition. On the other hand, let the functions $\mathrm{H} = \frac{\mathfrak{H}}{\sqrt{-g}}$, as well as

$G = \frac{\mathfrak{G}}{\sqrt{-g}}$, and $M = \frac{\mathfrak{M}}{\sqrt{-g}}$, be invariants in relation to any substitutions of space-time co-ordinates. From these assumptions follows the general covariance of the equations (7) and (8), deduced from (1). It further follows that G (apart from a constant factor) must be equal to the scalar of Riemann's tensor of curvature; because there is no other invariant with the properties required for G.† Thereby $\mathfrak{G}^*$ is also perfectly determined, and consequently the left-hand side of field equation (7) as well.‡

From the general postulate of relativity there follow certain properties of the function $\mathfrak{G}^*$ which we shall now deduce. For this purpose we carry through an infinitesimal transformation of the co-ordinates, by setting

$$x'_\nu = x_\nu + \Delta x_\nu \quad . \quad . \quad . \quad . \quad (10)$$

where the Δx_ν are arbitrary, infinitely small functions of the co-ordinates, and x'_ν are the co-ordinates, in the new system, of the world-point having the co-ordinates x_ν in the original system. As for the co-ordinates, so too for any other magnitude ψ, a law of transformation holds good, of the type

$$\psi' = \psi + \Delta\psi,$$

where $\Delta\psi$ must always be expressible by the Δx_ν. From the covariant property of the $g^{\mu\nu}$ we easily deduce for the $g^{\mu\nu}$ and $g^{\mu\nu}_\sigma$ the laws of transformation

$$\Delta g^{\mu\nu} = g^{\mu\alpha}\frac{\partial(\Delta x_\nu)}{\partial x_\alpha} + g^{\nu\alpha}\frac{\partial(\Delta x_\mu)}{\partial x_\alpha} \quad . \quad . \quad (11)$$

$$\Delta g^{\mu\nu}_\sigma = \frac{\partial(\Delta g^{\mu\nu})}{\partial x_\sigma} - g^{\mu\nu}_\alpha\frac{\partial(\Delta x_\alpha)}{\partial x_\sigma} \quad . \quad . \quad . \quad (12)$$

Since $\mathfrak{G}^*$ depends only on the $g^{\mu\nu}$ and $g^{\mu\nu}_\sigma$, it is possible, with the help of (11) and (12), to calculate $\Delta\mathfrak{G}^*$. We thus obtain the equation

$$\sqrt{-g}\,\Delta\left(\frac{\mathfrak{G}^*}{\sqrt{-g}}\right) = \mathfrak{S}^\nu_\sigma\frac{\partial(\Delta x_\sigma)}{\partial x_\nu} + 2\frac{\partial\mathfrak{G}^*}{\partial g^{\mu\sigma}_\alpha}g^{\mu\nu}\frac{\partial^2\Delta x_\sigma}{\partial x_\nu\partial x_\alpha}, \quad (13)$$

† Herein is to be found the reason why the general postulate of relativity leads to a very definite theory of gravitation.

‡ By performing partial integration we obtain

$$\mathfrak{G}^* = \sqrt{-g}\, g^{\mu\nu}[\{\mu\alpha, \beta\}\{\nu\beta, \alpha\} - \{\mu\nu, \alpha\}\{\alpha\beta, \beta\}].$$

where for brevity we have set

$$\mathrm{S}_\sigma^\nu = 2\frac{\partial \mathfrak{G}^*}{\partial g^{\mu\sigma}}g^{\mu\nu} + 2\frac{\partial \mathfrak{G}^*}{\partial g_\alpha^{\mu\sigma}}g_\alpha^{\mu\nu} + \mathfrak{G}^*\delta_\sigma^\nu - \frac{\partial \mathfrak{G}^*}{\partial g_\nu^{\mu\alpha}}g_\sigma^{\mu\alpha}. \qquad (14)$$

From these two equations we draw two inferences which are important for what follows. We know that $\frac{\mathfrak{G}}{\sqrt{-g}}$ is an invariant with respect to any substitution, but we do not know this of $\frac{\mathfrak{G}^*}{\sqrt{-g}}$. It is easy to demonstrate, however, that the latter quantity is an invariant with respect to any *linear* substitutions of the co-ordinates. Hence it follows that the right side of (13) must always vanish if all $\frac{\partial^2 \Delta x_\sigma}{\partial x_\nu \partial x_\alpha}$ vanish. Consequently $\mathfrak{G}^*$ must satisfy the identity

$$\mathrm{S}_\sigma^\nu \equiv 0 \quad . \quad . \quad . \quad . \quad (15)$$

If, further, we choose the Δx_ν so that they differ from zero only in the interior of a given domain, but in infinitesimal proximity to the boundary they vanish, then, with the transformation in question, the value of the boundary integral occurring in equation (2) does not change. Therefore $\Delta \mathrm{F} = 0$, and, in consequence,†

$$\Delta\int \mathfrak{G} d\tau = \Delta\int \mathfrak{G}^* d\tau.$$

But the left-hand side of the equation must vanish, since both $\frac{\mathfrak{G}}{\sqrt{-g}}$ and $\sqrt{-g}\,d\tau$ are invariants. Consequently the right-hand side also vanishes. Thus, taking (14), (15), and (16) into consideration, we obtain, in the first place, the equation

$$\int \frac{\partial \mathfrak{G}^*}{\partial g_\alpha^{\mu\sigma}} g^{\mu\nu} \frac{\partial^2 (\Delta x_\sigma)}{\partial x_\nu \partial x_\alpha} d\tau = 0 \quad . \quad . \quad . \quad (16)$$

Transforming this equation by two partial integrations, and having regard to the liberty of choice of the Δx_σ, we obtain

† By the introduction of the quantities $\mathfrak{G}$ and $\mathfrak{G}^*$ instead of $\mathfrak{H}$ and $\mathfrak{H}^*$.

the identity

$$\frac{\partial^2}{\partial x_\nu \partial x_\alpha}\left(g^{\mu\nu}\frac{\partial \mathfrak{G}^*}{\partial g^{\mu\sigma}_\alpha}\right) \equiv 0 \quad . \quad . \quad . \quad (17)$$

From the two identities (16) and (17), which result from the invariance of $\frac{\mathfrak{G}}{\sqrt{-g}}$, and therefore from the postulate of general relativity, we now have to draw conclusions.

We first transform the field equations (7) of gravitation by mixed multiplication by $g^{\mu\sigma}$. We then obtain (by interchanging the indices σ and ν), as equivalents of the field equations (7), the equations

$$\frac{\partial}{\partial x_\alpha}\left(g^{\mu\nu}\frac{\partial \mathfrak{G}^*}{\partial g^{\mu\sigma}_\alpha}\right) = -(\mathfrak{T}^\nu_\sigma + \mathfrak{t}^\nu_\sigma) \quad . \quad . \quad (18)$$

where we have set

$$\mathfrak{T}^\nu_\sigma = -\frac{\partial \mathfrak{M}}{\partial g^{\mu\sigma}}g^{\mu\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\mathfrak{t}^\nu_\sigma = -\left(\frac{\partial \mathfrak{G}^*}{\partial g^{\mu\sigma}_\alpha}g^{\mu\nu}_\alpha + \frac{\partial \mathfrak{G}^*}{\partial g^{\mu\sigma}}g^{\mu\nu}\right) = \tfrac{1}{2}\left(\mathfrak{G}^*\delta^\nu_\sigma - \frac{\partial \mathfrak{G}^*}{\partial g^{\mu\alpha}_\nu}g^{\mu\alpha}_\sigma\right) \quad (20)$$

The last expression for $\mathfrak{t}^\nu_\sigma$ is vindicated by (14) and (15). By differentiation of (18) with respect to x_ν, and summation for ν, there follows, in view of (17),

$$\frac{\partial}{\partial x_\nu}(\mathfrak{T}^\nu_\sigma + \mathfrak{t}^\nu_\sigma) = 0 \quad . \quad . \quad . \quad (21)$$

Equation (21) expresses the conservation of momentum and energy. We call $\mathfrak{T}^\nu_\sigma$ the components of the energy of matter, $\mathfrak{t}^\nu_\sigma$ the components of the energy of the gravitational field.

Having regard to (20), there follows from the field equations (7) of gravitation, by multiplication by $g^{\mu\nu}_\sigma$ and summation with respect to μ and ν,

$$\frac{\partial \mathfrak{t}^\nu_\sigma}{\partial x_\nu} + \tfrac{1}{2}g^{\mu\nu}_\sigma\frac{\partial \mathfrak{M}}{\partial g^{\mu\nu}} = 0,$$

or, in view of (19) and (21),

$$\frac{\partial \mathfrak{T}_\sigma^\nu}{\partial x_\nu} + \tfrac{1}{2} g_\sigma^{\mu\nu} \mathfrak{T}_{\mu\nu} = 0 \qquad \ldots \qquad (22)$$

where $\mathfrak{T}_{\mu\nu}$ denotes the quantities $g_{\nu\sigma}\mathfrak{T}_\mu^\sigma$. These are four equations which the energy-components of matter have to satisfy.

It is to be emphasized that the (generally covariant) laws of conservation (21) and (22) are deduced from the field equations (7) of gravitation, in combination with the postulate of general covariance (relativity) *alone*, without using the field equations (8) for material phenomena.

COSMOLOGICAL CONSIDERATIONS ON THE GENERAL THEORY OF RELATIVITY

BY

A. EINSTEIN

Translated from "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie," Sitzungsberichte der Preussischen Akad. d. Wissenschaften, 1917.

COSMOLOGICAL CONSIDERATIONS ON THE GENERAL THEORY OF RELATIVITY

By A. EINSTEIN

IT is well known that Poisson's equation

$$\nabla^2\phi = 4\pi K\rho \quad . \quad . \quad . \quad . \quad (1)$$

in combination with the equations of motion of a material point is not as yet a perfect substitute for Newton's theory of action at a distance. There is still to be taken into account the condition that at spatial infinity the potential ϕ tends toward a fixed limiting value. There is an analogous state of things in the theory of gravitation in general relativity. Here, too, we must supplement the differential equations by limiting conditions at spatial infinity, if we really have to regard the universe as being of infinite spatial extent.

In my treatment of the planetary problem I chose these limiting conditions in the form of the following assumption: it is possible to select a system of reference so that at spatial infinity all the gravitational potentials $g_{\mu\nu}$ become constant. But it is by no means evident *a priori* that we may lay down the same limiting conditions when we wish to take larger portions of the physical universe into consideration. In the following pages the reflexions will be given which, up to the present, I have made on this fundamentally important question.

§ 1. The Newtonian Theory

It is well known that Newton's limiting condition of the constant limit for ϕ at spatial infinity leads to the view that the density of matter becomes zero at infinity. For we imagine that there may be a place in universal space round about which the gravitational field of matter, viewed on a large scale, possesses spherical symmetry. It then follows from Poisson's equation that, in order that ϕ may tend to a limit at infinity, the mean density ρ must decrease toward zero more rapidly than $1/r^2$ as the distance r from the centre increases.* In this sense, therefore, the universe according to Newton is finite, although it may possess an infinitely great total mass.

From this it follows in the first place that the radiation emitted by the heavenly bodies will, in part, leave the Newtonian system of the universe, passing radially outwards, to become ineffective and lost in the infinite. May not entire heavenly bodies fare likewise? It is hardly possible to give a negative answer to this question. For it follows from the assumption of a finite limit for ϕ at spatial infinity that a heavenly body with finite kinetic energy is able to reach spatial infinity by overcoming the Newtonian forces of attraction. By statistical mechanics this case must occur from time to time, as long as the total energy of the stellar system—transferred to one single star—is great enough to send that star on its journey to infinity, whence it never can return.

We might try to avoid this peculiar difficulty by assuming a very high value for the limiting potential at infinity. That would be a possible way, if the value of the gravitational potential were not itself necessarily conditioned by the heavenly bodies. The truth is that we are compelled to regard the occurrence of any great differences of potential of the gravitational field as contradicting the facts. These differences must really be of so low an order of magnitude that the stellar velocities generated by them do not exceed the velocities actually observed.

If we apply Boltzmann's law of distribution for gas molecules to the stars, by comparing the stellar system with a gas in thermal equilibrium, we find that the Newtonian stellar system cannot exist at all. For there is a finite ratio of densities corresponding to the finite difference of potential between the centre and spatial infinity. A vanishing of the density at infinity thus implies a vanishing of the density at the centre.

* ρ is the mean density of matter, calculated for a region which is large as compared with the distance between neighbouring fixed stars, but small in comparison with the dimensions of the whole stellar system.

It seems hardly possible to surmount these difficulties on the basis of the Newtonian theory. We may ask ourselves the question whether they can be removed by a modification of the Newtonian theory. First of all we will indicate a method which does not in itself claim to be taken seriously; it merely serves as a foil for what is to follow. In place of Poisson's equation we write

$$\nabla^2\phi - \lambda\phi = 4\pi\kappa\rho \quad . \quad . \quad . \quad (2)$$

where λ denotes a universal constant. If ρ_o be the uniform density of a distribution of mass, then

$$\phi = -\frac{4\pi\kappa}{\lambda}\rho_0 \quad . \quad . \quad . \quad . \quad (3)$$

is a solution of equation (2). This solution would correspond to the case in which the matter of the fixed stars was distributed uniformly through space, if the density ρ_o is equal to the actual mean density of the matter in the universe. The solution then corresponds to an infinite extension of the central space, filled uniformly with matter. If, without making any change in the mean density, we imagine matter to be non-uniformly distributed locally, there will be, over and above the ϕ with the constant value of equation (3), an additional ϕ, which in the neighbourhood of denser masses will so much the more resemble the Newtonian field as $\lambda\phi$ is smaller in comparison with $4\pi\kappa\rho$.

A universe so constituted would have, with respect to its gravitational field, no centre. A decrease of density in spatial infinity would not have to be assumed, but both the mean potential and mean density would remain constant to infinity. The conflict with statistical mechanics which we found in the case of the Newtonian theory is not repeated. With a definite but extremely small density, matter is in equilibrium, without any internal material forces (pressures) being required to maintain equilibrium.

§ 2. The Boundary Conditions According to the General Theory of Relativity

In the present paragraph I shall conduct the reader over the road that I have myself travelled, rather a rough and winding road, because otherwise I cannot hope that he will take much interest in the result at the end of the journey. The conclusion I shall arrive at is that the field equations of gravitation which I have championed hitherto still need a slight modification, so that on the basis of the general theory of relativity those fundamental difficulties may be avoided which have been set forth in § 1 as confronting the Newtonian theory. This modification corresponds perfectly to the transition from Poisson's equation (1) to equation (2) of § 1. We finally infer that boundary conditions in spatial infinity fall away altogether, because the universal continuum in respect of its spatial dimensions is to be viewed as a self-contained continuum of finite spatial (three-dimensional) volume.

The opinion which I entertained until recently, as to the limiting conditions to be laid down in spatial infinity, took its stand on the following considerations. In a consistent theory of relativity there can be no inertia *relatively to "space,"* but only an inertia of masses *relatively to one another*. If, therefore, I have a mass at a sufficient distance from all other masses in the universe, its inertia must fall to zero. We will try to formulate this condition mathematically.

According to the general theory of relativity the negative momentum is given by the first three components, the energy by the last component of the covariant tensor multiplied by $\sqrt{-g}$

$$m\sqrt{-g}\quad g_{\mu\alpha}\frac{dx_\alpha}{ds} \quad . \quad . \quad . \quad . \quad (4)$$

where, as always, we set

$$ds^2 = g_{\mu\nu}dx_\mu dx_\nu \quad . \quad . \quad . \quad . \quad (5)$$

In the particularly perspicuous case of the possibility of choosing the system of co-ordinates so that the gravitational field at every point is spatially isotropic, we have more simply

$$ds^2 = -A(dx_1^2 + dx_2^2 + dx_3^2) + Bdx_4^2.$$

If, moreover, at the same time

$$\sqrt{-g} = 1 = \sqrt{A^3B}$$

we obtain from (4), to a first approximation for small velocities,

$$m\frac{A}{\sqrt{B}}\frac{dx_1}{dx_4},\ m\frac{A}{\sqrt{B}}\frac{dx_2}{dx_4},\ m\frac{A}{\sqrt{B}}\frac{dx_3}{dx_4}$$

for the components of momentum, and for the energy (in the static case)

$$m\sqrt{B}.$$

From the expressions for the momentum, it follows that $m\frac{A}{\sqrt{B}}$ plays the part of the rest mass. As m is a constant peculiar to the point of mass, independently of its position, this expression, if we retain the condition $\sqrt{-g} = 1$ at spatial infinity, can vanish only when A diminishes to zero, while B increases to infinity. It seems, therefore, that such a degeneration of the co-efficients $g_{\mu\nu}$ is required by the postulate of relativity of all inertia. This requirement implies that the potential energy $m\sqrt{B}$ becomes infinitely great at infinity. Thus a point of mass can never leave the system; and a more detailed investigation shows that the same thing applies to light-rays. A system of the universe with such behaviour of the gravitational potentials at infinity would not therefore run the risk of wasting away which was mooted just now in connexion with the Newtonian theory.

I wish to point out that the simplifying assumptions as to the gravitational potentials on which this reasoning is based, have been introduced merely for the sake of lucidity. It is possible to find general formulations for the behaviour of the $g_{\mu\nu}$ at infinity which express the essentials of the question without further restrictive assumptions.

At this stage, with the kind assistance of the mathematician J. Grommer, I investigated centrally symmetrical, static gravitational fields, degenerating at infinity in the way mentioned. The gravitational potentials $g_{\mu\nu}$ were applied, and from them the energy-tensor $T_{\mu\nu}$ of matter was calculated on the basis of the field equations of gravitation. But here it proved that for the system of the fixed stars no boundary conditions of the kind can come into question at all, as was also rightly emphasized by the astronomer de Sitter recently.

For the contravariant energy-tensor $T^{\mu\nu}$ of ponderable matter is given by

$$T^{\mu\nu} = \rho\frac{dx_\mu}{ds}\frac{dx_\nu}{ds},$$

where ρ is the density of matter in natural measure. With an appropriate choice of the system of co-ordinates the stellar velocities are very small in comparison with that of light. We may, therefore, substitute $\sqrt{g_{44}}\, dx_4$ for ds. This shows us that all components of $T^{\mu\nu}$ must be very small in comparison with the last component T^{44}. But it was quite impossible to reconcile this condition with the chosen boundary conditions. In the retrospect this result does not appear astonishing. The fact of the small velocities of the stars allows the conclusion that wherever there are fixed stars, the gravitational potential (in our case $\sqrt{B}$) can never be much greater than here on earth. This follows from statistical reasoning, exactly as in the case of the Newtonian theory. At any rate, our calculations have convinced me that such conditions of degeneration for the $g_{\mu\nu}$ in spatial infinity may not be postulated.

After the failure of this attempt, two possibilities next present themselves.

(*a*) We may require, as in the problem of the planets, that, with a suitable choice of the system of reference, the $g_{\mu\nu}$ in spatial infinity approximate to the values

$$\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

(*b*) We may refrain entirely from laying down boundary conditions for spatial infinity claiming general validity; but at the spatial limit of the domain under consideration we have to give the $g_{\mu\nu}$ separately in each individual case, as hitherto we were accustomed to give the initial conditions for time separately.

The possibility (*b*) holds out no hope of solving the problem, but amounts to giving it up. This is an incontestable position, which is taken up at the present time by de Sitter.* But I must confess that such a complete resignation in this fundamental question is for me a difficult thing. I should not make up my mind to it until every effort to make headway toward a satisfactory view had proved to be vain.

Possibility (*a*) is unsatisfactory in more respects than one.

* de Sitter, Akad. van Wetensch. te Amsterdam, 8 Nov., 1916.

In the first place those boundary conditions pre-suppose a definite choice of the system of reference, which is contrary to the spirit of the relativity principle. Secondly, if we adopt this view, we fail to comply with the requirement of the relativity of inertia. For the inertia of a material point of mass m (in natural measure) depends upon the $g_{\mu\nu}$; but these differ but little from their postulated values, as given above, for spatial infinity. Thus inertia would indeed be *influenced*, but would not be *conditioned* by matter (present in finite space). If only one single point of mass were present, according to this view, it would possess inertia, and in fact an inertia almost as great as when it is surrounded by the other masses of the actual universe. Finally, those statistical objections must be raised against this view which were mentioned in respect of the Newtonian theory.

From what has now been said it will be seen that I have not succeeded in formulating boundary conditions for spatial infinity. Nevertheless, there is still a possible way out, without resigning as suggested under (*b*). For if it were possible to regard the universe as a continuum which is *finite* (*closed*) *with respect to its spatial dimensions*, we should have no need at all of any such boundary conditions. We shall proceed to show that both the general postulate of relativity and the fact of the small stellar velocities are compatible with the hypothesis of a spatially finite universe; though certainly, in order to carry through this idea, we need a generalizing modification of the field equations of gravitation.

§ 3. The Spatially Finite Universe with a Uniform Distribution of Matter

According to the general theory of relativity the metrical character (curvature) of the four-dimensional space-time continuum is defined at every point by the matter at that point and the state of that matter. Therefore, on account of the lack of uniformity in the distribution of matter, the metrical structure of this continuum must necessarily be extremely complicated. But if we are concerned with the structure only on a large scale, we may represent matter to ourselves as being uniformly distributed over enormous spaces, so that its density of distribution is a variable function which varies extremely slowly. Thus our procedure will somewhat resemble that of the geodesists who, by means of an ellipsoid, approximate to the shape of the earth's surface, which on a small scale is extremely complicated.

The most important fact that we draw from experience as to the distribution of matter is that the relative velocities of the stars are very small as compared with the velocity of light. So I think that for the present we may base our reasoning upon the following approximative assumption. There is a system of reference relatively to which matter may be looked upon as being permanently at rest. With respect to this system, therefore, the contravariant energy-tensor $T^{\mu\nu}$ of matter is, by reason of (5), of the simple form

$$\left.\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \end{matrix}\right\} \quad . \quad . \quad . \quad (6)$$

The scalar ρ of the (mean) density of distribution may be *a priori* a function of the space co-ordinates. But if we assume the universe to be spatially finite, we are prompted to the hypothesis that ρ is to be independent of locality. On this hypothesis we base the following considerations.

As concerns the gravitational field, it follows from the equation of motion of the material point

$$\frac{d^2x_\nu}{ds^2} + \{\alpha\beta,\ \nu\}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0$$

that a material point in a static gravitational field can remain at rest only when g_{44} is independent of locality. Since, further, we presuppose independence of the time co-ordinate x_4 for all magnitudes, we may demand for the required solution that, for all x_ν,

$$g_{44} = 1 \quad . \quad . \quad . \quad . \quad (7)$$

Further, as always with static problems, we shall have to set

$$g_{14} = g_{24} = g_{34} = 0 \quad . \quad . \quad . \quad (8)$$

It remains now to determine those components of the gravitational potential which define the purely spatial-geometrical relations of our continuum ($g_{11}, g_{12}, \ldots g_{33}$). From our assumption as to the uniformity of distribution of the masses generating the field, it follows that the curvature of the required space must be constant. With this distribution of mass, therefore, the required finite continuum of the x_1, x_2, x_3, with constant x_4, will be a spherical space.

We arrive at such a space, for example, in the following way. We start from a Euclidean space of four dimensions, ξ_1, ξ_2, ξ_3 ξ_4, with a linear element $d\sigma$; let, therefore,

$$d\sigma^2 = d\xi_1^2 + d\xi_2^2 + d\xi_3^2 + d\xi_4^2 \quad . \quad . \quad . \quad (9)$$

In this space we consider the hyper-surface

$$R^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2, \quad . \quad . \quad . \quad (10)$$

where R denotes a constant. The points of this hyper-surface form a three-dimensional continuum, a spherical space of radius of curvature R.

The four-dimensional Euclidean space with which we started serves only for a convenient definition of our hyper-surface. Only those points of the hyper-surface are of interest to us which have metrical properties in agreement with those of physical space with a uniform distribution of matter. For the description of this three-dimensional continuum we may employ the co-ordinates ξ_1, ξ_2, ξ_3 (the projection upon the hyper-plane $\xi_4 = 0$) since, by reason of (10), ξ_4 can be expressed in terms of ξ_1, ξ_2, ξ_3. Eliminating ξ_4 from (9), we obtain for the linear element of the spherical space the expression

$$\left.\begin{aligned} d\sigma^2 &= \gamma_{\mu\nu} d\xi_\mu d\xi_\nu \\ \gamma_{\mu\nu} &= \delta_{\mu\nu} + \frac{\xi_\mu \xi_\nu}{R^2 - \rho^2} \end{aligned}\right\} \quad . \quad . \quad . \quad (11)$$

where $\delta_{\mu\nu} = 1$, if $\mu = \nu$; $\delta_{\mu\nu} = 0$, if $\mu \neq \nu$, and $\rho^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$. The co-ordinates chosen are convenient when it is a question of examining the environment of one of the two points $\xi_1 = \xi_2 = \xi_3 = 0$.

Now the linear element of the required four-dimensional space-time universe is also given us. For the potential $g_{\mu\nu}$, both indices of which differ from 4, we have to set

$$g_{\mu\nu} = -\left(\delta_{\mu\nu} + \frac{x_\mu x_\nu}{R^2 - (x_1^2 + x_2^2 + x_3^2)}\right) \quad . \quad (12)$$

which equation, in combination with (7) and (8), perfectly defines the behaviour of measuring-rods, clocks, and light-rays.

§ 4. On an Additional Term for the Field Equations of Gravitation

My proposed field equations of gravitation for any chosen system of co-ordinates run as follows :—

$$\left.\begin{aligned} G_{\mu\nu} &= -\kappa(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T), \\ G_{\mu\nu} &= -\frac{\partial}{\partial x_\alpha}\{\mu\nu, \alpha\} + \{\mu\alpha, \beta\}\,\{\nu\beta, \alpha\} \\ &\qquad + \frac{\partial^2 \log\sqrt{-g}}{\partial x_\mu \partial x_\nu} - \{\mu\nu, \alpha\}\frac{\partial \log\sqrt{-g}}{\partial x_\alpha} \end{aligned}\right\} \quad (13)$$

The system of equations (13) is by no means satisfied when we insert for the $g_{\mu\nu}$ the values given in (7), (8), and (12), and for the (contravariant) energy-tensor of matter the values indicated in (6). It will be shown in the next paragraph how this calculation may conveniently be made. So that, if it were certain that the field equations (13) which I have hitherto employed were the only ones compatible with the postulate of general relativity, we should probably have to conclude that the theory of relativity does not admit the hypothesis of a spatially finite universe.

However, the system of equations (14) allows a readily suggested extension which is compatible with the relativity postulate, and is perfectly analogous to the extension of Poisson's equation given by equation (2). For on the left-hand side of field equation (13) we may add the fundamental tensor $g_{\mu\nu}$, multiplied by a universal constant, $-\lambda$, at present unknown, without destroying the general covariance. In place of field equation (13) we write

$$G_{\mu\nu} - \lambda g_{\mu\nu} = -\kappa(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T) \quad . \quad . \quad (13a)$$

This field equation, with λ sufficiently small, is in any case also compatible with the facts of experience derived from the solar system. It also satisfies laws of conservation of momentum and energy, because we arrive at (13a) in place of (13) by introducing into Hamilton's principle, instead of the scalar of Riemann's tensor, this scalar increased by a universal constant; and Hamilton's principle, of course, guarantees the validity of laws of conservation. It will be shown in § 5 that field equation (13a) is compatible with our conjectures on field and matter.

§ 5. Calculation and Result

Since all points of our continuum are on an equal footing, it is sufficient to carry through the calculation for *one* point, e.g. for one of the two points with the co-ordinates

$$x_1 = x_2 = x_3 = x_4 = 0.$$

Then for the $g_{\mu\nu}$ in (13a) we have to insert the values

$$\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

wherever they appear differentiated only once or not at all. We thus obtain in the first place

$$G_{\mu\nu} = \frac{\partial}{\partial x_1}[\mu\nu, 1] + \frac{\partial}{\partial x_2}[\mu\nu, 2] + \frac{\partial}{\partial x_3}[\mu\nu, 3] + \frac{\partial^2 \log \sqrt{-g}}{\partial x_\mu \partial x_\nu}.$$

From this we readily discover, taking (7), (8), and (13) into account, that all equations (13*a*) are satisfied if the two relations

$$-\frac{2}{R^2} + \lambda = -\frac{\kappa\rho}{2}, \quad -\lambda = -\frac{\kappa\rho}{2},$$

or

$$\lambda = \frac{\kappa\rho}{2} = \frac{1}{R^2} \quad . \quad . \quad . \quad . \quad (14)$$

are fulfilled.

Thus the newly introduced universal constant λ defines both the mean density of distribution ρ which can remain in equilibrium and also the radius R and the volume $2\pi^2 R^3$ of spherical space. The total mass M of the universe, according to our view, is finite, and is in fact

$$M = \rho \cdot 2\pi^2 R^3 = 4\pi^2 \frac{R}{\kappa} = \pi^2 \sqrt{\frac{32}{\kappa^3 \rho}} \quad . \quad . \quad (15)$$

Thus the theoretical view of the actual universe, if it is in correspondence with our reasoning, is the following. The curvature of space is variable in time and place, according to the distribution of matter, but we may roughly approximate to it by means of a spherical space. At any rate, this view is logically consistent, and from the standpoint of the general theory of relativity lies nearest at hand; whether, from the standpoint of present astronomical knowledge, it is tenable, will not here be discussed. In order to arrive at this consistent view, we admittedly had to introduce an extension of the field equations of gravitation which is not justified by our actual knowledge of gravitation. It is to be emphasized, however, that a positive curvature of space is given by our results, even if the supplementary term is not introduced. That term is necessary only for the purpose of making possible a quasi-static distribution of matter, as required by the fact of the small velocities of the stars.

DO GRAVITATIONAL FIELDS PLAY AN ESSENTIAL PART IN THE STRUCTURE OF THE ELEMENTARY PARTICLES OF MATTER?

BY

A. EINSTEIN

Translated from "Spielen Gravitationsfelder im Aufber der materiellen Elementarteilchen eine wesentliche Rolle?" Sitzungsberichte der Preussischen Akad. d. Wissenschaften, 1919.

DO GRAVITATIONAL FIELDS PLAY AN ESSENTIAL PART IN THE STRUCTURE OF THE ELEMENTARY PARTICLES OF MATTER?

By A. EINSTEIN

NEITHER the Newtonian nor the relativistic theory of gravitation has so far led to any advance in the theory of the constitution of matter. In view of this fact it will be shown in the following pages that there are reasons for thinking that the elementary formations which go to make up the atom are held together by gravitational forces.

§ 1. *Defects of the Present View*

Great pains have been taken to elaborate a theory which will account for the equilibrium of the electricity constituting the electron. G. Mie, in particular, has devoted deep researches to this question. His theory, which has found considerable support among theoretical physicists, is based mainly on the introduction into the energy-tensor of supplementary terms depending on the components of the electro-dynamic potential, in addition to the energy terms of the Maxwell-Lorentz theory. These new terms, which in outside space are unimportant, are nevertheless effective in the interior of the electrons in maintaining equilibrium against the electric forces of repulsion. In spite of the beauty of the formal structure of this theory, as erected by Mie, Hilbert, and Weyl, its physical results have hitherto been unsatisfactory. On the one hand the multiplicity of possibilities is discouraging, and on the other hand those additional terms have not as yet allowed themselves to be framed in such a simple form that the solution could be satisfactory.

So far the general theory of relativity has made no change in this state of the question. If we for the moment disregard the additional cosmological term, the field equations take the form

$$G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G = -\kappa T_{\mu\nu} \quad . \quad . \quad . \quad (1)$$

where $G_{\mu\nu}$ denotes the contracted Riemann tensor of curvature, G the scalar of curvature formed by repeated contraction, and $T_{\mu\nu}$ the energy-tensor of "matter." The assumption that the $T_{\mu\nu}$ do *not* depend on the derivatives of the $g_{\mu\nu}$ is in keeping with the historical development of these equations. For these quantities are, of course, the energy-components in the sense of the special theory of relativity, in which variable $g_{\mu\nu}$ do not occur. The second term on the left-hand side of the equation is so chosen that the divergence of the left-hand side of (1) vanishes identically, so that taking the divergence of (1), we obtain the equation

$$\frac{\partial \mathfrak{T}_{\mu}^{\sigma}}{\partial x_{\sigma}} + \tfrac{1}{2}g_{\mu}^{\sigma\tau}\mathfrak{T}_{\sigma\tau} = 0 \quad . \quad . \quad . \quad (2)$$

which in the limiting case of the special theory of relativity gives the complete equations of conservation

$$\frac{\partial T_{\mu\nu}}{\partial x_{\nu}} = 0.$$

Therein lies the physical foundation for the second term of the left-hand side of (1). It is by no means settled *a priori* that a limiting transition of this kind has any possible meaning. For if gravitational fields do play an essential part in the structure of the particles of matter, the transition to the limiting case of constant $g_{\mu\nu}$ would, for them, lose its justification, for indeed, with constant $g_{\mu\nu}$ there could not be any particles of matter. So if we wish to contemplate the possibility that gravitation may take part in the structure of the fields which constitute the corpuscles, we cannot regard equation (1) as confirmed.

Placing in (1) the Maxwell-Lorentz energy-components of the electromagnetic field $\phi_{\mu\nu}$,

$$T_{\mu\nu} = \tfrac{1}{4}g_{\mu\nu}\phi_{\sigma\tau}\phi^{\sigma\tau} - \phi_{\mu\sigma}\phi_{\nu\tau}g^{\sigma\tau}, \quad . \quad . \quad (3)$$

we obtain for (2), by taking the divergence, and after some reduction,*

$$\phi_{\mu\sigma}\mathfrak{J}^{\sigma} = 0 \quad . \quad . \quad . \quad . \quad (4)$$

where, for brevity, we have set

$$\frac{\partial}{\partial x_{\tau}}(\sqrt{-g}\ \phi_{\mu\nu}g^{\mu\sigma}g^{\nu\tau}) = \frac{\partial \mathfrak{f}^{\sigma\tau}}{\partial x_{\tau}} = \mathfrak{J}^{\sigma} \quad . \quad . \quad (5)$$

In the calculation we have employed the second of Maxwell's systems of equations

$$\frac{\partial \phi_{\mu\nu}}{\partial x_{\rho}} + \frac{\partial \phi_{\nu\rho}}{\partial x_{\mu}} + \frac{\partial \phi_{\rho\mu}}{\partial x_{\nu}} = 0 \quad . \quad . \quad . \quad (6)$$

We see from (4) that the current-density $\mathfrak{J}^{\sigma}$ must everywhere vanish. Therefore, by equation (1), we cannot arrive at a theory of the electron by restricting ourselves to the electromagnetic components of the Maxwell-Lorentz theory, as has long been known. Thus if we hold to (1) we are driven on to the path of Mie's theory.†

Not only the problem of matter, but the cosmological problem as well, leads to doubt as to equation (1). As I have shown in the previous paper, the general theory of relativity requires that the universe be spatially finite. But this view of the universe necessitated an extension of equations (1), with the introduction of a new universal constant λ, standing in a fixed relation to the total mass of the universe (or, respectively, to the equilibrium density of matter). This is gravely detrimental to the formal beauty of the theory.

§ 2. The Field Equations Freed of Scalars

The difficulties set forth above are removed by setting in place of field equations (1) the field equations

$$G_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}G = -\kappa T_{\mu\nu} \quad . \quad . \quad . \quad (1a)$$

where $T_{\mu\nu}$ denotes the energy-tensor of the electromagnetic field given by (3).

The formal justification for the factor $-\frac{1}{4}$ in the second

* Cf. e.g. A. Einstein, Sitzungsber. d. Preuss. Akad. d. Wiss., 1916, pp. 187, 188.

† Cf. D. Hilbert, Göttinger Nachr., 20 Nov., 1915.

term of this equation lies in its causing the scalar of the left-hand side,

$$g^{\mu\nu}(G_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}G),$$

to vanish identically, as the scalar $g^{\nu\mu}T_{\mu\nu}$ of the right-hand side does by reason of (3). If we had reasoned on the basis of equations (1) instead of (1a), we should, on the contrary, have obtained the condition $G = 0$, which would have to hold good everywhere for the $g_{\mu\nu}$, independently of the electric field. It is clear that the system of equations [(1a), (3)] is a consequence of the system [(1), (3)], but not conversely.

We might at first sight feel doubtful whether (1a) together with (6) sufficiently define the entire field. In a generally relativistic theory we need $n - 4$ differential equations, independent of one another, for the definition of n independent variables, since in the solution, on account of the liberty of choice of the co-ordinates, four quite arbitrary functions of all co-ordinates must naturally occur. Thus to define the sixteen independent quantities $g_{\mu\nu}$ and $\phi_{\mu\nu}$ we require twelve equations, all independent of one another. But as it happens, nine of the equations (1a), and three of the equations (6) are independent of one another.

Forming the divergence of (1a), and taking into account that the divergence of $G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}G$ vanishes, we obtain

$$\phi_{\sigma\alpha}J^{\alpha} + \frac{1}{4\kappa}\frac{\partial G}{\partial x_{\sigma}} = 0 \quad . \quad . \quad . \quad (4a)$$

From this we recognize first of all that the scalar of curvature G in the four-dimensional domains in which the density of electricity vanishes, is constant. If we assume that all these parts of space are connected, and therefore that the density of electricity differs from zero only in separate "world-threads," then the scalar of curvature, everywhere outside these world-threads, possesses a constant value G_0. But equation (4a) also allows an important conclusion as to the behaviour of G within the domains having a density of electricity other than zero. If, as is customary, we regard electricity as a moving density of charge, by setting

$$J^{\sigma} = \frac{\mathfrak{J}^{\sigma}}{\sqrt{-g}} = \rho\frac{dx_{\sigma}}{ds}, \quad . \quad . \quad . \quad (7)$$

we obtain from (4a) by inner multiplication by J^σ, on account of the antisymmetry of $\phi_{\mu\nu}$, the relation

$$\frac{\partial G}{\partial x_\sigma}\frac{dx_\sigma}{ds} = 0 \quad . \quad . \quad . \quad . \quad (8)$$

Thus the scalar of curvature is constant on every world-line of the motion of electricity. Equation (4a) can be interpreted in a graphic manner by the statement: The scalar of curvature plays the part of a negative pressure which, outside of the electric corpuscles, has a constant value G_0. In the interior of every corpuscle there subsists a negative pressure (positive $G - G_0$) the fall of which maintains the electrodynamic force in equilibrium. The minimum of pressure, or, respectively, the maximum of the scalar of curvature, does not change with time in the interior of the corpuscle.

We now write the field equations (1a) in the form

$$(G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G) + \tfrac{1}{4}g_{\mu\nu}G_0 = -\kappa\left(T_{\mu\nu} + \frac{1}{4\kappa}g_{\mu\nu}(G - G_0)\right) \quad (9)$$

On the other hand, we transform the equations supplied with the cosmological term as already given

$$G_{\mu\nu} - \lambda g_{\mu\nu} = -\kappa(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T).$$

Subtracting the scalar equation multiplied by $\frac{1}{2}$, we next obtain

$$(G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G) + g_{\mu\nu}\lambda = -\kappa T_{\mu\nu}.$$

Now in regions where only electrical and gravitational fields are present, the right-hand side of this equation vanishes. For such regions we obtain, by forming the scalar,

$$-G + 4\lambda = 0.$$

In such regions, therefore, the scalar of curvature is constant, so that λ may be replaced by $\frac{1}{4}G_0$. Thus we may write the earlier field equation (1) in the form

$$G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G + \tfrac{1}{4}g_{\mu\nu}G_0 = -\kappa T_{\mu\nu} \quad . \quad . \quad (10)$$

Comparing (9) with (10), we see that there is no difference between the new field equations and the earlier ones, except that instead of $T_{\mu\nu}$ as tensor of "gravitating mass" there now occurs $T_{\mu\nu} + \frac{1}{4\kappa} g_{\mu\nu}(G - G_0)$ which is independent of the scalar of curvature. But the new formulation has this great advantage, that the quantity λ appears in the fundamental equations as a constant of integration, and no longer as a universal constant peculiar to the fundamental law.

§ 3. On the Cosmological Question

The last result already permits the surmise that with our new formulation the universe may be regarded as spatially finite, without any necessity for an additional hypothesis. As in the preceding paper I shall again show that with a uniform distribution of matter, a spherical world is compatible with the equations.

In the first place we set

$$ds^2 = -\gamma_{ik} dx_i dx_k + dx_4^2 \ (i, k = 1, 2, 3) \qquad (11)$$

Then if P_{ik} and P are, respectively, the curvature tensor of the second rank and the curvature scalar in three-dimensional space, we have

$$\begin{aligned} G_{ik} &= P_{ik} \ (i, k = 1, 2, 3) \\ G_{i4} &= G_{4i} = G_{44} = 0 \\ G &= -P \\ -g &= \gamma. \end{aligned}$$

It therefore follows for our case that

$$\begin{aligned} G_{ik} - \tfrac{1}{2} g_{ik} G &= P_{ik} - \tfrac{1}{2}\gamma_{ik} P \ (i, k = 1, 2, 3) \\ G_{44} - \tfrac{1}{2} g_{44} G &= \tfrac{1}{2} P. \end{aligned}$$

We pursue our reflexions, from this point on, in two ways. Firstly, with the support of equation (1a). Here $T_{\mu\nu}$ denotes the energy-tensor of the electro-magnetic field, arising from the electrical particles constituting matter. For this field we have everywhere

$$\mathfrak{T}_1^1 + \mathfrak{T}_2^2 + \mathfrak{T}_3^3 + \mathfrak{T}_4^4 = 0.$$

The individual $\mathfrak{T}_\mu^\nu$ are quantities which vary rapidly with position; but for our purpose we no doubt may replace them by their mean values. We therefore have to choose

$$\left.\begin{aligned} \mathfrak{T}_1^1 = \mathfrak{T}_2^2 = \mathfrak{T}_3^3 &= -\tfrac{1}{3}\mathfrak{T}_4^4 = \text{const.} \\ \mathfrak{T}_\mu^\nu &= 0 \ (\text{for } \mu \neq \nu), \end{aligned}\right\} \qquad (12)$$

and therefore

$$T_{ik} = \frac{1}{3}\frac{\mathfrak{T}_4^4}{\sqrt{\gamma}}\gamma_{ik},\quad T_{44} = \frac{\mathfrak{T}_4^4}{\sqrt{\gamma}}.$$

In consideration of what has been shown hitherto, we obtain in place of (1a)

$$P_{ik} - \frac{1}{4}\gamma_{ik}P = -\frac{1}{3}\gamma_{ik}\frac{\kappa\mathfrak{T}_4^4}{\sqrt{\gamma}} \quad . \quad . \quad . \quad (13)$$

$$\frac{1}{4}P = -\frac{\kappa\mathfrak{T}_4^4}{\sqrt{\gamma}} \quad . \quad . \quad . \quad (14)$$

The scalar of equation (13) agrees with (14). It is on this account that our fundamental equations permit the idea of a spherical universe. For from (13) and (14) follows

$$P_{ik} + \frac{4}{3}\frac{\kappa\mathfrak{T}_4^4}{\sqrt{\gamma}}\gamma_{ik} = 0 \quad . \quad . \quad . \quad (15)$$

and it is known * that this system is satisfied by a (three-dimensional) spherical universe.

But we may also base our reflexions on the equations (9). On the right-hand side of (9) stand those terms which, from the phenomenological point of view, are to be replaced by the energy-tensor of matter; that is, they are to be replaced by

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \end{matrix}$$

where ρ denotes the mean density of matter assumed to be at rest. We thus obtain the equations

$$P_{ik} - \frac{1}{2}\gamma_{ik}P - \frac{1}{4}\gamma_{ik}G_0 = 0 \quad . \quad . \quad (16)$$

$$\frac{1}{2}P + \frac{1}{4}G_0 = -\kappa\rho \quad . \quad . \quad . \quad . \quad (17)$$

From the scalar of equation (16) and from (17) we obtain

$$G_0 = -\frac{2}{3}P = 2\kappa\rho, \quad . \quad . \quad . \quad (18)$$

and consequently from (16)

$$P_{ik} - \kappa\rho\gamma_{ik} = 0 \quad . \quad . \quad . \quad (19)$$

* Cf. H. Weyl, "Raum, Zeit, Materie," § 33.

GRAVITATION AND ELECTRICITY*

By H. WEYL

ACCORDING to Riemann,† geometry is based upon the following two facts :—

1. *Space is a Three-dimensional Continuum.*—The manifold of its points may therefore be consistently represented by the values of three co-ordinates x_1, x_2, x_3.

2. (*Pythagorean Theorem*).—The square of the distance ds between two infinitely proximate points

$$P = (x_1, x_2, x_3) \text{ and } P' = (x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) \quad (1)$$

(any co-ordinates being employed) is a quadratic form of the relative co-ordinates dx_μ :—

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu, \quad (g_{\mu\nu} = g_{\nu\mu}) \quad . \quad . \quad (2)$$

The second of these facts may be briefly stated by saying that space is a *metrical* continuum. In complete accord with the spirit of the physics of immediate action we assume the Pythagorean theorem to be strictly valid only in the limit when the distances are infinitely small.

The special theory of relativity led to the discovery that *time* is associated as a fourth co-ordinate (x_4) on an equal footing with the three co-ordinates of space, and that the scene of material events, *the world*, is therefore *a four-dimensional, metrical continuum.* And so the quadratic form (2), which defines the metrical properties of the world, is not necessarily positive as in the case of the geometry of three-dimensional space, but has the index of inertia 3.‡ Riemann

* The footnotes in square brackets are later additions by the author.

† Math. Werke (2nd ed., Leipzig, 1892), No. XII, p. 282.

‡ That is to say that if the co-ordinates are chosen so that at one particular point of the continuum $ds^2 = \pm dx_1^2 \pm dx_2^2 \pm dx_3^2 \pm dx_4^2$, then in every case three of the signs will be + and one − (TRANS.).

himself did not fail to point out that this quadratic form was to be regarded as a physical reality, since it reveals itself, e.g. in centrifugal forces, as the origin of real effects upon matter, and that matter therefore presumably reacts upon it. Until then all geometricians and philosophers had looked upon the metrical properties of space as pertaining to space itself, independently of the matter which it contained. It is upon this idea, which it was quite impossible for Riemann in his day to carry through, that Einstein in our own time, independently of Riemann, has raised the imposing edifice of his general theory of relativity. According to Einstein the phenomena of *gravitation* must also be placed to the account of geometry, and the laws by which matter affects measurements are no other than the laws of gravitation: the $g_{\mu\nu}$ in (2) form the components of the gravitational potential. While the gravitational potential thus consists of an invariant *quadratic* differential form, *electromagnetic phenomena* are governed by a four-potential of which the components ϕ_μ together compose an invariant *linear* differential form $\Sigma\phi_\mu dx_\mu$. But so far the two classes of phenomena, gravitation and electricity, stand side by side, the one separate from the other.

The later work of Levi-Civita,* Hessenberg,† and the author‡ shows quite plainly that the fundamental conception on which the development of Riemann's geometry must be based if it is to be in agreement with nature, is that of the infinitesimal parallel displacement of a vector. If P and P* are any two points connected by a curve, a given vector at P can be moved parallel to itself along this curve from P to P*. But, generally speaking, this conveyance of a vector from P to P* is not integrable, that is to say, the vector at P* at which we arrive depends upon the path along which the displacement travels. It is only in Euclidean "gravitationless" geometry that integrability obtains. The Riemannian geometry referred to above still contains a residual element of finite geometry—without any substantial reason, as far as I can see.

* "Nozione di parallelismo . . .", Rend. del Circ. Matem. di Palermo, Vol. 42 (1917).

† "Vektorielle Begründung der Differentialgeometrie," Math. Ann., Vol. 78 (1917).

‡ "Space, Time, and Matter" (1st ed., Berlin, 1918), § 14.

It seems to be due to the accidental origin of this geometry in the theory of surfaces. The quadratic form (2) enables us to compare, with respect to their length, not only two vectors at the same point, but also the vectors at any two points. *But a truly infinitesimal geometry must recognize only the principle of the transference of a length from one point to another point infinitely near to the first.* This forbids us to assume that the problem of the transference of length from one point to another at a finite distance is integrable, more particularly as the problem of the transference of direction has proved to be non-integrable. Such an assumption being recognized as false, a geometry comes into being, which, when applied to the world, explains in a surprising manner *not only the phenomena of gravitation, but also those of the electromagnetic field.* According to the theory which now takes shape, both classes of phenomena spring from the same source, and in fact *we cannot in general make any arbitrary separation of electricity from gravitation.* In this theory *all physical quantities have a meaning in world geometry.* In particular the quantities denoting physical effects appear at once as pure numbers. The theory leads to a world-law which in its essentials is defined without ambiguity. It even permits us in a certain sense to comprehend why the world has four dimensions. I shall now first of all give a sketch of the structure of the amended geometry of Riemann without any thought of its physical interpretation. Its application to physics will then follow of its own accord.

In a given system of co-ordinates the relative co-ordinates dx_μ of a point P′ infinitely near to P—see (1)—are the components of the infinitesimal displacement PP′. The transition from one system of co-ordinates to another is expressed by definite formulæ of transformation,

$$x_\mu = x_\mu(x_1^*, x_2^* \ldots x_n^*) \quad \mu = 1, 2, \ldots n,$$

which determine the connexion between the co-ordinates of the same point in the two systems. Then between the components dx_μ and the components dx_μ^* of the same infinitesimal displacement of the point P we have the linear formulæ of transformation

$$dx_\mu = \sum_\nu a_{\mu\nu} dx_\nu^* \quad . \quad . \quad . \quad . \quad (3)$$

in which $a_{\mu\nu}$ are the values of the derivatives $\frac{\partial x_\mu}{\partial x^*_\nu}$ at the point P. A contravariant vector **x** at the point P referred to either system of co-ordinates has n known numbers ξ^μ for its components, which in the transition to another system are transformed in exactly the same way (3) as the components of an infinitesimal displacement. I denote the totality of vectors at the point P as the vector-space at P. It is, firstly, linear or affine, i.e. by multiplication of a vector at P by a number, and by addition of two such vectors, there always arises a vector at P; and, secondly, it is metrical, i.e. by the symmetrical bilinear form belonging to (2) a scalar product

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x} = \sum_{\mu\nu} g_{\mu\nu}\xi^\mu\eta^\nu$$

is invariantly assigned to each pair of vectors **x** and **y** with components ξ^μ, η^μ. We take it, however, that this form is determined only as far as to a positive factor of proportionality, which remains arbitrary. If the manifold of points of space is represented by co-ordinates x_μ, the $g_{\mu\nu}$ are determined by the metrical properties at the point P only to the extent of their proportionality. In the physical sense, too, it is only the ratios of the $g_{\mu\nu}$ that has an immediate tangible meaning. For the equation

$$\sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu = 0$$

is satisfied, when P is a given origin, by those infinitely proximate world-points which are reached by a light signal emitted at P. For the purpose of analytical presentation we have firstly to choose a definite system of co-ordinates, and secondly at each point P to determine the arbitrary factor of proportionality with which the $g_{\mu\nu}$ are endowed. Accordingly the formulæ which emerge must possess a double property of invariance: they must be invariant with respect to any continuous transformations of co-ordinates, and they must remain unaltered if $\lambda g_{\mu\nu}$, where λ is an arbitrary continuous function of position, is substituted for the $g_{\mu\nu}$. The supervention of this second property of invariance is characteristic of our theory.

If P, P* are any two points, and if to each vector **x** at P a vector **x*** at P* is assigned in such a way that in general $a\mathbf{x}$

becomes $a\mathbf{x}^*$, and $\mathbf{x} + \mathbf{y}$ becomes $\mathbf{x}^* + \mathbf{y}^*$ (a being any assigned number), and the vector zero at P is the only one to which the vector zero at P* corresponds, we then have made an affine or linear replica of the vector-space at P on the vector-space at P*. This replica has a particularly close resemblance when the scalar product of the vectors $\mathbf{x}^*$, $\mathbf{y}^*$ at P* is proportional to that of $\mathbf{x}$ and $\mathbf{y}$ at P for all pairs of vectors $\mathbf{x}$, $\mathbf{y}$. (In our view it is only this idea of a similar replica that has an objective sense; the previous theory permitted the more definite conception of a congruent replica.) The meaning of the parallel displacement of a vector at the point P to a neighbouring point P′ is settled by the two axiomatic postulates:

1. By the parallel displacement of the vectors at the point P to the neighbouring point P′ a similar image of the vector-space at P is made upon the vector-space at P′.

2. If P_1, P_2 are two points in the neighbourhood of P, and the infinitesimal vector PP_2 at P is transformed into P_1P_{12} by a parallel displacement to the point P_1, while PP_1 at P is transformed into P_2P_{21} by parallel displacement to P_2, then P_{12}, P_{21} coincide, i.e. infinitesimal parallel displacements are commutative.

That part of postulate 1 which says that the parallel displacement is an affine transposition of the vector-space from P to P′, is expressed analytically as follows: the vector ξ^μ at $P = (x_1, x_2, \ldots x_n)$ is by displacement transformed into a vector $\xi^\mu + d\xi^\mu$ at $P' = (x_1 + dx_1, x_2 + dx_2, \ldots x_n + dx_n)$ the components of which are in a linear relation to ξ^μ,—

$$d\xi^\mu = -\sum_\nu d\gamma^\mu_\nu \xi^\nu \qquad . \qquad . \qquad (4)$$

The second postulate teaches that the $d\gamma^\mu_\nu$ are linear differential forms

$$d\gamma^\mu_\nu = \sum_\rho \Gamma^\mu_{\nu\rho} dx_\rho,$$

the coefficients of which possess the symmetrical property

$$\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu} \qquad . \qquad . \qquad . \qquad (5)$$

If two vectors ξ^μ, η^μ at P are transformed by parallel displacement at P′ into $\xi^\mu + d\xi^\mu$, $\eta^\mu + d\eta^\mu$, then the postulate of similarity stated under 1 above, which goes beyond affinity, tells us that

$$\sum_{\mu\nu}(g_{\mu\nu} + dg_{\mu\nu})(\xi^\mu + d\xi^\mu)(\eta^\nu + d\eta^\nu)$$

must be proportional to

$$\sum_{\mu\nu} g_{\mu\nu}\xi^\mu\eta^\nu.$$

If we call the factor of proportionality, which differs infinitesimally from 1, $1 + d\phi$, and define the reduction of an index in the usual way by the formula

$$a_\mu = \sum_\nu g_{\mu\nu}a^\nu,$$

we obtain

$$dg_{\mu\nu} - (d\gamma_{\nu\mu} + d\gamma_{\mu\nu}) = g_{\mu\nu}d\phi \quad . \quad . \quad (6)$$

From this it follows that $d\phi$ is a linear differential form

$$d\phi = \sum_\mu \phi_\mu dx_\mu \quad . \quad . \quad . \quad . \quad (7)$$

If this is known, the equation (6) or

$$\Gamma_{\mu, \nu\rho} + \Gamma_{\nu, \mu\rho} = \frac{\partial g_{\mu\nu}}{\partial x_\rho} - g_{\mu\nu}\phi_\rho,$$

together with the condition for symmetry (5), gives unequivocally the quantities Γ. *The internal metrical connexion of space thus depends on a linear form* (7) *besides the quadratic form* (2)—which is determined except as to an arbitrary factor of proportionality.* If we substitute $\lambda g_{\mu\nu}$ for $g_{\mu\nu}$ with-

* [I have now modified this structure in the following points (cf. the final presentation in ed. 4 of "Raum, Zeit, Materie," 1921, §§ 13, 18). (*a*) In place of postulates 1 and 2, which the parallel displacement has to fulfil, there is now one postulate: Let there be a system of co ordinates at the point P, by the employment of which the components of every vector at P are not altered by parallel displacement to any point in infinite proximity to P. This postulate characterizes the essence of the parallel displacement as that of a transposition, concerning which it may be correctly asserted that it leaves the vectors "unaltered." (*b*) To the metrics at the single point P, according to which there is attached to every vector $\mathbf{x} = \xi^\mu$ at P a tract of such a kind that two vectors define the same tract when, and only when, they possess the same measure-number $l = \sum g_{\mu\nu}\xi^\mu\xi^\nu$, there must now be added the metrical connexion of P with the points in its neighbourhood: by congruent transposition to the infinitely near point P′ a tract at P passes over into a definite tract at P′. If we make a requirement of this concept of congruent transposition of tracts analogous to that which has just been postulated, under (*a*), of the concept of parallel displacement of vectors, we see that this process (in which the measure-number l of the tract is increased by dl) is expressed in the equations

$$dl = l d\phi\,; \quad d\phi = \sum \phi_\mu dx_\mu.$$

out changing the system of co-ordinates, the quantities $d\gamma^{\mu}_{\nu}$ do not change, $d\gamma_{\mu\nu}$ assumes the factor λ, and $dg_{\mu\nu}$ becomes $\lambda dg_{\mu\nu} + g_{\mu\nu}d\lambda$. Equation (6) then shows that $d\phi$ becomes

$$d\phi + \frac{d\lambda}{\lambda} = d\phi + d\,(\log \lambda).$$

What remains undetermined, therefore, in the linear form $\Sigma\phi_\mu dx_\mu$ is not a factor of proportionality which would have to be settled by an arbitrary choice of a unit of measurement, but, rather, the arbitrary element inherent in it consists in an additive total differential. For the analytical representation of geometry the forms

$$g_{\mu\nu}dx_\mu dx_\nu, \quad \phi_\mu dx_\mu \; . \qquad . \quad . \quad . \quad (8)$$

are on an equal footing with

$$\lambda \cdot g_{\mu\nu}dx_\mu dx_\nu \text{ and } \phi_\mu dx_\mu + d\,(\log \lambda) \quad . \quad . \quad (9)$$

where λ is any positive function of position. Hence there is invariant significance in the anti-symmetrical tensor with the components

$$F_{\mu\nu} = \frac{\partial\phi_\mu}{\partial x_\nu} - \frac{\partial\phi_\nu}{\partial x_\mu} \qquad . \quad . \quad . \quad (10)$$

i.e. the form

$$F_{\mu\nu} = dx_\mu \delta x_\nu = \tfrac{1}{2}F_{\mu\nu}\Delta x_{\mu\nu}$$

which depends bilinearly on two arbitrary displacements dx and δx at the point P—or, rather, depends linearly on the surface element with the components $\Delta x_{\mu\nu} = dx_\mu\delta x_\nu - dx_\nu\delta x_\mu$ which is defined by these two displacements. The special case of the theory as hitherto developed, in which the arbitrarily chosen unit of length at the origin allows itself to be transferred by parallel displacement to all points of space in a manner which is independent of the path traversed —this special case occurs when the $g_{\mu\nu}$ can be absolutely determined in such a way that the ϕ_μ vanish. The $\Gamma^{\mu}_{\nu\rho}$ are

In these circumstances the metrics and the metrical connexion determine the "affine" connexion (parallel displacement) without ambiguity—and indeed, according to my present view of the problem of space this is the most fundamental fact of geometry—whereas according to the presentation given in the text it is the linear form $d\phi$ that remains arbitrary in the given metrics at the parallel displacement.]

then nothing else than the Christoffel three-indices symbols. The necessary and sufficient invariant condition for the occurrence of this case consists in the identical vanishing of the tensor $F_{\mu\nu}$.

This naturally suggests interpreting ϕ_μ in world-geometry as the four-potential, and the tensor F consequently as electromagnetic field. For the absence of an electromagnetic field is the necessary condition for the validity of Einstein's theory, which, up to the present, accounts for the phenomena of gravitation only. If this view is accepted, it will be seen that the electric quantities are of such a nature that their characterization by numbers in a definite system of co-ordinates does not depend on the arbitrary choice of a unit of measurement. In fact, in the question of the unit of measurement and of dimension there must be a new orientation of the theory. Hitherto a quantity has been spoken of as, e.g., a tensor of the second rank, if a single value of the quantity determines a matrix of numbers $a_{\mu\nu}$ in each system of co-ordinates after an arbitrary unit of measurement has been selected, these numbers forming the coefficients of an invariant bilinear form of two arbitrary, infinitesimal displacements

$$a_{\mu\nu}dx_\mu\delta x_\nu \quad . \quad . \quad . \quad . \quad (11)$$

But here we speak of a tensor, if, with a system of co-ordinates taken as a base, and after definite selection of the factor of proportionality contained in the $g_{\mu\nu}$, the components $a_{\mu\nu}$ are determined without ambiguity and in such a way that on transforming the co-ordinates the form (11) remains invariant, but on replacing $g_{\mu\nu}$ by $\lambda g_{\mu\nu}$ the $a_{\mu\nu}$ become $\lambda^e a_{\mu\nu}$. We then say that the tensor has the weight e, or, ascribing to the linear element ds the dimension "length $= l$," that it is of dimension l^{2e}. Only those tensors of weight 0 are absolutely invariant. The field tensor with the components $F_{\mu\nu}$ is of this kind. By (10) it satisfies the first system of the Maxwell equations

$$\frac{\partial F_{\nu\rho}}{\partial x_\mu} + \frac{\partial F_{\rho\mu}}{\partial x_\nu} + \frac{\partial F_{\mu\nu}}{\partial x_\rho} = 0.$$

When once the idea of parallel displacement is clear, geometry and the tensor calculus can be established without difficulty.

(*a*) *Geodetic Lines.*—Given a point P and at that point a vector, the geodetic line from P in the direction of this vector is given by continuously moving the vector parallel to itself in its own direction. Employing a suitable parameter τ the differential equation of the geodetic line is

$$\frac{d^2x_\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx_\nu}{d\tau}\frac{dx_\rho}{d\tau} = 0.$$

(Of course it cannot be characterized as the line of smallest length, because the notion of curve-length has no meaning.)

(*b*) *Tensor Calculus.*—To deduce, for example, a tensor field of rank 2 by differentiation from a covariant tensor field of rank 1 and weight 0 with components f_μ, we call in the help of an arbitrary vector ξ^μ at the point P, form the invariant $f_\mu\xi^\mu$ and its infinitely small alteration on transition from the point P with the co-ordinates x_μ to the neighbouring point P′ with the co-ordinates $x_\mu + dx_\mu$ by shifting the vector along a parallel to itself during this transition. For this alteration we have

$$\frac{\partial f_\mu}{\partial x_\nu}\xi^\mu dx_\nu + f_\rho d\xi^\rho = \left(\frac{\partial f_\mu}{\partial x_\nu} - \Gamma^\rho_{\mu\nu}f_\rho\right)\xi^\mu dx_\nu.$$

The quantities in brackets on the right are therefore the components of a tensor field of rank 2 and weight 0, which is formed from the field f in a perfectly invariant manner.

(*c*) *Curvature.*—To construct the analogue to Riemann's tensor of curvature, let us begin with the figure employed above, of an infinitely small parallelogram, consisting of the points P, P_1, P_2, and $P_{12} = P_{21}$.* If we displace a vector $\mathbf{x} = \xi^\mu$ at P parallel to itself, to P_1 and from there to P_{12}, and a second time first to P_2 and thence to P_{21}, then, since P_{12} and P_{21} coincide, there is a meaning in forming the difference $\Delta\mathbf{x}$ of the two vectors obtained at this point. For their components we have

$$\Delta\xi^\mu = \Delta R^\mu_\nu\xi^\nu \quad . \quad . \quad . \quad . \quad (12)$$

where the ΔR^μ_ν are independent of the displaced vector $\mathbf{x}$, but

* [Here it is not essential that opposite sides of the infinitely small "parallelogram" are produced by parallel displacement one from the other; we are concerned only with the coincidence of the points P_{12} and P_{21}.]

on the other hand depend linearly on the surface-element defined by the two displacements $PP_1 = dx_\mu$, $PP_2 = \delta x_\mu$. Thus

$$\Delta R^\mu_\nu = R^\mu_{\nu\rho\sigma} dx_\rho \delta x_\sigma = \tfrac{1}{2} R^\mu_{\nu\rho\sigma} \Delta x_{\rho\sigma}.$$

The components of curvature $R^\mu_{\nu\rho\sigma}$, depending solely on the place P, possess the two properties of symmetry that (1) they change sign on the interchange of the last two indices ρ and σ, and (2), if we perform the three cylic interchanges $\nu\rho\sigma$, and add up the appropriate components, the result is 0. Reducing the index μ, we obtain at $R_{\mu\nu\rho\sigma}$ the components of a covariant tensor of rank 4 and weight 1. Even without calculation we see that R divides in a natural, invariant manner into two parts,

$$R^\mu_{\nu\rho\sigma} = P^\mu_{\nu\rho\sigma} - \tfrac{1}{2}\delta^\mu_\nu F_{\rho\sigma} \quad (\delta^\mu_\nu = 1 \text{ if } \mu = \nu : \; = 0 \text{ if } \mu \neq \nu), \quad (13)$$

of which the first, $P^\mu_{\nu\rho\sigma}$, is anti-symmetrical, not only in the indices $\rho\sigma$, but also in μ and ν. Whereas the equations $F_{\mu\nu} = 0$ characterize our space as one without an electromagnetic field, i.e. as one in which the problem of the conveyance of length is integrable, the equations $P^\mu_{\nu\rho\sigma} = 0$ are, as (13) shows, the invariant conditions for the absence of a gravitational field, i.e. for the problem of the conveyance of direction to be integrable. The Euclidean space alone is one which at the same time is free of electricity and of gravitation.

The simplest invariant of a linear copy like (12), which to each vector **x** assigns a vector Δ**x**, is its "spur"

$$\frac{1}{n}\Delta R^\mu_\mu.$$

For this, by (13), we obtain in the present case the form

$$-\tfrac{1}{2} F_{\rho\sigma} dx_\rho \delta x_\sigma$$

which we have already encountered above. The simplest invariant of a tensor like $-\tfrac{1}{2}F_{\rho\sigma}$ is the "square of its magnitude"

$$L = \tfrac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \quad . \quad . \quad . \quad . \quad (14)$$

L is evidently an invariant of weight -2, because the tensor F has weight 0. If g is the negative determinant of the $g_{\mu\nu}$, and

$$d\omega = \sqrt{g}\, dx_0 dx_1 dx_2 dx_3 = \sqrt{g}\, dx$$

the volume of an infinitely small element of volume, it is known that the Maxwell theory is governed by the quantity of electrical action, which is equal to the integral $\int L d\omega$ of this simplest invariant, extended over any chosen territory, and indeed is governed in the sense that, with any variations of the $g_{\mu\nu}$ and ϕ_μ, which vanish at the limits of world-territory, we have

$$\delta\int L d\omega = \int (S^\mu d\phi_\mu + T^{\mu\nu}\delta g_{\mu\nu})d\omega,$$

where

$$S^\mu = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}F^{\mu\nu})}{\partial x_\nu}$$

are the left-hand sides of the generalized Maxwellian equations (the right-hand sides of which are the components of the four-current), and the $T^{\mu\nu}$ form the energy-momentum tensor of the electromagnetic field. As L is an invariant of weight -2, whereas the volume-element in n-dimensional geometry is an invariant of weight $\frac{1}{2}n$, the integral has significance only when the number of dimensions $n = 4$. Thus on our interpretation the possibility of the Maxwell theory is restricted to the case of four dimensions. In the four-dimensional world, however, the quantity of electromagnetic action becomes a pure number. Nevertheless, the magnitude of the quantity 1 cannot be ascertained in the traditional units of the c.g.s. system until a physical problem, to be tested by observation (as for example the electron), has been calculated on the basis of our theory.

Passing now from geometry to physics, we have to assume, following the precedent of Mie's theory,* that all the laws of nature rest upon a definite integral invariant, the action-quantity

$$\int W d\omega = \int \mathfrak{W} dx, \quad \mathfrak{W} = W\sqrt{g},$$

in such a way that the real world is distinguished from all other possible four-dimensional metrical spaces by the characteristic that for it the action-quantity contained in any part of its domain assumes a stationary value in relation to such variations of the potentials $g_{\mu\nu}$, ϕ_μ as vanish at the limits of

* Ann. d. Physik, 37, 39, 40, 1912-13.

the territory in question. W, the world-density of the action, must be an invariant of weight -2. The action-quantity is in any case a pure number; thus our theory at once accounts for that atomistic structure of the world to which current views attach the most fundamental importance—the action-quantum. The simplest and most natural conjecture which we can make for W, is

$$W = R^{\mu}_{\nu\rho\sigma} R^{\nu\rho\sigma}_{\mu} = | R |^2.$$

For this we also have, by (13),

$$W = | P |^2 + 4L.$$

(There could be no doubt about anything here except perhaps the factor 4, with which the electric term L is added to the first.) But even without particularizing the action-quantity we can draw some general conclusions from the principle of action. For we shall show that as, according to investigations by Hilbert, Lorentz, Einstein, Klein, and the author,* the four laws of the conservation of matter (the energy-momentum tensor) are connected with the invariance of the action quantity (containing four arbitrary functions) with respect to transformations of co-ordinates, so in the same way the law of the conservation of electricity is connected with the "measure-invariance" [transition from (8) to (9)] which here makes its appearance for the first time, introducing a fifth arbitrary function. The manner in which the latter associates itself with the principles of energy and momentum seems to me one of the strongest general arguments in favour of the theory here set out—so far as there can be any question at all of confirmation in purely speculative matters.

For any variation which vanishes at the limits of the world-territory under consideration we have

$$\delta \int \mathfrak{W} dx = \int (\mathfrak{W}^{\mu\nu} \delta g_{\mu\nu} + \mathfrak{w}^{\mu} \delta \phi_{\mu}) dx \quad (\mathfrak{W}^{\mu\nu} = \mathfrak{W}^{\mu\nu}) \qquad (15)$$

* Hilbert, "Die Grundlagen der Physik," Göttinger Nachrichten, 20 Nov., 1915; H. A. Lorentz in four papers in the Versl. K. Ak. van Wetensch., Amsterdam, 1915-16; A. Einstein, Berl. Ber., 1916, pp. 1111-6; F. Klein, Gott. Nachr., 25 Jan., 1918; H. Weyl, Ann. d. Physik, 54, 1917, pp. 121-5.

The laws of nature then take the form

$$\mathfrak{W}^{\mu\nu} = 0, \mathfrak{w}^{\mu} = 0 \quad . \quad . \quad . \quad (16)$$

The former may be regarded as the laws of the gravitational field, the latter as those of the electromagnetic field. The quantities W^{μ}_{ν}, w^{μ} defined by

$$\mathfrak{W}^{\mu}_{\nu} = \sqrt{g}W^{\mu}_{\nu}, \quad \mathfrak{w}^{\mu} = \sqrt{g}w^{\mu}$$

are the mixed or, respectively, the contravariant components of a tensor of rank 2 or 1 respectively, and of weight -2. In the system of equations (16) there are five which are redundant, in accordance with the properties of invariance. This is expressed in the following five invariant identities, which subsist between their left-hand sides :—

$$\frac{\partial \mathfrak{w}^{\mu}}{\partial x_{\mu}} \equiv \mathfrak{W}^{\mu}_{\mu} \quad . \quad . \quad . \quad . \quad (17)$$

$$\frac{\partial \mathfrak{W}^{\mu}_{\nu}}{\partial x_{\mu}} - \Gamma^{\alpha}_{\nu\beta}\mathfrak{W}^{\beta}_{\alpha} \equiv \tfrac{1}{2}F_{\mu\nu}\mathfrak{w}^{\mu} \quad . \quad . \quad . \quad (18)$$

The first results from the measure-invariance. For if in the transition from (8) to (9) we assume for log λ an infinitely small function of position $\delta\rho$, we obtain the variation

$$\delta g_{\mu\nu} = g_{\mu\nu}\delta\rho, \quad \delta\phi_{\mu} = \frac{\partial(\delta\rho)}{\partial x_{\mu}}.$$

For this variation (15) must vanish. In the second place if we utilize the invariance of the action-quantity with respect to transformations of co-ordinates by means of an infinitely small deformation of the world - continuum,* we obtain the identities

$$\frac{\partial \mathfrak{W}^{\mu}_{\nu}}{\partial x_{\mu}} - \tfrac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x_{\nu}}\mathfrak{W}^{\alpha\beta} + \tfrac{1}{2}\left(\frac{\partial \mathfrak{w}^{\mu}}{\partial x_{\mu}}\phi_{\nu} - \Gamma_{\alpha\nu}\mathfrak{w}^{\alpha}\right) \equiv 0$$

which change into (18) when, by (17) $\partial\mathfrak{w}^{\mu}/\partial x_{\mu}$ is replaced by $g_{\alpha\beta}\mathfrak{W}^{\alpha\beta}$

From the gravitational laws alone therefore we already obtain

$$\frac{\partial \mathfrak{w}^{\mu}}{\partial x_{\mu}} = 0, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

* Weyl, Ann. d. Physik, 54, 1917, pp. 121-5; F. Klein, Gött. Nachr., 25 Jan., 1918.

and from the laws of the electromagnetic field alone

$$\frac{\partial}{\partial x_\mu}\mathfrak{W}^\mu_\nu - \Gamma^\alpha_{\nu\beta}\mathfrak{W}^\beta_\alpha = 0 \quad . \quad . \quad . \quad (20)$$

In Maxwell's theory $\mathfrak{w}^\mu$ has the form

$$\mathfrak{w}^\mu \equiv \frac{\partial(\sqrt{g}F^{\mu\nu})}{\partial x_\nu} - \mathfrak{s}^\mu, \quad \mathfrak{s}^\mu = \sqrt{g}s^\mu$$

where s^μ denotes the four-current. Since the first part here satisfies the equation (19) identically, this equation gives us the law of conservation of electricity

$$\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}s^\nu)}{\partial x_\mu} = 0.$$

Similarly in Einstein's theory of gravitation $\mathfrak{W}^\mu_\nu$ consists of two terms, the first of which satisfies equation (20) identically, and the second is equal to the mixed components of the energy-momentum tensor T^μ_ν multiplied by $\sqrt{g}$. Thus equations (20) lead to the four laws of the conservation of matter. Quite analogous circumstances hold good in our theory if we choose the form (14) for the action-quantity. The five principles of conservation are "eliminants" of the field laws, i.e. they follow from them in a twofold manner, and thus demonstrate that among them there are five which are redundant.

With the form (14) for the action-quantity the Maxwell equations run, for example :—

$$\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}F^{\mu\nu})}{\partial x_\nu} = s^\mu, \quad . \quad . \quad . \quad (21)$$

and the current is

$$s_\mu = \tfrac{1}{4}\left(R\phi_\mu + \frac{\partial R}{\partial x_\mu}\right),$$

where R denotes that invariant of weight -1 which arises from $R^\mu_{\nu\rho\sigma}$ if we first contract with respect to μ, ρ and then with respect to ν and σ. If R^* denotes Riemann's invariant of curvature constructed solely from the $g_{\mu\nu}$, calculation gives

$$R = R^* - \frac{3}{\sqrt{g}} \frac{\partial(\sqrt{g}\phi^\mu)}{\partial x_\mu} + \frac{3}{2}\phi_\mu\phi^\mu.$$

In the static case, where the space components of the electromagnetic potential disappear, and all quantities are independent of the time x_0, by (21) we must have

$$R = R^* + \frac{3}{2}\phi_0\phi^0 = \text{const.}$$

But in a world-territory in which $R \neq 0$ we may make $R = \text{const.} = \pm 1$ everywhere, by appropriate determination of the unit of length. Only we have to expect, under conditions which are variable with time, surfaces $R = 0$, which evidently will play some singular part. R cannot be used as density of action (represented by R^* in Einstein's theory of gravitation) because it has not the weight -2. The consequence is that though our theory leads to Maxwell's electromagnetic equations, it does not lead to Einstein's gravitation equations. In their place appear differential equations of order 4. But indeed it is very improbable that Einstein's equations of gravitation are strictly correct, because, above all things, the gravitation constant occurring in them is not at all in the picture with the other constants of nature, the gravitation radius of the charge and mass of an electron, for example, being of an entirely different order of magnitude (10^{20} or 10^{40} times as small) from that of the radius of the electron itself.*

It was my intention here merely to develop briefly the general principles of the theory.† The problem naturally

* Cf. Weyl, Ann. d. Physik, 54, 1917, p. 133.

† [The problem of defining all W invariants allowable as action-quantities, under the requirement that they should contain the derivatives of the $g_{\mu\nu}$ only to the second order at most, and those of the ϕ_μ only to the first order, was solved by R. Weitzenbock (Sitzungsber. d. Akad. d. Wissensch. in Wien, 129, 1920; 130, 1921). If we omit the invariants W for which the variation $\delta\int W d\omega$ vanishes identically, there remain according to a later calculation by R. Bach (Math. Zeitschrift, 9, 1921, pp. 125 and 169) only three combinations. The real W seems to be a linear combination of Maxwell's L and the square of R. This conjecture has been tested more carefully by W. Pauli (Physik. Zeitschrift., 20, 1919 pp. 457-67) and myself: in particular we succeeded in advancing so far on this basis as to deduce the equations of

presents itself of deducing the physical consequences of the theory on the basis of the special form for the action-quantity given in (14), and of comparing these with experience, examining particularly whether the existence of the electron and the peculiarities of the hitherto unexplained processes in the atom can be deduced from the theory.* The task is extraordinarily complicated from the mathematical point of view, because it is impossible to obtain approximate solutions if we restrict ourselves to the linear terms; for since it is certainly not permissible to neglect terms of higher order in the interior of the electron, the linear equations obtained by neglecting these may have, in general, only the solution 0. I propose to return to all these matters in greater detail in another place.

motion of a material particle. The invariant (14) selected above, at hazard in the first place, seems on the contrary to play no part in nature. Cf. Raum, "Zeit, Materie," ed. 4, §§ 35, 36, or Weyl, Physik. Zeitschr., 22, 1921, pp. 473-80.]

* [Meanwhile I have quite abandoned these hopes, raised by Mie's theory; I do not believe that the problem of matter is to be solved by a mere field theory. Cf. on this subject my article "Feld und Materie," Ann. d. Physik, 65, 1921, pp. 541-63.]