ANGULAR MOMENTUM IN GENERAL RELATIVITY

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Abstract

Although some progress has been made in generalizing the concept of angular momentum to general relativity, until now, no satisfactory definition of angular momentum at null infinity has been given. I here give the first such definition.

The definition applies to strongly asymptotically flat space-times. The angular momentum is given only in terms of physically measurable quantities. Further, the time rate of change of the angular momentum is given solely in terms of parameters measurable *directly* by a Michelson interferometer gravitational wave detector such as LIGO or LISA. I prove the uniqueness of the definition and give a minimum range of its validity. There is reason to believe that there is no physical restriction on the definition. I also give the physical interpretation of the preferred frames at null infinity that arise naturally in the process of finding the unambiguous expression for the angular momentum.

Acknowledgements

I would like to express many thanks to Demetrios Christodoulou for the use of his personal notes and for insightful conversations without which this work would literally not be possible. I, also, thank my wife, Susan, who supported this work by much encouragement and many prayers. Most especially, I express my thanks to God who has "ordered all things in measure, and number, and weight." (Wisdom 11:20)

For my wife and for my parents

Contents

Chapter 1

Introduction

Energy, momentum and angular momentum are pivotal concepts in physics. It is no surprise, therefore, that immediately after conceiving general relativity, Einstein addressed the problem of energy-momentum conservation. He introduced the idea of a *gravitational* energy tensor to pair with the *matter* energy tensor; together they formed a "conservation law." However, there were several problems, including the fact that the initial definition was not covariant and was not symmetric. Non-covariance violates the core principle of relativity and being non-symmetric makes it impossible to define the equally important concept of angular momentum. Einstein refined his idea further by introducing an integral over space and a symmetric tensor into his definition. It was already becoming clear what now is the common consensus; there is no local definition of energy-momentum and angular momentum in classical general relativity [1][2][3][4][5][6]. Hermann Weyl[7] put the nonlocal definition on solid mathematical ground in 1921.

Further light was shed on the problem when Dirac introduced[6] and applied his generalized Hamiltonian formulation to gravitation[8][9] in the 1950's. In the early sixties, Arnowitt, Deser and Misner[10][11][12], under the impetus of the Hamiltonian formulation, gave the definitions for energy and momentum at spatial infinity. There was, in actuality, a veritable renaissance of new work in the early sixties. Bondi, Van der Burg, Metzner[13] and Sachs[14][15](B(V)MS) gave the first results for null infinity, including a definition for energy at null infinity which was later generalized to energy-momentum within the BMS group framework. It was B(V)MS who introduced a group(called the BMS group) that preserved the metric in Bondi's form. In the mid sixties, Penrose[10][16][17] introduced the conformal definition of asymptotic flatness. He and others[18] exploited the fact that the light cone structure, a profound aspect of a space-time, is preserved under conformal transformation. The results were profitable insights into the nature of spatial and null infinity[19][20][21][22][23] [24][25][26][26]. For example, the BMS group was shown to be isomorphic to the group generated by the vectorfields that exist in the unphysical space-time at null infinity which are in some sense the closest approximation to Killing vectors that one can get in an asymptotically flat space-time. Figure 1.1 shows the 2-dimensional conformal diagram for Minkowski space; it brings infinity in to a finite "place" so that quantities at infinity can be treated in a similar manner to quantities at other "places." Newman and Penrose used their complex(spinor) tetrad formalism [27] to give the *peeling theorem* specifying the decay law of the Riemann tensor near null infinity.

Angular momentum took center stage when Penrose, Newman and collaborators made a push towards a definition at null infinity in the late sixties[28][29]. The effort bore fruit in interesting ideas; it lead Penrose to the idea of twistors and Newman to the idea of H-space[30]. However, while it gave important insights into the problem, the effort did not unlock the door to angular momentum. By 1982, Penrose lists finding the definition of angular momentum at null infinity as one of the unsolved problems of general relativity (problem 10 in "Unsolved Problems of Classical General Relativity" [31]).

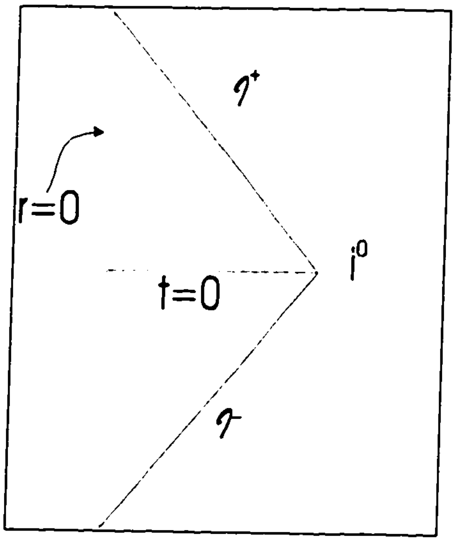

Figure 1.1: The 2-D conformal picture of Minkowski spacetime. It allows one to discuss quantities at infinity. $\Im^+$is future null infinity; $\Im^-$ is past null infintiy; i^0 is spatial infinity.

In 1990, Christodoulou[32] published the outline of the first proof of the nonlinear stability of Minkowski space. The entire proof, due to Christodoulou and Klainerman, was published in 1993[33]. Their research bore fruit for physics immediately; Christodoulou discovered the explicit nonlinear memory[34][35][36]which also gave a clear context for the linear[37][38] memory[1] effect. Their research revealed problems with the conformal picture. It showed the peeling theorem to be incorrect at order[2] $\frac{1}{r^5}$ —thus sending a reminder of the importance of the field equations. Most importantly, it opened up an alternate way of studying null and spatial infinity. This alternate way allows one to take limits of quantities as one goes to null and spatial infinity. These limits can be done in a natural way without having to introduce a conformal

[1]The memory effect refers to the fact that test masses in a Michelson Interferometer antenna will be displaced from their original distance after the passage of a gravitational wave.

[2]In Newman-Penrose's paper, $1/r^5$ decay is a fundamental assumption.

transformation and without assumptions about smoothness at null infinity. The conformal transformations are not uniquely defined so by using them one introduces, among other things(such as changes of topology), ambiguities. The smoothness assumptions maybe so strict that there are no non-trivial space-times that satisfy the standard conformal definition of asymptotically flat. The naturalness and the explicit harmony with the field equations make the Christodoulou-Klainerman approach very attractive. At the very least, it is a second window through which to glimpse the structure of general relativity at infinity.

In this thesis, the power and elegance of the Christodoulou formalism is used to advantage. In chapter 2, the space-time is foliated in a convenient and physically suggestive manner. Important quantities are brought to the fore, and the ground work is set for taking limits to null and spatial infinity. In chapter 3, the quantities and insights gained are applied to the concept of angular momentum, and the definition at null infinity is given. Also in this chapter, the limit mechanism is made mathematically precise, the angular momentum and the rate of change of angular momentum is given at null infinity in terms of variables at null infinity, and the meaning of these variable is elucidated. Finally, an ambiguity in the definition is revealed. Chapter 4 explores and resolves the ambiguity by finding a unique slicing having only the freedom natural to Minkowski space-time. Chapter 5 gives the physical interpretation of the remaining freedom and brings all the analysis together to give a complete picture of the definition of angular momentum at null infinity.

Chapter 2

The Foliation of Space-time

In defining angular momentum in the fully general relativistic regime[1], one needs to be able to have a way of talking about null infinity and spatial infinity in a natural way. A clever choice of foliation can be helpful in dealing with and formulating these two concepts mathematically. Null infinity[2] is of particular interest, for any radiation which travels at the speed of light "goes" to null infinity. In particular, gravitational wave experiments are done in this region. Further, it is at null infinity where a definition of angular momentum has been particularly hard to uncover. These two considerations force an emphasis on 2-D foliations of the 4-D space-time.

We now first give the general properties of such foliations, including a natural tetrad system on the foliation. This will then be followed by a detailed account of the two particular foliations used in this work.

[1] In what follows we will always be assuming that we are looking for a global(not local) definition in an asymptotically flat space-time.

[2] The null structure of the spacetime is also of fundemental interest in its own right. Recognizing the fundamental character of the null decomposition was crucial to Christodoulou and Klainerman's proof of non-linear stability of Minkowski space.

2.1 General Features of 2-D Foliations

A natural choice of frame for such a 2-D foliation, as will be seen, consists of two null vectors and two spatial vectors. Each 2-D surface in the foliations we choose, which I call an S^2 *foliation*, will have:

1. The topology of a sphere(S^2).
2. A tangent plane at each point that can be spanned by two spatial vectors (e_A such that $A \in 1, 2$). Such vectors are always indicated in this paper by an upper case Latin index.
3. A tangent space perpendicular to the e_A 's that can be spanned by two null vectors labeled by $e_4 = l$ and $e_3 = \underline{l}$. They will be normalized such that $l \cdot \underline{l} = -2$. A frame so defined is called a *null frame.*

In providing every point in space-time with two null vectors and two spatial vectors, one can already see a kinship with the Newman-Penrose formalism[39][27]. The null frames have the distinct advantage of having null vectors that point *in* the space-time not in a complex representation of the space-time. For instance, instead of dealing with the shear as a complex quantity, whose tensorial properties one must uncover, here one deals explicitly with a tensor on a 2-D surface (diffeomorphic to S^2).[3]

2.1.1 Ricci Rotation Coefficients

With a null frame established, it is natural to want to write down the Chrisoffel symbols and their relation to the curvature. In a null basis, one deals instead with

[3]Note: for dimension < 4 diffeomorphic $\Leftrightarrow$ homeomorphic

the null Ricci rotation coefficients[4] which are defined by:[5]

([33] pg 147)

$$
\begin{array}{ll}
H_{AB} = \langle D_A e_4, e_B \rangle & \underline{H}_{AB} = \langle D_A e_3, e_B \rangle \\
Z_A = \frac{1}{2}\langle D_3 e_4, e_A \rangle & \underline{Z}_A = \frac{1}{2}\langle D_4 e_3, e_A \rangle \\
Y_A = \frac{1}{2}\langle D_4 e_4, e_A \rangle & \underline{Y}_A = \frac{1}{2}\langle D_3 e_3, e_A \rangle \\
\Omega = \frac{1}{4}\langle D_4 e_4, e_3 \rangle & \underline{\Omega} = \frac{1}{4}\langle D_3 e_3, e_4 \rangle \\
V_A = \frac{1}{2}\langle D_A e_4, e_3 \rangle &
\end{array}
\tag{2.1}
$$

This implies:

$$
\begin{array}{ll}
D_A e_3 = \underline{H}_{AB} e_B + V_A e_3 & D_A e_4 = H_{AB} e_B - V_A e_4 \\
D_3 e_3 = 2\underline{Y}_A e_A - 2\Omega e_3 & D_3 e_4 = 2 Z_A e_A + 2\underline{\Omega} e_4 \\
D_4 e_3 = 2\underline{Z}_A e_A + 2\Omega e_3 & D_4 e_4 = 2 Y_A e_A - 2\Omega e_4
\end{array}
$$

and,

$$
\begin{aligned}
D_B e_A &= \nabla\!\!\!/_B e_A + \tfrac{1}{2} H_{AB} e_3 + \tfrac{1}{2}\underline{H}_{AB} e_4 \\
D_3 e_A &= D\!\!\!\!/_3 e_A + Z_A e_3 + \underline{Y}_A e_4 \\
D_4 e_A &= D\!\!\!\!/_4 e_A + Y_A e_3 + \underline{Z}_A e_4
\end{aligned}
$$

Lapse Transformation Properties:

Note that the normalization $l \cdot \underline{l} = -2$ is preserved under the *lapse transformation*:[6]

$$l \rightarrow a^{-1} l, \quad \underline{l} \rightarrow a\,\underline{l}$$

---

[4]The Ricci coefficients are often seen for the orthonormal basis (tetrad). Here, we deal with a null basis. The null Ricci coefficients given here are the null basis analogies to the Christoffel symbols. (Christoffel symbols apply for a coordinate basis).

[5]D represents the covariant derivative relative to the manifold M with metric, g written as (M,g). $\nabla\!\!\!/$ refers to the covariant derivative *intrinsic to* the surface S, and $D\!\!\!\!/$ refers to the derivative *projected to* the surface, S. $\langle , \rangle$ is the inner product with respect to the metric g.

[6]The physical meaning of this lapse function is discussed in chapter 5. In general, the lapse function, together with its counter part, the shift vector may be described as the non-dynamical variable that tells one how to move forward in time (cf. e.g. [5][10]). Refer to page #43 for further general information on the lapse function.

where : a is called the lapse function

and is any function on the S^2 surface.

Under this transformation the Ricci Coefficients transform as:

$$\begin{aligned} H &\to a^{-1}H & \underline{H} &\to a\,H \\ Z &\to Z & \underline{Z} &\to Z \\ Y &\to a^{-2}Y & \underline{Y} &\to a^{2}\underline{Y} \\ \Omega &\to \tfrac{1}{2}a^{-2}D_l a + a^{-1}\Omega & \underline{\Omega} &\to a\underline{\Omega} - \tfrac{1}{2}D_{\underline{l}}a \\ V &\to V + \nabla\!\!\!/ \ln a \end{aligned} \tag{2.2}$$

The Structure Equations[7], which will be used throughout this thesis since they specify the interrelation between the different Ricci coefficients, are given in Appendix I.

We note now the geometrical meaning of the most important null Ricci coefficients. We will make use of the fact that any symmetric twice covariant tensor, χ_{AB} , on the 2-D surface S can be decomposed as[33]:

$$\chi_{AB} = \hat{\chi}_{AB} + \frac{1}{2}\gamma_{AB}\, tr(\chi)$$

Shear

The shear, $\hat{H}_{AB} = \hat{\chi}_{AB}$ ($\hat{\underline{H}} = \hat{\underline{\chi}}_{AB}$), corresponds to the traceless part of the extrinsic curvature of S, χ_{AB} . It is the shear of the outgoing(respectively, ingoing) null rays. Geometrically, the shear tells one how the shape a small bundle of null rays changes during a short time period.

[7]The structure equations arise from commutation of covariant derivatives applied to the different basis vectors. Hence, they are analogous, in a coordinate basis, to expressing R_{abcd} as a function of the Christoffel symbols. In the appendix, use is also made of the Einstein vacuum equations.

Expansion

The expansion, $tr(\chi)$, $(tr(\underline{\chi}))$ gives the "trace" part of the extrinsic curvature. It tells the amount of expansion of a small bundle of outgoing(respectively, ingoing) null rays in a short time period.

Torsion

For a curve in R^3, with tangent vector $\hat{v}$, and principle normal $\hat{n} = \frac{\partial \hat{v}}{\partial s} / \left|\frac{\partial \hat{v}}{\partial s}\right|$, $\hat{b} = \hat{v} \times \hat{n}$, the binormal, $\{\hat{v}, \hat{n}, \hat{b}\}$ form an orthonormal set at each point. Using the Serret-Frenet Formulae, one defines the torsion as:

$$\frac{\partial \hat{b}}{\partial s} = \text{Torsion} * \hat{n} \quad \text{or} \quad \text{Torsion} = -\frac{\partial \hat{n}}{\partial s} \cdot \hat{b}$$

where : s is the natural parameter of the curve(arc length).

Here "torsion" measures the degree of twisting out of the initial plane of motion. In "2+1" space-time, this would correspond to the null rays twisting, like the grooves of a screw, as they move forward in time.

In a similar way, given a spacelike surface in (g, M), with normals l and $\underline{l}$, with $\underline{l}$ chosen to be a null geodesic field($D_{\underline{l}}\underline{l} = 0$) and l defined such that $l \cdot \underline{l} = -2$ and given $b = (l + \underline{l})/2$, the binormal defined in reference[34], and the principal normal $n = (\underline{l} - l)/2$, one defines the torsion, ζ :

$$\zeta_A = -(D_{e_A} n, b) = (D_{e_A} b, n) = \frac{1}{2}(D_{e_A} l, \underline{l})$$

where : quantities are defined in detail in the affine foliation section.

Remark 1: *This is a previously defined Ricci coefficient,* V_A.

In this thesis, two types of foliations will be used: the maximal and affine foliations. The maximal foliation covers all space-time; it will be discussed first. The affine foliation, an S^2 *foliation*, is not global; it is only a "near null infinity" foliation. It will be discussed last.

2.2 Specifics of the Two Foliations Used: Maximal and Affine

2.2.1 Maximal[33]

Since we are dealing, by choice, with a globally hyperbolic space-time (M, g_{ab}), the initial value problem is well posed and (M, g_{ab}) admits a global time function t(x). It is convenient and physically instructive to foliate the space-time into spacelike leaves, Σ_t, with t = constant on each leaf.[8] This foliation induces a natural coordinate system, (t, x^i). In this coordinate system, $g_{\mu\nu}$ is given by:

$$ds^2 = -\phi^2(t,x)dt^2 + \sum_{i,j=1}^{3} \tilde{g}_{ij}(t,x)dx^i dx^j$$

where : $\tilde{g}_{ij}$ is the induced metric on the leaf, Σ_t

ϕ is called the lapse function.

(a) $T \equiv \{\text{unit normal in direction of Dt}\}$, $T \cdot T = -1$, $T = \phi^{-1}Dt$ ($\tilde{T} = Dt \Rightarrow \tilde{T} \cdot \tilde{T} = \phi^{-2}$)

[8] We can then setup the Cauchy problem by giving the extrinsic curvature k_{ij} and metric $\tilde{g}_{ij}$ on an "initial" slice, say at t = 0. See the next pages for further description of k_{ij} and $\tilde{g}_{ij}$.

(b) We Fermi propagate the $e_i's$: $\overline{D}_T e_i = 0$.[9] This means one's coordinate system doesn't twist anymore than it must due to the space-time curvature.

We make the choice of foliation unique by:

1. Taking $trk = 0$.

   - This is what makes it a maximal foliation for this choice(in Einstein geometry) maximizes the volume of a given t = constant leaf([33] pg 13). For example, foliating Minkowski space[10] in this way, each leaf is a plane.

2. Choosing the lapse function, such that $\lim_{x\to\infty} \phi = 1$ on each leaf Σ_t.

   - This is referred to as normalizing the lapse function at infinity. Normalizing specifies a unique time function whenever there is a non-zero ADM energy(i.e., not Minkowski space). A normalized lapse function means that the maximal hyperplanes are parallel at spatial infinity, that is to say it is a true foliation of space-time. Physically, this means that one remains in the frame of the initial slice; if one starts in the center of momentum frame, where[11] $P^i_{mass} = 0$, one remains there.

3. Choosing the center of momentum frame($P^i_{mass} = 0$) for the initial data.

   - This is the same as saying that the initial data is strongly asyptotically flat(the major restriction put on all space-times in this thesis). Roughly speaking, asymptotic flatness gives us a way of describing isolated bodies.

[9] $\overline{D}$ denotes the projection of D to the tangent space of the foliation.

[10] As an aside, it is interesting to note that the {trace-free symmetric part of k}=0, called an umbilical surface, gives one a sphere.

[11] P is here the ADM linear momentum defined in Appendix II.

Since the mathematical details are somewhat of an aside, they are discussed in Appendix II.

Optical-Maximal Foliation

We may now construct the optical-maximal S^2 foliation. Although we will not use this foliation directly in this thesis, it is included for completeness and for its usefulness in understanding the affine foliation from a different perspective(cf. the end of this chapter). We thus precede to foliate further: to go from a 3-D leaf maximal leaf to a 2-D S^2 leaf. To do this, we make use of the structure provided by the maximal foliation and an "optical function."[12] This function, u, obeys the Eikonal equation $g^{\mu\nu}\partial_\mu u \partial_\nu u = 0$ and has level surfaces that are light cones—specifically outgoing null hypersurfaces. Each hypersurface, Σ_t , intersects a given $u =$ constant light cone in a 2-D surface, $S_{t,u}$, which has the topology of a sphere. One can construct a null frame for this foliation; this consists in picking two null and two spatial vectorfields on each $S_{t,u}$. The natural choice for this foliation is:

1. Two null vectors, l and $\underline{l}$ which are, respectively, the outgoing and ingoing null normals to $S_{t,u}$ and satisfy: $l \cdot \underline{l} = -2$.

 Choices for l and $\underline{l}$:

 (a) The simplest choice in the optical-maximal foliation is:

$$l_{ms} = T + N$$
$$\underline{l}_{ms} = T - N$$

[12]The 2-D foliation of space-time derived from the use of the maximal foliation and optical function as described in the above text and in[33] will be called the *optical-maximal* foliation.

where N is the unit normal to $S_{t,u}$ in Σ_t and the subscript "ms" stands for *m*aximal *s*imple.

(b)

$$l = a^{-1} l_{ms}$$
$$\underline{l} = a \underline{l}_{ms}$$

where a in general could be any function, but most naturally it is chosen such that it is the lapse function of the $S_{t,u}$ in Σ_t. In this case, $l^\mu = g^{\mu\nu}\partial_\nu u$.

2. Two spatial vectors, e_A where $A \in 1, 2$, orthogonal to the above and tangent to $S_{t,u}$.

With choice a, called the standard null pair, one gets the following Ricci rotation parameters:

([33] pg 171)

$$\begin{array}{ll} H = \chi' & \underline{H} = \underline{\chi}' \\ Z = \zeta' & \underline{Z} = \underline{\zeta}' \\ Y = 0 & \underline{Y} = \underline{\xi}' \\ \Omega = \frac{1}{2}\nu & \underline{\Omega} = \frac{1}{2}\underline{\nu} \\ V = \epsilon & \end{array} \tag{2.3}$$

where[13]

[13] ∇ is covariant derivative in Σ_t.

$$\chi'_{AB} = \theta_{AB} - k_{AB} \quad \underline{\chi}'_{AB} = -\theta_{AB} - k_{AB}$$
$$\zeta'_A = a^{-1}\nabla\!\!\!/_A a + \epsilon_A \quad \underline{\zeta}'_A = \phi^{-1}\nabla\!\!\!/_A \phi - \epsilon_A$$
$$\underline{\xi}'_A = \phi^{-1}\nabla\!\!\!/_A \phi - a^{-1}\nabla\!\!\!/_A a$$
$$v = -\phi^{-1}\nabla_N \phi + \delta \quad \underline{v} = \phi^{-1}\nabla_N \phi + \delta$$
$$\varepsilon_A = k_{AN}$$

where :

$a = \frac{1}{|\nabla u|} =$ {the lapse function of the foliation induced by u on each Σ_t}

$\theta_{AB} =$ {the extrinsic curvature of the surfaces $S_{t,u}$ relative to Σ_t}

$k_{ij} =$ {extrinsic curvature of the maximal slice}

$= \frac{1}{2}$Spatial Components$(\mathcal{L}_T\, \tilde{g}_{ij}) = -(2\phi)^{-1}\partial_t \tilde{g}_{ij}$

Decomposition of k : $\eta_{AB} = k_{AB},\ \ \varepsilon_A = k_{AN},\ \ \delta = k_{NN}$

With choice b, called the l-null pair or l-geodesic pair, one gets the following: ([33] pg 351)

$$\begin{aligned} H &= \chi & \underline{H} &= \underline{\chi} \\ Z &= \zeta & \underline{Z} &= \underline{\zeta} \\ Y &= 0 & \underline{Y} &= \underline{\xi} \\ \Omega &= 0 & \underline{\Omega} &= -\omega \\ V &= \zeta \end{aligned} \tag{2.4}$$

where :

$$\chi = a^{-1}\chi' \quad \underline{\chi} = a\underline{\chi}'$$
$$\zeta = \zeta' \quad \underline{\zeta} = \underline{\zeta}'$$
$$\underline{\xi} = a^2\underline{\xi}' \quad \omega = \tfrac{1}{2}a(a^{-1}D_3 a - \underline{v})$$

2.2.2 Affine Foliation

An affine foliation[35][34] begins with a maximal hypersurface Σ_0 with {center of mass momentum}=P=0. This foliation is best understood pictorially(refer to Figure 2.1).

A basic outline of the construction of the affine foliation follows:

1. Construct a surface S_0 in Σ_0.

2. Send out light rays from S_0 in the outgoing direction to generate a forward light cone; designate it the u=0 light cone or C_0^+.

3. Travel to an affine "distance," labeled s, ($l(s) = 1$) from S_0 along the null rays of C_0^+ to a second maximal hypersurface. This 2-surface, $S_{0,s}$, is by definition a surface of constant s with the value of s defined to be[14] $s = \sqrt{\frac{S_{u,s}}{4\pi}}$. This procedure eliminates the ambiguity in the affine parameter and in the normalization of l. Note that S_0 will not, in general, be a surface of constant s since each null ray travels a different affine "distance."

   (a) Construct a 2-D spatial vectorfield orthonormal basis, e_A, on $S_{0,s}$.

   (b) Use the tangent vector to C_0^+ for l.

[14]Here the variable S plays a second role; it represents also the area of the surface S . Recall, up to here, it referred to the surface S itself.

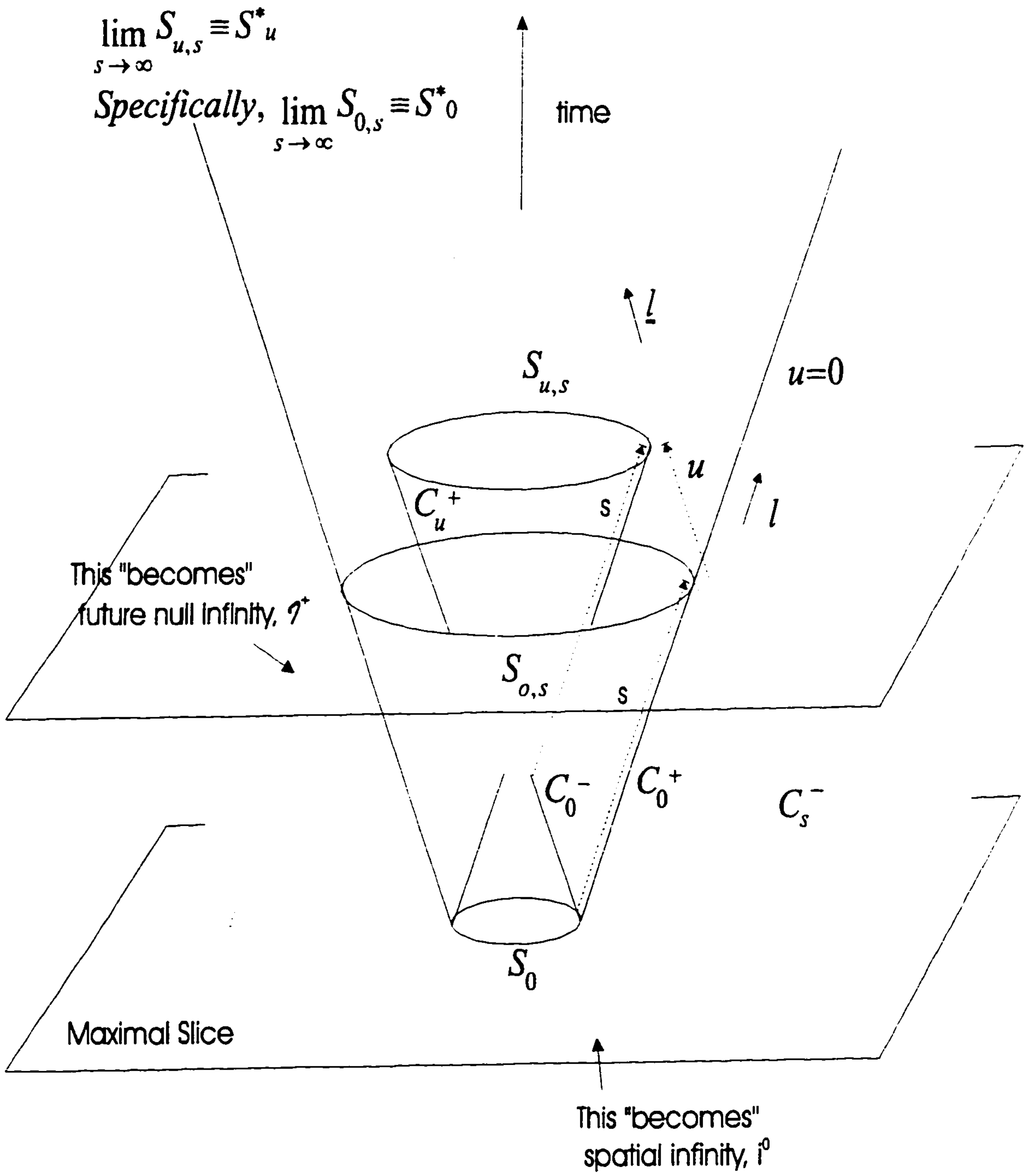

Figure 2.1: Affine Foliation of Space-time (Only valid for s large. Hence, in general, C_0^- is a fictitious device useful for illustration; on the other hand, $C_{s'}^-$ would actually exist for large s'.)

(c) For each point of $S_{0,s}$ construct an ingoing vectorfield $\underline{l}$ such that $l \cdot \underline{l} = -2$.[15]

4. Send a light ray from $S_{0,s}$ in the direction of $\underline{l}$. The past and future of this congruence creates a cone C_s^-.

   (a) Extend $\underline{l}$ to C_s^- by taking it to be the tangent to C_s^-.

   (b) Extend l to C_s^- by taking it to be the conjugate of $\underline{l}$ such that $l \cdot \underline{l} = -2$.

5. Travel an affine "distance" u along the ingoing ray from $S_{0,s}$. This leads one to a new surface $S_{u,s}$. Note that rays coming in from the past to $S_{u,s}$ create a second u =constant forward light cone; call it C_u^+ .

   (a) For later convenience, define $r_{u,s} \equiv \sqrt{\frac{S_{u,s}}{4\pi}}$, and note that one can choose the affine parameter, u, such that $u = 2(r_{0,s} - r_{u,s})$. [16]

   (b) Note the naming convention is:

      i. The central letter denotes shape of surface; in this case "C" for cone.

      ii. The superscript, in the case of a light cone, indicates backward or forward cone.

      iii. The subscript indicates what parameter(s) is held constant to form the surface.

6. For complete rigor work only at $s \to \infty$. It is convenient to label the S^2 leafs of null infinity by defining $\lim_{s\to\infty} S_{u,s} = S_u^*$.

[15]Strictly speaking, off null infinity, specifying that l is a geodesic vectorfield(orthogonal to the surface of constant s as above) and taking $\underline{l}$ to be the conjugate such that $l \cdot \underline{l} = -2$ uniquely specifies $\underline{l}$ and leaves no freedom to also globally set $D_{\underline{l}}\underline{l} = 0$. This can only be done in an approximate sense in space-time close to null infinity.

[16]Note the analogy with Minkowski space:
at event $u = 0$: $2r_0 = s - 0$; and at event $u = u$: $2r = s - u$. Since "s" is a constant, $u = 2(r_0 - r)$.

A *frame* set up in the previous way will be called an *affine frame*. The affine foliation is only useful under the following conditions:

1. There exists a compact spatial region of Σ_t ,$\forall t$, outside of which there is no mass; that is, the space-time is Ricci flat there.

2. One must be suitably near null infinity. This foliation is most useful when one is interested in dealing solely with the 3-D surface, i.e., null infinity, that one obtains in the limit of infinite affine parameter "s." It is roughly valid near null infinity. If one tries to go far in from null infinity, say "down" along C_u^+, the null rays may develop caustics and destroy the foliation. What is more, if one gets so close as to be in the near zone, then the wave approach is no longer even appropriate.

The Ricci rotation parameters for such an affine frame are:

$$\begin{array}{cc} H = \chi_{affine} & \underline{H} = \chi_{affine} \\ Z = \zeta_{affine} & \underline{Z} = -\zeta_{affine} \\ Y = 0 & \underline{Y} = 0 \\ \Omega = 0 & \underline{\Omega} = 0 \\ V = \zeta_{affine} & \end{array}$$

Note that the optical- maximal foliation goes into the affine foliation at null infinity as can be seen by noting that, in equations (2.3) and (2.4) above:

$Y, \underline{Y}, \Omega, \underline{\Omega} \stackrel{r\to\infty}{\longrightarrow} 0$ (because $k_{ij} \to 0, a \to 1, \phi \to 1$)

where : r is the geodesic distance in Σ_t from some origin in Σ_t.

This is relevant for $t \to \infty$, u= constant, because in this limit $r \to \infty$.

Chapter 3

The Null Definition of Angular Momentum

Any definition of angular momentum should satisfy certain reasonable properties. We give the properties in an order that motivates the discussion rather than in order of importance.

3.1 Properties of Angular Momentum

1. At null infinity, it[1](using its time rate of change and initial value) should be expressible solely in terms of measurable properties of gravitational radiation.

2. It should be associated with a measure of "rotation."

3. It should give the intuitive answer expected in Minkowski and Kerr space-times and in some appropriate Newtonian limit.

[1]This will be done assuming one is given the initial value and using the time rate of change of the angular momentum. Hence, one is fundamentally discussing conservation and a link between property 1 and 5 is established.

4. It should be unique, so that there is not more than one way to specify the angular momentum of a given physical thing.

5. It should, in some sense, be conserved.

Since the definition at null infinity is of the most interest for reasons already stated, we will start with property one. For concreteness, picture an experimental apparatus setup to measure gravitational radiation far from the source—i.e., at null infinity. Since the angular momentum carried away from the source by the radiation depends on the radiation pattern over the entire two-sphere around the source, an integral over S^2 must be performed. Several questions immediately arise. Is it really possible to express the angular momentum of the radiation in parameters natural to the experimental setup? What quantity should one integrate over to get the angular momentum? Is there ambiguity in choice of integration surface? If so, what surface should one integrate over? The last two questions are the subject of the next chapter.

The physical intuition that angular momentum *should* be encoded in the stretching[2] of space-time that the Michelson interferometer measures suggests that the answer to the first question is yes. Further, the same experimental considerations lead one to take the role of the shear seriously; theoretical considerations had already driven theoretical investigators[28][29][40] to see the importance of the shear. What is the specific dependence of the angular momentum on the shear(and any other quantities)? This is part of the already posed second question.

[2]Given by the shear in an appropriate coordinate system.

3.2 The Definition

The second question can be answered by consideration of property two above. This property is obviously at the core of what we mean when we say *"angular"* momentum. Specifically, angular momentum should measure the quantity of twisting of the surface along the null rays of the gravitational radiation. This is nothing but the previously discussed Ricci rotation coefficient referred to as torsion, ζ. Hence, we arrive at the *null* definition of angular momentum for strongly asymptotically flat spaces proposed by this thesis:

$$L(\Omega_{(i)}) = \frac{1}{8\pi} \lim_{s\to\infty} \int_{S^2} \zeta_A \Omega^A_{(i)} d\mu_\gamma \tag{3.1}$$

where:

s is the affine parameter illustrated in Figure 2.1 pg 16.

$d\mu_\gamma$ and $\Omega_{(i)}$ [41] are given in pulled back coordinates defined below.

$\Omega^A_{(i)}$ is the e_Ath component in the tangent space of the S^2 surface.

The factor of $\frac{1}{8\pi}$ is obtained from correspondence with the quadropole-Newtonian case(cf. pg 33).

The three rotation vector fields $\Omega_{(i)}$ $(i \in \{1, 2, 3\})$ are defined as follows:

1. On null infinity, take a surface S_0^* (cf. Figure 2.1) and define "rotation" vector fields on it in the following manner:

   (a) Define a diffeomorphism $\phi_0 : S^2 \to S_0^*$. Later, it is shown that this can be taken as simple identity transformation where the S^2 surface is isometric to a unit sphere.

   (b) Define the standard rotation fields $\Omega_{(i)}$ on S^2 (recall $[\Omega_{(i)}, \Omega_{(j)}] = \varepsilon_{ijk}\Omega_{(k)}$).

   (c) Push forward these vector fields onto S_0^*, using $\Omega' = \phi_{0*}(\Omega)$.

2. Extend this definition away from S_0^*:

   (a) Use $\underline{l}$ to move the definition of Ω along null infinity for all S_u^* by setting $\pounds_{\underline{l}}\Omega = 0$.

   (b) Use l to move the definition of Ω into space-time for any $S_{u,s}$ by setting $\pounds_l \Omega = 0$.

   (c) Or, equivalently[3], use the diffeomorphisms defined by the flow $\underline{l}$ and l (resp. $\psi_{0,u} : S_0^* \to S_u^*$, $\Psi_{u,s} : S_u^* \to S_{u,s}$) coupled with $\phi_0 : S^2 \to S_0^*$ to construct the diffeomorphism $\phi_{u,s} : S^2 \to S_{u,s}\, \forall u, s$, (in particular one can construct $\phi_u : S^2 \to S_u^*$). Then, use $\phi_{u,s}$ to push $\Omega_{(i)}$ into any point in space-time.

We already see that the limit involved in the definition given in equation (3.1) is a limit of variables pulled back to an S^2 surface. More precisely, the definition involves taking the limit of all variables pulled back to S^2. To make further progress in understanding the definition we must study the pulled back variables of interest and their interrelation. Hence, the next section discusses the interesting pulled back variables and the section after that gives the equations relating them.

3.3 Pulled Back Variables

Some interesting facts appear when we look at quantities pulled back to S^2 in the limit as one approaches null infinity. First, note that reference[33] proves, for $r \equiv \sqrt{\frac{S_{u,s}}{4\pi}}$, on

[3]To see the equivalence, recall that:
$\pounds_X Y = Lim_{\epsilon \to 0} \frac{Y|\phi(p) - \phi_{\epsilon *} Y|p}{\epsilon}$ where ϕ_ϵ is the flow of X.

a cone of constant u in an optical-maximal foliation, that $s \approx r$ for sufficiently large s. Thus, any limits of large s can be replaced by limits of large r. In the following summary of results, all variables with tildes are in pulled back coordinates.

1. The pulled back metric: $\tilde{\gamma} \equiv \phi^{*}_{u,s}(r^{-2}\gamma)$; $\lim_{s\to\infty} \tilde{\gamma} = \overset{0}{\gamma}$. That is, in these pulled back coordinates, this "cut" of null infinity has the standard metric of the unit sphere. Further, the pulled back gauss curvature, $\lim_{s\to\infty} \tilde{K} = 1$.

2. The pulled back area element with respect to the metric $\tilde{\gamma}$: $\lim_{s\to\infty} d\mu_{\tilde{\gamma}} = d\mu_{\overset{0}{\gamma}}$ which is the area element on a standard unit sphere. Furthermore, the covariant derivative with respect to $\tilde{\gamma}, \tilde{\nabla}$, becomes ∇^0. Here the zero superscript refers to with respect to $\overset{0}{\gamma}$.

In this same manner, all variables can be pulled back to S^2. Quantities of interest will be given followed by equations in coordinates natural to an affine foliation. The Key Quantities[35] and their definitions are:

Name	**Description**	**"Pulled Back" Limit** $\lim_{s\to\infty}\phi^*_{u,s}(\ )$
γ	Metric on 2-surface	$(r^{-2}\gamma)\to\gamma^O$
$\hat{\chi}$	Traceless part of Extrinsic Curvature (ℓ)	$(\hat{\chi})\to\Sigma$ Tracefree relative to γ^o
$\hat{\underline{\chi}}$	Traceless part of Extrinsic Curvature ($\underline{\ell}$)	$(r^{-1}\hat{\underline{\chi}})\to\Xi$ Tracefree relative to γ^o
$tr(\chi)$	measures change in dA as move along s (ℓ)	$(r\,tr\chi)\to 2$ $\chi(X,Y)\equiv\nabla_X\ell\cdot Y$
$tr(\underline{\chi})$	Measures change in dA as move along u ($\underline{\ell}$)	$(r\,tr\underline{\chi})\to -2$
h	$h\equiv rtr\chi-2$	$(rh)\to H$
ζ	Torsion $\zeta(X)=\frac{1}{2}g(\nabla_X\ell,\underline{\ell})$	$(r\zeta)\to Z$
$\alpha,\underline{\alpha}$	$\underline{\alpha}(X,Y)\equiv R(X,\underline{\ell},Y,\underline{\ell})$ $\alpha(X,Y)\equiv R(X,\ell,Y,\ell)$	$(r^{-1}\underline{\alpha})\to A$
$\underline{\beta}$	$\underline{\beta}(X)\equiv\frac{1}{2}R(X,\underline{\ell},\underline{\ell},\ell)$	$(r\underline{\beta})\to B$
β	$\beta\ (X)\equiv\frac{1}{2}R(X,\ell,\underline{\ell},\ell)$	$r^3\beta\to I$ (Kerr)
ρ	$\rho\equiv\frac{1}{4}R(\underline{\ell},\ell,\underline{\ell},\ell)$	$(r^3\rho)\to P$
σ	$\sigma\,\epsilon(X,Y)=\frac{1}{2}R(X,Y,\underline{\ell},\ell)$	$(r^3\sigma)\to Q$
μ	Mass aspect function $\int_s\mu dA_\gamma=\frac{8\pi m}{r}=\int_S\underline{\mu}dA_\gamma$	$(r^3\underline{\mu})\to\underline{N}$
$\underline{\mu}$	Conjugate mass function	$r^3\underline{\mu}\to\underline{N}$
m	m = hawking mass $=\frac{r}{2}(1+\frac{1}{16\pi}\int tr\chi\,tr\underline{\chi}dA_\gamma)$	$m\to M(u)$
$X,Y\equiv$ arbitrary vectors tangent to S		

3.4 Pulled Back Equations at Null Infinity in Affine Foliation

One gets the following equations that relate the various pulled back variables by making use of the Structure Equations(pg 57) in $\lim_{s\to\infty}$ with the appropriate r-weighting:[4]

The null Codazzi equation and its conjugate:

$$\nabla\!\!\!/^0 \cdot \Sigma = \frac{1}{2}\nabla\!\!\!/^0 H + Z \tag{3.2}$$

$$\nabla\!\!\!/^0 \cdot \Xi = B \tag{3.3}$$

The Hodge system for the torsion, which is a statement of geometry:[5]

$$\nabla\!\!\!/^0 \times Z = Q - \frac{1}{2}\Sigma \wedge \Xi \tag{3.4}$$

$$where : \nabla\!\!\!/^0 \times Z \equiv \varepsilon^{AB}\nabla\!\!\!/^0 Z_B$$

$$\nabla\!\!\!/^0 \cdot Z = \underline{N} + P - \frac{1}{2}\Sigma \cdot \Xi \tag{3.5}$$

The statement that H is independent of retarded time:

$$\frac{\partial H}{\partial u} = 0 \tag{3.6}$$

The relation between the two null shears:

[4]Confer Chapter 17 of [33]. The equations given here are found in [35]. Recall that all results depend on the assumption of strong asymptotic flatness.

[5]The divergence equation represents a new geometrical insight found by Christodoulou. The wedge is defined as: for S-tangent symmetric, traceless 2-forms: $u \wedge v = \varepsilon^{AB} u_{AC} v^C_B$. For S-tangent 1-forms: $w \wedge z = \varepsilon^{AB} w_A z_B$.

$$\frac{\partial \Sigma}{\partial u} = -\frac{1}{2}\Xi \tag{3.7}$$

Conservation of Energy in differential and integral form respectively:

$$\frac{\partial \underline{N}}{\partial u} = -\frac{1}{4}|\Xi|^2 \tag{3.8}$$

$$\text{where} \quad : \quad M = \frac{1}{8\pi}\int \underline{N}\, d\mu_{\gamma^0} \tag{3.9}$$

$$\frac{\partial M}{\partial u} = -\frac{1}{32\pi}\int_{S^2} |\Xi|^2\, d\mu_{\gamma^0} \tag{3.10}$$

Hence, $|\Xi|^2 \propto$ gravitational wave power per unit solid angle.

Remark 2: *Given either $\{H$ and $\Sigma\}$ or $\{H$ and $Z\}$, one can calculate all other variables.*

At this point, a couple interesting observations can already be made:

1. Equation (3.8) together with the decay rate of ($\Xi = O(|u|^{-1.5})\, as\ |u| \to \infty$) imply that there is no radiation at the "beginning" or "end" of time (*i.e.*, $u \to \pm\infty$)[34][33].

2. If one knows something about the source[6], then from knowledge of $\Xi(u)$ at one point or a small patch on S^2, one can calculate $\Xi(u)$ over the whole sphere.

3.5 The Functional Form of $L(\Omega)$

Now we are ready to begin to calculate the null angular momentum as a function of measurable parameters. In standard pulled back coordinates, it will be useful to recall that, for large r:

[6] For example, if one knew the quadrupole moment($l = 2$) approximation applied, one could calculate the angular momentum from knowledge of $\Xi(u)$ (thus Σ) at one point on the sphere.

$$\begin{aligned} \hat{\chi}_{AB} &\approx \Sigma_{AB} \\ \zeta_A &\approx \frac{Z_A}{r} + \frac{Z_A^{(2)}}{r^2} \\ tr(\chi) &\approx \frac{2}{r} + \frac{H}{r^2} \end{aligned} \tag{3.11}$$

Using equation "k" of the structure equations in Appendix I (pg 57), the relation between the Lie and covariant derivatives, and the value of the Ricci coefficients in the affine foliation one obtains:

$$D\zeta = \frac{\partial\zeta}{\partial s} = -\chi \cdot \zeta - \beta \tag{3.12}$$

Other relevant equations:

$$A(S) \equiv Area\,of\ S = 4\pi r^2 = \int_S d\mu_\gamma \tag{3.13}$$

$$\begin{aligned} D\gamma &= \frac{\partial\gamma}{\partial s} = 2\chi; \quad (\underline{D}\gamma = 2\underline{\chi}) \\ Dd\mu_\gamma &= tr\chi d\mu_\gamma; \ (\underline{D}d\mu_\gamma = tr\underline{\chi}d\mu_\gamma) \end{aligned} \tag{3.14}$$

where D and $\underline{D}$ are respectively the projection to S of $\mathcal{L}_l$ and $\mathcal{L}_{\underline{l}}$.

Using these equations one can obtain:

$$d\mu_\gamma = r^2\{1 - \frac{1}{r}(H - \bar{H}) + O(r^{-1.5})\}d\mu_{\gamma^0} \tag{3.15}$$

One then derives, taking advantage of the fact that $\mathcal{L}_\Omega\gamma^0 = 0$ at null infinity and calculating $\frac{\partial(r\zeta)}{\partial s}$ to obtain ζ to order $1/r^2$ in known variables.

$$L(\Omega_{(i)}) = \frac{1}{8\pi}\int(\Sigma_{AB}\nabla\!\!\!/_C \cdot \Sigma^{CB} + I_A)\Omega^A_{(i)}d\mu_{\gamma^0} \tag{3.16}$$

Here one assumes that $\lim_{s\to\infty} r^3\beta_A = I_A$ which, in general, is not true(in general, $|\beta| = O(r^{-3.5)}))$, but is true for Kerr space-time; Kerr is an important test of the equation (3.16) covered later in this chapter. More to the point, one can show, in general, that the above integrand (equation (3.1)) is well defined in(whether I exists or not).[7]

3.5.1 Gauge Freedom in the Definition

Are there any "gauge" freedoms in equation (3.16)? That is, are there any non-physical degrees of freedom? Yes. They are:

1. One can change $l \to a^{-1}l$ and $\underline{l} \to a\,\underline{l}$ in such a way that in the limit one still has an affine foliation; i.e., $\lim_{s\to\infty} a = 1$. It is obvious that such a transformation does not change the value of $L(\Omega)$. This is just the type of mathematical gauge freedom that one desires; that is, the equation representing the physical quantity is invariant under the gauge transformation.

2. One can pick different starting values of Σ. The definition obviously does depend on this choice. Here the transformations have $\lim_{s\to\infty} a(u,\theta,\phi) \neq 1$; the transformations of the type described in (1.) are a subset of these transformations. As will be seen, "a" is closely connected with specifying a rest frame. To make sense of the definition, we are forced to speak of coordinates and rest frames. We will be faced with the fact that angular momentum, even classically, is inherently a coordinate dependent quantity. This classical freedom is not, however, the only degree of freedom hiding in this general transformation.

[7] The procedure to show this is to use the null Codazzi equation(structure equation h in Appendix III in an affine frame) to eliminate β in equation (3.12) and show that the derivative of $\partial L(\Omega)/\partial s$ falls off as $1/r^{-3/2}$. This involves using the decay rates $\mathcal{L}_\Omega tr(\chi) \approx (\mathcal{L}_\Omega H)/r^2 + O(r^{-5/2})$, $\mathcal{L}_\Omega \hat{\chi} \approx \mathcal{L}_\Omega \Sigma + O(r^{-1/2})$, $\mathcal{L}_\Omega l = 0$ and Ω killing and calculation similar to Example one.

The notion of Σ freedom has occupied investigators for some time[28][29][31][40]. These compelling issues will be explored in detail later(especially with respect to $\frac{\partial L(\Omega)}{\partial u}$).

For now, let us explore other features of equation (3.16).

3.5.2 Minkowski Space

In Minkowski space a good definition should give $L(\Omega) = 0$. In this case, on any given S^{*}_{u} there is no curvature; neither "static"($I = 0$) nor gravity waves ($\Xi = 0$). It will be shown later[8] that this implies $\nabla^{C} \cdot \Sigma_{CA} = \nabla_A \phi_{odd}$ for some function ϕ_{odd} on S^2. Proving $L(\Omega) = 0$ for Minkowski space is a short calculation, and will serve as a representative sample of the type of calculations needed throughout this thesis.

Example 1: *We start with:* $L(\Omega_{(i)}) = \frac{1}{8\pi}\int(\Sigma_{AB}\nabla_C \cdot \Sigma^{CB})\Omega^{A}_{(i)} d\mu_{\gamma^0}$
substituting the odd part only restriction (i.e., $\nabla_C\Sigma^{CB} = \nabla^B\phi_{odd}$*) yields:* [9]

$$L(\Omega) = \int_S \Sigma_{AB}\nabla^B\phi_{odd}\Omega^A d\mu_{\gamma^0}$$

Integration by parts yields:

$$L(\Omega) = -\int_S \nabla^B\Sigma_{AB}\,\phi_{odd}\,\Omega^A d\mu_{\gamma^0} - \int_S \Sigma_{AB}\,\phi_{odd}\nabla^B\Omega^A d\mu_{\gamma^0}$$

Because Σ_{AB} *is symmetric, the second integral becomes:*

$$\frac{1}{2}\int_S \phi_{odd}\,\Sigma^{AB}\,(\nabla_A\Omega_B + \nabla_B\Omega_A) d\mu_{\gamma^0}$$

Since Ω *is killing and* Σ *is symmetric, the integrand is zero.*

[8]The part of Σ which can be expressed in this way is called electric "part" or "odd parity" part of the shear.

[9]Anticipating the irrelevance of the factor $1/8\pi$, it will be dropped to simplify the exposition.

This leaves:

$$L(\Omega) = -\int_S \nabla_A \phi_{odd}\ \phi_{odd} \Omega^A d\mu_{\gamma^0} = \int_S \frac{1}{2} \nabla_A (\phi_{odd}^2)\ \Omega^A d\mu_{\gamma^0}$$

Hence,

$$L(\Omega) = 0.$$

3.6 The Functional Form of $\dot{L}(\Omega)$

We note that, in principle, one can experimentally measure all the parameters in the definition given by equation (3.16). Although Σ is measurable from gravitational radiation experiments, "I ", the curvature term, is only measurable by other methods. In short, property one (cf. pg 19) has not yet been met; we cannot calculate the angular momentum by a gravity wave experiment using this definition. Here we must make an important distinction turning on the question: are we discussing the angular momentum of the source or of the wave? We must be discussing the source because, taking the most important reason, the angular momentum of a wave at a point in space-time, like its energy, is not a well defined concept. By contrast, the rate of angular momentum transport[10], $\dot{L}(\Omega)$, is well defined. Further, if one knows $\dot{L}(\Omega)$ as a function of measurable parameters of the gravity wave, one then has, as far as is possible, L_{Source} as a function of measurable gravity wave parameters. One proceeds as follows:

1. Before the source begins to radiate, measure $L_{initial}(\Omega)$ using equation (3.16) or other method.

[10] Of course, $\dot{L}(\Omega)$ is, by conservation of L, the same for the source and the wave.

2. Measure gravity wave parameters during the emission phase and calculate angular momentum carried away by gravitational radiation using $\dot{L}(\Omega$, measurable parameters of gravity waves). Then use the following simple equation to get the instantaneous L of the source:

$$L(t) = L_{initial} + \int_0^t \dot{L}\, dt'$$

There remains a major defect in the above argument. We do not yet have an expression that gives the rate of change of $L(\Omega)$, $\dot{L}(\Omega)$, as a function of easily measurable parameters. Such an expression can be obtained by using the Bianchi identities (ref. [33] pg 161equation (7.3.11c)) and a long series of manipulations similar to those illustrated in example one and in the previous section. One obtains:

$$\begin{aligned} \frac{\partial L(\Omega)}{\partial u} &\equiv \\ \dot{L}(\Omega) &= \frac{1}{8\pi}\int_S (-\Xi_{AB}\nabla\!\!\!/_C\Sigma^{CB} + \frac{1}{2}(\Sigma_{AB}\nabla\!\!\!/_C\Xi^{CB} - \Sigma^C_B\nabla\!\!\!/^B\Xi_{CA}))\Omega^A d\mu_{\gamma^0} \\ where &: \text{u is the previously defined retarded time.} \end{aligned} \tag{3.17}$$

Remark 3: *Recall that, in a free fall Fermi-normal frame, changes in Σ correspond to changes in separation of the Michelson interferometer test masses and Ξ is the speed of that separation.*

3.7 Tests of the Null Definition

3.7.1 Minkowski

Remark 4: *One verifies by inspection the requirement that $\dot{L}(\Omega) = 0$ for Minkowski space.*

3.7.2 Kerr

We will make use of equation (3.16) to check the angular momentum of the Kerr solution.

1. Take the spatial S^2 surfaces to be the surfaces $r = t =$ constant slices of Kerr metric in standard coordinates.

   (a) Define the outgoing(l) and ingoing($\underline{l}$) null normals perpendicular to each such surface and normalize in the standard fashion($l \cdot \underline{l} = -2$).

   (b) The spatial coordinates on the surface can be taken to be the standard angular ones in the Kerr metric.

2. One can then show that Σ, the shear at null infinity of the null geodesics with tangent l, vanishes.

3. One is left with $L(\Omega_{(i)}) = \frac{1}{8\pi} \int I_A \Omega^A_{(i)} d\mu_{\gamma^0}$ where I is defined on page 23. If one defines the "axis" of symmetry along the z direction, and substitutes the appropriate expressions for $\Omega_{(i)}$ and I, one gets only the one component of angular momentum:

$$
\begin{aligned}
L_z &= M\, a \\
\textit{where} &: M \text{ is mass} \\
& \quad a \text{ is Kerr parameter}
\end{aligned}
$$

Remark 5: *Hence, the proposed definition gives the anticipated result for the Kerr solution.*

3.7.3 Quadrupole Approximation Case

In this case, we seek a test of the radiative aspect. We make use of a modified form of equation (3.17). We consider two, for simplicity, equal mass bodies rotating around each other in obedience to the laws of Newtonian gravity, except emitting quadrupole radiation.

1. We use the quadrupole approximation and take:

$$\Sigma_{ij}^{Linear} = Q_{ij}^{TT}$$

where : Σ_{ij}^{Linear} is the linear part of the shear

Q^{TT} is the transverse traceless

component of the quadropole moment

2. We use equation (3.7) and $F = \frac{1}{8}\int_{-\infty}^{+\infty} |\Xi|^2 \, d\mu_{\gamma^0}$ to calculate F which appears as a source term for the non-linear part of the shear. Substituting into the equation given in references [34] and [35] gives the nonlinear part, $\Sigma^{NonLinear}$.

3. Substituting into the equation for $\dot{L}(\Omega)$, using an algebraically identical version of equation (3.17), one reproduces the "Newtonian" result that one would expect for a source radiating in a quadrupole fashion.

Remark 6: *Hence, the proposed definition of angular momentum gives the anticipated result for the quadrupole approximation case.*

So, having found property one, two and three to be satisfied, we come to property four: Is the definition unique? This raises the specter of the ambiguity of what surface one integrates over and what to do with the freedom in l already mentioned. These two problems are really the same problem. We now turn to these considerations.

Chapter 4

A Unique Slicing of Null Infinity

The goal of this chapter is to find a way to reduce the degrees of freedom of the null definition and thereby find a unique choice of surface to do the surface integrals over. Since the *affine frame* previously defined (using l, $\underline{l}, e_1, e_2$ with $e_{1,2}$ tangent to surfaces of constant u) gives quantities that are intrinsic to the null infinity cone, $\Im^+$, we will express all quantities in this frame. However, we will not confine ourselves to integrating over just surfaces of constant affine parameter, u, we will allow more general integration surfaces.

4.1 The Decompositon of the Shear

To begin, if one does choose to integrate over surfaces of constant affine parameter *in the affine frame,* and is given the "news" function, $\Xi(\theta, \phi, u)$, one still has to specify one functional degree of freedom that is entangled in three functions:[1]

4.1.1 The Degree of Freedom in Affine Frame

1. A scalar function $H(\theta, \phi)$, given once and valid for all times.

[1]Here all unprimed variables represent variables with respect to the affine frame.

2. Two functions in the form of a 2-covariant symmetric traceless tensor on the pulled-back to S^2 2-D slice of null infinity, $S_u^\bullet$, at one instant of time, u. That tensor is:

$$\Sigma_{AB}(\theta,\phi) = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & -\sigma_1 \end{pmatrix}$$
$$where : \sigma_1 \text{ and } \sigma_2 \text{ are arbitrary functions}$$

The two degrees of freedom of Σ can also be expressed in terms of two different functions, ϕ_{odd}, ϕ_{even}, as follows:[2]

(a) Note $\nabla^B \cdot \Sigma_{BA}$, which is a one-form on (S^2, γ^0_{AB}), can be decomposed into two pieces using Hodge Decomposition Theorem on (M, g):

$$V_r = d\alpha_{r-1} + d^\dagger \beta_{r+1} + \gamma_r \tag{4.1}$$

where :

$$d^\dagger \equiv (-1)^{mr+m+1} * d * \text{ (for } (M, g_{\alpha\beta}) \text{ Riemannian)}$$

subscripts on V, α, β, γ indicate the degree of the form.

$$m = \dim M$$

and γ_r is harmonic : $\Delta\gamma_r = 0$

i. In the chosen case, m={dimension of manifold}=2, take $r = 1$. Also, $\gamma_1 = 0$.

One, thus, obtains the sought after decomposition:

$$\nabla^B \cdot \Sigma_{BA} = \nabla_A \phi_{odd} + \varepsilon_{AB} \nabla^B \phi_{even} \tag{4.2}$$

[2]There is no general connection between $\sigma 1$, $\sigma 2$ and respectively ϕodd, $\phi even$. However, if all functions depend on the single angular variable θ, i.e. no ϕ dependence: upon setting $\phi even$ (ϕodd) to zero, one can put all the dependance into $\sigma 1$ ($\sigma 2$).

$$where \quad : \quad \text{the } \phi_{odd}\,,\ \phi_{even} \text{ are functions on } S^2$$

A. The first term corresponds to that portion of the divergence of the shear that has odd parity[3] and thus behaves like a standard vector, for example in E&M, the electric field. This part is called the *odd* or *electric* part of the shear.

B. The second term corresponds to that portion of the divergence of the shear that has even parity and thus behaves like an axial vector, for example in E&M, the magnetic field. This part is called the *even* or *magnetic* part of the shear.

C. Hence, the shear can be written:

$$\begin{aligned} \Sigma_{total} &= \Sigma_{odd} + \Sigma_{even} \\ where &: \ div\Sigma_{odd} = \nabla\!\!\!/_A \phi_{odd} \\ div\Sigma_{even} &= \varepsilon_{AB} \nabla\!\!\!/^B \phi_{even} \end{aligned}$$

4.1.2 The Degree of Freedom in a Non-Affine Frame

In a general *non-affine frame*, specified by $l \to a^{-1} l$, $\underline{l} \to a\,\underline{l}$, $a(\theta, \phi, u)$, called the lapse function for reasons discussed later, is a new variable. A transformation of this kind is referred to as a *lapse transformation.* We can use this new variable to set H to some desired function of θ and ϕ at each value of the u: mathematically, H becomes $H(u)(\theta, \phi)$. We have no qualms about doing this because H has no direct physical meaning. What functional dependence shall we choose for H? Or, said equivalently,

[3]Under parity change: a vector with tail at the origin transforms as $\theta \to \pi - \theta, \phi \to \pi + \phi$, so that $\frac{\partial}{\partial\theta} \to -\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\phi} \to \frac{\partial}{\partial\phi}$.

what is $a(u, \theta, \phi)$? An examination of the null Codazzi equation (equation (3.2)) gives the clue to the solution; H enters the equation as an electric("odd") part of the shear. Let us make this analysis explicit.

First note that the null Codazzi equation at null infinity and its conjugate are invariant under change of lapse function. Further, the Hodge system for the torsion (pg 25)is also invariant under lapse transformation. Hence, we can work directly with these four equations as already given. To begin, re-express the Hodge system equations in terms of the functions ϕ_{odd} and ϕ_{even}.

1. Substitute the expression for $\not{\nabla}^B \cdot \Sigma_{BA}$ into the null Codazzi equation (equation (3.2))

2. Solve this equation for Z and substitute this expression for Z into the Hodge system equations. One obtains:

The Hodge curl equation becomes:[4]

$$-\not{\Delta}\phi_{even} = Q - \frac{1}{2}\Sigma \wedge \Xi \tag{4.3}$$

The Hodge divergence equation becomes:

$$\not{\Delta}(\phi_{odd} - \frac{1}{2}H) = \underline{N} + P - \frac{1}{2}\Sigma \cdot \Xi \tag{4.4}$$

For further insight, let us use these facts to look at the simplest possible case. In Minkowski space, $Q = 0$, $\Xi = 0$. Hence, the even(magnetic), part of the shear *must* be zero, ϕ_{even} = constant or $\Sigma_{even} = 0$. On the other hand, the odd(electric) part of the shear is free because of the H freedom. A convenient value for a free parameter

[4] Recall that in equations (4.3) and (4.4), Σ carries an implicit functional dependence on ϕ_{odd} and ϕ_{even}.

is zero so we set $\phi_{odd} = 0$ or $\Sigma_{odd} = 0$ for all times. Thus, in Minkowski space we have $\Sigma = 0$ at all times.

Equation (4.4) with $\phi_{odd} = 0$ gives the sought after expression for H, but the lapse function is really the interesting physical quantity, and we have not determined it yet. We will do this by using the propagation equation for the shear along the direction $\underline{l}' = a\,\underline{l}$. In other words, we will find the lapse function on $\Im^+$ that makes the electric part of the (ingoing) shear, using the tetrad with l and $\underline{l}$ (not the transformed ones), zero at each point on all surfaces of constant u' (where u' is defined by $\underline{l}'(u') = 1$).

4.2 Set the Odd Parity Part of the Shear to Zero

Using the structure equation[5] "1" in Appendix I on page 57, with $l \to a^{-1} l$ and $\underline{l} \to a\,\underline{l}$, gives in pulled-back coordinates:

$$
\begin{aligned}
2\frac{\partial \Sigma_{AB}}{\partial u'} &= 2\nabla\!\!\!/_A \nabla\!\!\!/_B \psi - \gamma_{AB} \Delta\!\!\!/ \psi - \psi \Xi_{AB} \qquad (4.5) \\
\textit{where} &: \lim_{s \to \infty} a = \psi \\
& \Sigma, \Xi \text{ are with respect to affine frame.}
\end{aligned}
$$

To find the equation for ψ when the odd part of the shear is zero, operate on equation (4.5) from the right with $\nabla\!\!\!/^A \nabla\!\!\!/^B$. Then, use the fact that $\Sigma_{odd} = 0 \iff \nabla\!\!\!/^A \nabla\!\!\!/^B \Sigma_{AB} = 0$. One finds ψ must satisfy the following equation if $\Sigma_{odd} = 0$ along each u'=constant surface:

$$
\Delta\!\!\!/^2 \psi + 2\Delta\!\!\!/ \psi = \nabla\!\!\!/^A \nabla\!\!\!/^B (\psi \cdot \Xi_{AB}) \qquad (4.6)
$$

[5] $H \to \chi$, $Z \to \zeta$, $Y \to 0$. One gets:
$D\!\!\!/_l \hat{\chi}_{AB} + \frac{1}{2} tr\underline{\chi}\, \hat{\chi}_{AB} = (\nabla\!\!\!/_B \zeta_A + \nabla\!\!\!/_A \zeta_B - \nabla\!\!\!/_C \zeta^C \gamma_{AB}) + 2\underline{\Omega} \hat{\chi}_{AB}$
$-\frac{1}{2} tr\chi \hat{\underline{\chi}}_B + +(\zeta \hat{\otimes} \zeta)_{AB}$

4.2.1 Special Case of Minkowski Space

In Minkowski space-time, $\Xi = 0$; hence, the equation reduces to:

$$\Delta^2\psi = -2\Delta\psi \tag{4.7}$$

It is easily seen that the solutions to this equation are:

$$\psi(\theta,\phi,u) = \sum_{j=0}^{j=1}\sum_{m} c_{j,m}(u)\, E_{j,m} = \psi_{sourceless} \tag{4.8}$$

$$where : \Delta E_{j,m} = -j(j+1)E_{j.m} \tag{4.9}$$

($i.e.$, $E_{l.m}$ are real analogies to $Y_{l,m}$'s)

c_j are arbitrary functions of u

Written out explicitly the solution is:

$$\psi_{sl}(\theta,\phi,u) = c_0(u) + c_1(u)\,\cos(\theta) + c_2(u)\sin(\theta)\cos(\phi) + c_3(u)\sin(\theta)\sin(\phi) \tag{4.10}$$

In Chapter #5, we will illustrate pictorially how this lapse function arises from the geometry of Minkowski space-time..

4.2.2 General Case

In general, one would like to show uniqueness of the solution up to the freedom available in Minkowski space. The proof of uniqueness is given in Appendix III. The proof is limited to cases where $\sup_{S^2}\Xi_{AB}\Xi^{AB} < 16$. In a coordinate basis constructed of the eigenvectors of Ξ one can write:

$$\Xi_{AB} = \begin{pmatrix} f & \\ & -f \end{pmatrix}$$

$$where \quad : \quad |f| < \sqrt{8}$$

Putting equation (3.10) in "nongeometrized" units, one gets:

$$\left|\frac{\partial M}{\partial u}\right| = \frac{1}{32\pi}\frac{c^3}{G}\int_{S^2} |\Xi|^2 \, d\mu_{\gamma^0} < 2\frac{c^3}{G} \approx 10^{36}\frac{Kg}{\text{sec}}$$

or

$$\frac{\partial M}{\partial u} < 10^6 M_{\odot}/\,\text{sec} \tag{4.11}$$

An extremely reasonable restriction on the gravitational radiation.

The general solution is:

$$\psi(\theta, \phi, u) = \psi_{sl}(\theta, \phi, u) + \Psi(u)(\theta, \phi) \tag{4.12}$$

$$where \quad : \quad \Psi \text{ is unique and inherits its time dependence from } \Xi.$$

The above notation, $\Psi(u)(\theta, \phi)$, is meant to point to the fact that equation (4.6) is a different equation for a different $\Psi(\theta, \phi)$ at each moment of retarded time, u. This is because the source terms in the equation for Ψ can be new functions of θ and ϕ at each new moment of time. The source terms can be seen in equation (8.2) on page 63; they are: Ξ, which is called the news function, and ψ_{sl}.

4.3 The Unique Slicing

With the lapse function calculated, we now can identify a surface of constant u' with the property that the odd(electric) part of Σ is zero everywhere along it. To make the surface of integration unique so that we can use it in the null definition of $\dot{L}(\Omega)$, we will only have to choose values for the $c_i(u)$ and to pick a Lorentz frame. In Minkowski space and in the general case, $c_{1,2,3} = 0$ and $c_0 = 1$ are singled out. Hence, in summary, the unique slicing of $\Im^+$ is given by the following steps:

1. Find u' = constant surfaces such that $\nabla^B \cdot \Sigma_{BA} = \varepsilon_{AB}\nabla^B \phi_{even}$ (i.e., odd part of shear is zero).

2. Choose $c_{1,2,3} = 0$ and $c_0 = 1$

Remark 7: *One still has the freedom to boost once initially. In the affine slicing, this means picking the initial maximal slice with $P \neq 0$. For elaboration on this point see chapter #5.*

Remark 8: *Above is valid* at least *when* $\sup_{S^2} \Xi_{AB}\Xi^{AB} < 16$.

Remark 9: *There are arguments which indicate that there is a limit on $|\Xi|^2$ above which one must have white holes.*[6] *Further, white holes can not be present in an evolutionary space-time unless they are given on the initial data set. Since no evidence or reason currently warrant supposing white holes, the mathematical restriction on the uniqueness* ($\sup_{S^2} \Xi_{AB}\Xi^{AB} < 16$) *appears again (cf. equation (4.11)), from a different perspective, to be* not *a physical restriction.*

[6] D. Christodoulou has shown rigourously that there is such a limit in the case of radiating spherically symmetric scalar fields.

Chapter 5

The Interpretation of the Preferred Cuts of Null Infinity

To understand the preferred foliation of $\Im^+$, summarized on page 40, we must first understand the lapse function, $a(\theta, \phi, u)$. A complete interpretation of the lapse function requires us to manifest the link between the foliation at $\Im^+$ and the "interior" space-time. Essentially, one would like to know the link between the slicing of null infinity and the source of gravitational radiation that one is trying to analyze. There are two basic steps in specifying the preferred foliation:

1. Set the odd(electric) part of the shear to zero.

2. Choose $c_{1,2,3} = 0$ and $c_0 = 1$

We would like to uncover the physical significance of each restriction. Both restrictions involve the lapse function. The lapse function definition is an appropriate point to begin the process of digging for physical significance.

5.1 The Lapse Function

Mathematically, the lapse function is the a in the following transformation that preserves $l \cdot \underline{l} = -2$:

$$\begin{aligned} l' &= a^{-1} l \\ \underline{l}' &= a\, \underline{l} \\ \textit{where} &: \ l \text{ and } \underline{l} \text{ are the vectors tangent to the} \\ & \quad \text{previously discussed null geodesics} \end{aligned}$$

This implies:

$$\frac{du}{du'} = \underline{l}'(u) = a\, \underline{l}(u) = a \tag{5.1}$$

In words, $1/a$ specifies[1] "how far," at each angular position, one traverses in u' for a small displacement in u. To find a surface $u'=$ constant designated by its affine "distance", u, from a $u = 0$ surface we use:

- Use equation (5.1) to write:

$$u'(u, \theta, \phi) - u'(0, \theta, \phi) = \int_0^u \frac{1}{a\,(u_{dummy}, \theta, \phi)}\, du_{dummy}$$

Hence,

$$\begin{aligned} u'(u) &= A(u, \theta, \phi) - A(0, \theta, \phi) + u'(\theta, \phi, 0) \equiv f(u, \theta, \phi) \qquad (5.2) \\ \textit{where} &: \ u'(\theta, \phi, 0) \text{ describes the shape of the surface} \end{aligned}$$

[1] Other references define the lapse function, $a(u)$, inverted from that given here. It was defined in the present manner to simplify the form of equation (4.5) and of its solution. One could also take $a(u')$ thus avoiding the need to take the inverse in order to do the integral of equation (5.1).

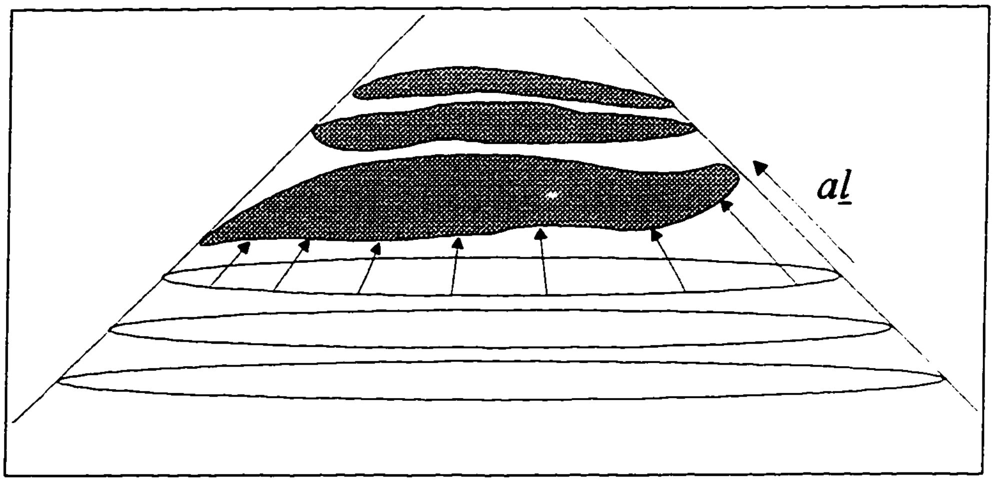

Figure 5.1: Three slices of the affine fiduciary foliation of $\Im^+$ and three of a non-affine foliation. Note that for the gray shaded figures a depends on u (or equivalently u'); this is the defining characteristic of a non-affine foliation.

the initial u' surface.

A is $\int \frac{1}{a} du$

If f describes a foliation of $\Im^+$, then f must be a monotonically increasing function of u at each (θ, ϕ). Hence, $\forall u\ \exists$ a unique u' and $\forall u'\ \exists$ a unique u. Thus, one can find the inverse function that gives:

$$u(u') = g(u', \theta, \phi) \tag{5.3}$$

For each $u' =$ constant we get a new surface $u = g(\theta, \phi)$.

Figure 5.1 illustrates, how starting from a nearby affine slice (say with $u = u' = 0$) of $\Im^+$, one can generate a slice($u' =$ constant) with u varying along it.

Remark 10: *The* lapse function *of itself specifies only a* lapse, $\frac{\delta u}{\delta u'}$, *not an absolute value of the affine parameter. One also needs a constant of integration at each* (θ, ϕ).

One must specify a starting surface and *the lapse function for all u to completely specify the foliation of* $\Im^+$.

5.2 $\Im^+$ Foliations in Minkowski Space

5.2.1 Vanishing Odd Part of Shear

Recall that in Minkowski space there can be no even (magnetic) part of the shear at $\Im^+$. So, setting $\Sigma_{odd} = 0$ implies the total shear is zero. This means that, in Minkowski space-time, any ingoing (from $\Im^+$) group of light rays of vanishing electric shear from a slice of $\Im^+$ will converge to a point in the interior. Conversely, any point emitting light rays in all directions will not only cut $\Im^+$, it will cut it with a zero electric(and magnetic) shear slice. Figure 5.2 illustrates the connection between the affine foliation and the interior space-time that obtains in Minkowski space.

Since every point of Minkowski space generates a cut of $\Im^+$, every worldline, not necessarily a geodesic, generates a foliation of $\Im^+$. This is shown in Figure 5.3.

5.2.2 The Lapse Function and Motion in the Interior

To facilitate discussion, we make an explicit choice of rest frame and origin (cf. Figure 2.1):

1. Pick an initial spatial hypersurface, Σ_0, (in general space-time this slice will be a maximal (ADM linear momentum)=P=0 slice) in Minkowski space.

2. In Σ_0, pick an origin and draw a geodesic sphere of radius r.

3. Send inward $(T - N)$ directed null rays to create a backward light cone.

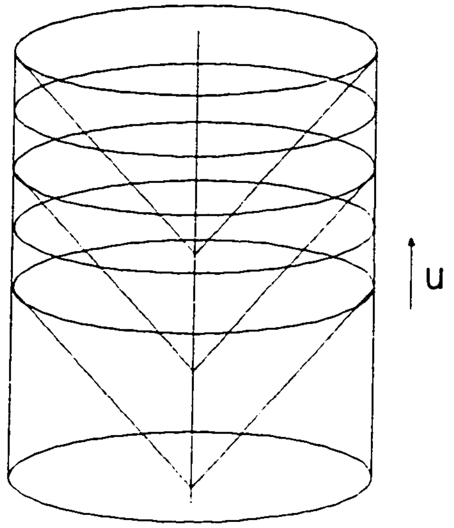

Figure 5.2: Generation of an affine slicing of $\Im^+$ in Minkowski space. Start with an observer on a geodesic; have him send light pulses at regular intervals of his proper time.

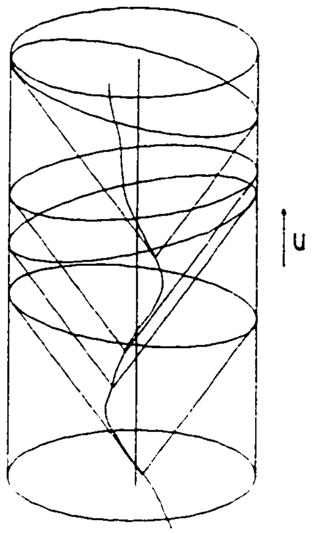

Figure 5.3: Foliation of $\Im^+$ induced by non-geodesic worldline in Minkowski space.

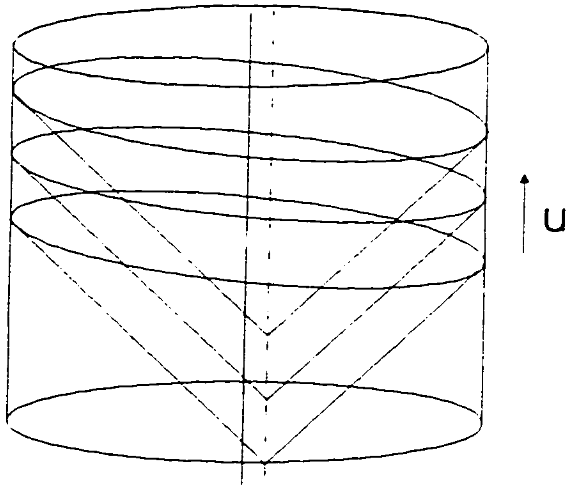

Figure 5.4: Foliation of $\Im^+$ in Minkowski space-time induced by a geodesic translated from the fudiciary geodesic. The lapse function is unchanged by translation.

In Minkowski space-time, if the r sent to infinity, the backward light cone will become null infinity($\Im^+$in the conformal picture). Furthermore, the vertex of this cone will be i^+, future time-like infinity. The geodesic normal to Σ_0 that issues from the origin will intersect i^+; this geodesic will be our fiduciary geodesic. Using this construction, we can now begin to study the connection between the interior motion and the cut of null infinity.

First, consider the simplest case, a geodesic translated with respect to the fiduciary geodesic, shown in Figure 5.4. The lapse function remains unchanged from the fiduciary affine (u' = constant) slices: $c_0 = 1, c_i = 0$, but the new slices are tilted with respect to the affine ones. The distance between an affine slice and a "translated" slice is also given by the right hand side of equation (4.10).[2]

[2]This is shown in a rigorous manner through the use of straightforward algebra and geometry using figures like Figure 4.10.

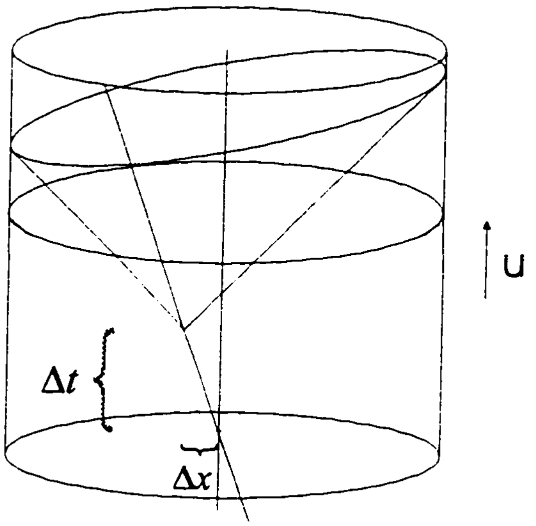

Figure 5.5: A non-fiduciary slice corresponding to a small simultaneous space and time translation.

Figure 5.5 shows the cuts induced by a point on the fiduciary and a point translated in time; it shows the affine slice relative to the translated slice. The construction is as follows:

1. Given the timelike fiduciary geodesic; it will induce an affine foliation on $\Im^+$.

2. Draw a second geodesic moving at finite velocity relative to the first.

3. At the point of intersection of the two geodesics:

   (a) Send out a light pulse to generate one affine slice.

   (b) Place oneself on the second geodesic and wait an infinitesimal proper time and then emit another light pulse.

One begins to see the foliation of $\Im^+$ induced by a geodesic boosted from the fiduciary. The foliation induced by a boosted geodesic which passes through the

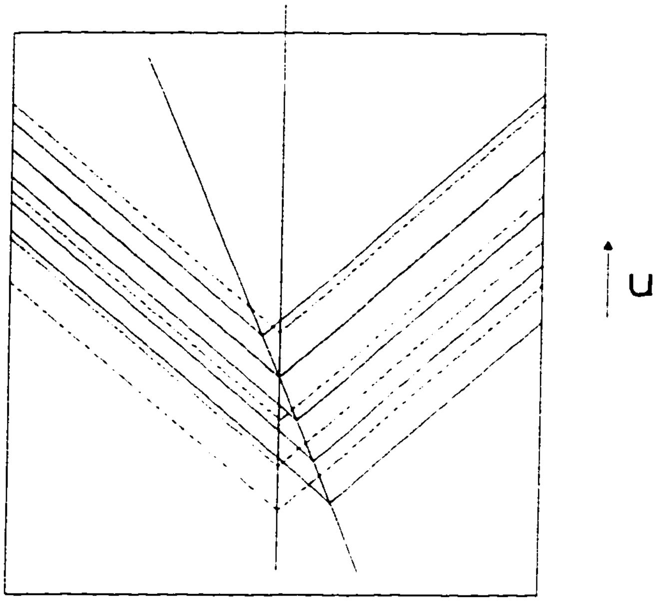

Figure 5.6: Two geodesics in Minkowski space inducing their respective foliations of $\Im^+$. One can clearly see the non-trivial relationship between $\delta u'$ and δu.

origin is shown in Figure 5.6. Figure 5.7 shows the general case: the foliation of $\Im^+$induced by a boosted geodesic not passing through the origin. Since the boosted line is a linearly increasing translation in time and space, it is not surprising that it generates a lapse function given by equation (4.10).

More generally, one notes that two points related by a timelike vector (such points induce a boost) generate a lapse function of the form of equation (4.10).[3]

An Example to Further One's Intuition in Minkowski Space-time

As an example to further elucidate the connection between interior motion and the foliation of $\Im^+$ consider the case of a boosted geodesic in more detail. For definiteness,

[3]This result can be proved rigorously; since the demonstration adds no new understanding, it is omitted.

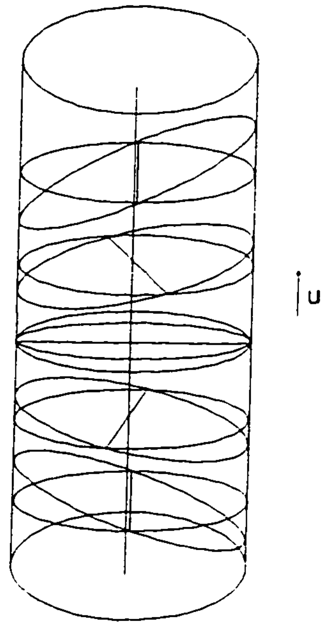

Figure 5.7: The fiduciary geodicesic induces the usual parallel foliation shown. The boosted geodesic that does not pass through the origin generates the tilted foliation. Note that the interior intersection of the fiduciary cut and the boosted cut(called the line of nodes) rotates as one proceeds from the past to the future. The center boosted cut is generated by the point of the boosted geodesic which comes closest to the origin.

we will discuss a boost in the z-direction; of course, the same reasoning applies to each spatial axis.

1. The fiduciary geodesic observer induces an affine foliation of null infinity by emitting light pulses at regular intervals, say δt, of his proper time; confer again Figure 5.2.

2. The construction of the boosted foliation is:

   (a) At the intersection of the two geodesics, let the observer on each geodesic emit a pulse. Both will create the same slice of null infinity. Now, let the second observer emit a light pulse after $\delta t'$ of his proper time elapses. The geometry of the situation gives:

   $$\delta u = \{1 - \beta\cos(\theta)\}\delta t$$

   where δu is the affine "distance," the elapsed retarded time between fiduciary affine slice and the boosted slice (produced by the 2nd signal), and δt [4] is the time elapsed in the lab (1^{st} geodesic frame). Using the relation of the local times between the two frames($\delta t = \gamma\,\delta t'$), one can express the retarded time between pulses in the lab frame at null infinity as function of the time between pulses in the frame in which the light pulses were emitted.

   $$\delta u = \gamma\{1 - \beta\cos(\theta)\}\delta t'$$

   Note: compare this equation with equations (4.10) and (5.1).

[4] This is the variable that is constant on the boosted foliation.

Hence, viewed from null infinity constructed relative to the fiduciary geodesic, one gets the classic Doppler formula:

$$\begin{aligned}
\nu &= \frac{\nu'}{\gamma(1-\beta\cos(\theta))} \\
where &: \gamma = \frac{1}{\sqrt{1-\beta^2}} \\
\nu &= \text{frequency in frame created} \\
\nu' &= \text{frequency in lab frame} \\
\theta &= \text{angle of incoming light given} \\
&\quad \text{by azimuthal coordinate at orgin}
\end{aligned}$$

The above suggests that $\delta u'$ might be the null equivalent[5] of local time on the foliation inducing geodesic in general space-times. Whether or not it actually is does not effect the substance of the analysis and will not be discussed further. It is however, interesting to consider and is helpful in exploring the limits of the analogy.

5.2.3 Summary

We have seen that the lapse function in Minkowski is intimately connected with boosts and/or shifts off one's initial choice of reference frame[6] and/or origin, respectively. Thus, in Minkowski space:[7]

1. Setting the odd(electric) shear to zero means one only foliates $\Im^+$ with zero shear surfaces. This means surfaces are generated from points in the interior.

[5]This, of course, assumes that the right lapse function is chosen.

[6]"Frame" is here taken in the physical sense; more precisely, in Minkowski space, it corresponds to choosing a time direction that will be used throughout by parallel propagation.

[7]In what follows, it will be helpful to refer to equation (4.10) .

2. One then picks an initial fiduciary *geodesic* by constructing the backward light cone issuing from spatial infinity as previously described.[8]

   (a) This determines a rest frame and a center for the coordinate system.

   (b) As described earlier, null infinity can be affine foliated by the fiduciary geodesic.

   (c) A particle on the fiduciary geodesic induces a lapse function $a = c_0 = 1$ ($c_i = 0$, ($i \in \{1, 2, 3\}$)).

3. One can view c_0 as the running rate of the observer's clock. It can be changed by altering the rate of one's chosen clock by hand, or by boosting to a non-affine frame.

4. If one travels on a stationary geodesic off origin, one generates a foliation tilted with respect to the fiduciary affine foliation with the lapse function remaining $a = c_0 = 1$.

5. A geodesic with constant speed relative to the fiduciary introduces c_μ's of the form given in equation (4.10) (e.g., above). In fact, there is a one-to-one correspondence between the speed one travels with respect to the fiduciary in the interior and the lapse function at $\Im^+$. For example, $a = 1$ for all times corresponding to no boosts off the initial frame.

6. One should recall the fiduciary choice could have been made boosted or shifted from the original and all that is said here would remain unchanged.

[8]In Minkowski space-time, one could also proceed by picking the fiduciary geodesic, creating a backward light cone issuing from a vertex on this geodesic and then send the vertex to time-like infinity making the cone become null infinity.

Remark 11: *In short, we still have, after all the c_μ' s are set, a freedom of one global Poincaré transformation..*

5.3 $\Im^+$ Foliations in *General* Asymptotically Flat Space-times

In a general space-time, $\Xi \neq 0$ and there can be static curvature as well. This means that if one trys to send rays out from a point, say from the center of a source, these rays will, in general, develop caustics and *not* induce a slice of null infinity. It is thus fortunate that Minkowski space has left one with what appears to be a strong intuition about what is going on so that we can extrapolate. In short, the ansatz is that in a general space-time:

1. Setting the electric(odd) part of the shear to zero is the closest one can do to directing rays back to a point in the interior.

2. Create the backward light cone as described on page 45. Setting[9] the $c_i = 0$ and $c_0 = 1$ corresponds to, in some sense, staying in initial center of momentum frame. It further corresponds to, in some sense, keeping the initial choice of origin.

   In the limit of a vanishingly small source mass and no radiation, this clearly makes sense. In this case, one is arbitrarily close to Minkowski space so that the odd part of shear is close to zero, we can foliate with "points," and proceed with the arguments as already given for Minkowski space-time.

[9]Note, in the general case(cf. equation (4.12), page 40), the $\dot{c}_\mu$'s do not represent the entire lapse function, but they are, as in the Minkowski case, the only degrees of freedom remaining after the electric(odd) part of the shear is set to zero.

5.4 The Preferred Cuts and the Null Definition of Angular Momentum

Given the above we can now formulate more precisely the complete null definition of angular momentum:[10]

$$\begin{aligned} L(\Omega_{(i)})(u) &= \frac{1}{8\pi}\lim_{s\to\infty}\int_{\text{cut of } \Im^+} \zeta_A \Omega^A_{(i)} d\mu_\gamma \\ &= \frac{1}{8\pi}\int_{\text{cut of } \Im^+} (\Sigma_{AB}\not{\nabla}_C \cdot \Sigma^{CB} + I_A)\Omega^A_{(i)} d\mu_\gamma \end{aligned} \tag{5.4}$$

$$\begin{aligned} \frac{\partial L(\Omega)}{\partial u}(u) &= \\ & \frac{1}{8\pi}\int_{\text{cut of } \Im^+} (-\Xi_{AB}\not{\nabla}_C \Sigma^{CB} \\ & +\frac{1}{2}(\Sigma_{AB}\not{\nabla}_C \Xi^{CB} - \Sigma^C_B \not{\nabla}^B \Xi_{CA}))\Omega^A d\mu_{\gamma^0} \end{aligned} \tag{5.5}$$

The angular momentum of the source is the quantity obtained by doing the above integral over the preferred cut ($u' =$ constant[11]) described previously.

To bring home the physical import, we give an example of how an idealized experiment to determine the angular momentum might be conducted. For simplicity we assume the angular momentum before the experiment starts is given, so that we only need measure $\dot{L}$.

1. Setup the Michelson interferometer in the free-fall frame at null infinity, placing test masses at constant affine parameter u around the source.

[10] Note that it does not matter whether one defines ζ with respect to an affine frame(l and $\underline{l}$) or a non-affine frame($(1/a)\,l$ and $a\,\underline{l}$), the answer is the same. Refer to page 8 for ζ transformation properties.

[11] Note that because of the one-to-one correspondence between u and u', one can assign the angular momentum at a moment of retarded time u as well as at u'.

2. In the free-fall frame, $\Sigma(\theta, \phi, u)$ is proportional to the separation of the test masses at the retarded time, u and the given angular position.

3. The data will thus be a continuous table of values of $\Sigma(\theta, \phi)$ for different u's.

4. Calculate Ξ and use equation (4.6) to determine the lapse function.

5. Carry out the integral in equation (5.5) over the values of u's specified by the appropriate lapse functions and initial frame and origin choice.

Remark 12: *As expected, the angular momentum is dependent on a choice of "origin" and rest frame as in Minkowski space.*[12]

5.5 Conclusion

In summary, the definition of angular momentum given above meets all five properties specified on page 19. Further, it gives the correct answer for Minkowski space-time, Kerr and quadrupole approximation case. Also, the Minkowski freedom in the definition of angular momentum with respect to choice of origin and rest frame remains as expected. The generally strong physical intuition about what is happening also counts in favor of the definition.

Questions that remain to be explored further are the relation of this definition to the Bondi linear momentum, the relation to the spatial infinity definition, the relation of this conserved quantity to an invariance of the action and other more general(than quadrupole) tests of the definition. Work has begun on these fronts.

[12] One notes a unique choice can be picked out by using the initial rest frame and putting the origin at the source. However, this will only represent the coordinate system where one measures the spin of the source if the gravitational radiation carries away no net linear momentum.

Chapter 6

Appendix I: Structure Equations

The Structure Equations are given here:[1] (ref. [33] pg 168 equations (7.4.2a-u))

a.

$$\begin{aligned} \not{D}_3 \hat{\underline{H}}_{AB} + (tr\underline{H})\hat{\underline{H}}_{AB} &= (\not{\nabla}_A \underline{Y}_B + \not{\nabla}_B \underline{Y}_A - div\underline{Y}\ \gamma_{AB} \\ &\quad -2\Omega\, \hat{\underline{H}}_{AB} + ((Z + \underline{Z} - 2V)\hat{\otimes}\underline{Y})_{AB} - \underline{\alpha}_{AB} \end{aligned}$$

b.

$$curl\,\underline{Y} = \underline{Y} \wedge (Z + \underline{Z} - 2V)$$

c.

$$D_3(tr\underline{H}) + \frac{1}{2}(tr\underline{H})^2 = 2div\underline{Y} - 2\underline{\Omega}\, tr\underline{H} + 2\underline{Y} \cdot (Z + \underline{Z} - 2V) - \hat{\underline{H}} \cdot \hat{\underline{H}}$$

d.

$$\begin{aligned} \not{D}_4 \hat{\underline{H}}_{AB} + \frac{1}{2} trH\ \hat{\underline{H}}_{AB} &= (\not{\nabla}_B \underline{Z}_A + \not{\nabla}_A \underline{Z}_B - div\underline{Z}\,\gamma_{AB}) + 2\Omega\, \hat{\underline{H}}_{AB} \\ &\quad -\frac{1}{2} tr\underline{H}\, \hat{H}_{AB} + (Y\hat{\otimes}\underline{Y})_{AB} + (\underline{Z}\hat{\otimes}\underline{Z})_{AB} \end{aligned}$$

---

[1] ($\hat{\otimes}$) refers to symmetric trace-free part.
All *curl* and *div* 's are on S^2.

e.

$$curl\, \underline{Z} = \frac{1}{2}\hat{H} \wedge \hat{\underline{H}} - Y \wedge \underline{Y} - \sigma$$

f.

$$\begin{aligned} \not{D}_4 tr\underline{H} + \frac{1}{2}(trH)tr\underline{H} &= 2div\underline{Z} + 2\Omega\ tr\underline{H} - \hat{H}\cdot\hat{\underline{H}} \\ &+2(Y\cdot\underline{Y} + \underline{Z}\cdot\underline{Z}) + 2\rho \end{aligned}$$

g.

$$(div\underline{H})_A - V_B\underline{H}_{AB} = \nabla\!\!\!/_A tr\underline{H} - V_A\, tr\underline{H} + \underline{\beta}_A$$

h.

$$(divH)_A + V_B\, H_{AB} = \nabla\!\!\!/_A trH + V_A trH - \beta_A$$

i.

$$K = -\frac{1}{4}trH\ tr\underline{H} + \frac{1}{2}\hat{H}\cdot\hat{\underline{H}} - \rho$$

j.

$$\begin{aligned} \not{D}_3 V_A &= -2\nabla\!\!\!/_A\underline{\Omega} - \underline{H}_{AB}(V_B + Z_B) + 2\underline{\Omega}\,(V_A - Z_A) \\ &+H_{AB}\underline{Y}_B + 2\Omega\,\underline{Y}_A - \underline{\beta}_A \end{aligned}$$

k.

$$\begin{aligned} \not{D}_4 V_A &= 2\nabla\!\!\!/_A\Omega + H_{AB}\,(-V_B + \underline{Z}_B) + 2\Omega\,(V_A + \underline{Z}_A) \\ &-\underline{H}_{AB}Y_B - 2\underline{\Omega}Y_A - \beta_A \end{aligned}$$

l.

$$\begin{aligned} \not{D}_3\hat{H}_{AB} + \frac{1}{2}tr\underline{H}\,\hat{H}_{AB} &= (\nabla\!\!\!/_B Z_A + \nabla\!\!\!/_A Z_B - divZ\,\gamma_{AB}) + 2\underline{\Omega}\,\hat{H}_{AB} \\ &-\frac{1}{2}trH\ \hat{\underline{H}}_{AB} + (\underline{Y}\hat{\otimes}Y)_{AB} + (Z\hat{\otimes}Z)_{AB} \end{aligned}$$

m.

$$curl\, Z = -\frac{1}{2}\hat{H} \wedge \underline{\hat{H}} + Y \wedge \underline{Y} + \sigma$$

n.

$$\begin{aligned} \not{D}_3 tr H + \frac{1}{2}(tr\underline{H})\, trH &= 2\, div Z + 2\underline{\Omega}\, trH - \hat{H} \cdot \underline{\hat{H}} \\ &+2(\underline{Y} \cdot Y + Z \cdot Z) + 2\rho \end{aligned}$$

o.

$$\begin{aligned} D_4 \hat{H}_{AB} + (trH)\hat{H}_{AB} &= (\not{\nabla}_A Y_B + \not{\nabla}_B Y_A - divY\ \gamma_{AB}) - 2\Omega\, \hat{H}_{AB} \\ &+((Z + \underline{Z} + 2V)\hat{\otimes} Y)_{AB} - \alpha_{AB} \end{aligned}$$

p.

$$curlY = Y \wedge (Z + \underline{Z} + 2V)$$

q.

$$\begin{aligned} D_4(trH) + \frac{1}{2}(trH)^2 &= 2divY - 2\Omega\, trH \\ &+ 2Y \cdot (Z + \underline{Z} + 2V) - \hat{H} \cdot \hat{H} \end{aligned}$$

r.

$$\not{D}_4 \underline{Y}_A - \not{D}_3 \underline{Z}_4 = 4\Omega \underline{Y}_A + \underline{H}_{AB}(\underline{Z}_B - Z_B) - \underline{\beta}_A$$

s.

$$\not{D}_3 Y_A - \not{D}_4 Z_4 = 4\underline{\Omega}\, Y_A + H_{AB}(Z_B - \underline{Z}_B) + \beta_A$$

t.

$$\begin{aligned} D_4\underline{\Omega} + D_3\Omega &= Y \cdot \underline{Y} + V \cdot (Z - \underline{Z}) - Z \cdot \underline{Z} \\ &+4\Omega\underline{\Omega} + \rho \end{aligned}$$

Chapter 7

Appendix II: Asymptotic Flatness

The general notions will be given followed by their implications.

Globally asymptotically flat means, loosely, space-times which are flat as one approaches infinity in any direction. Rigorously, it means the magnitude of the Riemann curvature tensor along any given spacelike geodesic in the initial spacelike hypersurface approaches zero faster than $1/s^2$, as the affine parameter, s, tends to infinity. The initial hypersurface being asymptotically flat then implies, for an isolated source, that all the hypersurfaces of the foliation are asymptotically flat.

Specifically, we start(cf. ref. [33] pg 11) with a strongly asymptotically flat initial data set which is an initial data set, (Σ, g, k_{ij}), such that

as $r \rightarrow$infinity

$$\begin{aligned} g_{ij} &= (1 + \frac{2M}{r})\delta_{ij} + o_4(r^{-\frac{3}{2}}) \\ k_{ij} &= o_3(r^{-\frac{5}{2}}) \end{aligned}$$

Strongly asymptotically flat initial data sets satisfy the following important properties:

1. "The complement of a finite set in Σ is diffeomorphic to the complement of a ball in R^3 (i.e., Σ is diffeomorphic to R^3 at infinity)"

2. E,P and J are well defined and finite and E,P,J are defined as[33]:

$$
\begin{aligned}
E &= \frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r}\sum_{i,j}(\partial_i g_{ij} - \partial_j g_{ii})\, N^j dA \\
P_i &= \frac{1}{8\pi}\lim_{r\to\infty}\int_{S_r}(k_{ij} - trk\, g_{ij})N^j dA; \quad i = 1,2,3 \\
J_i &= \frac{1}{8\pi}\lim_{r\to\infty}\int_{S_r}\in_{iab}\, x^a(k^{bj} - g^{bj}trk)\, N_j\, dA; \quad i = 1,2,3
\end{aligned}
$$

1. E=M, P=0. That is, the initial data set is in center of mass frame.

2. Under evolution equations of Einstein vacuum in a normal foliation such a strongly asymptotically flat initial data set, the following are true:

   (a) The definitions for E, P and J are preserved. (cf. [33] pg 12)

   (b) If, in addition, a global smallness assumption is satisfied, there is a "globally hyperbolic, smooth and geodesically complete solution of Einstein vacuum."

Chapter 8

Appendix III: Uniqueness Theorem

Theorem 1: *The general solution of:*

$$\Delta^2\psi + 2\Delta\psi = \nabla^A\nabla^B(\psi \cdot \Xi_{AB}) \tag{8.1}$$

given Ξ *such that:*

$$\sup_{S^2} \Xi_{AB}\Xi^{AB} < 16$$

is

$$\psi = \psi_{sl} + \Psi$$

$$\begin{aligned} where &: \\ \Psi &= \sum_{i=2,m}^{\infty} c_{i,m} E_{i,m} \\ & E_{i,m} \text{ is the } i^{th} \text{ eigen function of the laplacian} \\ & \text{as previously defined} \\ \psi_{sl} &= \psi_{sourceless} = \sum_{i=0,m}^{1} c_{i,m}\, E_{i,m} \end{aligned}$$

and where Ψ *is unique.*

Proof. [1]

[1]The fundamentals of this proof are due to D. Christodoulou.

Substituting ψ into equation (8.1), gives:

$$\Delta^2\Psi + 2\Delta\Psi = +\nabla\cdot\nabla\cdot\{\Xi\,(\Psi + \psi_{sl})\} \tag{8.2}$$

Note: since ψ_{sl} is solution to the sourceless equation (i.e. Minkowski space, cf. pg 39) it does not contribute to the left hand side.

Start with an iterative equation:

$$\Delta^2\Psi_{n+1} + 2\Delta\Psi_{n+1} = \nabla\cdot\nabla\cdot\{\Xi\,(\psi_{sl}\} + \nabla\cdot\nabla\cdot\{\Xi\,\Psi_n\} \tag{8.3}$$

(a) We take $\Psi_0 = 0$ so that the n=0 term of the equation becomes:

$$\Delta^2\Psi_1 + 2\Delta\Psi_1 = \nabla\cdot\nabla\cdot\{\Xi\,\psi_{sl}\}$$

We will show that the $\Psi_{j>1}$'s converge to a unique Ψ. We are thus interested $h_{n+1} = \Psi_{n+1} - \Psi_n$ which obeys the following equation:

$$\Delta^2 h_{n+1} + 2\Delta h_{n+1} = +\nabla\cdot\nabla\cdot\{\Xi\,h_n\}$$

Let us then proceed by analyzing:

$$\Delta^2 g + 2\Delta g = +\nabla\cdot\nabla\cdot\eta \tag{8.4}$$

1. Decompose the function g:

$$g = \sum_{l=2,m}^{\infty} c_{l,m}E_{l,m} \equiv \sum_{l=2}^{\infty} g_l$$

where : the subscript notation, g_l, which indicates an l^{th} eigen function of the laplacian is not used for Ψ_l ; i.e., the Ψ_l 's are *not*, in general, eigenfunctions of the laplacian.

(a) Because of orthogonality of g_l's:

$$\int_{S^2} |\Delta g|^2 = \sum_{l=2}^{\infty} l^2(l+1)^2 \int_{S^2} g_l^2 \tag{8.5}$$

(b)

$$-\int_{S^2} g\, \Delta g = \sum_{l=2}^{\infty} l(l+1) \int_{S^2} g_l^2$$

(c) We also have $l(l+1) \geq 6$,since $l \geq 2$.

Thus,

$$-\int_{S^2} g\, \Delta g \leq \frac{1}{6} \int_{S^2} |\Delta g|^2 \tag{8.6}$$

2. Multiplying equation (8.4) by g and using:

$$\int_{S^2} g\, \Delta^2 g = \int_{S^2} |\Delta g|^2$$

(a) One obtains:

$$\int_{S^2} |\Delta g|^2 = -2 \int_{S^2} g\, \Delta g + \int_{S^2} \eta \cdot \nabla^2 g$$

3. Using the inequality (8.6), one gets:

(a)

$$\int_{S^2} |\Delta g|^2 \leq \frac{1}{3} \int_{S^2} |\Delta g|^2 + \int_{S^2} \eta \cdot \nabla^2 g \tag{8.7}$$

$$or \tag{8.8}$$

$$\frac{2}{3} \int_{S^2} |\Delta g|^2 \leq \int_{S^2} \eta \cdot \nabla^2 g$$

(8.9)

(b) From reference [33]:

$$\int_{S^2} (|\nabla^2 g|^2 + |\nabla g|^2) = \int_{S^2} |\Delta g|^2$$

(c) Using this equation one can write:

$$\int_{S^2} \eta \cdot \nabla^2 g \leq \sqrt{\int_{S^2} |\eta|^2 \cdot \int_{S^2} |\nabla^2 g|^2} \leq \sqrt{\int_{S^2} |\eta|^2 \cdot \int_{S^2} |\Delta g|^2} \tag{8.10}$$

(d) Thus, inequality (8.8) and inequality (8.10) imply:

$$\int_{S^2} |\Delta g|^2 \leq \frac{9}{4} \int_{S^2} |\eta|^2 \tag{8.11}$$

4. Using equation (8.5) and that $l \geq 2$.

$$\int_{S^2} |\Delta g|^2 \geq 36 \sum_{l=2}^{\infty} \int_{S^2} g_l^2 = 36 \int_{S^2} |g|^2 \tag{8.12}$$

5. Now re-substituting;

$$g \rightarrow h_{n+1}$$

$$\eta \rightarrow \Xi\, h_n$$

(a)

$$\int_{S^2} |\Delta h_{n+1}|^2 \leq \frac{9}{4} \sup_{S^2} |\Xi|^2 \int_{S^2} |\, h_n|^2 \tag{8.13}$$

(b) Using inequality (8.12) with $g \rightarrow h_n$, this becomes:

$$\int_{S^2} |\Delta h_{n+1}|^2 \leq \frac{1}{16} \sup_{S^2} |\Xi|^2 \int_{S^2} |\Delta h_n|^2 \tag{8.14}$$

Thus, the condition for geometric convergence in the L^2 norm is:

$$\sup_{S^2} |\Xi|^2 \leq 16$$

Q.E.D.

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