Historia Mathematica 9 (1982) 392-412

EARLY CRITICISM OF THE SYMBOLICAL APPROACH TO ALGEBRA

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SUMMARIES

For a long time, historians have believed that in the 1830s Sir William Rowan Hamilton was a lone critic of symbolical algebra. Using published and unpublished documents, this article shows that symbolical algebra was a considerably controversial subject among British mathematicians of the 1830s and 1840s. Special attention is paid to William Frend's and Osborne Reynolds' criticism of symbolical algebra. The article ends with a brief discussion of reservations concerning symbolical algebra expressed by Augustus De Morgan, William Whewell, and Philip Kelland.

Depuis longtemps, les historiens ont cru que Sir William Rowan Hamilton fut le seul critique de l'algèbre symbolique dans les années 1830. En s'appuyant sur des documents publiés et inédits, cette étude montrera que l'algèbre symbolique fut un sujet de controverse considérable parmi les mathématiciens britanniques dans les années 1830 et 1840. Nous examinerons surtout la critique que firent William Frend et Osborne Reynolds de l'algèbre symbolique. L'étude s'achève sur une courte discussion des doutes qu'exprimèrent Augustus De Morgan, William Whewell, et Philip Kelland sur l'algèbre symbolique.

It is common knowledge that in the 1830s George Peacock, Augustus De Morgan, and Duncan F. Gregory adopted the symbolical approach to algebra, while Sir William Rowan Hamilton rejected it [Nagel 1935, 448-466; Nový 1973, 127, 189-199]. But, contrary to the impression conveyed by previous histories of early-19th-century British algebra, Hamilton was not a lone opponent of the symbolical approach. This paper analyzes three hitherto-unexplored documents, dating from the late 1830s, in which other critics described symbolical algebra as an art rather than a

0315-0860/82/040392-21$02.00/0

science and condemned the lack of concern for meaning and the arbitrariness which it evidenced. The first document is a letter from William Frend to De Morgan [Frend 1836]; the second, composed most likely by Frend or his daughter, Sophia Elizabeth, is a spoof-play on De Morgan's version of symbolical algebra [Anon. n.d.]. The third and most exciting is *Strictures on Certain Parts of "Peacock's Algebra"* [Anon. 1837], published anonymously in 1837 by Osborne Reynolds, then a recent graduate of Queens' College, Cambridge. This document is a critique of Peacock's *Treatise on Algebra* of 1830.

The paper ends with a few comments on the early reception accorded symbolical algebra by De Morgan, William Whewell, and Philip Kelland. These remarks are tentative and necessarily brief, intended basically to supplement analysis of the three above-described documents which form the core of the paper. All parts considered, the paper suggests that from its first formal presentation in Peacock's *Treatise* the symbolical approach to algebra met with widespread skepticism and criticism, rather than with near-universal endorsement. Such strong early negative reaction to symbolical algebra at least partially explains the fizzling-out of the British symbolical movement by the mid-19th century [1].

1. WILLIAM FREND, AUGUSTUS DE MORGAN, AND SOPHIA ELIZABETH FREND DE MORGAN

William Frend, author of the first and possibly the second document studied in this paper, was the last of the major British opponents of the negative numbers [2]; he was also the only critic of the negatives who lived to study and evaluate algebra of the symbolical period. Frend graduated as second wrangler from Christ's College, Cambridge, in 1780 and then moved to Jesus College where he became a fellow and tutor. In 1793, however, he was banished from Cambridge for his Unitarian beliefs and the radical political views he expressed in *Peace and Union Recommended to the Associated Bodies of Republicans and Anti-Republicans* [Frend 1793]. After his expulsion from the University, his religious heresy led to mathematical heresy--rejection of the negative (and hence imaginary) numbers. As a recent convert to Unitarianism, Frend maintained that earlier in life he had been duped by religious authorities into believing in the false doctrine of the Trinity. He resolved, therefore, to accept nothing else on the trust or authority of others. Among the widely accepted but poorly understood ideas which he rejected at this point was that of the negative number, defined until the early 19th century as a quantity less than nothing. Frend could not comprehend negative numbers and declared that he was unwilling to work with these numbers merely because their use was sanctioned by such mathematical authorities as Newton [1728, 3]. He called, therefore, for abandonment of the negative numbers and for the reduction of algebra to universal arithmetic in the strictest sense [Frend 1796, Vol. 1, esp. x-xii] [3].

Despite Frend's eccentricities, he and Augustus De Morgan were friends and mathematical discussants from 1828 on. Their friendship was built on a mutual distaste for compulsory religious orthodoxy and a mutual respect for clear thinking. At the end of his undergraduate career at Trinity College, Cambridge, De Morgan refused to take the oath of subscription necessary for securing a fellowship at the University. He thus shared Frend's position as a religious outcast from Cambridge. Although he disagreed with Frend's idea of the necessity of abandoning the negatives, De Morgan admired the clarity of Frend's thought and writings. Frend's *Principles of Algebra,* he once wrote, "is on the points which it treats, the clearest book in our language" [A. De Morgan 1842a, 467].

Conversation between the two men during the late 1820s and early 1830s appears to have centered at least occasionally on the problem of the negative numbers and the nature of algebra. De Morgan's many published works from this period show that, although he never joined Frend in rejecting the negatives, he worried about the problem and proposed the symbolical approach as a resolution thereof [A. De Morgan 1831, 1832, 1835, for example]. As early as 1828, the year of his appointment as professor of mathematics at London University (which became University College London in 1836), he thus wrote to Frend:

> *I am quite agreed with you, that the extension of algebraical symbols which form impossible quantities, abstract negative Quantities & c. should never be allowed to be made by a beginner. All his Algebra should be strictly "universal Arithmetic." But you will not perhaps agree with me in asserting that the part of Algebra which I would call the "science of symbols" ought, when its mathematical meaning & bearings are fully explained, to form part of the elementary Course.* [A. De Morgan 1828]

The last sentence is significant as a statement of rejection of Frend's solution to the problem of the negative numbers and an early expression of De Morgan's belief in the appropriateness of some sort of symbolical approach to algebra. It also conveys an impression of uneasiness about symbolical algebra--especially about "its mathematical meaning & bearings." This impression is reinforced by De Morgan's review [1835] of the first edition of Peacock's *Treatise on Algebra.* As the review shows, Peacock's work did not immediately make a convert of De Morgan. In the review (another hitherto-untapped source for the history of early-19th-century British algebra) De Morgan tried to explain why the *Treatise on Algebra* had "not been noticed before, seeing that it was published in 1830." The reason for its seeming neglect was, he maintained, "the very great difficulty of forming fixed opinions upon views so new and so extensive. At first sight it appeared to us [he admitted] something like symbols bewitched, and running about the world in search of a meaning" [A. De Morgan 1835, 311]. Thus through around 1835 De Morgan himself was wary of the possible meaninglessness of symbolical algebra.

Sometime in the summer of 1835, shortly after publication of the first edition of *Elements of Algebra Preliminary to the Differential Calculus* [A. De Morgan 1837]--by which point De Morgan had clearly come to grips with symbolical algebra and accepted the symbolical approach as legitimate--he gave a copy of the work to Sophia Elizabeth Frend, William Frend's eldest child and De Morgan's future bride. This gift appears to have occasioned the writing of two of the documents treated in this paper. Most likely father and daughter studied De Morgan's book together; it is quite possible that they also collaborated on the writing of the second document analyzed in the paper.

Sophia Frend was an intelligent woman with some interest in mathematics and science. As the daughter of Unitarians, she had received the kind of education which set her and many of her Unitarian sisters apart from other early-19th-century British women. From childhood she had been tutored by her father and in later years treated as his intellectual companion. Her father had taught her Hebrew, since he had a special attachment to and respect for the language and traditions of the Jews, and later some Greek and Latin. He had also "encouraged her in metaphysical and philosophical reading, for which she had a natural bent" [Mary A. De Morgan in S. De Morgan 1895, xxvii]. As a very young child Sophia Frend had listened to mathematical discussions between her father and Francis Maseres, another of the major opponents of the negative numbers [S. De Morgan 1895, 29]. As a teenager she had written an account of astronomical lectures. Her father had replied to her account by criticizing the concept of centripetal force [Frend 1824]. In 1835 she was reportedly "delighted" by one of William Frend's newer algebraic results [Frend 1835a].

In short, as the recipient of a presentation copy of De Morgan's *Elements of Algebra,* Sophia Frend was well chosen. The scanty available source materials on her indicate that she was a rare early-19th-century woman with interest in and some basic understanding of algebra, even if only the strict universal arithmetic which her father exclusively tolerated. It probably therefore came as no surprise to De Morgan when, in September

1835, William Frend conveyed his daughter's appreciation for the copy of the *Elements of Algebra* and mentioned that she was preparing for De Morgan an account of the work. She was studying the book while on vacation at the seashore, a circumstance which prompted her father to ask De Morgan's indulgence: "if she does not give you a good account of it on her return attribute it to the sea air which is very injurious to such pursuits" [Frend 1835b].

2. DOCUMENT I: LETTER OF JUNE 22, 1836, FROM WILLIAM FREND TO DE MORGAN

In the letter of June 22, 1836, Frend told De Morgan:

> *I desire certainty not uncertainty science not art.... I have great respect for your art & have no doubt that it will be the means of introducing us to many valuable discoveries in science as in mechanicks how much are we not indebted to men who without the aid of science have produced various machines of great benefit to mankind. I am very much inclined to believe that your figment $\sqrt{-1}$ will keep its hold among mathematicians not much longer than the Trinity does among theologians.* [Frend 1836; Frend's punctuation]

This statement shows that neither discussions with De Morgan nor the opportunity to study the latter's *Elements of Algebra* had substantially changed Frend's views on the negative numbers and on algebra in general.

In 1836 (and probably until his death in 1841) Frend refused to recognize as a science an algebra including the negative and imaginary numbers. As he expressed in his earlier works written in opposition to the negatives, he believed that mathematical science involved the application of deductive reasoning to true principles concerning clear and distinct ideas. In his *Principles of Algebra* he had explained: "The ideas of number are the clearest and most distinct in the human mind; the acts of the mind upon them are equally simple and clear. There cannot be confusion in them, unless numbers too great for the comprehension of the learner are employed, or some *arts* are used which are not justifiable" [Frend 1796, Vol. 1, ix-x; my italics]. Since, as he reasoned, the idea of a negative number was not clear and distinct, the introduction of such numbers into algebra vitiated the scientific work of the algebraist and reduced algebra to an art. As an art, in Frend's opinion, any algebra (except universal arithmetic in the strictest sense) led but to uncertain conclusions. Science alone led to certainty.

Yet, by 1836 Frend was appreciative of the potential usefulness of an algebra (such as symbolical) which included the negative and imaginary numbers. In the above-quoted letter, he compared the practitioners of such an algebra with mechanical artisans. Without understanding the principles of the science of mechanics, these artisans produced much of the machinery of the first Industrial Revolution. Their inventions, in turn, led to the further development of the mechanical sciences. Frend predicted a similar relationship between what he described as algebraic art and science.

In short, even after exposure to De Morgan's defense of the negatives and imaginaries, Frend continued his opposition to these numbers, describing them in the above-quoted letter as "figments." He still clung to his "contentual" view of the mathematical sciences, according to which symbols stood only for clear and distinct ideas. Like most of his contemporaries, however, he realized the potential "fruitfulness" of an algebra including the negative and imaginary numbers. Wanting to banish these numbers from the mathematical sciences and still permit their use (analogous to eating his cake and yet having it), Frend proposed as a compromise the distinction of algebra into scientific and artistic parts, a distinction which (as shown later in this paper) was developed in De Morgan's algebraic writings of the 1840s.

3. DOCUMENT II: SPOOF-PLAY ON SYMBOLICAL ALGEBRA

The second newly discovered critique of symbolical algebra is a short play which spoofs not only the symbolical approach but De Morgan and Cambridge students as well. This document is undated and unsigned, but internal analysis and general knowledge about William and Sophia Frend lend support to the hypotheses that it was written between 1836 and 1838 [4], by either or both of the Frends.

The tentative conclusion concerning authorship of the play rests on the following evidence. The play is presently found in manuscript form in the University College Library, University College London, in a folder containing letters from William and Sophia Frend as well as other assorted manuscripts. The play is written in the hand of Sophia Frend [5]. In addition, the play's content identifies it as a product of the tradition of opposition to the negative numbers, a tradition of which, by the 1830s, William Frend was the only known living representative. The dedication which appears at the very beginning of the manuscript criticizes De Morgan's use of the negative sign. De Morgan's approach to algebra, it claims, "enables the learner to adapt the operations & the symbols of Algebra to every possible elevation of sense & every impossible depth of nonsense for ... it is only necessary to place the negative sign before a word or an expression to alter its entire meaning" [Anon. n.d., 1-2]. The play ends with a ludicrous application of the principle that the greater may always be taken from the lesser [Anon. n.d., 12], a principle to which, of course, opponents of the negatives objected.

Sophia Frend may be the sole author of the play. Evidence in favor of this hypothesis is by no means overwhelming; yet, in lieu of positive identification of the author of the manuscript it deserves at least brief exploration. The manuscript, after all, survives in her handwriting. There are also three pieces of circumstantial evidence which lend support to such speculation: (1) Sophia Frend studied De Morgan's *Elements of Algebra* at the beginning of the period during which, I have conjectured, the play was composed; (2) in the autumn of 1836 she regularly corresponded with De Morgan and in one letter of the period even used language resembling that of the dedication to the play; and (3) the play was written "after the manner of Miss Martineau" [Anon. n.d., 1], another woman. The first point has already been discussed. Extant letters between Sophia Frend and De Morgan show that their correspondence in fall 1836, when the Frend family was temporarily living in western England and De Morgan was sick in London [S. De Morgan 1882, 69], was lengthy and sometimes even turned to scientific matters. In September, for example, Sophia Frend provided De Morgan with a written account of a recent meeting of the British Association for the Advancement of Science, which she and her father, but not De Morgan, had attended. Like the dedication to the play, the preface to the account juxtaposed "sense" and "nonsense," perhaps terms with special connotations for Sophia Frend and De Morgan. Thus the letter containing the account of the BAAS meeting began: "If I can only manage to manufacture any nonsense, you shall have it at the expense of postage. It does not follow, let me tell you, that those who can neither originate nor comprehend sense, can do the reverse...." [S. De Morgan 1836]. The play under discussion was possibly some of the "nonsense" which Sophia Frend composed for her future husband. Finally, it is plausible that only a woman would have chosen to copy the style of Harriet Martineau's *Illustrations of Political Economy* [Martineau 1832-1835], writing a critique of De Morgan's symbolical approach to algebra in dialogue form.

While the conclusion that the play was written by a woman is alluring and intriguing, the evidence offered above is inconclusive. William Frend still remains a possible author. Support for this alternative hypothesis is not difficult to uncover. Unlike his daughter, William Frend was, after all, a famous algebraist and opponent of the negatives. As already noted, his debate with De Morgan concerning the negative numbers and the nature of algebra went back as far as 1828. Furthermore, as a champion of the rights of all persons--even women--William Frend would probably not have hesitated to adopt the literary style of a woman, especially one of Unitarian background as was Martineau. Even more compelling evidence in favor of this solution to the riddle of the identity of the play's author, moreover, is an article written by William Frend in 1803 and published in *The Gentleman's Monthly Miscellany,* a journal of a few months' duration of which Frend was the editor. Entitled "Pantagruel's Decision of the Question about Nothing," this article was a spoof of the use of the number zero in the form of a parody of François Rabelais' *Gargantua and Pantagruel.* In his *Budget of Paradoxes* De Morgan described this piece as a good imitation of Rabelais and proof "that an impugner of algebra could attempt ridicule" [A. De Morgan 1915, Vol. 1, 208]. The play under discussion, which at points rather closely parodies De Morgan's *Elements of Algebra,* was perhaps another product of Frend's power of satire. Explanation of the survival of the play in Sophia Frend's handwriting is easy: William Frend sometimes employed his daughter as an amanuensis. At the 1836 meeting of the BAAS, for example, Frend handed Sir William Rowan Hamilton "a paper very prettily written out by Sophia" [Knight 1971, 301]. In the absence of any further evidence, we must then conclude that the play in question was most likely written by either or both of the Frends.

Like the letter of June 22, 1836, from William Frend to De Morgan (Document I of the present paper), the play is significant as an early response to the symbolical approach to algebra written from the tradition of opposition to the negative numbers. These two documents show, in particular, that such opposition survived elaboration of the symbolical approach. But even more fundamentally, the play demonstrates that a basic objection to the approach concerned the arbitrariness which it introduced into algebra. The play's author(s) could see no rhyme or reason behind the symbolical approach and therefore misinterpreted symbolical algebra as an expression of the mere whims of certain algebraists.

Such a reaction is understandable when viewed in the context of early-19th-century British algebra. Through the late 1830s the principle of mathematical freedom had not been clearly and consistently enunciated; Hamilton's classic proclamation of the principle came only in 1843 with the invention of the quaternions. Yet acceptance of at least a certain degree of mathematical freedom was implicit and occasionally explicit in the early formulations of symbolical algebra. In his "Report on the Recent Progress and Present State of Certain Branches of Analysis," Peacock acknowledged such freedom:

> *in symbolical algebra, the rules determine the meaning of the operations.... we might call them* arbitrary *assumptions, in as much as they are* arbitrarily *imposed upon a science of symbols and their combinations, which might be adapted to any other assumed system of consistent rules.* [Peacock 1833, 200-201]

Peacock's endorsements of mathematical freedom, however, lay buried in the "Report" and in his *Treatise* of 1830. In both works he raised the freedom issue only when justifying his decision to apply the rules of arithmetic to symbolical algebra [Pycior 1981, 36-40]. While the principle of mathematical freedom was implicit and only occasionally and tentatively made explicit, the legitimate limits to the exercise of such freedom also remained largely unexplored. As far as present historical scholarship indicates, British mathematics, by the late 1830s, had produced no clear discussion of such desirable properties of postulate systems as consistency, independence, and fertility. Again, isolated comments on one or the other of the properties had appeared; the just-quoted passage from Peacock's "Report," for example, refers to consistency.

The author(s) of the spoof-play, then, explored two fundamental issues raised by the formulation of the symbolical approach--mathematical freedom and limits to the exercise of such freedom. Put on a more personal level, the play's author(s) perceived arbitrariness as basic to the symbolical approach. Manifestations of such arbitrariness were manifold, including Peacock's and De Morgan's assertion of the right to study and manipulate undefined symbols and the right to use the laws of arithmetic as those of algebra. The author(s) had no idea of the degree to which this arbitrariness would or could be carried. In the play, they caricatured mathematical freedom by carrying it to the extreme.

Discussion of two of the play's main episodes will suffice to convey a basic appreciation of the level and flavor of its attack on algebraic arbitrariness. The play opens as twenty Cambridge undergraduates ascribe to De Morgan superhuman powers, including the ability to teach one thousand times as quickly as any other professor. The students conclude that as a group they can learn in one day of study under De Morgan what would take fifty days at Cambridge. (Some sort of individual tutoring seems to be implied.) Having journeyed to London to study under De Morgan, the students then negotiate with him concerning his salary for instructing them.

These negotiations constitute a satire on De Morgan's handling of the appearance of negative numbers in arithmetical calculations. In the *Elements of Algebra* and earlier works, De Morgan explained that negatives sometimes arose in arithmetical problems because of misinterpretation of the original conditions of the problem. He argued further that correction of such arithmetical absurdity required no more than a reversal of sign [A. De Morgan 1837, esp. 56]. Scene two of the play satirizes this principle of sign reversal. The students, who describe themselves as Cambridge men who "always do things liberally," decide that, since De Morgan is able to teach one thousand times as fast as any other professor, he deserves but one-thousandth of a normal professor's salary. When faced with such an offer, De Morgan, however, turns the tables. "It is evident," he declares, "that we have made this equation wrong. All that we have to do, is, to suppose the case the direct reverse of what we have stated it. The numbers will continue the same, but the sign of every term will be altered." Thus, where the students have divided, De Morgan chooses to multiply, demanding a thousand times the normal salary. Justification for this reversal of sign is brief and not particularly convincing, perhaps reflecting the author's opinion of the explanations found in the *Elements of Algebra:*

> *It is plain* [*De Morgan tells the students*] *that you have misstated your problem. Suppose that instead of receiving 1th/1000 part of what other Professors have I were to receive 1000 times as much. The equation then becomes: Fee for one course = £7 × 1000. This renders the problem rational for it is evident that a man who has the power of 1000 men must receive as much as 1000 others.* Vide *page 34 of my Elements of Algebra wherein I explain the meaning of half a horse, two men & three quarters & c.* [pp. 37-38 of [A. De Morgan 1837]]

To this the students simply reply: "What a Sell!... Can't be helped. We must pay" [Anon. n.d., 6-8]. Thus scene two ends with an easy (albeit temporary) victory of De Morgan and algebraic arbitrariness over the students.

This notion of arbitrariness also becomes the butt of a fantastic problem framed in the third and final scene of the play. The problem is posed and solved by De Morgan. It reads as follows: "*A*, the lesser = 1; *B*, the greater = 20. What value of *A* will make it fifty times as great as *B*?" Clearly there is no algebraic solution to this problem. But in the play De Morgan solves it: *A* equals De Morgan and *B* is the group of twenty Cambridge students come to study under him [Anon. n.d., 9-10]. Fifty times twenty, of course, equals one thousand, and De Morgan has the power of one thousand men. Q.E.D. But it is all nonsense! Even in symbolical algebra (with all its arbitrariness) one does not equal fifty times twenty. Again the play clearly misrepresents the symbolical approach, carrying its arbitrariness too far. Yet the author or authors of the play were probably depicting symbolical algebra exactly as they saw it--an art dependent above all on the caprice of the algebraist. If the symbolical algebraist could use the negative and imaginary numbers, which William and possibly Sophia Frend regarded as nonsense, why could he not also set one equal to a thousand? In short, this part of the play captures the essence of early discontent and uneasiness not only with algebraic arbitrariness but also with the lack of any explicit limits on it.

At the very end of the play, the arbitrariness of symbolical algebra finally turns the students away from De Morgan. De Morgan claims to derive two general rules from problems worked in the play: "That unless there is any reason to suppose the contrary, the greater number may always be taken from the lesser--2nd. That the values of terms are entirely arbitrary as well as every operation in Algebra." Now the Cambridge men turn the tables on De Morgan. Loosely applying the second rule, they declare that the terms of De Morgan's salary are arbitrarily high. Application of the first rule results in the students' return to Cambridge. "As the greater number may always be taken from the lesser," one student puts it, "I vote that we all take ourselves away to Cambridge." This is a suggestion the other students, also bewildered by the symbolical approach, readily follow [Anon. n.d., 12].

4. DOCUMENT III: *STRICTURES ON CERTAIN PARTS OF "PEACOCK'S ALGEBRA"*

The third document, *Strictures on Certain Parts of "Peacock's Algebra,"* was published anonymously at Cambridge in 1837. It was written by Osborne Reynolds [A. De Morgan 1849, v; Sylvester n.d.], whom J. J. Sylvester has identified as a twelfth wrangler at Cambridge in 1837 and later a fellow of Queens' College, Cambridge [Sylvester n.d.]. (A rare later reference to Reynolds describes him as a thirteenth wrangler and the father of Osborne Reynolds, the famous British engineer of the late 19th and early 20th centuries [Allen 1970, 2].) This tract was basically a review of Peacock's *Treatise on Algebra*. Since Peacock wrote his *Treatise* to make symbolical algebra "perfectly accessible" to students [Peacock 1830, xxii], Reynolds' work deserves special attention as the considered judgment on symbolical algebra of a member of Peacock's intended audience. A critique of symbolical algebra much more detailed than either Document I or Document II, the *Strictures* shows that Reynolds was, in particular, unable to accept (and possibly even understand) Peacock's transference of emphasis in algebra from the meaning of symbols and signs to the laws of operation. Like the young De Morgan and William Frend, Reynolds demanded that algebra be basically "contentual," that interpretation of algebraic symbols and signs precede rather than follow their manipulation.

In Section II of the *Strictures* Reynolds explored the meaning of the symbols and signs of algebra; he concluded that, because it is devoid of meaning, symbolical algebra is not a science. Considering algebraic symbols first, he correctly noted that Peacock's remarks on the meaning of such symbols were ambiguous. They permitted two interpretations: (1) that the symbols "are intended to represent nothing but themselves, and are in meaning, as in appearance, symbols only," and (2) "that the symbols denote indifferently every species of quantity abstract and concrete" [Anon. 1837, 7-8]. For Peacock, of course, the symbols (and signs) were for a time viewed as arbitrary or meaningless, so that the results of their manipulations could later be interpreted as applicable to many sorts of quantities. But Reynolds ignored this relationship between what we might call "arbitrary" (standing for nothing in particular) and "universal" (standing for many different things) symbols. Instead he attacked the two interpretations separately and dismissed both as unphilosophical and unscientific.

A science of algebra, Reynolds argued, could not be developed around arbitrary or meaningless symbols. *"A symbol,* or anything *merely symbolical,"* he declared, "is *nothing,* until some representation is given to it." Furthermore, if a symbol were merely a symbol--that is, uninterpreted--no mathematical operation could be defined on it. As Reynolds explained, "a symbol as such is not susceptible ... of any *operation.* What is the meaning of adding *a* to *b*, or of subtracting *a* from *b*, if *a* and *b* be nothing more than their forms designate?" Reiterating this point, he (like Frend) compared an algebra of arbitrary symbols to mechanics:

> *To speak of a* mathematical *operation on a symbol as such only, does violence to our ideas of things. The formation of* symbols *is mechanical, and the only operations, of which as such they are susceptible, are also mechanical, and regard the relative local arrangement of themselves or their parts.* [Anon. 1837, 7]

Having disposed of the possibility of an algebraic science built on arbitrary symbols, Reynolds turned to the question of universal symbols, or those standing for all the different kinds of quantity. Certain parts of the *Treatise on Algebra* clearly promoted this second view of algebraic symbols. "The symbols of Algebra," Peacock had at one point stated, "may be made the representatives of every species of quantity, whether abstract or concrete" [Peacock 1830, 1]; in another section of the *Treatise* he had written: "In one system [arithmetical algebra], the symbols represent numerical quantities only: in the other [symbolical algebra] they are perfectly general in their representation" [Peacock 1830, 68]. Reynolds attacked first the assumption upon which the incorporation of universal symbols into algebra appeared to be based--as he put it, the premise "that there are properties *common to every* species of abstract and concrete quantity." "This," he declared, "seems to me an *unphilosophical assumption,* for no attempt is made to prove it by any evidence of the fact either previous or subsequent to the assumption itself" [Anon. 1837, 8]. This criticism was well taken. Peacock sought a symbolical algebra applicable to *all* quantity--numbers, lines, and so on--but he did not know, and certainly was unable to prove, that such a common science existed. Reynolds bolstered his attack on universal symbols with examples of specific algebraic situations in which symbols could not represent all species of quantity. He noted, as an illustration, the case of $a\sqrt{-1}$, where a stood for time. This expression, he declared, was uninterpretable [Anon. 1837, 9]. Similarly, he appealed to Peacock's own admission of agreement with Frend that "there is no such quantity as $-b$" [Peacock 1830, 66], where b represents a number. "Symbolical algebra," Reynolds summarized, "... professes to consider only the *common* properties and relations of *every* species of quantity abstract and concrete, and yet recognises and uses relations, of which various species are not susceptible" [Anon. 1837, 10].

Reynolds next criticized Peacock's use of signs representing indeterminate or undefined operations. In the *Treatise on Algebra* Peacock had refused to define the operations of symbolical algebra beyond giving the rules governing the use of the signs denoting them. "The definitions of those operations," he had declared, "must regard the laws of their combination *only*" [Peacock 1830, ix]. But Peacock's remarks neither satisfied nor removed Reynolds' desire for definitions that dealt with the meanings rather than the manipulations of algebraic signs. Thus, in the *Strictures* Reynolds declared: "I cannot understand how symbols, whether general or not, can be *operated on* (i.e., how they can be used in any system of Algebra) without the *operations* to which they are subjected being both determinate and known." As an illustration of the general problem of algebraic operations, he attacked Peacock's use of an undefined multiplicative operation. He concurred with Peacock's contention that the definition of arithmetical multiplication was inapplicable to symbolical algebra. But then he demanded to know what multiplication meant in the latter algebra. As he put it: "Our first inquiry naturally is, what is this operation?" Peacock's failure to define (in the traditional way) the multiplicative operation left Reynolds frustrated. "It is not sufficient to be told," he complained,

> *"that it [the multiplicative operation] is some one of which all species of quantity are susceptible," nor "that it is one indicated by the sign ×": we are not* thus *informed of its real nature,--we are not* thus *told what it is. Moreover, we in vain attempt to discover the nature of* this operation *from Mr. Peacock's Treatise; it is left indeterminate, it cannot be determined, and for this reason,--that we know and can conceive of no operation which is equally applicable to every species of quantity.... Hence therefore we are driven to this unavoidable conclusion, that it is no operation at all, and neither means nor can mean any thing more than the mechanical interposition of × between the symbols on which it is supposed to be performed.* [Anon. 1837, 11-12]

Satisfied with his demonstration of Peacock's failure to define the operations of symbolical algebra--and of the impossibility of doing so, Reynolds concluded at the end of Section II that "the operations of Symbolical Algebra are no operations at all, and the science, consequently, which professes to use them can be no science at all" [Anon. 1837, 13]. To Peacock, symbolical algebra was the study of arbitrary or universal symbols and indeterminate, undefined operations. To Reynolds, a representative of the old view, mathematics was supposed to be meaningful in more than a strictly logical way; thus, to him, symbolical algebra, with what Peacock saw as its temporary rejection of meaning, did not deserve to be ranked among the mathematical sciences.

One could, Reynolds argued in Section III of the *Strictures,* justify symbolical algebra neither theoretically nor practically. Not only was the symbolical approach unscientific, as explained above, but it did not even resolve the problem in answer to which it had been created. "The Theory of Symbolical Algebra appears to have been invented solely for the sake of avoiding a supposed difficulty in the *independent use of the sign* -" [Anon. 1837, 14], Reynolds wrote. Peacock had regarded $-b$, where b stood for an abstract number, as unintelligible, rejecting as nonsense the definition of such (negative) numbers as "quantities less than nothing." Peacock had therefore introduced the symbol $-b$ into symbolical algebra by assumption and without definition. But, Reynolds argued, such a step only compounded the problem of the negatives. Returning to the interpretation of Peacock's symbols as universal, Reynolds claimed that symbolical algebra raised the specter not only of numbers "less than nothing" but also of lengths, weights, and all sorts of other quantities "less than nothing." For *b* in *-b* now stood for all quantities. Section III concluded with the declaration that: "Were there ... *no other* objections to Symbolical Algebra, it is surely a *fatal one,* that it altogether fails to effect that for which it was devised" [Anon. 1837, 19].

In Section III Reynolds formulated his response to the problem of the negative numbers by dismissing it as no problem at all. In a nutshell, he argued that the distinction between negative and positive numbers was based on "a relation actually existing" in nature. Negative and positive numbers were therefore as intelligible as unsigned abstract numbers. Just as the mathematician derived his idea of an unsigned abstract number from consideration of concrete magnitude, so he came to the notion of signed numbers from consideration of "an invariable and mutually opposite relation existing between the two classes in which quantity usually appears whenever it presents itself to the contemplation of the mind." He noted, for example, that travellers think of distances covered in opposite directions and physicists, of forces acting on bodies in opposite directions [Anon. 1837, 15-16].

Thus, in Reynolds' opinion, Peacock's work on symbolical algebra was not only flawed but also for naught. Reynolds believed that he himself had finally vindicated traditional algebra against the objections of the opponents of the negatives; he had finally penetrated through to the essence of the negative and positive numbers. In short, unable to accept the meaninglessness of the symbolical approach, Reynolds rejected the approach in toto, arguing that the meaningfulness of the negative numbers removed the rationale for introducing undefined algebraic symbols and signs. Peacock's *Treatise on Algebra* had clearly failed to convince at least this one Cambridge graduate.

Scanty evidence indicates that reaction to the *Strictures* was mildly favorable. Praise came from an unlikely source--J. J. Sylvester, traditionally identified with the 19th-century British movement toward abstract algebra. In an undated [6] letter of recommendation for Reynolds, Sylvester wrote:

> *Mr. Osborne Reynolds has been long known to me and there is no one to whose intellectual and engaging qualities I can speak with more confidence....*
>
> *In evidence of Mr. Reynolds' propriety of composition and perspicacious judgment I may mention that a Work partly Mathematical and partly Philosophical coming from a very high authority at Cambridge [Peacock's* Treatise on Algebra], *of which a second edition was long advertised has been since suppressed*

> *in deference it is generally believed to certain severe but fair "Strictures" attributed to Mr. Reynolds.* [7]

De Morgan, into whose possession the letter fell, was of a different opinion on the matter. In the margin next to the final paragraph just quoted, he wrote: "Pooh Pooh Pooh $(\text{Pooh})^n$ $n = \infty$" [Sylvester n.d.]. There are two possible interpretations of De Morgan's marginal annotation. Either De Morgan was objecting to Sylvester's appeal to the *Strictures* as the explanation for Peacock's delay in publishing a revised edition of his *Treatise*, or he was protesting Sylvester's flattering characterization of Reynolds' work. Even if the latter interpretation is correct, there are indications that De Morgan at least eventually deemed Reynolds' critique of symbolical algebra somewhat meretorious. The *Strictures*, for example, appeared in a bibliography of works on "the peculiar Symbols of Algebra" at the beginning of De Morgan's *Trigonometry and Double Algebra* [A. De Morgan 1849, v-vi]. Furthermore, possibly as a result of contact with the ideas of Reynolds (and other early critics of the symbolical approach), De Morgan, in the late 1830s, abandoned his earlier designation of symbolical algebra as a science.

5. CONCLUSION

What does all this prove? First, resistance to the symbolical approach to algebra was more widespread among members of the early-19th-century British mathematical community than hitherto thought. Second, opposition came from old as well as young mathematicians, from those who encountered symbolical algebra at the end of their lives as well as from those who studied it as undergraduates. For example, at the time of the publication of Peacock's *Treatise on Algebra*, William Frend was seventy-three years old and had been an opponent of the negatives and all algebra beyond universal arithmetic for over thirty years. Hamilton, on the other hand, was only twenty-five in 1830. Osborne Reynolds and possibly Augustus De Morgan (who studied under Peacock at Trinity [S. De Morgan 1882, 16]) came into contact with the approach as undergraduates. Third, despite differences in age and mathematical experience, these early critics of symbolical algebra shared a common conception of algebra as a science of symbols and signs which stood for *something*--be it in the human mind or in the physical universe. All, therefore, rejected Peacock's symbolical approach which involved the development of an algebra of basically meaningless symbols and signs, which were interpreted only after manipulation. Expressed differently, Hamilton, William (and possibly

Sophia) Frend, Reynolds, and even the young De Morgan shrank before abandoning meaning in mathematics.

Fourth, in the 1830s mathematical freedom was an idea whose time had not yet come. The author(s) of the play chose to satirize it, carrying it to extremes and subjecting it to laughter and ridicule. Mathematicians of the period failed to exercise it. The idea gained acceptance only after Hamilton invented the quaternions *and* demonstrated their usefulness and (to a limited extent) consistency. Hamilton, in fact, devoted the final twenty-two years of his life to legitimating the quaternions by showing that they were physically applicable and that their manipulation did not lead to contradictions [Pycior 1976, 160-181].

Finally, these three documents point to the need for a detailed, comprehensive study of British reception of symbolical algebra, particularly in the late 1830s and early 1840s. (The year of the invention of the quaternions, 1843, provides a natural terminus for such a project.) Preliminary research indicates that much work on this topic remains to be done. Once alerted to the phenomenon of early resistance to symbolical algebra, the scholar of early-19th-century British mathematics quickly discovers evidence of such resistance throughout the period's literature--in the writings of De Morgan, William Whewell, and Philip Kelland, for example. A firm advocate of the science of symbolical algebra in 1835 [A. De Morgan 1835], De Morgan maintained in the 1840s that symbolical algebra was an art, not a science (see, e.g., [A. De Morgan 1842b, 173-177]). During this later period he reserved the rank of science for that part of algebra--which he called "logical"--"which investigates the method of giving meaning to the primary symbols, and of interpreting all subsequent symbolic results" [A. De Morgan 1842b, 173].

Whewell and Kelland voiced different reservations about the symbolical approach. As early as 1835, Whewell, then a fellow and tutor of Trinity College, Cambridge, seemed to argue (without directly referring to Peacock) against inclusion of symbolical algebra in the liberal arts curriculum. In his *Thoughts on the Study of Mathematics as a Part of a Liberal Education,* for example, he condemned "mathematics ... taught in such a manner that its foundations appear to be laid in arbitrary definitions without any corresponding act of the mind." In considering "the rival claims of geometrical and algebraical modes of reasoning in cases where either may be used," he added, "we should require that the mode which is selected be so presented as to shew that the meaning of the expressions employed is distinctly understood by the student." Later in the same book, he took a stronger stand against undergraduate algebra in general, concluding that "if necessary, let the knowledge of Algebra be required no longer" [Whewell 1835, 8, 42, 44]. In England of the early 19th century, when mathematics was pursued as much for its mind-disciplining qualities as for its inherent interest and potential usefulness, Whewell's pedagogical criticism probably proved a strong deterrent to acceptance of symbolical algebra.

In 1839, Kelland, the professor of mathematics at the University of Edinburgh and a former fellow of Queens' College, Cambridge [8], attacked Peacock's work from a technical perspective. In the preface to his *Elements of Algebra* of that year, Kelland referred directly to Peacock's *Treatise,* objecting to its simultaneous declarations that symbolical algebra was independent of arithmetic and yet that arithmetical algebra was the science of suggestion for symbolical algebra [Kelland 1839, vii]. Having maintained the inappropriateness of including a detailed criticism of Peacock's work in his elementary treatise of 1839, Kelland produced a more substantial critique in 1843. Briefly, in his later book, Kelland insisted on establishing the meaning of symbols and signs prior to manipulation as a possible guarantee of both applicability and consistency [Kelland 1843, 111-114].

The full story of the early British reception of symbolical algebra, beginning possibly with the three documents studied in this paper, still remains to be told. Careful analysis of all surviving comments on the approach is needed. Many questions remain: Why did De Morgan change his opinion on symbolical algebra in the late 1830s? Were the Frends, Reynolds, Whewell, Kelland, Hamilton, or even presently unknown early critics of the symbolical approach, responsible for De Morgan's turnabout? To what extent did early criticism affect the timing and content of the second edition of Peacock's *Treatise*? Finally, was there a British school of symbolical algebraists in the late 1830s and the early 1840s?

ACKNOWLEDGMENTS

I thank The Syndics of Cambridge University Library, the University of London Library, and The Library, University College London, for use of their manuscripts collections and for permission to quote selected passages therefrom. I also wish to thank the anonymous reviewers of an earlier version of this paper for helpful comments and suggestions.

NOTES

1. Nový maintains that the movement died by midcentury. Although stating that the causes of its decline are outside the scope of his work, he suggests that some of the ideas associated

with early-19th-century British symbolical algebra "did not admit of further treatment in their time, or ... were not necessary in view of the state of algebraic research at the time" [Nový 1973, 199].

2. Two other 18th-century British opponents of the negatives were Robert Simson and Francis Maseres. For a brief discussion of their views on the problem of the negative numbers, see [Nagel 1935, 435-437].

3. An important source on the origins and content of Frend's iconoclastic mathematical views is [A. De Morgan 1842a].

4. The document was written no earlier than 1836. It refers to De Morgan's *Elements of Algebra*, which was originally published in 1835, and to the "University College London," the name taken by London University upon the founding of the University of London in 1836 [S. De Morgan 1882, 91-92]. More tentatively, I propose 1838 as the last possible year of the play's composition. The play introduces as a minor character a Mr. Kennell, who was identified for me by Janet Percival, archivist for the D. M. S. Watson Library, University College London, as the accountant of University College London through 1838 when his embezzlement of school funds was discovered. It seems unlikely that a known embezzler would have been immortalized in the play. Furthermore, as I argue below, the play was probably written by William or Sophia Frend. There are compelling reasons against their composing the play later than 1838. By that year William Frend had suffered two strokes, the second of which left him "hardly able to speak or to move" [Knight 1971, 306], and Sophia Frend was De Morgan's wife.

5. I am indebted to Janet Percival for recognizing the handwriting as Sophia Frend's.

6. This letter was written between 1838 and 1841. It describes Sylvester as Professor of Natural Philosophy at University College London, a position which he occupied only during this short three-year span.

7. Shortly after Sylvester wrote this recommendation, Peacock published a two-volume, revised edition of the *Treatise on Algebra* [Peacock 1842-1845].

8. It is possible that during the late 1830s Queens' College was a focal point of Cambridge opposition to the symbolical approach. Both Reynolds and Kelland were associated with Queens' College, the former as a student and the latter as a fellow.

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