Principles of Electrodynamics

Carl Neumann

Editor's Note: English translation of Carl Neumann's 1868 paper "Die Principien der Elektrodynamik", [Neu68a].

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By Carl Neumann[1,2,3]

The individual areas of physical science could aptly be subdivided into *two* parts, according to the nature of the elementary forces which are assumed to explain the relevant phenomena. On one side stands *celestial mechanics, elasticity, capillarity*, in general those areas for which the direction and magnitude of the force is fully determined by the relative position of the material parts; on the other side are to be considered the investigations of *friction, electricity* and *magnetism*, and perhaps also *optics*, in general those areas of physics in which the known forces depend upon other conditions in addition to their relative positions – their velocities and accelerations, for example.

Now, if the *law (or principle) of vis viva*[4] rules completely over the natural phenomena (and all previous experience speaks for this), as appears to apply for the first subdivision as a direct consequence of the underlying ideas, for the second subdivision it seems to be a *matter of chance*. For the elementary forces of the first type subject themselves to the rule of that law, but those of the second type do not.

> "It seems"–says Fechner in his *Psychophysik* (1860),[5] Vol. I, page 34–"that these (last) elementary forces work together in such a way that the law[6] remains applicable to all actions of nature. In the case of the magnetic forces (and therefore electric currents as well) this is self-evident, insofar as they actually can be represented as the effects of central forces, which are independent of velocity and acceleration. Moreover, Prof. W. Weber has responded orally to my questioning, that in all cases to which his investigation has led, even beyond the limit of the latter forces,

[1] [Neu68a].

[2] Translated by Laurence Hecht, larryhecht33@gmail.com, and U. Frauenfelder, urs.frauenfelder@math.uni-augsburg.de. Edited by A. K. T. Assis, www.ifi.unicamp.br/~assis

[3] The Notes by Carl Neumann are represented by [Note by CN:]; the Notes by Laurence Hecht are represented by [Note by LH:]; while the Notes by A. K. T. Assis are represented by [Note by AKTA:].

[4] [Note by AKTA:] The Latin expression *viv viva* (*living force* in English or *lebendige Kraft* in German) was coined by G. W. Leibniz (1646-1716).

Originally the *vis viva* of a body of mass m moving with velocity v relative to an inertial frame of reference was defined as mv^2, that is, twice the modern kinetic energy. However, during the XIXth century many authors like Weber and Helmholtz defined the *vis viva* as $mv^2/2$, that is, the modern kinetic energy.

[5] [Note by AKTA:] [Fec60, p. 34].

[6] [Note by AKTA:] Fechner is referring here to the law of the conservation of energy.

> the law is found to be valid, even if its full applicability to the region of these forces still requires strict proof."

But this is actually not a proof, but a discovery. Because that law represents a relation between the *vis viva* and the potential, and thus a relation between two magnitudes, the latter of which is known as an elementary force of the first type, *but is completely unknown for the second type.* Respecting the latter forces, it is therefore not the proof of the law, but the *discovery of its content*, and the determination of its magnitude, which would be regarded as the potential of those forces.

Three years ago, stimulated by the just cited words of Fechner, I began to interest myself in this question and thus directed my attention to *those* elementary forces of the second kind which Weber assumed between two electric particles, and I soon found that the potential of such a force could be viewed with certain authority by the following expression:

$$W = \frac{mm_1}{r} + G\frac{mm_1}{r}\left(\frac{dr}{dt}\right)^2 ,$$

where m, m_1 indicate the masses of the two particles, r their distance apart, t the moment of time under consideration, and G a constant.[7] Then it is seen that the force assumed by Weber can be derived from this expression by *variation* of the coordinates in exactly the same way in which an elementary force of the *first* kind is obtained from its potential though a *differentiation* of its coordinates.

And simultaneously it resulted that during the motion of both particles a very simple relationship prevails between the *vis viva* and the two parts of the expression W adopted as potential, namely:

$$(\text{vis viva}) + \frac{mm_1}{r} - G\frac{mm_1}{r}\left(\frac{dr}{dt}\right)^2 = \text{constant} .$$

It can scarcely be doubted that this relationship represents the law to be discovered for the force assumed by Weber.

I also had already back then, in accordance with the expression for W, formulated the potential for *two elements of electric current*, and found that from the potential so obtained, both the repulsive and the inductive action of the two elements on one another could be derived in a very simple way, namely the former could be deduced by variation according to the distance, the latter by variation according to direction of an element.

[7][Note by AKTA:] By "electric masses" m and m_1 we should understand here the charges of the particles, [Arc86, p. 787].

Amazing as it may seem at first sight, and in some contrast to the hitherto prevailing view, variation must take the place of differentiation. However, as I want to remark right away, this contrast is to some extent tempered, when one observes that a similar treatment already applies in the area of elementary forces of the *first* type, for example in investigation into elasticity. Namely, let u, v, w be those functions of the coordinates through which the internal displacement of a given elastic body are represented, and Φ the potential which the particles of the body collectively exert on any *one* of them, then the force acting on the latter is found through *variation* of Φ according to u, v, w (as was developed in detail by me in an essay on elasticity, Borchardt's *Journal*, Vol. 57, page 304).[8]

Some time ago I was prompted to resume and continue my investigations into the subject in question by a posthumous essay of Riemann's, published in Poggendorff's *Annalen* (Vol. 131, page 237),[9] in which the attempt (which, however was not very successful, and perhaps, as a result of the too brief presentation should not be judged) is made to explain the repulsive action of two current elements on each other by elementary forces of the *first* kind, under the assumption that the potential of this force – similar to light – is propagated through space with a certain constant velocity. To my surprise I found that this assumption leads directly to my conjecture, namely by assuming such a progressive propagation, the ordinary potential $\frac{mm_1}{r}$ (corresponding to the Newtonian gravitational force), transforms into a magnitude whose effective constituent is completely identical with the previously mentioned expression W.

Already in May of this year I made a short communication to the Göttingen Scientific Society about the starting point and results of the investigation in question (*Nachrichten der Gesellschaft*, June 16, 1868).[10] If I now intend to present these investigations, or at least a part of them, in detail and as carefully as possible, this is not because I consider these investigations to be completely thorough, but rather because of the *extraordinary importance of the subject at hand*, and because I am of the opinion that my researches may be necessary, or at least not without use, for a deeper penetration into this subject.

[8] [Note by AKTA:] [Neu60, p. 304].
[9] [Note by AKTA:] [Rie67b] with English translation in [Rie67a] and [Rie77a].
[10] [Note by AKTA:] [Neu68b].

1 Section 1. Overview

1.1 Basis of the Investigation

In the present investigation I will share the nomenclature of those authors who understand *vis viva (Lebendige Kraft)* as the sum of the masses multiplied by *one half* the square of their velocity, and who further understand the potential as that function of the coordinates whose *negative* differential coefficient represents the force.[11,12] By applying this nomenclature (which in hindsight seems especially appropriate to the mechanical theory of heat) the *Principle of Vis Viva* assumes the form

$$(\text{vis viva}) + (\text{potential}) = \text{constant} .$$

At the same time another general principle of mechanics, the *Hamiltonian Principle*, finds its expression in the formula

$$\delta \int [(\text{vis viva}) - (\text{potential})] dt = 0 ,$$

where the integration is carried out over any chosen time interval, and where δ designates the *internal variation*, that is a variation which does not affect the limits but only the inside of that time interval.

If I now notice that we know the potential by the given forces, but also that, inversely, by specifying the potential the forces are determined, and if I accordingly allow myself to consider the potential as primary, as the actual *driver of impulse to motion*, and the forces as secondary, as the *form* in which the impulse manifests itself, this is not a real but at best a formal innovation. On the other hand, what is essentially new (albeit related to the conjecture already made by Riemann) is my assumption that *the motive impulse represented by the potential does not pass from one mass point to the other instantaneously, but progressively, that it propagates in space with a certain albeit extremely great velocity.* This velocity is considered constant and will be designated by c.[13]

[11] [Note by CN:] By this definition *vis viva* and *potential* are identical to the magnitudes the English call *actual* and *potential energy*. *Potential* is also identical to the magnitude Helmholtz calls *Spannkraft*.

[12] [Note by AKTA:] Helmholtz introduced the concept *Spannkraft* in 1847, [Hel47, p. 14]. It was translated as *tension*, [Hel66, p. 122]. According to Elkana, Helmholtz coined the phrase "Spannkraft" for the clearly defined mechanical entity that we call "potential energy", [Elk70, p. 280]. Caneva translated it as "tensional force", [Can19].

[13] [Note by LH:] Weber's constant c is not the speed of light, being equal to $\sqrt{2}$ times the speed of light.

The idea just mentioned, and the assumption that *Hamilton's Principle* is applicable without restriction, form the basis of my investigation; they form the source from which the laws of electric phenomena (discovered by Ampère, Weber, and my father)[14] come out on their own, without bringing in any further assumption.

It is scarcely necessary to remark that the *ordinary* conception of an instantaneous propagation of the potential is contained as a special case in the conception *put forth here* of a progressive propagation, namely that this conception goes over to the ordinary one as soon as one sets the constant $c = \infty$.

1.2 Weber's Law

First consider only *two* points m and m_1, which move under their mutual influence. Then, proceeding from the conception of a progressive propagation of the potential, for each given instant of time t, two different potentials appear, the emissive and the receptive.

The *emissive potential* is that which *is sent out at the time* t from each of the two points, and which therefore reaches the other point a little later. Let r represent the distance of the two points at time t, and $\tilde{\omega}$ the emissive potential corresponding to the same time, then according to Newton's law $\tilde{\omega} = \frac{mm_1}{r}$, or generally:[15]

$$\tilde{\omega} = mm_1\varphi \ , \tag{1}$$

where $\varphi = \varphi(r)$ represents any given function of r.

The *receptive potential* on the other hand, is that which *is received at time* t and which therefore was already sent out a little earlier from the other point. The receptive potential belonging to the *given* time is accordingly identical to the emissive potential of an *earlier* time. The distance at time

[14][Note by AKTA:] Carl Neumann is referring to André-Marie Ampère (1775-1836), Wilhelm Eduard Weber (1804-1891) and Franz Ernst Neumann (1798-1895). Ampère's main work on electrodynamics containing his force between current elements is from 1826, [Amp23] and [Amp26] with a complete and commented English translation in [AC15]. Weber's force between point charges was published in 1846, [Web46] with partial French translation in [Web87] and a complete English translation in [Web07]. Franz Neumann's works on induction can be found in [Neu46] and [Neu47], with French translation in [Neu48a]; [Neu48b] and [Neu49].

[15][Note by AKTA:] Neumann numbered the equations in each Section of his paper beginning with (1). This creates a possible misunderstanding related to which specific equation he might be referring to in later portions of the work. In this English translation we numbered sequentially the equations of the whole paper.

t is again designated as r, and the receptive potential corresponding to that time is ω, so there results after some calculation

$$\omega = w + \frac{d\mathfrak{w}}{dt} \ , \tag{2}$$

where

$$\begin{aligned} w &= mm_1 \left[\varphi + \left(\frac{d\psi}{dt} \right)^2 \right] \ , \\ \mathfrak{w} &= mm_1 \left[\chi + \frac{d\Phi}{dt} \right] \ . \end{aligned} \tag{3}$$

Here φ is the function contained in the emissive potential; and at the same time ψ, χ, Φ are certain other functions, also only depending upon r, which allow derivation out of the given function φ through fairly simple operations. So, for example

$$\psi = \frac{1}{c} \int \sqrt{-r \frac{d\varphi}{dr}} dr \ . \tag{4}$$

The function φ is, as emerges directly from its definition, independent of the propagation velocity c; φ, χ however are affected by the factor $\frac{1}{c}$ and Φ with the factor $\frac{1}{c^2}$.

$$\left. \begin{array}{c} \textit{Still to be noticed is that for the case of the} \\ \textit{Newtonian emission law, namely for } \varphi = \frac{1}{r}, \textit{ the function} \\ \psi \textit{ assumes the value:} \psi = \frac{2\sqrt{r}}{c} \ . \end{array} \right\} \tag{5}$$

Of the two parts of the receptive potential, we denote w as the *effective potential* and the other one $\frac{d\mathfrak{w}}{dt}$ the *ineffective potential.*

Since Hamilton's principle is considered to be valid without limitation, one has to be able to derive the dynamics of the points m and m_1 from the formula

$$\delta \int (\tau - \omega) dt = 0 \ ,$$

where τ is the vis viva of the two points and ω the already mentioned receptive potential.

Substituting for ω its value (2), the formula reduces to

$$\delta \int (\tau - w) dt = 0 \ .$$

If one carries out the variation from this expression, the six differential equations needed to determine the dynamics follow. These equations explain *how* the dynamics take place, i.e., they explain the *force* acting between the two points. The result obtained in this way is the following:

I. *A force, R, acts between the two points as they move, the force acting along the straight line r connecting the points at each point in time.*

II. *If one considers this force as a repulsive one and if w is the (already mentioned) effective potential of the two points, then R equals the negative variation coefficient of w with respect to r.*[16]

An immediate consequence of this is

$$R = mm_1 \left[-\frac{d\varphi}{dr} + 2\frac{d\psi}{dr}\frac{d^2\psi}{dt^2} \right] \ . \tag{6}$$

In the special case mentioned in (5), namely $\varphi = \frac{1}{r}$ and $\psi = \frac{2\sqrt{r}}{c}$, this formula becomes

$$R = mm_1 \left[\frac{1}{r^2} + \frac{4}{c^2\sqrt{r}}\frac{d^2\sqrt{r}}{dt^2} \right] \ . \tag{7}$$

Formula (6) precisely coincides with the law on which I based ten years ago my study dealing with the magnetic rotation of the plane of polarization of light. And formula (7) is literally *identical to Weber's law*.

A closer examination leads to the following additional results

III. *If W is the effective potential of an arbitrary system of points and if x, y, z are the coordinates of the point having the mass m, the components of the force acting on m become equal to the negative variation coefficient of W with respect to x, y, z.*

IV. *If P is the component of that force in an arbitrary given direction p, then P equals the negative variation coefficient of W with respect to p.*

The term variation coefficient used several times needs a short explanation. Suppose that u, v, ... w are *undetermined* functions of a base variable (for example the time), or *undetermined* functions of any number of base variables

[16] [Note by AKTA:] In the original: *dem negativen Variationscoefficienten von w nach r.*

α_1, α_2, ... α_n, and G is a *given* expression from the variables α_1, α_2, ... α_n, from the functions u, v, ... w and from some derivatives of these functions with respect to these variables. Then it is well-known that the *internal variation* coming from a change of u, v, ... w

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n$$

always can be put into the form

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \cdots d\alpha_n = \int^{(n)} \big(a\delta u + b\delta v + \cdots + c\delta w\big) d\alpha_1 d\alpha_2 \cdots d\alpha_n \ ,$$

in which the coefficients a, b, ... c only depend on α_1, α_2, ... α_n, u, v, ... w, being independent of the variations δu, δv, ... δw. These coefficients $a, b, \ldots c$ I call *variation coefficients with respect to* u, v, ... w.

1.3 The Laws of Electric Repulsion and Induction

Since the hypotheses employed led to Weber's universal law of electrical action, they obviously must also guide us to those special laws regarding the repulsion and induction of electric currents, which were discovered earlier and later unified under Weber's law. Nevertheless I examined this topic more closely and found,[17] that for the deduction of the known special laws it almost does not matter if one starts from the *two-fluid* or *one-fluid* theory of electrical current. A difference in this respect only shows up in the laws of induction and here as well in the (probably still not sufficiently examined) cases dealing with induction of *non-closed* currents.

Let ds denote an element of an electric current. Moreover, $+eds$ and $-eds$ are the quantities of positive and negative electric fluids contained in it. Finally $s' = \frac{\partial s}{\partial t}$ and S' are the velocities of these quantities with respect to *one and the same* direction s.

Putting $S' = -s'$, then both fluids move with the same speed in opposite directions. This is in full correspondence with the two-fluid theory one usually starts with.

However, if one puts $S' = 0$ the negative fluid is considered to be attached to the ponderable matter or even to be identical with this matter. In this case only *one* fluid is moving. This latter point of view I denoted before as the unitary one.

[17][Note by CN:] What I state concerning electric repulsion and induction as a result of my studies will not be justified and carried further in the current article. I plan to do this in a later note.

If one follows simultaneously *both* concepts and keeps the function φ in the emissive potential undetermined, one obtains the following results. Here ds, eds, $s' = \frac{\partial s}{\partial t}$ have the already mentioned meaning and $d\sigma$, $\eta d\sigma$, $\sigma' = \frac{\partial \sigma}{\partial t}$ have an analogous meaning with respect to a second current element.

I. *Assume that W is the effective potential of the two current elements and r their distance, then*

$$W = \frac{(2n)^2 ds d\sigma \cdot es' \, \eta\sigma'}{2} \frac{\partial \psi}{\partial s} \frac{\partial \psi}{\partial \sigma} \,, \tag{8}$$

where ψ represents the function mentioned in (4) and n is an integer, which $= 2$ or $= 1$ depending on whether one assumes the two-fluid or one-fluid theory.

As mentioned in (5) for the special case $\varphi = \frac{1}{r}$ one has $\psi = \frac{2\sqrt{r}}{c}$. In this case the value of the potential becomes

$$W = \left(\frac{2n}{c}\right)^2 \frac{ds d\sigma \cdot es' \, \eta\sigma'}{2r} \frac{\partial r}{\partial s} \frac{\partial r}{\partial \sigma} \,. \tag{9}$$

II. *The repulsive force $\mathfrak{R}$ by which the two current elements act on each other equals the negative variation coefficient of the potential W with respect to r.*

From that follows the formula

$$\mathfrak{R} = (2n)^2 ds d\sigma \cdot es' \, \eta\sigma' \, \frac{\partial \psi}{\partial r} \frac{\partial^2 \psi}{\partial s \partial \sigma} \,, \tag{10}$$

which in the case $\varphi = \frac{1}{r}$, $\psi = \frac{2\sqrt{r}}{c}$ becomes

$$\mathfrak{R} = \left(\frac{2n}{c}\right)^2 \frac{2 ds d\sigma \cdot es' \, \eta\sigma'}{\sqrt{r}} \frac{\partial^2 \sqrt{r}}{\partial s \partial \sigma} \,. \tag{11}$$

However, this last formula is *identical to Ampère's law*, as follows easily.

III. *If $d\sigma$ and ds are two elements of closed currents, and $\mathfrak{E}$ denotes the electromotive force exerted by $d\sigma$ on ds along the direction s, then $\mathfrak{E}$ equals the negative variation coefficient of W with respect to s.*

This formula, which is valid in general, whether the induction is due to a change of relative position or a change of the intensity of the current, immediately leads to the formula

$$\mathfrak{E} = \frac{d\overline{W}}{dt} \ , \tag{12}$$

if one understands by $\overline{W}$ the value of the potential W for $s' = 1$. This formula precisely represents *the induction law as stated by my father.*

IV. Up to this point there is a complete correspondence between the results obtained from the two-fluid and one-fluid theories. However, I examined as well the case of induction between *non-closed* currents and found that in this case there is quite a *difference* between the results obtained from the two points of view.

1.4 The Principle of Vis Viva

Our assumption is that Hamilton's principle is valid without restriction. An immediate consequence of this assumption is that the principle of vis viva always holds as well. However, it might change its usual form.

If only *two* points m and m_1 are given and w is the effective potential of the two points, then according to (3) we have

$$w = mm_1\left[\varphi + \left(\frac{d\psi}{dt}\right)^2\right] \ , \tag{13}$$

or equivalently

$$w = u + v \ , \tag{14}$$

where u and v are given by

$$\begin{aligned} u &= mm_1\varphi \ , \\ v &= mm_1\left(\frac{d\psi}{dt}\right)^2 \ . \end{aligned} \tag{15}$$

From the meaning of φ and ψ (compare (1) and (4)) it follows that u is independent of the propagation velocity c, whereas v is affected by the factor $\frac{1}{c^2}$. On the other hand one immediately sees from (15), that v vanishes, as soon as the two points are at *rest*, and that in this case the potential w becomes u. For this reason I call u the *static* and v the *motive potential.*[18]

[18] [Note by AKTA:] *Das motorische Potential* in the original.

It is worth noticing that the static potential coincides with the emissive potential, which follows not only from the formulas, but also directly from the definition of these potentials.

We consider now the dynamics of an arbitrary system of points and let W be its effective potential. We decompose W (as was done for w) into two terms

$$W = U + V \ . \tag{16}$$

The term U independent from c represents the static potential, while the term V affected by the factor $\frac{1}{c^2}$ represents the motive potential. Using these notions we will show the validity of the following theorem for the vis viva:

During the movement of an arbitrary system of points the vis viva, increased by the static and decreased by the motive potential, always has the same value. Mathematically one has

$$T + U - V = \text{constant} \ , \tag{17}$$

where T is the vis viva of the system. In the case of instantaneous propagation, i.e., for $c = \infty$, the expression V affected by the factor $\frac{1}{c^2}$ vanishes. In this case the formula (17) simplifies to the well-known formula $T + U = \text{constant}$.

As regards the expressions T, U, V, we remark that the first one only depends on the velocities of the points, the second one only on their relative position and the third one simultaneously on *both* the velocities and the relative position.

2 The Variation Coefficients

2.1 Preliminary Remark

If f and φ are functions of the three variables α, β, γ the following equations hold

$$\begin{aligned}
f\frac{\partial^3\varphi}{\partial\alpha\partial\beta\partial\gamma} &= \frac{\partial}{\partial\alpha}\left(f\frac{\partial^2\varphi}{\partial\beta\partial\gamma}\right) - \frac{\partial f}{\partial\alpha}\frac{\partial^2\varphi}{\partial\beta\partial\gamma} \ , \\
\frac{\partial f}{\partial\alpha}\frac{\partial^2\varphi}{\partial\beta\partial\gamma} &= \frac{\partial}{\partial\beta}\left(\frac{\partial f}{\partial\alpha}\frac{\partial\varphi}{\partial\gamma}\right) - \frac{\partial^2 f}{\partial\alpha\partial\beta}\frac{\partial\varphi}{\partial\gamma} \ , \\
\frac{\partial^2 f}{\partial\alpha\partial\beta}\frac{\partial\varphi}{\partial\gamma} &= \frac{\partial}{\partial\gamma}\left(\frac{\partial^2 f}{\partial\alpha\partial\beta}\varphi\right) - \frac{\partial^3 f}{\partial\alpha\partial\beta\partial\gamma}\varphi \ .
\end{aligned}$$

If these equations are multiplied by $(-1)^0$, $(-1)^1$, $(-1)^2$, respectively, and than added together one gets

$$\begin{aligned} \frac{\partial^3\varphi}{\partial\alpha\partial\beta\partial\gamma} &= \frac{\partial}{\partial\alpha}\left((-1)^0 f\frac{\partial^2\varphi}{\partial\beta\partial\gamma}\right) + \frac{\partial}{\partial\beta}\left((-1)^1\frac{\partial f}{\partial\alpha}\frac{\partial\varphi}{\partial\gamma}\right) \\ &+ \frac{\partial}{\partial\gamma}\left((-1)^2\frac{\partial^2 f}{\partial\alpha\partial\beta}\varphi\right) + (-1)^3\frac{\partial^3 f}{\partial\alpha\partial\beta\partial\gamma}\varphi \ . \end{aligned} \tag{18}$$

Analogously, if f and φ are functions of arbitrarily many (for instance p) variables α, β, $\dots$ π, one obtains a formula of the following form

$$f\frac{\partial^p\varphi}{\partial\alpha\partial\beta\dots\partial\pi} = \frac{\partial A}{\partial\alpha} + \frac{\partial B}{\partial\beta} + \dots + \frac{\partial P}{\partial\pi} + (-1)^p\frac{\partial^p f}{\partial\alpha\partial\beta\dots\partial\pi}\varphi \ . \tag{19}$$

Let there be in total n variables α_1, α_2, $\dots$ α_n on which f, φ depend, and let α, β, $\dots$ π represent any number of these n variables each one with arbitrary many repetitions, then the formula (19) is still valid. If one multiplies that formula by $d\alpha_1 d\alpha_2\dots d\alpha_n$ and integrates over an arbitrary domain, it follows that

$$\begin{aligned} &\int^{(n)} f\frac{\partial^p\varphi}{\partial\alpha\partial\beta\dots\partial\pi}d\alpha_1 d\alpha_2\dots d\alpha_n \\ &= \Sigma + (-1)^p\int^{(n)}\frac{\partial^p f}{\partial\alpha\partial\beta\dots\partial\pi}\varphi d\alpha_1 d\alpha_2\dots d\alpha_n \ , \end{aligned} \tag{20}$$

where Σ is a sum of $(n-1)$-fold integrals on the *boundary* of the domain of integration. Moreover, it follows from the meaning of A, B, $\dots$ P that *these integrals vanish when the function φ and its derivatives vanish at that boundary.*

2.2 Definition of the Variation Coefficients

Assume that u is an *undetermined* function in the variables α_1, α_2, $\dots$ α_n. As before α, β, $\dots$ π is an arbitrary selection of these variables each with arbitrary many repetitions. We abbreviate

$$\frac{\partial^p u}{\partial\alpha\partial\beta\dots\partial\pi} = u' \ . \tag{21}$$

Moreover,

$$G = G(\alpha_1,\, \alpha_2,\, \ldots\, \alpha_n,\, u,\, u') \tag{22}$$

is a *given* expression from those variables as well as from u and u'. We have to examine the variation of the integral

$$\int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n \tag{23}$$

over an arbitrary given domain for a modification of u, under the simplifying assumption that the function u and all its derivatives are fixed at the boundary. In the future we refer to this as *internal variation* of this integral. For that we immediately obtain

$$\begin{aligned} \delta \int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n &= \int^{(n)} \delta G \cdot d\alpha_1 d\alpha_2 \ldots d\alpha_n \\ &= \int^{(n)} \left(\frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial u'}\delta u' \right) d\alpha_1 d\alpha_2 \ldots d\alpha_n . \end{aligned} \tag{24}$$

By (21) we have

$$\frac{\partial G}{\partial u'}\delta u' = \frac{\partial G}{\partial u'} \frac{\partial^p \delta u}{\partial\alpha\partial\beta\ldots\partial\pi} \;, \tag{25}$$

hence according to (20)

$$\begin{aligned} &\int^{(n)} \frac{\partial G}{\partial u'}\delta u' d\alpha_1 d\alpha_2 \ldots d\alpha_n \\ = \; &\Sigma + (-1)^p \int^{(n)} \frac{\partial^p}{\partial\alpha\partial\beta\ldots\partial\pi} \frac{\partial G}{\partial u'} \cdot \delta u d\alpha_1 d\alpha_2 \ldots d\alpha_n \;. \end{aligned} \tag{26}$$

The previously mentioned case where Σ vanishes takes place here. In fact the function δu vanishes with all its derivatives at the *boundary* of the integration domain, since the variation is an *inner* one. Consequently, by substituting (26) into (24) one has

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \cdots d\alpha_n = \int^{(n)} a \delta u d\alpha_1 d\alpha_2 \cdots d\alpha_n \;, \tag{27}$$

where a is given by

$$a = \frac{\partial G}{\partial u} + (-1)^p \frac{\partial^p}{\partial\alpha\partial\beta\ldots\partial\pi} \frac{\partial G}{\partial u'} \;, \tag{28}$$

i.e.,

$$a = \frac{\partial G}{\partial u} + (-1)^p \frac{\partial^p}{\partial\alpha\partial\beta\ldots\partial\pi} \frac{\partial G}{\partial \frac{\partial^p u}{\partial\alpha\partial\beta\ldots\partial\pi}} \ . \tag{29}$$

We abbreviate this quantity by

$$a = \frac{\partial G}{\partial u} + \varepsilon_{u'} D_{u'} \frac{\partial G}{\partial u'} \ , \tag{30}$$

where $D_{u'}$ indicates the differentiation with respect to all variables used to build the derivative u'. The symbol $\varepsilon_{u'}$ denotes a number, which is either $+1$ or -1 depending if u' is a derivative of even or odd order.

Analogously a more general task can be carried out. Assume that u is an *undetermined* function of the variables $\alpha_1,\ \alpha_2,\ \ldots\ \alpha_n$ and $u',\ u'',\ \ldots$ are arbitrarily many derivatives of this function of arbitrarily high degree. Moreover,

$$G = G(\alpha_1,\ \alpha_2,\ \ldots\ \alpha_n,\ u,\ u',\ u''\ \ldots) \tag{31}$$

is a *given* expression of those variables, functions and derivatives. Then for the *internal variation* of the integral

$$\int^{(n)} G d\alpha_1 d\alpha_2 \ldots \alpha_n \tag{32}$$

one obtains the following value:

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n = \int^{(n)} a \delta u d\alpha_1 d\alpha_2 \ldots d\alpha_n \ , \tag{33}$$

where using the notion introduced in (30) one can express a by

$$a = \frac{\partial G}{\partial u} + \varepsilon_{u'} D_{u'} \frac{\partial G}{\partial u'} + \varepsilon_{u''} D_{u''} \frac{\partial G}{\partial u''} + \cdots \ . \tag{34}$$

With the same ease an even more general task can be treated. Assume that $u,\ v,\ \ldots\ w$ are arbitrarily many *undetermined* functions of the variables $\alpha_1,\ \alpha_2,\ \ldots\ \alpha_n$. Moreover, let G be a *given* expression composed from the variables α, the functions $u,\ v,\ \ldots\ w$ and arbitrary many derivatives of these functions with respect to α. The task at hand is to determine the *internal variation* of the integral

$$\int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n \tag{35}$$

by simultaneous perturbation of u, v, $\ldots$ w. It is easy to see that the result in this case is

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n = \int^{(n)} \big(a\delta u + b\delta v + \ldots + c\delta w\big) d\alpha_1 d\alpha_2 \ldots d\alpha_n \ , \tag{36}$$

where a, b, $\ldots$ c are given by

$$\begin{aligned} a &= \frac{\partial G}{\partial u} + \varepsilon_{u'} D_{u'} \frac{\partial G}{\partial u'} + \varepsilon_{u''} D_{u''} \frac{\partial G}{\partial u''} + \cdots \ , \\ b &= \frac{\partial G}{\partial v} + \varepsilon_{v'} D_{v'} \frac{\partial G}{\partial v'} + \varepsilon_{v''} D_{v''} \frac{\partial G}{\partial v''} + \cdots \ , \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c &= \frac{\partial G}{\partial w} + \varepsilon_{w'} D_{w'} \frac{\partial G}{\partial w'} + \varepsilon_{w''} D_{w''} \frac{\partial G}{\partial w''} + \cdots \ . \end{aligned} \tag{37}$$

Here it is understood that

$$\begin{matrix} u',\, u'',\, \ldots \ , \\ v',\, v'',\, \ldots \ , \\ \ldots\ldots\ldots \\ w',\, w'',\, \ldots \end{matrix}$$

are the derivatives of u, v, $\ldots$ w, on which G depends.

It seems appropriate to call the quantities a, b, $\ldots$ c used to represent the variation of the integral of G the *variation coefficients of G with respect to u, v, $\ldots$ w* (cf. page 10). We denote them in an analogous way to the differential coefficients with the only difference that we use Δ instead of ∂.[19] With this convention we have

$$\begin{aligned} a &= \frac{\Delta G}{\Delta u} \ , \\ b &= \frac{\Delta G}{\Delta v} \ , \\ & \ldots\ldots \\ c &= \frac{\Delta G}{\Delta w} \ . \end{aligned} \tag{38}$$

[19] [Note by CN:] According to my knowledge the term *differential coefficient* is seldom used, always in the same meaning as *derivative* or *differential quotient*. Analogously like the term *differential coefficient*, we refer here to *variation coefficient*. (In the original article from 1868 instead of Δ a reversed ρ was used).

The lowercase letter δ is reserved to denote the variation itself.

As follows from (37) the variation coefficients of G with respect to u, v, $\dots$ w transform to the differential coefficients $\frac{\partial G}{\partial u}$, $\frac{\partial G}{\partial v}$, $\dots$ $\frac{\partial G}{\partial w}$ as soon as the expression G only contains the functions u, v, $\dots$ w themselves, but not their derivatives.

2.3 A Theorem on Variation Coefficients

For the following discussion we need to derive a theorem which in many cases simplifies computations involving variation coefficients. I noted this result before in "Untersuchungen über Elasticität" which appeared in Crelle's *Journal*, Vol. 57, p. 299.[20]

Apart from the variables α_1, α_2, $\dots$ α_n and the m undetermined functions u, v, $\dots$ w we might have an additional M new undetermined functions U, V, $\dots$ W, which also only depend on α_1, α_2, $\dots$ α_n, but are connected to the previous functions u, v, $\dots$ w by certain prescribed relations

$$\begin{aligned} U &= \varphi(\alpha_1, \alpha_2, \dots \alpha_n, u, v, \dots w) , \\ V &= \psi(\alpha_1, \alpha_2, \dots \alpha_n, u, v, \dots w) , \\ &\dots\dots\dots\dots\dots\dots\dots \\ W &= \chi(\alpha_1, \alpha_2, \dots \alpha_n, u, v, \dots w) . \end{aligned} \tag{39}$$

M might be bigger or smaller than m or the two integers might be equal.

We assume that G is a *given* expression of the variables α_1, α_2, $\dots$ α_n, of the functions U, V, $\dots$ W and of the derivatives of arbitrary high degree of these functions. We want to determine the *internal variation* of the integral

$$\int^{(n)} G d\alpha_1 d\alpha_2 \dots d\alpha_n \tag{40}$$

subject to a perturbation of u, v, $\dots$ w. This task can be solved in two ways.

First way. As soon as u, v, $\dots$ w are varied by arbitrary given quantities δu, δv, $\dots$ δw, the functions U, V, $\dots$ W contained in G are varied by quantities δU, δV, $\dots$ δW which in view of the relations (39) can be expressed as

[20] [Note by AKTA:] [Neu60, p. 299].

$$\begin{aligned}
\delta U &= \frac{\partial U}{\partial u}\delta u + \frac{\partial U}{\partial v}\delta v \ldots + \frac{\partial U}{\partial w}\delta w \ , \\
\delta V &= \frac{\partial V}{\partial u}\delta u + \frac{\partial V}{\partial v}\delta v \ldots + \frac{\partial V}{\partial w}\delta w \ , \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\delta W &= \frac{\partial W}{\partial u}\delta u + \frac{\partial W}{\partial v}\delta v \ldots + \frac{\partial W}{\partial w}\delta w \ .
\end{aligned} \tag{41}$$

As a consequence of these perturbations δU, δV, ... δW the integral (40) will be subject to a variation described by

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n = \int^{(n)} \big(A\delta U + B\delta V \ldots + C\delta W\big) d\alpha_1 d\alpha_2 \cdots d\alpha_n, \tag{42}$$

where A, B, ... C are the variation coefficients of G with respect to U, V, ... W.

Second way. One can *eliminate* the functions U, V, ... W contained in G and their derivatives by replacing them by the functions u, v, ... w and their derivatives using the relations (39). Doing this, the change in the integral (39) which arises on the basis of the given changes δu, δv ... δw, is represented by the formula

$$\delta \int^{(n)} G d\alpha_1 d\alpha_2 \ldots d\alpha_n = \int^{(n)} \big(a\delta u + b\delta v \ldots + c\delta w\big) d\alpha_1 d\alpha_2 \cdots d\alpha_n \ , \tag{43}$$

where a, b, ... c are the variation coefficients of G with respect to u, v, ... w.

Comparison of the results. The results obtained in (42) and (43) have to agree for arbitrary values of δu, δv, ... δw under the hypothesis that by δU, δV, ... δW one understands the expressions found in (41). For example the coefficient of δu in (43) has to be the same as in (42). Therefore

$$a = A\frac{\partial U}{\partial u} + B\frac{\partial V}{\partial u} \cdots + C\frac{\partial W}{\partial u}.$$

Analogous formulas one gets by equating the coefficients of δv, ... δw.

Using the notion just introduced for the variation coefficients, then these formulas become

$$
\begin{aligned}
\frac{\Delta G}{\Delta u} &= \frac{\Delta G}{\Delta U}\frac{\partial U}{\partial u} + \frac{\Delta G}{\Delta V}\frac{\partial V}{\partial u} \cdots + \frac{\Delta G}{\Delta W}\frac{\partial W}{\partial u}\ , \\
\frac{\Delta G}{\Delta v} &= \frac{\Delta G}{\Delta U}\frac{\partial U}{\partial v} + \frac{\Delta G}{\Delta V}\frac{\partial V}{\partial v} \cdots + \frac{\Delta G}{\Delta W}\frac{\partial W}{\partial v}\ , \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{\Delta G}{\Delta w} &= \frac{\Delta G}{\Delta U}\frac{\partial U}{\partial w} + \frac{\Delta G}{\Delta V}\frac{\partial V}{\partial w} \cdots + \frac{\Delta G}{\Delta W}\frac{\partial W}{\partial w}\ .
\end{aligned}
\tag{44}
$$

These formulas are the theorem we wanted to prove . It can be seen as a generalization of a known theorem in calculus. In case the expression G only depends on U, V, $\ldots W$, but not on their derivatives, then after the elimination by the relations (39) it depends as well only just on u, v, $\ldots w$, but not on the derivatives of these functions. In such a case the variation coefficients appearing in (44) become the corresponding differential coefficients and the formulas themselves turn into well-known formulas of calculus.

To make the general theorem contained in (44) clear, we remark that if there is just one function u, v, $\ldots w$ and just one function U, V, $\ldots W$ as well, then the assertion becomes the following.

If G depends on an undetermined function U and its derivative, and if the function U in turn depends on a different undetermined function u, then the variation coefficient of G with respect to u is obtained by building the variation coefficient of G with respect to U and multiplying it by the differential coefficient of U with respect to u. The formula

$$
\frac{\Delta G}{\Delta u} = \frac{\Delta G}{\Delta U}\frac{\partial U}{\partial u}
\tag{45}
$$

holds. Here u and U are functions in arbitrary many variables α_1, α_2, $\ldots\ \alpha_n$ and the derivatives of these functions are derivatives with respect to α_1, α_2 $\ldots\ \alpha_n$ where the differentiation with respect to each of these variables can be repeated as many times as one likes.

3 The Emissive and Receptive Potential

We[21,22] consider two points m and m_1 moving under their mutual interaction. We denote their distance for a *given* moment of time t by r, and for a *previous* moment of time $t - \Delta t$ by $r - \Delta r$. Putting

[21] [Note by CN:] *More details* about this (probably a bit too short) Section can be found in these *Annalen*, Vol. 1, pp 317–324.

[22] [Note by AKTA:] [Neu69] with English translation in [Neu20].

$$r = f(t) \ , \tag{46}$$

the function f is to be understood as *unknown*, like the dynamics of the points. Anyway one has to put as well

$$r - \Delta r = f(t - \Delta t) \ , \tag{47}$$

or equivalently

$$r - \Delta r = f(t) - \frac{\Delta t}{1} f'(t) + \frac{\Delta t^2}{1 \cdot 2} f''(t) - \ldots \ . \tag{48}$$

From (46) we have the equations

$$\frac{dr}{dt} = f'(t) \ , \quad \frac{d^2 r}{dt^2} = f''(t) \ , \quad \ldots$$

so that the formula above becomes

$$r - \Delta r = r - \frac{\Delta t}{1} \frac{dr}{dt} + \frac{\Delta t^2}{1 \cdot 2} \frac{d^2 r}{dt^2} - \ldots \ . \tag{49}$$

Using the notions introduced before (see page 7) we denote by $\tilde{\omega}$ the *emissive potential* of the two points at time t. We have

$$\tilde{\omega} = m m_1 \varphi(r) \ , \tag{50}$$

where $\varphi(r)$ is any *given* function which in the case of Newton's law would be $\frac{1}{r}$.

On the other hand we denote the *receptive potential* of the two points at time t by ω. To fix the ideas we think of m as the absorber and m_1 as the emitter. Then ω is the potential which m receives at time t and which therefore at a *previous time* $t - \Delta t$ was emitted by m_1. In this case ω coincides with the emissive potential at this previous time and has therefore the value

$$\omega = m m_1 \varphi(r - \Delta r) \ . \tag{51}$$

Using (49) this value becomes

$$\omega = m m_1 \varphi\left(r - \frac{\Delta t}{1} \frac{dr}{dt} + \frac{\Delta t^2}{1 \cdot 2} \frac{d^2 r}{dt^2} - \ldots \right) \ . \tag{52}$$

The expression Δt here represents that time, which the potential needs to pass through the path r. Since we denoted the propagation velocity of the potential by c (page 6), i.e., we understand by c the distance through which the potential propagates in time 1, we have $\Delta t : r = 1 : c$, implying

$$\Delta t = \frac{r}{c} \ . \tag{53}$$

In the following we assume that the velocity c is huge and therefore the fraction $\frac{r}{c}$ is tiny, so that we can ignore its *third* power. Substituting the value (53) into (52) we obtain

$$\omega = mm_1 \varphi\left(r - \frac{r}{c}\frac{dr}{dt} + \frac{r^2}{2c^2}\frac{d^2r}{dt^2} \right) , \tag{54}$$

which leads to the expansion

$$\omega = mm_1 \left[\varphi - \frac{r}{c}\frac{dr}{dt}\varphi' + \frac{r^2}{2c^2}\frac{d^2r}{dt^2}\varphi' + \frac{r^2}{2c^2}\left(\frac{dr}{dt}\right)^2 \varphi'' \right] , \tag{55}$$

or after rearrangement

$$\omega = mm_1 \left[\varphi + \frac{r^2\varphi''}{2c^2}\left(\frac{dr}{dt}\right)^2 + \frac{r^2\varphi'}{2c^2}\frac{d^2r}{dt^2} - \frac{r\varphi'}{c}\frac{dr}{dt} \right] . \tag{56}$$

Here we abbreviated $\varphi(r) = \varphi$, $\frac{d\varphi(r)}{dr} = \varphi'$, $\frac{d^2\varphi(r)}{dr^2} = \varphi''$. If Φ is an arbitrary function of r, we have the following general formulas

$$\begin{aligned} \Phi\frac{d^2r}{dt^2} &= \frac{d}{dt}\left(\Phi\frac{dr}{dt}\right) - \frac{d\Phi}{dr}\left(\frac{dr}{dt}\right)^2 , \\ \Phi\frac{dr}{dt} &= \frac{d}{dt}\left(\int \Phi dr\right) . \end{aligned}$$

Applying these formulas to the last two terms in the expression (56) for ω one gets

$$\begin{aligned} \omega = mm_1 &\left[\varphi + \frac{r^2\varphi''}{2c^2}\left(\frac{dr}{dt}\right)^2 - \frac{(r^2\varphi')'}{2c^2}\left(\frac{dr}{dt}\right)^2 \right] \\ &+ mm_1 \frac{d}{dt}\left[\frac{r^2\varphi'}{2c^2}\frac{dr}{dt} - \frac{\int r\varphi' dr}{c} \right] , \end{aligned} \tag{57}$$

where we put $(r^2\varphi')'$ for $\frac{d(r^2\varphi')}{dr}$ which equals $r^2\varphi'' + 2r\varphi'$. Substituting this value and noting further that $\int r\varphi' dr = r\varphi - \int \varphi dr$, then the expression for ω takes the following form

$$\omega = mm_1\left[\varphi - \frac{r\varphi'}{c^2}\left(\frac{dr}{dt}\right)^2\right] + mm_1\frac{d}{dt}\left[\frac{(\int \varphi dr) - r\varphi}{c} + \frac{r^2\varphi'}{2c^2}\frac{dr}{dt}\right] . \qquad (58)$$

We thought so far m_1 as emitter and m as absorber of the potential. As one easily sees, the *same* consideration involving the *same* formulas can be carried out in the opposite case where m is the emitter and m_1 the absorber of the potential.

It follows from this that the potential value ω found in (58) not only is the one which *reaches* m in the moment t *emitted from* m_1, but simultaneously the one which *reaches* in that instant m_1 *emitted from* m.

We obtained the following result:

If two points m and m_1 are moving under their common interaction, r denoting the distance at time t and moreover ω the receptive potential of the two points corresponding to the same time, then

$$\omega = w + \frac{d\mathfrak{w}}{dt} , \qquad (59)$$

where w and $\mathfrak{w}$ represent the following expressions:

$$\begin{aligned} w &= mm_1\left[\varphi - \frac{r}{c^2}\frac{d\varphi}{dr}\left(\frac{dr}{dt}\right)^2\right] , \\ \mathfrak{w} &= mm_1\left[\frac{(\int \varphi dr) - r\varphi}{c} + \frac{r^2}{2c^2}\frac{d\varphi}{dr}\frac{dr}{dt}\right] . \end{aligned} \qquad (60)$$

Here φ abbreviates $\varphi(r)$ and moreover c is a huge constant speed by which the potential propagates through space.

We further remark that the value of the expression w can be represented more easily by

$$w = mm_1\left[\varphi + \left(\frac{d\psi}{dt}\right)^2\right] , \qquad (61)$$

where ψ is the function

$$\psi = \int \sqrt{-r\frac{d\varphi}{dr}} \cdot \frac{dr}{c} . \qquad (62)$$

According to (59) the receptive potential consists of the two terms w and $\frac{d\mathfrak{w}}{dt}$. We refer to the first term, namely w, as the effective potential, and the other one, namely $\frac{d\mathfrak{w}}{dt}$, as the ineffective potential.

The notions introduced here seem quite necessary in order to avoid that the following discussion becomes cumbersome. *How* the notions are chosen should become clearer during the exposition.

For the case of Newton's law, namely $\varphi = \frac{1}{r}$, one obtains $\psi = \frac{2\sqrt{r}}{c}$. In this case the formulas (59), (60), and (61) become

$$\omega = w + \frac{d\mathfrak{w}}{dt} \ , \tag{63}$$

$$w = mm_1\left[\frac{1}{r} + \frac{4}{c^2}\left(\frac{d\sqrt{r}}{dt}\right)^2\right] = \frac{mm_1}{r}\left[1 + \frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right] \ , \tag{64}$$

$$\mathfrak{w} = mm_1\left[\frac{\log r}{c} - \frac{1}{2c^2}\frac{dr}{dt}\right] \ . \tag{65}$$

4 Weber's Law

4.1 Derivation of the Law

The task at hand is to determine the dynamics of two points m and m_1 under the hypothesis that the potential emitted by one point reaches the other point at a later time.

For a time t the coordinates of the points are denoted by x, y, z, x_1, y_1, z_1 and their distance to each other by r. Moreover, for that moment, ω is the receptive potential derived in (59) up to (61):

$$\omega = w + \frac{d\mathfrak{w}}{dt} \ , \tag{66}$$

and τ is their vis viva

$$\begin{aligned}\tau = \frac{m}{2}&\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] \\ + \frac{m_1}{2}&\left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dz_1}{dt}\right)^2\right] \ .\end{aligned} \tag{67}$$

As mentioned on page 7, we consider Hamilton's principle applicable without restriction. Therefore the dynamics of the points m and m_1 is characterized by the formula

$$\delta \int (\tau - \omega) dt = 0 \ . \tag{68}$$

According to page 6, the integration is carried out over an arbitrary interval of time. By δ we understand the *internal variation*, i.e., the variation which is only concerned with the interior of the time interval, but not its boundaries.

Through the substitution of (66) the formula (68) takes the form

$$\delta \int \tau dt = \delta \int \left(w + \frac{d\mathfrak{w}}{dt} \right) dt = \delta \left(\mathfrak{w}_{''} - \mathfrak{w}_{'} + \int w dt \right) , \tag{69}$$

or, since δ is an *internal* variation and therefore $\delta\mathfrak{w}_{''} = \delta\mathfrak{w}_{'} = 0$, one obtains

$$\delta \int \tau dt = \delta \int w dt \ . \tag{70}$$

Taking into account that the undetermined functions contained in τ and w are represented by x, y, z and x_1, y_1, z_1, we obtain the following six equations using our previous notation introduced on page 17:

$$\begin{aligned} \frac{\Delta\tau}{\Delta x} &= \frac{\Delta w}{\Delta x} , & \frac{\Delta\tau}{\Delta x_1} &= \frac{\Delta w}{\Delta x_1} , \\ \frac{\Delta\tau}{\Delta y} &= \frac{\Delta w}{\Delta y} , & \frac{\Delta\tau}{\Delta y_1} &= \frac{\Delta w}{\Delta y_1} , \\ \frac{\Delta\tau}{\Delta z} &= \frac{\Delta w}{\Delta z} , & \frac{\Delta\tau}{\Delta z_1} &= \frac{\Delta w}{\Delta z_1} . \end{aligned} \tag{71}$$

If one computes the variation coefficients on the left hand side using the value of τ given in (67), then the six equations become

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -\frac{\Delta w}{\Delta x} , & m_1\frac{d^2x_1}{dt^2} &= -\frac{\Delta w}{\Delta x_1} , \\ m\frac{d^2y}{dt^2} &= -\frac{\Delta w}{\Delta y} , & m_1\frac{d^2y_1}{dt^2} &= -\frac{\Delta w}{\Delta y_1} , \\ m\frac{d^2z}{dt^2} &= -\frac{\Delta w}{\Delta z} , & m_1\frac{d^2z_1}{dt^2} &= -\frac{\Delta w}{\Delta z_1} . \end{aligned} \tag{72}$$

These equations show, *that the negative variation coefficients of the right-hand side represent the components of that forces, which act on the points during their movement.* To explicitly determine these variation coefficients we observe that by (61) the effective potential w has the value

$$w = mm_1 \left[\varphi + \left(\frac{d\psi}{dt} \right)^2 \right] = mm_1 \left[\varphi + \left(\frac{d\psi}{dr}\frac{dr}{dt} \right)^2 \right] . \tag{73}$$

In particular, it depends on r and $\frac{dr}{dt}$, where r itself depends on the undetermined functions x, y, z, x_1, y_1, z_1 through the equation

$$r^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \ . \tag{74}$$

Therefore the variation coefficients can be computed by the theorem stated on page 20, namely using the formulas

$$\begin{aligned}
\frac{\Delta w}{\Delta x} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial x} \ , & \frac{\Delta w}{\Delta x_1} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial x_1} \ , \\
\frac{\Delta w}{\Delta y} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial y} \ , & \frac{\Delta w}{\Delta y_1} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial y_1} \ , \\
\frac{\Delta w}{\Delta z} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial z} \ , & \frac{\Delta w}{\Delta z_1} &= \frac{\Delta w}{\Delta r}\frac{\partial r}{\partial z_1} \ .
\end{aligned} \tag{75}$$

Substituting these expressions into (72) and using the values for $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$, ..., which follow from (74), one obtains the equations

$$\begin{aligned}
m\frac{d^2x}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{x - x_1}{r} \ , & m_1\frac{d^2x_1}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{x_1 - x}{r} \ , \\
m\frac{d^2y}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{y - y_1}{r} \ , & m_1\frac{d^2y_1}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{y_1 - y}{r} \ , \\
m\frac{d^2z}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{z - z_1}{r} \ , & m_1\frac{d^2z_1}{dt^2} &= -\frac{\Delta w}{\Delta r}\frac{z_1 - z}{r} \ .
\end{aligned} \tag{76}$$

It remains to compute the variation coefficient $\frac{\Delta w}{\Delta r}$. Abbreviating $\frac{dr}{dt}$ by r' and $\frac{d^2r}{dt^2}$ by r'', it follows from (73) that

$$w = mm_1\left[\varphi + \left(\frac{d\psi}{dr}r'\right)^2\right] \ . \tag{77}$$

Therefore

$$\begin{aligned}
\frac{\partial w}{\partial r} &= mm_1\left[\frac{d\varphi}{dr} + 2\frac{d\psi}{dr}\frac{d^2\psi}{dr^2}(r')^2\right] \ , \\
\frac{\partial w}{\partial r'} &= mm_1 \cdot 2\left(\frac{d\psi}{dr}\right)^2 r' \ ,
\end{aligned}$$

or equivalently

$$\frac{\partial w}{\partial r} = mm_1\left[\frac{d\varphi}{dr} + 2\frac{d\psi}{dt}\frac{d\frac{d\psi}{dr}}{dt}\right] , \tag{78}$$

$$\frac{\partial w}{\partial r'} = mm_1 \cdot 2\frac{d\psi}{dr}\frac{d\psi}{dt} . \tag{79}$$

Differentiating the last formula we obtain

$$\frac{d\frac{\partial w}{\partial r'}}{dt} = mm_1\left[2\frac{d\psi}{dr}\frac{d^2\psi}{dt^2} + 2\frac{d\psi}{dt}\frac{d\frac{d\psi}{dr}}{dt}\right] . \tag{80}$$

Since w only depends on r and r' by (77), one has

$$\frac{\Delta w}{\Delta r} = \frac{\partial w}{\partial r} - \frac{d\frac{\partial w}{\partial r'}}{dt} . \tag{81}$$

Therefore by (78) and (80)

$$\frac{\Delta w}{\Delta r} = mm_1\left[\frac{d\varphi}{dr} - 2\frac{d\psi}{dr}\frac{d^2\psi}{dt^2}\right] . \tag{82}$$

From (76) and (82) the following theorems follow:

Between two points m and m_1 a force R is acting during their movement, which at each moment coincides with the connecting line r.

If one considers this force R as a repulsive one and if w is the effective potential of the two points with respect to each other, then R equals at each moment the negative variation coefficient of w with respect to r, so that it has the value

$$R = -\frac{\Delta w}{\Delta r} . \tag{83}$$

In case the emission law of the potential is arbitrary, i.e., the emissive potential equals $mm_1\varphi(r)$, where φ is an arbitrary function, using the abbreviation

$$\begin{aligned} \varphi(r) &= \varphi , \\ \frac{1}{c}\int\sqrt{-r\frac{d\varphi}{dr}}dr &= \psi(r) = \psi , \end{aligned} \tag{84}$$

the values of the effective potential w and the force R become

$$w = mm_1 \left[\varphi + \left(\frac{d\psi}{dt} \right)^2 \right] ,$$

$$R = -\frac{\Delta w}{\Delta r} = mm_1 \left[-\frac{d\varphi}{dr} + 2\frac{d\psi}{dr}\frac{d^2\psi}{dt^2} \right] . \tag{85}$$

In the special case of Newton's emission law one has

$$\varphi = \frac{1}{r} ,$$

$$\psi = \frac{2\sqrt{r}}{c} , \tag{86}$$

and therefore

$$w = mm_1 \left[\frac{1}{r} + \frac{4}{c^2} \left(\frac{d\sqrt{r}}{dt} \right)^2 \right] ,$$

$$R = -\frac{\Delta w}{\Delta r} = mm_1 \left[\frac{1}{r^2} + \frac{4}{c^2\sqrt{r}}\frac{d^2\sqrt{r}}{dt^2} \right] , \tag{87}$$

i.e.,

$$R = \frac{mm_1}{r^2} \left[1 - \frac{1}{c^2} \left(\frac{dr}{dt} \right)^2 + \frac{2r}{c^2}\frac{d^2r}{dt^2} \right] .$$

Here c always represents the constant but huge speed through which the potential propagates in space.[23]

[23][Note by CN:] The value of R in (85) can also be deduced from the formula

$$w = mm_1 \left[\varphi + \left(\frac{d\psi}{dt} \right)^2 \right]$$

by the following reasoning. According to the theorem on variation coefficients on page 20 we have

$$\frac{\Delta w}{\Delta r} = \frac{\Delta w}{\Delta \varphi}\frac{\partial \varphi}{\partial r} + \frac{\Delta w}{\Delta \psi}\frac{\partial \psi}{\partial r} = mm_1\frac{\partial \varphi}{\partial r} - mm_1 \cdot 2\frac{d^2\psi}{dt^2}\frac{\partial \psi}{\partial r} ,$$

hence

$$R = mm_1 \left[-\frac{\partial \varphi}{\partial r} + 2\frac{\partial \psi}{\partial r}\frac{d^2\psi}{dt^2} \right] .$$

The general formula (85) coincides completely with the law I supposed in my PhD thesis "*Explicare tentatur quomodo fiat, ut lucis planum polarisationis per vires electricas vel magneticas declinetur. Halis Saxonum 1858*,"[24] which discussed the mutual interaction of an *electric* and an *aether particle*. In fact formula (85) can be written as follows

$$R = mm_1 \left[-\frac{d\varphi}{dr} + 2\frac{d\psi}{dr}\frac{d^2\psi}{dr^2}\left(\frac{dr}{dt}\right)^2 + 2\left(\frac{d\psi}{dr}\right)^2\frac{d^2r}{dt^2} \right] . \tag{88}$$

Putting

$$-\frac{d\varphi}{dr} = F , \qquad 2\left(\frac{d\psi}{dr}\right)^2 = \Phi , \tag{89}$$

it becomes

$$R = mm_1 \left[F + \frac{1}{2}\frac{d\Phi}{dr}\left(\frac{dr}{dt}\right)^2 + \Phi\frac{d^2r}{dt^2} \right] . \tag{90}$$

However, this is the law supposed in that thesis on page 3.[25,26]

[24] [Note by AKTA:] [Neu58].

[25] [Note by CN:] According to (84), the formulas (89) can as well be written as

$$-\frac{d\varphi}{dr} = F , \qquad -\frac{2r}{c^2}\frac{d\varphi}{dr} = \Phi . \tag{91}$$

Therefore one has between F and Φ the relation

$$\frac{2F}{c^2} = \frac{\Phi}{r} . \tag{92}$$

In the mentioned thesis I kept the relation between F and Φ undetermined, so that there is not the slightest contradiction between that thesis and the theory developed in this paper. The mentioned optical phenomenon I have treated later in more depth in my note "Ueber die Magnetische Drehung der Polarisationsebene des Lichtes. Halle. 1863." Unfortunately I assumed there in order to make the exposition simpler a certain relation between F and G, namely

$$\frac{2F}{c^2} = -\frac{d\Phi}{dr} . \tag{93}$$

For the *special case* $\varphi = \frac{1}{r}$, i.e., $F = \frac{1}{r^2}$, this is identical to the relation (92) and leads as well to the value $\Phi = \frac{2}{c^2 r}$. *But in general* it contradicts (92). I remark that the assumption of the relation (93) in the above-mentioned paper was not motivated by internal reasons, but just to give the exterior form more simplicity. In fact the function F does not play a role at all in my investigation of the rotation of the plane of polarization. It drops out of the computations quite at the beginning. Therefore the results in that investigation are *the same* whatever relation between F and Φ we assume.

[26] [Note by AKTA:] [Neu63].

The most important point however is the fact that (87) coincides literally with the well-known *law of Weber*.

4.2 Addenda

We denote by R the force which acts on m during the movement of the two points m and m_1. Its components we abbreviate by X, Y and Z. According to (72) we have the equations

$$\begin{aligned} X &= -\frac{\Delta w}{\Delta x} \;, \\ Y &= -\frac{\Delta w}{\Delta y} \;, \\ Z &= -\frac{\Delta w}{\Delta z} \;. \end{aligned} \tag{94}$$

We think that there is a line through m whose direction is determined by the direction cosine α, β, γ. We denote the component of the force R in this direction by P. Then

$$P = X\alpha + Y\beta + Z\gamma = -\left[\frac{\Delta w}{\Delta x}\alpha + \frac{\Delta w}{\Delta y}\beta + \frac{\Delta w}{\Delta z}\gamma\right] \;. \tag{95}$$

We think for a moment that the motion of the point m or x, y, z is constrained to that line. We thus put

$$x = a + p\alpha \;, \qquad y = b + p\beta \;, \qquad z = c + p\gamma \;,$$

where a, b, c is a fixed point of the line and p is the distance between this point and the point x, y, z. We then have

$$\alpha = \frac{\partial x}{\partial p} \;, \qquad \beta = \frac{\partial y}{\partial p} \;, \qquad \gamma = \frac{\partial z}{\partial p} \;.$$

Consequently the formula (95) becomes

$$P = -\left[\frac{\Delta w}{\Delta x}\frac{\partial x}{\partial p} + \frac{\Delta w}{\Delta y}\frac{\partial y}{\partial p} + \frac{\Delta w}{\Delta z}\frac{\partial z}{\partial p}\right] \;. \tag{96}$$

As regards the dependence between w and p, we first remark that w depends on x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, while on the other hand x, y, z depend on p. The expression in square brackets in (96) is then nothing else than the variation coefficient of w with respect to p as follows from the theorem on page 10. It follows that

$$P = -\frac{\Delta w}{\Delta p} , \tag{97}$$

which is analogous to the formulas (94) and contains these as special cases.

In case we have arbitrary many points m, m_1, m_2, m_3, $\cdots$, and denote by w_1, w_2, w_3, $\ldots$ the effective potentials for each pair of points (m, m_1), (m, m_2), (m, m_3), $\cdots$, then we obtain from (97) that the expression

$$-\left(\frac{\Delta w_1}{\Delta p} + \frac{\Delta w_2}{\Delta p} + \frac{\Delta w_3}{\Delta p} + \cdots\right) \tag{98}$$

represents that force through which the point m is driven along the direction p by all other points together. This expression can be written more compactly by using the effective potential of the whole system of points, W, in the form

$$-\frac{\Delta W}{\Delta p} . \tag{99}$$

Hence the theorem follows:

If W is the effective potential of an arbitrary system of points, the force by which any of these points is driven along a given direction, is always equal the negative variation coefficient of W in that direction.

5 The Principle of Vis Viva

5.1 Consideration of Two Points

We start with a rather easy case, namely the one where only *two* points m and m_1 exist. Moreover, we assume that m is *moveable*, while m_1 is *fixed*.

Let x, y, z and x_1, y_1, z_1 be the coordinates of the two points, r their distance and furthermore ω the *receptive potential* of the two points. Finally τ is their vis viva.

According to page 23 the receptive potential consists of two parts

$$\omega = w + \frac{d\mathfrak{w}}{dt} . \tag{100}$$

We refer to the first term as the *effective* and the last term as the *ineffective* potential. Moreover, according to page 23 the effective potential w has the value

$$w = mm_1\left[\varphi(r) + \left(\frac{d\psi(r)}{dt}\right)^2\right] , \tag{101}$$

where $\varphi(r)$ and $\psi(r)$ are given functions of r. In the case of Newton's emission law they are represented by $\frac{1}{r}$ and $\frac{2\sqrt{r}}{c}$, where c is the propagation velocity which was mentioned several times. We denote the two parts of w by u and v, namely

$$\begin{aligned} w &= u + v \ , \\ u &= mm_1\varphi(r) = mm_1\varphi \ , \\ v &= mm_1\left(\frac{d\psi(r)}{dt}\right)^2 = mm_1\left(\frac{d\psi}{dt}\right)^2 \ . \end{aligned} \tag{102}$$

In the state of rest, i.e., when r is constant, v vanishes and w becomes u. We refer to the first part u of the *effective potential* w as the *static potential* and to the second part v as the *motive potential*.

Since m_1 is *fixed*, the vis viva τ is given by

$$\tau = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] \ . \tag{103}$$

We denote the derivatives with respect to the time by primes. Since x_1, y_1, z_1 are *constant*, we can also write formulas (102) and (103) as

$$\begin{aligned} w &= u + v \ , \\ u &= mm_1\varphi \ , \\ v &= mm_1\left(\frac{\partial\psi}{\partial x}x' + \frac{\partial\psi}{\partial y}y' + \frac{\partial\psi}{\partial z}z'\right)^2 \ , \end{aligned} \tag{104}$$

$$\tau = \frac{m}{2}\left((x')^2 + (y')^2 + (z')^2\right) \ . \tag{105}$$

According to Hamilton's principle for the dynamics of the points the formula

$$\delta\int(\tau - \omega)dt = 0 \tag{106}$$

holds, i.e., according to (100):

$$\delta\int \tau dt \ = \ \delta\int\left(w + \frac{d\mathfrak{w}}{dt}\right)dt = \delta\mathfrak{w}_{''} - \delta\mathfrak{w}_{'} + \delta\int w dt \ , \tag{107}$$

or since the boundaries of integrals are considered as fixed with respect to position and velocity:

$$\delta \int \tau dt = \delta \int w dt \ . \tag{108}$$

Since x_1, y_1, z_1 are constant and only x, y, z variable, we obtain three equations after carrying out the variation δ. These are

$$\begin{aligned} -mx'' &= \frac{\partial w}{\partial x} - \frac{d}{dt}\frac{\partial w}{\partial x'} \ , \\ -my'' &= \frac{\partial w}{\partial y} - \frac{d}{dt}\frac{\partial w}{\partial y'} \ , \\ -mz'' &= \frac{\partial w}{\partial z} - \frac{d}{dt}\frac{\partial w}{\partial z'} \ , \end{aligned} \tag{109}$$

where the primes indicate differentiation with respect to time. Multiplying the equations of (109) by $-x'$, $-y'$, $-z'$ and adding them together, one obtains in view of (106):

$$\frac{d\tau}{dt} = -\left(x'\frac{\partial w}{\partial x} + y'\frac{\partial w}{\partial y} + z'\frac{\partial w}{\partial z}\right) + \left(x'\frac{d}{dt}\frac{\partial w}{\partial x'} + y'\frac{d}{dt}\frac{\partial w}{\partial y'} + z'\frac{d}{dt}\frac{\partial w}{\partial z'}\right) , \tag{110}$$

or in abbreviated form

$$\frac{d\tau}{dt} = -\left(x'\frac{\partial w}{\partial x} + \cdots\right) + \left(x'\frac{d}{dt}\frac{\partial w}{\partial x'} + \cdots\right) \ . \tag{111}$$

Differentiating the effective potential w (104) with respect to time and noting that this w not only depends on x, y, z, but as well on x', y', z', one gets the formula:

$$\frac{dw}{dt} = \left(x'\frac{\partial w}{\partial x} + \cdots\right) + \left(x''\frac{\partial w}{\partial x'} + \cdots\right) , \tag{112}$$

or equivalently

$$\frac{dw}{dt} = \left(x'\frac{\partial w}{\partial x} + \cdots\right) + \frac{d}{dt}\left(x'\frac{\partial w}{\partial x'} + \cdots\right) - \left(x'\frac{d}{dt}\frac{\partial w}{\partial x'} + \cdots\right) . \tag{113}$$

Adding (111) and (113) it follows that:

$$\frac{d(\tau + w)}{dt} = \frac{d}{dt}\left(x'\frac{\partial w}{\partial x'} + y'\frac{\partial w}{\partial y'} + z'\frac{\partial w}{\partial z'}\right) \ . \tag{114}$$

By (104) we have $w = u + v$, moreover v is independent of x', y', z' and on the other hand v is a homogeneous expression of degree two in x', y', z', so that:

$$x'\frac{\partial w}{\partial x'} + y'\frac{\partial w}{\partial y'} + z'\frac{\partial w}{\partial z'} = x'\frac{\partial v}{\partial x'} + y'\frac{\partial v}{\partial y'} + z'\frac{\partial v}{\partial z'} = 2v \ .$$

Therefore equation (114) becomes

$$\frac{d(\tau + w)}{dt} = \frac{d(2v)}{dt} \ . \tag{115}$$

This implies

$$\tau + w - 2v = \text{constant}, \tag{116}$$

or using $w = u + v$:

$$\tau + u - v = \text{constant}. \tag{117}$$

This means that if one adds the static and subtracts the motive potential from the vis viva one gets a constant of motion.

5.2 Examination of an Arbitrary System of Points

The same discussion can be applied to a system of arbitrary many points, say n, not only in the case where the system is freely movable, but also in the case where there are some constraints. However, if there are constraints, we assume that they can be expressed by equations involving *only the coordinates* of the points, but not their velocities. These equations we denote by

$$B_1 = 0 \ , \qquad B_2 = 0 \ , \qquad B_3 = 0 \ , \qquad \cdots \ . \tag{118}$$

We denote the vis viva of the system by T and the receptive potential by Ω. In this case T is a sum of n terms each having the form

$$\tau = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = \frac{m}{2}\big((x')^2 + (y')^2 + (z')^2\big) \ . \tag{119}$$

On the other hand Ω is a sum of $\frac{n(n-1)}{2}$ terms, each belonging to two points of the form

$$\omega = w + \frac{d\mathfrak{w}}{dt} = u + v + \frac{d\mathfrak{w}}{dt} \ . \tag{120}$$

In this case Ω itself has an analogous form, namely:

$$\Omega = W + \frac{d\mathfrak{W}}{dt} = U + V + \frac{d\mathfrak{W}}{dt} \ , \tag{121}$$

where W represents the effective and $\frac{d\mathfrak{W}}{dt}$ the ineffective potential of the system. As regards the two parts of W, we refer to U as the static and to V as the motive potential of the system.

The effective potential $W = U + V$ of the system consists of $\frac{n(n-1)}{2}$ terms of the form $w = u + v$. If m and m_1 are any two points of the system, r their distance and, moreover, x, y, z and x_1, y_1, z_1 their coordinates, the term $w = u+v$ belonging to these two points has the value [cf. formula (102)]

$$\begin{aligned} w &= u + v \ , \\ u &= mm_1\varphi(r) = mm_1\varphi \ , \\ v &= mm_1\left(\frac{d\psi(r)}{dt}\right)^2 = mm_1\left(\frac{d\psi}{dt}\right)^2 \ , \end{aligned} \tag{122}$$

or written in more detail

$$\begin{aligned} w &= u + v \ , \\ u &= mm_1\varphi \ , \\ v &= mm_1\left(\frac{\partial\psi(r)}{\partial x}(x' - x_1') + \frac{\partial\psi}{\partial y}(y' - y_1') + \frac{\partial\psi}{\partial z}(z' - z_1')\right)^2 \ . \end{aligned} \tag{123}$$

The following formula holds for an unconstrained dynamical system

$$\delta\int T dt = \delta\int \Omega dt \ .$$

However, if the system is constrained by the conditions (118), the above formula has to be replaced by

$$\delta\int T dt = \delta\int(\Omega + \lambda_1 B_1 + \lambda_2 B_2 + \cdots)dt \ , \tag{124}$$

where λ_1, λ_2, $\cdots$ are unknown functions of time. Putting $\Omega = W + \frac{d\mathfrak{W}}{dt}$ this formula becomes:

$$\delta\int T dt = \delta\int(W + \lambda_1 B_1 + \lambda_2 B_2 + \cdots)dt \ . \tag{125}$$

If one carries out the variation δ one obtains $3n$ differential equations, namely the same number of equations as one has of variables x, y, z. The equations which belong to the point m with coordinates x, y, z read:

$$\begin{aligned} -mx'' &= \frac{\partial W}{\partial x} - \frac{d}{dt}\frac{\partial W}{\partial x'} + \lambda_1 \frac{\partial B_1}{\partial x} + \lambda_2 \frac{\partial B_2}{\partial x} + \cdots \,, \\ -my'' &= \frac{\partial W}{\partial y} - \frac{d}{dt}\frac{\partial W}{\partial y'} + \lambda_1 \frac{\partial B_1}{\partial y} + \lambda_2 \frac{\partial B_2}{\partial y} + \cdots \,, \\ -mz'' &= \frac{\partial W}{\partial z} - \frac{d}{dt}\frac{\partial W}{\partial z'} + \lambda_1 \frac{\partial B_1}{\partial z} + \lambda_2 \frac{\partial B_2}{\partial z} + \cdots \,. \end{aligned} \tag{126}$$

After multiplication by $-x', -y', -z'$ and addition, one obtains in view of (119) the equation

$$\begin{aligned} \frac{d\tau}{dt} \;=\; & -\left(x'\frac{\partial W}{\partial x} + y'\frac{\partial W}{\partial y} + z'\frac{\partial W}{\partial z}\right) \\ & + \left(x'\frac{d}{dt}\frac{\partial W}{\partial x'} + y'\frac{d}{dt}\frac{\partial W}{\partial y'} + z'\frac{d}{dt}\frac{\partial W}{\partial z'}\right) \\ & - \lambda_1\left(x'\frac{\partial B_1}{\partial x} + \cdots\right) - \lambda_2\left(x'\frac{\partial B_2}{\partial x} + \cdots\right) - \cdots \,. \end{aligned} \tag{127}$$

There are as many of these equations as there are points. Adding all these equations together, one obtains in view of (118) the formula:

$$\frac{\partial T}{\partial t} = -\sum\left(x'\frac{\partial W}{\partial x} + \cdots\right) + \sum\left(x'\frac{d}{dt}\frac{\partial W}{\partial x'} + \cdots\right) . \tag{128}$$

The effective potential W depends on the coordinates and the velocities. If one differentiates it with respect to time one gets

$$\frac{dW}{dt} = \sum\left(x'\frac{\partial W}{\partial x} + \cdots\right) + \sum\left(x''\frac{\partial W}{\partial x'} + \cdots\right) ,$$

or equivalently

$$\begin{aligned} \frac{dW}{dt} = \sum\left(x'\frac{\partial W}{\partial x} + \cdots\right) + \frac{d}{dt}\sum\left(x'\frac{\partial W}{\partial x'} + \cdots\right) \\ - \sum\left(x'\frac{d}{dt}\frac{\partial W}{\partial x'} + \cdots\right) . \end{aligned} \tag{129}$$

Adding (128) and (129) one obtains

$$\frac{d(T+W)}{dt} = \frac{d}{dt}\sum\left(x'\frac{\partial W}{\partial x'} + y'\frac{\partial W}{\partial y'} + z'\frac{\partial W}{\partial z'}\right) . \tag{130}$$

With the help of (123) one sees that U is independent of the $3n$ magnitudes x', y', z' and, on the other hand, V is a homogeneous expression of degree two in these $3n$ magnitudes. With $W = U + V$ it follows that:

$$\sum\left(x'\frac{\partial W}{\partial x'} + y'\frac{\partial W}{\partial y'} + z'\frac{\partial W}{\partial z'}\right) = 2V .$$

Equation (130) therefore becomes

$$\frac{d(T+W)}{dt} = \frac{d(2V)}{dt} . \tag{131}$$

This implies

$$T + W - 2V = \text{constant}, \tag{132}$$

or, since $W = U + V$:

$$T + U - V = \text{constant}. \tag{133}$$

This formula reduces in the case of an *instantaneous* propagation of the potential, i.e., $c = \infty$, to the well-known formula $T + U =$ constant (cf. page 6). The general formula (133) contains the following theorem:

The vis viva increased by the static and decreased by the motive potential is a constant of motion for an arbitrary system of points. This holds not only in the unconstrained case, but also in the case where the coordinates of the points are constrained by some conditions.

This theorem was derived under the assumption that in the system of points only *internal* forces are acting. In the case where a system consisting of points m_1, m_2, $\cdots$ m_n is subject not only to its *internal* forces, but also to *external* forces, one can always find *fixed points* M_1, M_2, $\cdots$ M_p which are the centers of these latter forces. The system consisting of all these $n + p$ points is then subject only to *internal* forces and therefore the theorem above applies to it. The fact that among the $n + p$ points some are fixed is not a problem for the utilization of the theorem.

5.3 Afterword

If one assumes (as almost always happens since Newton) that *spatially* separate objects act directly on one another, it should be just as permissible to assume a direct mutual action between two objects which are *temporally* separated from one another; provided naturally that such an assumption leads to equally happy consequences as the first. Accordingly Professor Weber, to whom I am indebted for his gracious communication, remarks that the hypothesis put forth by me (for the case $\varphi = \frac{1}{r}$) can be formulated in this way:

> "*The potential values stemming from a particle of matter are inversely proportional to the distances, and are valid for later moments of time in proportion to the distance.* The reason why they are valid for later moments of time, *may* lie in a propagation, of which it is only possible to speak under the assumption of a *higher mechanics* (as for example, the propagation of waves in air can only be treated with knowledge of fluid mechanics), from which it would follow that the propagation can be disturbed and interrupted at every point of the medium."

If the question raised here, whether the presumed effect between *temporally* separated objects should be regarded as primary (not further explicable) or as something secondary (derivable from simpler processes), would have to be decided *right away*, I would safely give preference to the first conception. But even in this case, the mode of expression I have chosen should be legitimate at least as a *figurative* and not inappropriate one.

Tübingen, in May 1868.

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6 Supplementary Remarks of Carl Neumann in the Year 1880

The *last words* of this article *considered by itself* make it already quite clear that the criticism of *Clausius* in the year 1869 (in Poggendorff's *Annalen*, Vol. 135, page 606)[27] against it is not applicable. Concerning this point one should also compare it to my note in *Math. Ann.*, Vol. 1, page 317–324.[28]

A brief look at the *first few pages* of this article (pages 400-402)[29] show that in the year 1868, when I wrote it, I was not aware of two important considerations of *Weber* and *Riemann*.

[27] [Note by AKTA:] [Cla68] with English translation in [Cla69].
[28] [Note by AKTA:] [Neu69] with English translation in [Neu20].
[29] [Note by AKTA:] Pp. 400-402 of the 1880 reprint of Neumann's 1868 paper, [Neu68a].

An *argument by Weber* (which appeared as a short note in Poggendorff's *Annalen*, Vol. 73, page 229 in the year 1848)[30] *shows in an elementary way, that the principle of vis viva continues to be valid for Weber's fundamental law.* — I regret, that at the time of writing I did not know this note. In my later publications (like for example in the *Abhandlungen der Kgl. Sächs. Ges. d. Wiss.*, Vol. 11, 1874, page 115)[31] I made an effort to bring to light the argument by Weber.

On the other hand the considerations by Riemann (compare the work of Hattendorff about weight, electricity and magnetism, Hannover, Rümpler, 1876, pag. 316–336)[32] *already contain the idea to introduce an electrodynamic potential and to deduce from it the electric forces by variation.* This idea is crucial in this article and is developed in great detail. There is no need to apologize that *I did not know* the considerations by Riemann when I wrote the article in the year 1868. Although part of it were already contained in a lecture by Riemann in the year 1861 as Hattendorff mentions, they appeared in print only in 1876 (in the work by Hattendorff referred to above).

Other people might decide if under these circumstances one should just call *Riemann* the author of these ideas, or if it is not more appropriate to give credit as well to *the person* who, independently of Riemann, had this same idea and published it *first* (and in greater detail). On the other hand *Clausius* in a recent note used the idea of introducing an electrodynamic potential and deriving the forces from it by variation, *without mentioning my work.* I think that those who do not know the literature well could easily get quite a wrong impression from this.

Leipzig, in November 1880.

[30] [Note by AKTA:] [Web48] with English translation in [Web52], [Web66] and [Web19].
[31] [Note by AKTA:] [Neu74].
[32] [Note by AKTA:] [Rie76] with partial English translation in [Rie77b]. See also [Rie67b] with English translation in [Rie67a] and [Rie77a].

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