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I. *Electrodynamic Measurements. By* Professor WILHELM WEBER.—Sixth Memoir, *relating specially to the Principle of the Conservation of Energy**.

THE law of electrical action announced in the First Memoir on Electrodynamic Measurements (*Elektrodynamische Maassbestimmungen*, Leipzig, 1846) has been tested on various sides and been modified in many ways. It has also been made the subject of observations and speculations of a more general kind; these, however, cannot by any means be regarded as having as yet led to definite conclusions. The First Part of the following Memoir is limited to a discussion of the relation which this law bears to the *Principle of the Conservation of Energy*, the great importance and high significance of which have been brought specially into prominence in connexion with the Mechanical Theory of Heat. In consequence of its having been asserted that the law referred to is in contradiction with this principle, an endeavour is here made to show that no such contradiction exists. On the contrary, the law enables us to make an addition to the Principle of the Conservation of Energy, and to alter it so that its application to each pair of particles is no longer limited solely to the time during which the pair does not undergo either increase or diminution of *vis viva* through the action of other bodies, but always holds good independently of the manifold relations to other bodies into which the two particles can enter.

Besides this, in the Second Part the law is applied to the de-

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velopment of *the equations of motion of two electrical particles subjected only to their mutual action.* Albeit this development does not lead directly to any comparisons or exact control by reference to existing experience (on which account it has hitherto received little attention), it nevertheless leads to various results which appear to be of importance as furnishing clues for the investigation of the molecular conditions and motions of bodies which have acquired such special significance in relation to Chemistry and the theory of Heat, and to offer to further investigation interesting relations in these still obscure regions.

On the Relation between the Laws of Electricity and the Principle of the Conservation of Energy.

1. *Electrical Particles and Electrical Masses.*

Particles of positive and of negative electricity are denoted by the same letters, for instance by e or e' &c., but a positive or a negative value is assigned to e or e' ... according to whether it represents a particle of the positive or of the negative fluid.

If the measurable force of repulsion exerted by the first particle e upon another exactly equal particle e at the constant measurable distance r be denoted by f, and also the measurable force of repulsion exerted by the second particle e' upon another exactly equal particle e', at the same distance r, be denoted by f', then $\pm r\sqrt{f}$ is taken as the measure of e, and $\pm r\sqrt{f'}$ as the measure of e', where the upper or the lower sign is to be taken according to whether the particle is a particle of positive or of negative fluid. The unit of force which is here adopted for the measurement of f and f' is the unit recognized in Mechanics, namely the force which, when it acts upon the unit of mass recognized in Mechanics (1 milligramme), imparts to this mass unit of velocity in unit of time. The repulsive force of the two particles e, e', so long as their distance r remains unchanged, is, in acccordance with the electrostatical law,

$$=\frac{ee'}{rr}.$$

A negative value of this expression denotes attractive force.

In this mode of denoting particles of the electric fluids, however, e, e' have not the signification of *masses* in the mechanical sense, as appears from the simple consideration that e, e' may have at one time positive and at another time negative values; but nevertheless the values of e, e' are closely related to the masses of the particles. For if we denote the *masses* of the particles e, e' (in the mechanical sense, according to which the unit of mass [1 milligramme] is determined by the mass of *one* ponderable body, and different masses are compared with each other in proportion to the reciprocals of the accelerations produced in them by the same force) by ϵ, ϵ', of which the values are always positive, we get for *positive* values of e, e',

$$\frac{e}{\epsilon}=\frac{e'}{\epsilon'}=a;$$

and for negative values of e, e',

$$\frac{e}{\epsilon}=\frac{e'}{\epsilon'}=b,$$

where a has a definite *positive* and b a definite *negative value.* Whether or not we have here $aa=bb$, or what ratio aa bears to bb, has not as yet been made out, any more than the numerical value of a or b. In many cases the electrical mass ϵ is connected with a ponderable mass m, so that it is impossible for it to be moved independently of it; in such cases, only the combined mass $m+\epsilon$ comes into account, and in general ϵ may be regarded as vanishingly small in comparison with m. Consequently it is only seldom that the masses ϵ, ϵ' have to be considered.

The distinction here indicated between the particles e, e' and their masses ϵ, ϵ' is not always made; on the contrary, the symbols of the particles e, e' are also used to denote the corresponding masses. It is, however, to be observed that, when this is done, no regard can be had to the signs of e, e'. The omission of the unknown factors a and b is always allowable when we are dealing only with the *relative values* of masses of positive or of negative electricity.

2. *The Law of Electrical Force.*

The Law of Electrical Force is thus stated in 'Electrodynamic Measurements' (Leipzig, 1846, p. 119:—

> If e and e' denote two electrical particles, the repulsive force exerted by the two particles on each other at the distance r is represented by
>
> $$\frac{ee'}{rr}\left(1-\frac{1}{cc}\frac{dr^2}{dt^2}+\frac{2r}{cc}\frac{ddr}{dt^2}\right),$$

where c is the constant denoted at the place quoted by $\frac{4}{a}$.

But this expression for the force which the particles e and e' mutually exert upon each other, it is easy to see, is dependent on a magnitude which contains as a factor the very force that is to be determined. This is readily seen when the relative acceleration of the two particles, namely $\frac{ddr}{dt^2}$, is broken up into two parts, thus,

$$\frac{ddr}{dt^2}=\frac{ddr'}{dt^2}+\frac{ddr''}{dt^2};$$

where the first part, $\frac{ddr'}{dt^2}$, is the relative acceleration due to the mutual action of the two particles, and the second part, $\frac{ddr''}{dt^2}$, is the acceleration due to other causes (namely to the acquired velocity of the particles perpendicular to r, and to the mutual action between them and other bodies). The first part, however, or that due to the mutual action of the two particles, is proportional to the force arising from this mutual action, and is represented by the quotient of this force by the mass upon which it acts.

Hence there easily follows, as was shown in the memoir already quoted (page 168), another expression for the force which the particles e and e' mutually exert upon each other, containing only terms which are independent of the force to be determined, namely the expression

$$\frac{ee'}{rr-\frac{2r}{cc}(e+e')}\left(1-\frac{1}{cc}\frac{dr^2}{dt^2}+\frac{2rf}{cc}\right)$$

(in which f is put for $\frac{ddr''}{dt^2}$), or, if the electrical particles e and e' are distinguished from their masses ϵ and ϵ' in accordance with the previous section (a distinction which was not made in the memoir quoted above), the expression

$$\frac{ee'}{rr-\frac{2r}{cc}\cdot\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}ee'}\left(1-\frac{1}{cc}\frac{dr^2}{dt^2}+\frac{2rf}{cc}\right).$$

From this it results that the law of electrical force is by no means so simple as we expect a fundamental law to be; on the contrary, it appears in two respects to be particularly complex.

In the first place, it is a consequence of this expression for the force, that, as was pointed out in the memoir referred to, the force which two electrical particles exert upon each other does not depend exclusively upon these particles themselves, but also upon the portion of their relative acceleration denoted by f, which is in part due to the action of other bodies. It was also pointed out that, inasmuch as the forces exerted by two bodies upon each other have been called by Berzelius *catalytic forces* when they depend upon the presence of a third body, electrical forces considered generally are, in this sense, catalytic forces.

In the second place, another noteworthy result follows from this expression for the force—namely, that when the particles e and e' are of the same kind, *they do not by any means always* *repel each other*; thus when $\frac{dr^2}{dt^2} < cc + 2rf$, they repel only so long as $r > \frac{2}{cc}\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}ee'$, and, on the contrary, they attract when

$$r < \frac{2}{cc}\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}ee'.$$

An exception to this rule occurs only in the case in which $\left(r - 2\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}\frac{ee'}{cc}\right)$, which is always a factor of the denominator, becomes likewise a factor of the numerator. This case occurs when the two electrical particles are at *permanent relative rest*, so that $\frac{dr}{dt} = 0$ and $\frac{ddr}{dt^2} = 0$.

The general expression for the force given above becomes in fact

$$\frac{ee'}{r\left(r - 2\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}\frac{ee'}{cc}\right)} \cdot \left(1 + \frac{2r}{cc}f\right)$$

when $\frac{dr}{dt} = 0$; and by dividing this by the mass $\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}$, we find the part of the acceleration which is due to the forces exerted upon each other by the two electrical particles, namely

$$\frac{(\epsilon+\epsilon')ee'}{\epsilon\epsilon' r\left(r - 2\frac{\epsilon+\epsilon'}{\epsilon\epsilon'} \cdot \frac{ee'}{cc}\right)} \cdot \left(1 + \frac{2r}{cc}f\right).$$

By adding to this the other part of the acceleration, namely f, which is due to the acquired motion of the particles at right angles to r and to the action of other bodies, we obtain the *total* acceleration, namely

$$\frac{ddr}{dt^2} = f + \frac{(\epsilon+\epsilon')ee'}{\epsilon\epsilon' r\left(r - 2\frac{\epsilon+\epsilon'}{\epsilon\epsilon'} \cdot \frac{ee'}{cc}\right)} \cdot \left(1 + \frac{2r}{cc}f\right),$$

which, when the particles are at permanent relative rest, $=0$. Hence for permanent relative rest we have

$$f = -\frac{\epsilon+\epsilon'}{\epsilon\epsilon'} \cdot \frac{ee'}{rr}.$$

If this value of f be substituted in the expression for the force

$$\frac{ee'}{r\left(r - 2\frac{\epsilon+\epsilon'}{\epsilon\epsilon'} \cdot \frac{ee'}{cc}\right)} \cdot \left(1 + \frac{2r}{cc}f\right),$$

the latter becomes

$$\frac{ee'}{r\left(r-2\,\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}\cdot\frac{ee'}{cc}\right)}\cdot\frac{1}{r}\left(r-2\,\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}\cdot\frac{ee'}{cc}\right).$$

Hence it appears that, in the case of permanent relative rest, the factor $\left(r-2\,\frac{\epsilon+\epsilon'}{\epsilon\epsilon'}\cdot\frac{ee'}{cc}\right)$ is common to numerator and denominator. The value of the quotient, which is thus independent of this factor, namely $\frac{ee'}{rr}$, consequently gives the expression for the force, in the case of permanent relative rest, in complete agreement with the fundamental laws of electrostatics, according to which this force has a *positive* value for particles of the *same kind* at *all distances.*

3. *The Law of Electrical Potential.*

In the previous section the law of electrical force is shown to be, in two respects, of a very complicated character, namely :—in the first place, in that the repulsive force between two electrical particles is dependent on things that do not appertain either to the nature of the particles which exert the force upon each other, or to their relative positions in space, or their existing relative motion, but *depends upon other bodies;* and secondly, in that *repulsion* may be exerted upon each other at certain distances by the same particles, and *attraction* at other distances.

Compared with this complicated law of *electrical force,* the law of *electrical potential* is very simple.

The value of the potential V of two electrical particles e and e', in fact, as I pointed out as long ago as the year 1848 in Poggendorff's *Annalen* (vol. lxxiii. p. 229), is determined by the following law,

$$V=\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2}-1\right).$$

Observing that both r and $\frac{dr}{dt}$ have different values at different times for both the particles e and e', and that consequently both are functions of the time, it follows that $\frac{dr}{dt}$ may also be regarded as a function of r, which may be denoted by fr. We thus obtain

$$V=\frac{ee'}{r}\left(\frac{1}{cc}\cdot(fr)^2-1\right),$$

and from this, by differentiation, the expression for the force

$$\frac{dV}{dr} = -\frac{ee'}{rr}\left(\frac{1}{cc}\cdot(fr)^2 - 1\right) + 2\frac{ee'}{rcc}\cdot fr\cdot\frac{dfr}{dr},$$

or, if we again put $\frac{dr}{dt}$ for fr,

$$\frac{dV}{dr} = \frac{ee'}{rr}\left(1 - \frac{1}{cc}\cdot\frac{dr^2}{dt^2} + \frac{2r}{cc}\cdot\frac{dr}{dt}\cdot\frac{d\frac{dr}{dt}}{dr}\right),$$

for which we may write

$$\frac{dV}{dr} = \frac{ee'}{rr}\left(1 - \frac{1}{cc}\,\frac{dr^2}{dt^2} + \frac{2r}{cc}\cdot\frac{ddr}{dt^2}\right).$$

From this it appears that

$$\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2} - 1\right)$$

is a function whose differential coefficient with respect to r represents the repulsive force between the two particles e and e', where r and $\frac{dr}{dt}$ denote respectively their distance and relative velocity regarded as functions of the time. But since $\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2} - 1\right)$ becomes equal to nothing when e and e' are separated infinitely far from each other, $\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2} - 1\right)$ is the *potential* of the electrical particles e and e'—that is to say, the *work* which is expended in causing the particles to approach each other from an infinite distance while under the action of their mutual repulsion, and to arrive at the distance r with the relative velocity $\frac{dr}{dt}$ *.

It likewise results from the foregoing that the *work*, which is expended when a given relative arrangement and state of motion of a system of particles e, e' are changed to another arrangement and another state of motion, depends only on the initial and

* This law of electrical *potential* has also been taken as his starting-point by Beer in his 'Introduction to Electrodynamics' (see *Einleitung in die Elektrostatik, die Lehre vom Magnetismus und die Elektrodynamik*, von August Beer. Nach dem Tode des Verfassers herausgegeben von Julius Plücker: Braunschweig, 1865. S. 250). The placing of the law of *potential* in the foreground as the fundamental law, and deriving the law of force from it, ought not to give rise to any misgiving. We have in many respects a better justification for speaking of the *physical existence of the work expressed by the potential* than for speaking of the *physical existence of a force*, as to which all we can say is that it *tends to change the physical relations of bodies.*

final arrangements and movements of the particles, and is independent of the way by which the transition has been effected, and also independent of states of motion which may have existed during the transition.

4. *Fundamental Electrical Laws.*

The law of *electrical potential* certainly appears to stand, in view of its simplicity, in a much closer relation to the true fundamental laws of electricity than the far more complex law of *electrical force*; but the expression of the former law may still be resolved into two simpler laws, which may be stated in the following manner:—

First Law.—If two particles e and e' are at relative rest or possess the same relative motion at two different distances r and ρ, the quantities of work V and U which are expended in separating the particles, while mutually acting on each other, from these distances to an infinite distance, are to each other inversely as these two distances, that is,

$$V : U = \rho : r. \quad \quad \quad (1)$$

Second Law.—The work U, which is expended in separating the particles e and e' while subject to the force exerted by them on each other from *a given distance* ρ $\left(=\frac{ee'}{a}\right)$ *proportional to the quantity* ee' to an infinite distance, makes *together with the vis viva* x, which belonged to the particles in consequence of their relative motion at the distance ρ, *a constant sum,* namely a; that is,

$$U + x = a. \quad \quad \quad (2)$$

For from equation (1) it follows that

$$U = \frac{r}{\rho} V;$$

and hence, by equation (2),

$$\frac{r}{\rho} V + x = a,$$

or, since $\rho = \frac{ee'}{a}$,

$$V = \frac{ee'}{r}\left(1 - \frac{x}{a}\right).$$

But the relative *vis viva* x is proportional to the square of the relative velocity $\frac{dr}{dt}$, so that we may substitute for a a new constant cc, such that

$$\frac{x}{a}=\frac{1}{cc}\cdot\frac{dr^2}{dt^2}*.$$

* If ϵ and ϵ' denote the masses of the particles e and e', α and β the velocities of ϵ in the direction of r and at right angles thereto, and α' and β' the same velocities for ϵ', so that $\alpha-\alpha'=\frac{dr}{dt}$ is the relative velocity of the two particles, then

$$\tfrac{1}{2}\,\epsilon(\alpha\alpha+\beta\beta)+\tfrac{1}{2}\,\epsilon'(\alpha'\alpha'+\beta'\beta')$$

is the total *vis viva* of the two particles. If we now put for α

$$\frac{\epsilon\alpha+\epsilon'\alpha'}{\epsilon+\epsilon'}+\frac{\epsilon'(\alpha-\alpha')}{\epsilon+\epsilon'},$$

and for α'

$$\frac{\epsilon\alpha+\epsilon'\alpha'}{\epsilon+\epsilon'}-\frac{\epsilon'(\alpha-\alpha')}{\epsilon+\epsilon'},$$

we get the total *vis viva* of the two particles represented as the sum of two parts in the following manner—namely,

$$=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2}+\frac{1}{2}\left[\frac{(\epsilon\alpha+\epsilon'\alpha')^2}{\epsilon+\epsilon'}+\epsilon\beta\beta+\epsilon'\beta'\beta'\right],$$

the *first* part of which, or $\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2}$, is the *relative* vis viva of the particles which was denoted above by x. But a is also a relative *vis viva* of the same particles, namely that which corresponds to a definite relative velocity c, so that $a=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot cc$. Hence we get $\frac{x}{a}=\frac{1}{cc}\cdot\frac{dr^2}{dt^2}$, as was given above.

It may be further observed that the *second* part of the above sum, namely $\frac{1}{2}\left[\frac{(\epsilon\alpha+\epsilon'\alpha')^2}{\epsilon+\epsilon'}+\epsilon\beta\beta+\epsilon'\beta'\beta'\right]$, may be again represented, after another subdivision, as the sum of two parts, thus

$$=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{ds^2}{dt^2}+\frac{1}{2}\left[\frac{(\epsilon\alpha+\epsilon'\alpha')^2}{\epsilon+\epsilon'}+(\epsilon+\epsilon')\gamma\gamma\right],$$

where $\frac{ds}{dt}$ represents the velocity with which the two particles move relatively to each other in space perpendicularly to r, while γ represents the velocity, perpendicular to r, of the centre of gravity of the two particles. We thus get the total *vis viva* of the two particles divided into three parts—namely,

i. $\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2}$,

ii. $\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{ds^2}{dt^2}$,

iii. $\frac{1}{2}\left[\frac{(\epsilon\alpha+\epsilon'\alpha')^2}{\epsilon+\epsilon'}+(\epsilon+\epsilon')\gamma\gamma\right]$;

the *first* of which, namely $\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2}$, represents the *relative* vis viva of

We thus obtain

$$V=\frac{ee'}{r}\left(1-\frac{1}{cc}\cdot\frac{dr^2}{dt^2}\right).$$

Here V denotes the work expended in separating the two particles from the distance r to an infinite distance. If V is to denote the work done in bringing the particles from an infinite distance to the distance r, as it is usually understood to do, so that positive values of $\frac{dV}{dr}$ may indicate *repulsion*, we obtain

$$V=\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2}-1\right);$$

that is to say, the *law of electrical potential.*

5. *Principle of the Conservation of Energy for Two Particles which form a detached system.*

The two fundamental laws laid down in the foregoing section, which may be called

The Law of the dependence of the Potential on the distance for *a constant relative motion,* and

The Law of the dependence of the Potential on the relative motion for *a constant distance,*

require to be further discussed in relation to their bearing upon the principle of the Conservation of Energy.

In accordance with the principle of the conservation of energy, three forms of energy are to be distinguished from each other—namely, *energy of motion* (kinetic energy), *potential energy,* and *energy of heat* (thermal energy).

The *energy of motion* is that part of the energy which depends upon the existing movements; and a special determination is given of the way in which it depends upon movement—namely, partly upon the magnitude of the moving mass, and partly upon the velocity with which this mass moves.

The same determination also appies to *thermal energy,* if this is regarded, in accordance with the mechanical theory of heat,

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the two particles; while the *first* two parts taken together, namely

$$\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\left(\frac{dr^2}{dt^2}+\frac{ds^2}{dt^2}\right),$$

represent the total *internal* vis viva, or the total *internal kinetic energy of the system*; and the *third* part, namely $\frac{1}{2}\left[\frac{(\epsilon\alpha+\epsilon'\alpha')^2}{\epsilon+\epsilon'}+(\epsilon+\epsilon')\gamma\gamma\right]$, represents the *external* vis viva, or the *external kinetic energy of the system* (that is, the *vis viva* of the centre of gravity of the two particles).

as an *internal motion of the particles of bodies.* But if we are dealing with a system of two *elementary particles* (that is to say, particles such that there can be no motion *within* them), it is obvious that in the case of such a system thermal energy has no existence, and *energy of motion* and *potential energy* alone remain.

Lastly, the *potential energy* is that part of the energy which depends on the existing potential; and a special determination is needed of the way in which potential energy *depends upon the potential,* exactly as, in the case of the energy of motion, it is needful to determine the special way in which it depends on movement.

Now this special determination has been made by *equating potential energy* (without regard to the sign) *and potential*.*

The justification for this proceeding has been found in the fact that the potential is a magnitude which is homogeneous with kinetic energy, which, when taken with the negative sign and added to the kinetic energy, gives always the same sum, so long as the two particles constitute a detached system which does not undergo either gain or loss of energy from without.

For instance, if we have a system of two ponderable particles m, m', its *potential* is

$$V = \frac{mm'}{r};$$

and the internal *vis viva,* or the *internal kinetic energy of the system,* is

$$W = \frac{1}{2}\,\frac{mm'}{m+m'}\,(uu+\alpha\alpha),$$

where $u = \frac{dr}{dt}$ is the relative velocity of the two particles, and α the difference of the velocities in space perpendicularly to r. But, for such a *detached system,* if we put $r = r_0$ and $\alpha = \alpha_0$

* The sign of the *potential,* V, is so determined that positive values of $\frac{dV}{dr}$ indicate repelling forces; the sign of the *potential energy* is fixed by the sign of the work which is done, in consequence of the mutual action of the particles, when the two particles are separated from the distance r to an infinite distance. Consequently, for two ponderable particles m, m', the potential is $V = \frac{mm'}{r}$, and the potential energy $= -\frac{mm'}{r}$. For two electrical particles e, e' the potential is $= \frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2}-1\right)$, and the potential energy $= \frac{ee'}{r}\left(1-\frac{1}{cc}\cdot\frac{dr^2}{dt^2}\right)$.

when $u=0$, the following value is easily got, namely

$$uu=\frac{r_0-r}{r_0}\left[\frac{2(m+m')}{r}-\frac{r_0+r}{r_0}\alpha\alpha\right]^*,$$

and consequently the sum

$$\mathrm{W}-\mathrm{V}=-\frac{mm'}{r_0}+\frac{1}{2}\frac{mm'}{m+m'}\cdot\alpha_0\alpha_0.$$

This sum always retains the same value as long as the values of r_0 and α_0 remain unchanged—that is, so long as the system of the two particles undergoes neither loss nor gain of energy from without. The *external kinetic energy* of such a detached system amounts *separately to a constant sum.*

Now the same thing holds good also for two *electrical* particles e, e'; for their potential, taken with the negative sign and added to their kinetic energy, gives in like manner always the same sum so long as the particles constitute a *detached* system.

* The force with which the two particles mutually act on each other, namely $\frac{d\mathrm{V}}{dr}$, divided by m, gives the acceleration of the particle m—that is, $\frac{1}{m}\cdot\frac{d\mathrm{V}}{dr}$; divided by m' it gives the acceleration of the particle m', namely $\frac{1}{m'}\cdot\frac{d\mathrm{V}}{dr}$. Consequently that part of the relative acceleration of the two particles which arises from their mutual action is $\left(\frac{1}{m}+\frac{1}{m'}\right)\frac{d\mathrm{V}}{dr}$, while that part of the relative acceleration of the two particles which arises from their rotation about one another is represented by $\frac{\alpha\alpha}{r}$. If now this last portion be subtracted from the total acceleration $\frac{du}{dt}$, the following equation results:

$$\frac{du}{dt}-\frac{\alpha\alpha}{r}=\left(\frac{1}{m}+\frac{1}{m'}\right)\frac{d\mathrm{V}}{dr}.$$

Putting $r=r_0$ and $\alpha=\alpha_0$ for the instant at which $u=0$, we obtain the expression

$$\alpha r=\alpha_0 r_0$$

as applicable for the case in which the only forces acting on the two particles are those due to their mutual action. Accordingly we get, by integrating the above differential equation after it has been multiplied by $2dr=2udt$,

$$uu+\alpha_0\alpha_0 r_0 r_0\left(\frac{1}{rr}-\frac{1}{r_0 r_0}\right)=2\left(\frac{1}{m}+\frac{1}{m'}\right)\left(\frac{mm'}{r}-\frac{mm'}{r_0}\right),$$

and hence

$$uu=\frac{r_0-r}{r}\left(\frac{2(m+m')}{r_0}-\frac{r_0+r}{r}\alpha_0\alpha_0\right)=\frac{r_0-r}{r_0}\left(\frac{2(m+m')}{r}-\frac{r_0+r}{r_0}\alpha\alpha\right).$$

We have, for the *potential* of such a system of two electrical particles,

$$V=\frac{ee'}{r}\left(\frac{uu}{cc}-1\right),$$

and, for the *internal kinetic energy of the system,*

$$W=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}(uu+\alpha\alpha)=\frac{ee}{\rho cc}(uu+\alpha\alpha),$$

if $u=\frac{dr}{dt}$ denotes the relative velocity of the two particles, and α the difference of their velocities in space at right angles to r. But, for such a *detached* system, when we put $r=r_0$ and $\alpha=\alpha_0$ for $u=0$, it is easy to obtain

$$\alpha=\frac{r_0}{r}\alpha_0,$$

$$uu=\frac{r-r_0}{r-\rho}\left(\frac{\rho}{r_0}cc+\frac{r_0+r}{r}\alpha_0\alpha_0\right)^*,$$

and consequently the sum

$$W-V=\frac{ee'}{r_0}+\frac{ee'}{\rho}\cdot\frac{\alpha_0\alpha_0}{cc}=\frac{ee'}{r_0}+\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\alpha_0\alpha_0.$$

This sum likewise retains the same value so long as the values of r_0 and α_0 remain unchanged—that is, so long as the system of two particles undergoes neither loss nor gain of energy from without†. *The same principle holds good in relation to the external kinetic energy of a detached system of two electrical particles and to that of two ponderable particles.*

* See Section 11.

† In Professor Tait's very instructive work, 'A Sketch of Thermodynamics' (Edinburgh, 1868), the following passage occurs at page 76, in reference to the investigations of Riemann and Lorenz which appeared in Poggendorff's *Annalen* for 1867 [Phil. Mag. S. 4. vol. xxxiv. pp. 368 and 287]:—"But the investigations of these authors are entirely based on Weber's inadmissible theory of the forces exerted on each other by *moving electric particles,* for which the conservation of energy is not true, while Maxwell's result is in perfect consistence with that great principle." This assertion of Professor Tait's seems to be in contradiction with the above. At page 56 of the same work Mr. Tait mentions that Helmholtz has based the doctrine of energy on Newton's principle and on the following postulate:—"Matter consists of ultimate particles which exert upon each other forces whose directions are those of the lines joining each pair of particles, and whose magnitudes depend solely on the distances between the particles." The contradiction between the fundamental law of electricity and *this postulate* is evident; but the contradiction between it and the *principle of the conservation of energy* is by no means evident,—a distinction which Professor Tait seems to have overlooked.

6. *Extension of the Principle of the Conservation of Energy to two electrical particles which do not form a detached system.*

If potential energy is taken, as is done in the previous section, as equal and opposite to potential, the principle of the conservation of energy holds good for two particles only so long as these two particles constitute a *detached* system—that is, so long as the system formed of the two particles undergoes neither gain nor loss of energy from without.

If the *total* energy of such a detached system of two particles were at first $=A$, but, the system ceasing to be detached, it received from without a quantity of kinetic energy $=a$, it seems to follow that, if the system were now again to become detached, the *total* energy would again become and remain constant so long as it remained detached, but that the total energy of the system in its final detached state would have the value $A+a$ (that is, a value exceeding that corresponding to its previous detached state by a). This, however, does not by any means conclusively prove the impossibility of extending the principle of the conservation of energy to two electrical particles which do not constitute a detached system.

For, strictly speaking, this has only been proved on the assumption that the *potential energy* of the system depends solely on the *distance* between the two particles; while if, on the other hand, the potential energy does not depend simply on the distance of the two particles, but also on their relative *motion*, it is evident that while the system receives from without an amount of *kinetic energy* $=a$, a change in its *potential energy* may be indirectly produced thereby. It is thus possible that the change of *potential energy*, so caused indirectly from without, might be $=-a$, so that the *total* energy (kinetic energy and potential energy together) of the two particles, even if they did not constitute a detached system, would retain always the same value.

This, however, certainly does not occur in reality for a system of two electrical particles, if the *potential energy* is taken as *equal and opposite to the potential*; but this assumption, which would thus make the extension of the principle impossible, has by no means been proved to be a necessary one. In general, all that is required is a *special determination of the way in which the potential energy depends upon the potential*; and here all that is self-evident is, that inasmuch as potential and potential energy are homogeneous magnitudes, a purely numerical relation must exist between them. But whether this numerical relation is always that of $+1$ to -1, or whether it is to be fixed otherwise, must still be regarded as in general doubtful; so that the possibility of the extension of the principle still remains.

We understand, in fact, by the *potential* of two particles, the amount of *work* which, in consequence of the mutual action of the two particles, is done when they are transferred in any way whatever from an infinite distance to the actually existing distance r with the actually existing relative velocity $\frac{dr}{dt}$.

It is, however, evident that *work* is done, in consequence of the mutual action of the two particles, not only during their transference from a *greater* distance to the distance r, but also during their transference from a *smaller* distance to the distance r. And there is no obvious reason why the *energy ascribed to the system* should be made to depend on the work done in the *former* case, and not on that done in the *latter* case also.

For example, if the *first* quantity of work were denoted, according to Section 4, by V, and the *second* by $\frac{\rho-r}{\rho}V$, the potential energy ascribed to the system might be taken as the *difference of these two amounts of work*, namely $=\frac{\rho-r}{\rho}V-V=-\frac{r}{\rho}V$. This difference of the two amounts of work is evidently the quantity of work which is done, in consequence of the mutual action of the two particles, during their transference from the limiting value of *small* distances to the limiting value of *great* distances—that is to say, the value which $-V=\frac{ee'}{r}\left(1-\frac{uu}{cc}\right)$ assumes when r is taken therein as equal to the limiting value of *small* distances, or when we put $r=\rho$, where ρ denotes the limiting value of small distances. According to this, therefore, this *difference of the two quantities of work* $=\frac{ee'}{\rho}\left(1-\frac{uu}{cc}\right)=-\frac{r}{\rho}V$.

In order to determine in this way the potential energy of a system of two electrical particles when the *first* quantity of work above referred to is

$$V=\frac{ee'}{r}\left(\frac{uu}{cc}-1\right),$$

it is only necessary further, for the determination of the *second* quantity of work, to determine the value of ρ—that is, of the *smaller distance* which is to be taken account of in that portion of the work.

Now this *smaller distance,* equally with the *greater distance,* must be determined *on its own account, independently of the actually existing conditions* of the two particles. This was done in the case of the *greater* distance by assigning to it an infinitely great value; in the case of the *smaller* distance the same thing is accomplished if we assign to it the value $2\frac{\epsilon+\epsilon'}{\epsilon\epsilon}\cdot\frac{ee'}{cc}$, a distance which is given by the particles e, e', by their masses ϵ, ϵ', and by the known electrical constant c.

If we now put the smaller distance equal to the value of ρ, we get, in virtue of the equations

$$V=\frac{ee'}{r}\left(\frac{uu}{cc}-1\right),$$

$$\frac{\rho-r}{\rho}V=\frac{\rho-r}{\rho}\cdot\frac{ee'}{r}\left(\frac{uu}{cc}-1\right),$$

the required value of the *potential energy*, namely

$$-\frac{r}{\rho}V=-\frac{ee'}{\rho}\left(\frac{uu}{cc}-1\right)=\frac{1}{2}\frac{\epsilon\epsilon}{\epsilon+\epsilon'}(cc-uu).$$

In accordance with the distinction which is here drawn between the *potential* and the *potential energy* of two electrical particles and with the corresponding determination of their relation to each other, an analogous distinction may also be made between the *vis viva* and the *kinetic energy* of two particles. For there is no necessity that the *kinetic energy* of two particles should be taken as being equal to *the total* vis viva *of the two particles*; all that is generally essential is a *definite determination of the relation subsisting between the kinetic energy of two particles and the total* vis viva *belonging to them both.*

Now the total *vis viva* possessed by the two particles was represented in the note to section 4 as the sum of two parts, of which the *first* part, namely $\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot\frac{dr^2}{dt^2}$, was called the *relative* vis viva. The *second* part was that which the two particles possessed in virtue of their revolution about each other in space, and in virtue of the motion of their centre of gravity in space.

If now, in order to establish the conception of the *energy* of two particles, we take it as our starting-point that the *principle of the conservation of energy* of two particles must be based upon the essential characters of the two particles, and in fact upon what is *essential to them when regarded as constituting a detached system,* it is obvious that for this purpose the conception of the *energy* of two particles must be made to depend only on the relations presented by the system of the two particles as such, quite irrespectively of the relations in which these particles may stand to all other bodies in space.

Applying this fundamental principle to the *kinetic energy* of two particles in the same way as it has just been done in respect of the *potential energy,* we see that the *kinetic energy* must be taken as dependent upon the *first* part of the total *vis viva* belonging to the two particles—that is to say, upon their *relative* vis viva—and not upon the *second* part of the total *vis viva*, or that which the two particles possess in virtue of their revolution about one another in space or of the motion of their centre of gravity in space; for this latter part depends upon relations which the two particles do not of themselves directly present. For the two particles taken by themselves do not directly present any relation to space except their distance apart, from which no knowledge can be had of their rotation or of the motion of their centre of gravity in space.

Consequently, in what follows, by the *kinetic energy* of two particles is to be understood, not the total *vis viva* possessed by the two particles, but only their *relative* vis viva.

But it is easy to see that, in accordance with this, while a system of two electrical particles e, e' receives from without an amount of kinetic energy $=a$, it really undergoes an alteration of its *potential energy* $=-a$; so that the *whole* energy of the system must always retain the same value not only when the two particles constitute a detached system, but also when they do not do so. For if we represent the *kinetic energy* communicated from without by

$$a=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}vv,$$

while the kinetic energy of the particles *before* the communication of this portion was

$$=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot u_0u_0,$$

the kinetic energy existing *after* the communication is

$$\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}\cdot uu=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}(u_0u_0+vv).$$

Consequently the *potential energy before the communication* is

$$-\frac{r}{\rho}\mathrm{V}=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}(cc-u_0u_0),$$

whereas the *potential energy after the communication* is

$$-\frac{r}{\rho}\mathrm{V}=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}(cc-uu)=\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}(cc-u_0u_0)-\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}vv;$$

so that, in consequence of the communication from without of *kinetic energy* equal to $+a$, a change of *potential energy* has occurred which is represented by

$$-\frac{1}{2}\frac{\epsilon\epsilon'}{\epsilon+\epsilon'}vv=-a.$$

7. *Application to other Bodies.*

If we distinguish, in accordance with the last section, between the potential and the potential energy of two particles—that is to say, if we define

Potential as the amount of *work* which, in consequence of the mutual action of the two particles, is done during the transference of the particles from an infinite distance to the actual distance r with the existing relative velocity $\frac{dr}{dt}$; and

Potential energy as that amount of *work*, taken *negatively*, which, in consequence of the mutual action of the two particles, is done during the transference of the particles *from the greater distance* $r=\infty$ *to the smaller distance* $r=\rho$ determined by the particles e, e', their masses ϵ, ϵ', and by the constant c, with the existing relative velocity $\frac{dr}{dt}$,—

the latter (that is to say, the *potential energy in the sense that has been indicated*) may be resolved into two parts, one of them equal and opposite to the *potential*, and therefore identical with the magnitude which has *hitherto* been *alone called potential energy*, but which, regarded henceforward as only a part of the potential energy, we may call the *free potential energy*; the remainder is the *second* part, which may be called the *latent potential energy*.

Hence the principle of the conservation of energy may be enunciated in the first place in the *earlier* wider sense as follows:—

For a *detached* system of two particles the sum of the *kinetic energy* and of the *free potential energy* is always the same.

For so long as no kinetic energy is either lost or communicated from without, every change in the free potential energy will be compensated by an equal and opposite change in the kinetic energy.

But the principle of the conservation of energy may also be enunciated, secondly, in the *narrower sense* as follows (potential energy and kinetic energy being understood in the sense that has just been defined):—

The *relative kinetic energy* of two particles, and the *total potential energy* which they possess along with this kinetic energy, together give always the same sum.

Upon this the following remarks may be made:—

(1) One particle regarded by itself can only possess *kinetic energy*.

(2) Two particles likewise possess in the first place kinetic energy, which is the sum of those which they possess when considered separately.

(3) This sum consists of a part A, which may be ascribed partly to the motion of their centre of gravity, and partly to their rotation about one another in space—and of another part B, which the particles possess relatively to each other when considered by themselves. This latter part, B, is called the *relative kinetic energy,* or *that belonging to the system formed by the two particles.*

(4) But in the *system of two particles* there is a something, in addition to its kinetic energy, which does not belong to the two particles taken separately, namely a greater or less *capacity for doing work* in virtue of the mutual action of the two particles upon each other. The *measure* of this capacity for doing work is termed the *potential energy of the system,* or the *relative potential energy of the two particles*; and that quantity of work serves as the *measure of this working-power* which is done in consequence of the mutual action of the two particles during their transference *from the smaller distance* $r=\rho$ *to the greater distance* $r=\infty$, where ρ is determined by the particles themselves e, e', by their masses ϵ, ϵ', and by the constant c.

(5) The principle of the conservation of energy, however, when specially defined as above, is only applicable to two particles when their *potential* is of the same form as that of two electrical particles, namely

$$V=\frac{ee'}{r}\left(\frac{1}{cc}\cdot\frac{dr^2}{dt^2}-1\right).$$

The potential of two ponderable masses m, m', on the contrary, is

$$V=\frac{mm'}{r},$$

which (neglecting the sign) can be included under the above general form only if the value of the constant c for ponderable masses is infinitely great. It is evident, however, that it would in reality suffice for the constant c to have only a very great value instead of an infinite value, in order that there might not be any thing perceptibly inconsistent with the results of experiment. And, considering the extraordinarily high value which must be ascribed to the constant c in the case of electrical particles, it does not seem at all necessary, for the avoidance of all sensible contradictions, to adopt any other value for ponderable bodies; consequently it must be permissible to represent the *potential* of two ponderable particles m, m' by

$$V=\frac{mm'}{r}\left(1-\frac{1}{cc}\cdot\frac{dr^2}{dt^2}\right),$$

where the constant c retains *the same value as in the potential of two electrical particles.*

But even if it should hereafter result from more accurate experimental results that it is not permissible thus to ascribe the same value to the constant c in the case of ponderable particles, the possibility would always remain of assigning to the constant c a still greater value for ponderable particles; and this could easily be taken so great that any sensible disagreement with experiment would completely vanish.

[To be continued.]

II. *Further Notes on the Theory of the Tides.* *By the* Rev. T. K. Abbott, *Trinity College, Dublin**.

IN the demonstrations given in two previous papers in this Magazine (January 1870 and February 1871), we have supposed the water to be limited to an equatorial canal, the moon also being in the equator. It is desirable to consider what modifications will be introduced, first, by supposing the earth to be uniformly covered with water, and, secondly, by taking into account the moon's declination.

It will save repetition if we state once for all certain general principles which we shall have to employ. First, suppose an accelerating force acts alternately in opposite directions, the effect (measured by velocity) increases as long as the force acts in either direction, and therefore the velocity in that direction is greatest at the moment that the force changes its direction. Secondly, the velocity (diminishing under the counteraction of the new force) continues to be in the same direction until this counter force has undone all the work done in that direction by the previous force. When the circumstances are alike in both directions, this will be when the force has done half its work. This is precisely the case of the common pendulum. Thirdly, in the case before us, the water rises when the particles behind are moving faster than those before. The rate of rise is greatest when this difference is greatest; but as the effect is cumulative, the whole amount of the rise is greatest at the moment when the difference = 0, and is about to change to the opposite. Fourthly, as in 2, this difference ceases to increase (*i. e.* is greatest) when the force (or difference of forces) producing it ceases to act; but it is not reduced to 0 until the opposite force has done half its work. At this moment the accumulation is greatest. Fifthly, in the case which we are now considering, the effective force depends on the form of the surface, and *vice*

* Communicated by the Author.