Sophus Lie and Harmony in Mathematical Physics, on the 150th Anniversary of His Birth

Nail H. Ibragimov

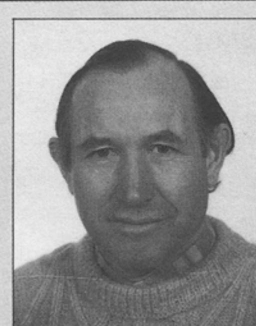
"The extraordinary significance of Lie's work for the general development of geometry can not be overstated; I am convinced that in years to come it will grow still greater" — so wrote Felix Klein [13] in his nomination of the results of Sophus Lie on the group-theoretic foundations of geometry to receive the N. I. Lobachevskii prize. This prize was established by the Physical-Mathematical Society of the Imperial University of Kazan in 1895 and was to recognize works on geometry, especially non-Euclidean geometry, chosen by leading specialists. The first three prizes awarded were to the following:

1897: S. Lie	(Nominator: F. Klein)
1900: W. Killing	(Nominator: F. Engel)
1904: D. Hilbert	(Nominator: H. Poincaré).

There can be no doubt that the work of Lie in differential equations merits equally high evaluation. One of Lie's striking achievements in this domain was the discovery that the majority of the known methods of integration, which until then had seemed artificial and not intrinsically related to one another, could be introduced all together by means of group theory. Further, Lie gave a classification of ordinary differential equations of arbitrary order in terms of the admitted group, thereby identifying the full set of equations which could be integrated or reduced to lower-order equations by group-theoretic considerations. But these and a rich store of other results of his did not lend themselves to popular expositions and remained for a long time the special preserve of a few. Today we find that this is the case with methods of solution of the problems of mathematical physics: Many of them have a group-theoretic nature yet are taught as though they were the result of a lucky guess.

It was my good fortune to get interested in application of groups to differential equations at the very beginning of my university work, and to write my first paper under the direction of Professor L. V. Ovsiannikov, who has done so much to awaken interest in this discipline and establish it as a contemporary scientific field. In my later

Friedrich Engel



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work I saw over and over how effective a tool Lie theory is for solving complicated problems. It significantly widens and sharpens the intuitive notion of symmetry, supplies concrete methods to apply it, guides one to the proper formulation of problems, and often discloses possible approaches to solving them.

This article presents my view of the role of Lie group theory in mathematical physics, drawing on parts of some of my lectures over the years at Moscow University and Moscow Institute of Physics and Technology.

His Life Story

Marius Sophus Lie was born 17 December 1842 in the town of Nordfjordeid, Norway, the sixth and youngest child of the Lutheran pastor Johann Herman Lie. He studied in Christiania (now Oslo) from 1857, first in gymnasium and then (1859–1865) at the University. Among the events of Lie's life which set his creative course, these stand out: his independent study in 1868 of the geometric works of Chasles, Poncelet, and Plücker; his travels in Germany and France in 1869–1870; his contacts there with Felix Klein, Chasles, Jordan, and Darboux; and his close friendship with Klein, leading to a long collaboration. Lie worked at the University of Christiania from 1872 to 1886, then from 1886 to 1898 at Leipzig. He died 18 February 1899 in Christiania.

The life and intellectual development and works of the greatest Norwegian mathematician are described in reminiscences of his colleagues and later biographies (see, for example, [7, 22, 27, 29], and references therein). I call special attention to the painstaking introduction of F. Engel to Lie's Collected Works [21]. These give detailed insight into the essence of Lie's ideas and a picture of him as a person.

Symmetry of Differential Equations

The notion of differential equations really has two components. For an ordinary first-order differential equation, for example, it is necessary

1. to specify a surface $F(x, y, y') = 0$ in the space of the three variables x, y, y'; we will call this surface the *skeleton* of the differential equation;
2. to define the class of solutions; for example, a smooth solution is a continuously differentiable function $\varphi(x)$ such that the curve

$$y = \varphi(x), \qquad y' = \frac{\partial\varphi(x)}{\partial x}$$

lies on the surface, i.e.,

$$F\left(x, \varphi(x), \frac{\partial\varphi(x)}{\partial x}\right) = 0$$

identically in x; going over to discontinuous or generalized solutions (keeping the same skeleton) changes the situation altogether.

A decisive move in integrating differential equations is simplifying the skeleton by means of a suitable change of variables. For this purpose, one uses the *symmetry group* of the differential equation (or its *admissible group*),

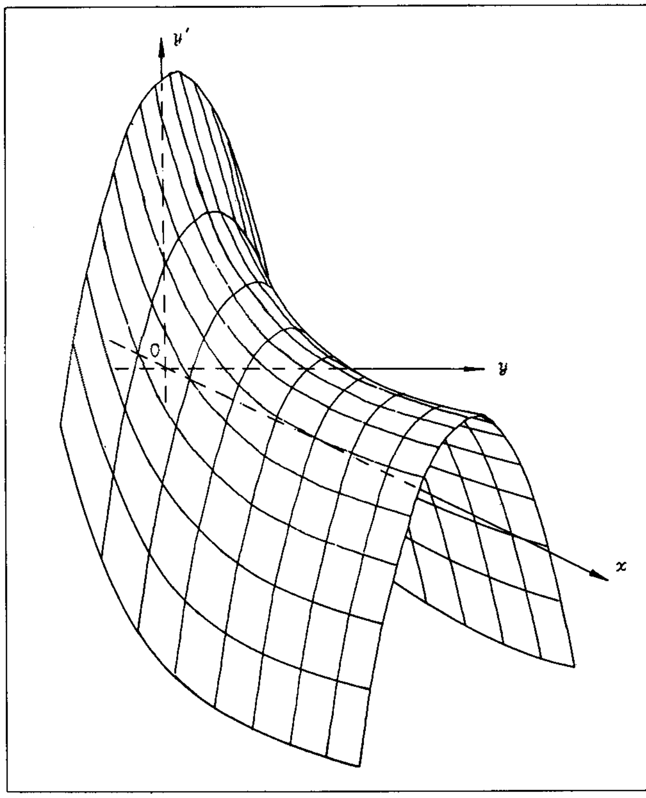

Figure 1. The skeleton of the Riccati equation $y' + y^2 - 2/x^2 = 0$ is a surface invariant under the group of inhomogeneous deformations $\bar{x} = xe^a, \bar{y} = ye^{-a}, \bar{y}' = y'e^{-2a}$.

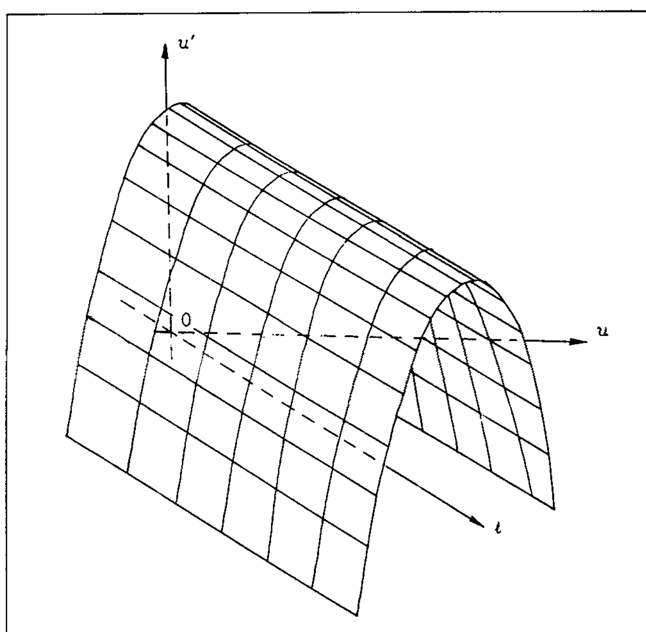

Figure 2. The skeleton of the equation $u' + u^2 - u - 2 = 0$, obtained from the Riccati equation $y' + y^2 - 2/x^2 = 0$ by the change of variables $t = \ln x, u = xy$.