

**ADDITIONAL CONSERVED QUANTITIES OF
WEBER'S FORCE LAW?**

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Abstract

Weber's force law, derived from Ampère's force law for electrical currents, gives the electrodynamic force between two moving charges. It reduces to Coulomb's law in the static case, but in general it is dependent on the relative velocities and accelerations of the interacting charges. Although it conserves linear and angular momentum and obeys Newton's 3rd Law, the Runge-Lenz vector is not conserved for it (thus it can explain Mercury's perihelion shift). We investigate whether there are additional symmetries and, by Noether-like theorems, additional conserved quantities of Weber's force law.

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1 Introduction

Ampère's force law [1, 15] expresses the force between electrical current elements $I_1 d\vec{\ell}_1$ and $I_2 d\vec{\ell}_2$ a distance r_{12} apart:

$$I_1 I_2 \frac{\hat{r}_{12}}{r_{12}^2} \left[2(d\vec{\ell}_1 \cdot d\vec{\ell}_2) - 3(\hat{r}_{12} \cdot d\vec{\ell}_1)(\hat{r}_{12} \cdot d\vec{\ell}_2) \right]. \quad (1)$$

From this law Weber derived his relative velocity- and acceleration-dependent electrodynamic force between two moving charges [1, 28, 29]; expressed as a system of nonlinear ordinary differential equations (ODEs), it is:

$$\ddot{x}_i = \frac{q_1 q_2 x_i}{r^3} \left[1 - \frac{\zeta}{2c^2} \dot{r}^2 + \frac{\zeta}{c^2} r \ddot{r} \right], \quad (2)$$

where q_1, q_2 are the interacting charges; $i = 1, 2, 3$; over-dots are time derivatives; x_i the spatial coordinates of the charge; $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$; c the speed of light; and ζ a dimensionless quantity that determines how "Weberian" the force is ($\zeta = 0$ is the Coulomb case); and c is the speed of light *in vacuo*.

Weber's force law is derivable from Neumann's velocity-dependent potential [2, 478]:

$$\frac{q_1 q_2}{r} \left[1 + \frac{\zeta}{2c^2} \dot{r}^2 \right]. \quad (3)$$

It is repulsive or attractive, depending on the magnitudes of the relative velocities or accelerations; Weber used this fact in his planetary model of the atom [4] to explain—a half-century before Bohr and Rutherford and without any need for a new, nuclear force—how positively charged nucleons can exist in close proximity without being repulsed.

More recently, Frauenfelder and Weber [16] use a Weber Hamiltonian to derive the fine-structure energy levels of a hydrogen atom.

They also show how, according to Carl Neumann, Weber's force can be derived from retarded potentials [14, 148-149; cf. 23, 24].

Weber's force law obeys Newton's 3rd Law.¹ It conserves linear and angular momentum and energy [3, 63-7]. It explains Mercury's perihelion shift [2, 480], so the Runge-Lenz vector is not conserved. But does it have other conservation laws?

2 Symmetry Finding

Sophus Lie originally intended his transformation groups to be a powerful tool for solving differential equations [21, 22], such as by reducing their order. Previously, Seegers [26] used elliptic functions to solve 2, but we investigate whether symmetry methods are more powerful.

A differential equation admits a group of point transformations if it remains invariant under the operation of the transformation given by the group generators [27, 93-94; 5, 46]:

$$X_n = \xi_n(t, q_i) \frac{\partial}{\partial t} + \eta_n^a(t, q_i) \frac{\partial}{\partial q^a}, \quad (4)$$

where $a = 1, 2, 3$ in the case of 2, corresponding to the x, y, z coordinates. Lie showed that $1 \leq n \leq 8$ [22].

The determining equations [20, 202] — partial differential equations (PDEs) that have to be solved to find the symmetry group corresponding to a system of, in this case, second-order differential equations

$$\ddot{q}^a = \omega^a(t, q^i, \dot{q}^i), \quad (5)$$

where $a, i = 1, \dots, N$, the number of equations; $,t = \frac{\partial}{\partial t}$; and $,c = \frac{\partial}{\partial q^c}$

¹cf. 8 for how Ampère's force law obeys Newton's 3rd Law

— are [27, 95]:

$$\begin{aligned} & \xi\omega_{,t}^a + \eta^b\omega_{,b}^a + (\eta_{,t}^b + \dot{q}^c\eta_{,c}^b - \dot{q}^b\xi_{,t} + \dot{q}^b\dot{q}^c\xi_{,c}) \frac{\partial\omega^a}{\partial\dot{q}^b} \\ & + 2\omega^a (\xi_{,t} + \dot{q}^b\xi_{,b}) + \omega^b (\dot{q}^a\xi_{,b} - \eta_{,b}^a) \\ & + \dot{q}^a\dot{q}^b\dot{q}^c\xi_{,bc} + 2\dot{q}^a\dot{q}^c\xi_{,tc} - \dot{q}^c\dot{q}^b\eta_{,bc}^a + \dot{q}^a\xi_{,tt} - 2\dot{q}^b\eta_{,tb}^a - \eta_{,tt}^a = 0. \end{aligned} \quad (6)$$

Recommended by Hereman [18, 19] as the best symmetry-finding software, we use the Maple package GeM² [9–13] to solve for the ξ_n and η_n^a of admitted generators 4 of 2:

```
restart;
read("gem32_12.mpl");

gem_decl_vars(indeps=[t], deps=[x(t),y(t),z(t)],
freeconst=[c,k,zeta]);

r := t -> sqrt(x(t)^2+y(t)^2+z(t)^2);
rp := t -> diff(r(t),t);
rpp := t -> diff(rp(t),t);
fac := t -> k/r(t)^3
(1 - zeta/(2c^2)r(t)^2 + zeta/c^2r(t)rpp(t));

#Coulomb problem:
#zeta := 0;

gem_decl_eqs([
diff(x(t),t$2) = x(t)fac(t),
diff(y(t),t$2) = y(t)fac(t),
diff(z(t),t$2) = z(t)fac(t)],
solve_for=[diff(x(t),t$2),diff(y(t),t$2),diff(z(t),t$2)]);

#This step uses over 6GB of memory:
det_eqs:=gem_symm_det_eqs([t,x(t),y(t),z(t)]);
```

²<https://math.usask.ca/~shevyakov/gem/>

GeM finds 60 determining equations. Solving them with Maple's built-in PDE solver,

```
sym_components:=gem_symm_components();
symm_sol:=pdsolve(det_eqs);
gem_output_symm(symm_sol);
```

GeM outputs these admitted generators:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t} \\ X_2 &= -x \frac{\partial}{\partial z} + z \frac{\partial}{\partial x} \\ X_3 &= -x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x} \\ X_4 &= -y \frac{\partial}{\partial z} + z \frac{\partial}{\partial y}, \end{aligned} \tag{7}$$

which are time and rotation symmetries [7, 476]; however, when $\zeta \neq 0$, pdsolve does not find the dilation symmetry

$$X_5 = \frac{2}{3} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) + t \frac{\partial}{\partial t}. \tag{8}$$

This is strange because the determining equations 6 depend on ζ , so it would seem that 8 would depend on ζ , too.

We also tried solving for higher-order (Lie-Bäcklund) symmetries by making the ξ_n and η_n^a depend not only on t, x, y, z but also on $\dot{x}, \dot{y}, \dot{z}$, by changing

```
det_eqs:=gem_symm_det_eqs([t,x(t),y(t),z(t)]);
```

to

```
det_eqs := gem_symm_det_eqs([t, x(t), y(t), z(t),
diff(x(t), t), diff(y(t), t), diff(z(t), t)])
```

After running for about a day, Maple did not find any solutions for additional symmetries.

2.1 Conservation laws

We also tried to find conservation laws [6, 38-120, 363-4] using the following GeM functions:

```
det_eqs:=gem_conslaw_det_eqs([t,x(t),y(t),z(t)]);
CL_multipliers:=gem_conslaw_multipliers();
multipliers_sol:=pdsolve(det_eqs);
gem_get_CL_fluxes(multipliers_sol);
```

But Maple was unable to find any conservation laws; unless we initially set $\zeta = 0$, the Coulomb case.

3 Conclusion

Weber's force law might have additional symmetries and conservation laws, but we were not able to find them. Perhaps a new mathematics, like Lie-Santilli theory [17], is necessary to make finding solutions to the nonlinear ODE 2 more tractable. Santilli [25, 9:18] seems to allude to the possibility that isomathematics can linearize [cf. 20, 214-8] all(?) nonlinear DEs.

Other future possibilities include applying Weber's force law to cold fusion; cf. the lectures by Indrani B. Das Sarma and Simone Beghella Bartoli at this conference.

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