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Motion of Poles and Zeros of Special Solutions of Nonlinear and Linear Partial Differential Equations and Related « Solvable » Many-Body Problems.

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Summary. — The motion of the poles and zeros of special solutions of certain nonlinear and linear partial differential equations is shown to be interpretable in terms of equivalent many-body problems. Several solvable many-body models are thus introduced and discussed. The treatment is limited to problems involving a finite number of particles moving in one space dimension.

1. – Introduction.

The first idea to investigate the time evolution of the positions of the poles of special solutions of the Korteweg-de Vries (KdV) equation is due to KRUSKAL (1). This investigation was pursued by THICKSTUN (2), and it was greatly advanced by AIRAULT, MCKEAN and MOSER (3), who uncovered and discussed a remarkable connection between the motion of the poles of rational and elliptic solutions of the KdV equation and the time evolution of certain one-dimensional many-body problems that had been introduced some years ago in the quantal context (4) and whose integrable character in the classical

(1) M. D. KRUSKAL: *Lectures in Appl. Math.*, **15**, *Amer. Math. Soc.* (1974), pp. 61-83.
(2) W. R. THICKSTUN: *Journ. Math. Anal. Appl.*, **55**, 335 (1976).
(3) H. AIRAULT, H. P. MCKEAN and J. MOSER: *Comm. Pure Appl. Math.*, **30**, 95 (1977), hereafter referred to as AMM.
(4) F. CALOGERO: *Journ. Math. Phys.*, **12**, 419 (1971).

case had been recently demonstrated [5], leading to the discovery of a number of remarkable properties and to the development of various generalizations [6]. AMM also investigated, in a similar manner, the Boussinesq equation. Results analogous to and, in certain respects, more advanced than those of AMM were obtained by the CHOODNOVSKY brothers, who moreover investigated the Burgers-Hopf (BH) equation and thereby considerably enlarged the scope of many-body problems whose time evolution can be shown to coincide with the time evolution of the poles of (solvable) nonlinear partial differential equations [7]. It should also be mentioned that certain equations, suggestive, at least in some special cases, of some of the developments studied in depths by AMM and CC, were previously given in a review paper by DUBROVIN, MATVEEV and NOVIKOV [8].

The display of these relationships between finite-dimensional integrable dynamical systems and solvable nonlinear evolution equations (that may themselves be considered infinitely dimensional instances of integrable dynamical systems, as first pointed out by FADDEEV and ZAKHAROV [9]) is of great interest, especially in the light of the beautiful findings of AMM and CC; and no doubt much remains to be uncovered, as emphasized by AMM. Moreover, these relationships may be used to evince information on one type of system from the known properties of the other. In particular, one may discover in this way many-body problems that are (in some sense) solvable, being related to partial differential equations whose time evolution is amenable to analysis. Let us recall in this connection that the number of exactly solvable many-body problems with pair interactions is, even in one-dimensional space, extremely scarce; while their interest is clearly considerable, both from a purely mathematical point of view and as a tool for the investigation of physical applications.

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(5) J. MOSER: *Adv. Math.*, **16**, 197 (1975).

(6) F. CALOGERO: *Lett. Nuovo Cimento*, **13**, 411, 507 (1975); **16**, 22, 35, 77 (1976); F. CALOGERO, C. MARCHIORO and O. RAGNISCO: *Lett. Nuovo Cimento*, **13**, 383 (1975); M. A. OLSHANETSKY and A. M. PERELOMOV: *Lett. Nuovo Cimento*, **16**, 333 (1976); **17**, 97 (1976); *Lett. Math. Phys.*, **1**, 187 (1976); *Invent. Math.*, **37**, 93 (1976); A. M. PERELOMOV: *Lett. Math. Phys.*, **1**, 531 (1977); M. ADLER: *Some finite-dimensional integrable systems*, in *Proceedings of the Conference on Solitons, Tucson, Ar., January 1976*; T. KOTERA and K. SAWADA: *Journ. Phys. Soc.* (*Japan*), **39**, 1614 (1975); S. M. WOJCIECHOWSKI: *Phys. Lett.*, **59** A, 84 (1976); *Lett. Nuovo Cimento*, **18**, 103 (1977); G. CASATI and J. FORD: *Journ. Math. Phys.*, **17**, 494 (1976); G. V. CHOODNOVSKY and D. V. CHOODNOVSKY: *Lett. Nuovo Cimento*, 291, 300 (1977).

(7) D. V. CHOODNOVSKY and G. V. CHOODNOVSKY: *Nuovo Cimento*, **40** B, 339 (1977), hereafter referred to as CC.

(8) B. A. DUBROVIN, V. B. MATVEEV and S. P. NOVIKOV: *Usp. Math. Nauk*, **31**, 55 (1976).

(9) L. D. FADDEEV and V. E. ZAKHAROV: *Funk. Anal. Priloz*, **5**, 18 (1971).

In this respect, the problems discussed by AMM and CC are, however, not very useful. The many-body problems discussed by them are generally problems whose solvability had been demonstrated previously; indeed, this is one highlight of their findings. The relationship between the previously known many-body problems and the motion of the poles of special solutions of the nonlinear partial differential equations they consider is, however, sometimes not quite direct, so that in some cases the motion of the poles does indeed provide novel examples of solvable many-body problems. For instance, CC have shown that the poles $x_j(t)$ of an appropriate rational solution of the KdV equation evolve in time according to the equations of motion [10]

$$\ddot{x}_j(t) = 2 \sum_{k=1}^{n}{}' [x_j(t) - x_k(t)]^{-5}, \qquad j = 1, 2, \ldots, n, \tag{1.1}$$

corresponding to n one-dimensional particles interacting via a pair potential inversely proportional to the fourth power of the interparticle distance. The solvability of this many-problem is a novel finding [11]. However, it holds only in a very restricted subset of phase space, characterized by the $2n$ conditions [12]

$$\sum_{k=1}^{n}{}' (x_j - x_k)^{-3} = 0, \qquad j = 1, 2, \ldots, n, \tag{1.2}$$

$$\dot{x}_j = \sum_{k=1}^{n}{}' (x_j - x_k)^{-2}, \qquad j = 1, 2, \ldots, n; \tag{1.3}$$

a subset that is nonvoid only for $n = \frac{1}{2}p(p+1)$, p being a positive integer, and that, moreover, requires the x_j's and $\dot{x}_j$'s not to be all real. Clearly this last condition greatly reduces the relevance of this result to physics [13].

The existence of constraints that limit the co-ordinates x_j (such as (1.2)) and/or the velocities $\dot{x}_j$ (such as (1.3)) is characteristic of the approaches of AMM and CC; indeed, as we discuss below, the presence of such constraints is an almost universal feature of the time evolution of the poles of nonlinear partial differential equations (the single exceptional case that violates this rule is discussed in detail below); in fact, one advantage of the BH equation considered by CC is that in that case only a limitation on the velocities occurs, but no limitation on the pole positions; and the many-body problem being

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([10]) A prime appended to the symbol of summation always indicates that the singular term must be omitted.

([11]) This finding had also been derived from the results of AMM by Calogero and Degasperis (unpublished).

([12]) Dots always indicate time differentiation.

([13]) One could, however, consider the time evolution of the real and imaginary parts of the pole positions. See, for instance, the third paper of ref. ([6]).

reproduced [7] is just that with a pair potential proportional to the inverse square of the interparticle distance, namely the one that has played a central role in these developments [3-6]. Also important in this connection, indeed most relevant to the developments discussed in sect. **3** below, is the fact that the BH equation can be linearized by a simple change of dependent variable; it is, therefore, substantially simpler than the KdV or Boussinesq equations.

The main focus of this paper is on the derivation and discussion of *many-body problems without constraints on the initial data*, whose time evolution can be shown to coincide with the motion of the poles or zeros of (special) solutions of partial differential equations. Since these differential equations can generally be solved in rather explicit form, it is thereby generally possible to analyse in rather explicit detail the corresponding many-body problems that we call, therefore, « solvable ».

In sect. **2** we discuss the motion of poles of nonlinear partial differential equations, beginning from a simple case (subsect. **2**˙1), extending it in various ways (subsect. **2**˙2, **2**˙3 and **2**˙4) and finally (subsect. **2**˙5) outlining a classification of nonlinear equations that implies that, in the framework of the approach of AMM and CC that we also follow, no other example besides our very simple (but rather rich) one yields equivalent many-body problems without constraints, and only the BH equation treated by CC, as well as the higher-order BH cases also considered by CC, yield equivalent many-body problems without any constraint on the positions of the poles.

In sect. **3** we discuss mainly the motion of the zeros of linear partial differential equations; this investigation, suggested by the results of CC for the BH equation and by the results discussed in sect. **2**, turns out to be very fruitful in the sense of generating several interesting examples of many-body problems, of which only a few are discussed in detail. The treatment starts with a presentation of the basic formulae and procedure (subsect. **3**˙1); equations of motion of first order (that also yield many-body models characterized by equations of motion of second order, but with constraints) are then considered (subsect. **3**˙2); finally, various models involving equations of motion of second order (without constraints) are treated, including cases with translation-invariant forces (subsect. **3**˙3 and **3**˙4), with nontranslation-invariant interactions (subsect. **3**˙5) and with forces involving circular or hyperbolic functions (subsect. **3**˙6). In the discussion of the many-body models of subsect. **3**˙5 an important role is played by the classical polynomials of Hermite, Laguerre and Jacobi; indeed certain properties, presumably new, of the zeros of these polynomials are also uncovered.

Section **4** summarizes tersely the main results and mentions the directions of research that are suggested by these findings. It may be a good idea to glance through this last section before delving into the body of the paper.

The notation is defined as the paper unfolds; suffice here to note that a variable (not an index!) appended as a subscript indicates partial differentiation, and to reiterate that the dots indicate differentiation with respect to the time variable t and that a prime appended to the symbol of summation indicates that the singular term in the sum must be omitted ([10,12]).

2. – Motion of poles of nonlinear evolution equations and related many-body problems.

In this section, that is divided in 5 subsections, we discuss the topic indicated in the title. Much of the discussion is based on a simple example, analysed in subsect. 2˙1, and extended in various directions in the subsequent 3 subsections. This discussion originates several interesting many-body models, whose solutions are analysed in some detail. The last subsection discusses more general nonlinear partial differential equations and explains why it should not be expected that solvable many-body problems without constraints be obtainable by these techniques used in connection with such equations. Thus for these equations the study of the motion of the poles (of special solutions of the kind considered here, and previously by AMM, CC and others ([1-3,7])) is not a convenient starting point to generate many-body problems that are both interesting and, in some sense, solvable.

2˙1. *A simple example.* – Consider the nonlinear partial differential equation ([14])

$$\varphi_t + \varphi_x + \alpha\varphi + \varphi^2 = 0\,, \qquad \varphi \equiv \varphi(x, t)\,. \tag{2.1.1}$$

It is immediately solvable by the substitution $\varphi = 1/\psi$, since the equation for ψ is the linear wave equation $\psi_t + \psi_x - \alpha\psi = 1$; thus the solution of the initial-value problem for the nonlinear equation (2.1.1) is given by the explicit formula

$$\varphi(x, t) = \exp[-\alpha t]\,\varphi_0(x-t)/\{1 + \varphi_0(x-t)[1-\exp[-\alpha t]]/\alpha\}\,, \tag{2.1.2}$$

where, of course,

$$\varphi_0(x) = \varphi(x, 0)\,. \tag{2.1.3}$$

([14]) Of course arbitrary constants can be inserted in front of each term of this equation; this corresponds to an appropriate rescaling of x, t and/or φ itself. The arbitrariness implied by the possibility to rescale variables is generally used in the following in order to write equations as simply as possible; it is obvious how it could be exploited in each case to get more general formulae. Here we have kept the constant α for convenience (see below).

Consider now a special solution of eq. (2.1.1) having the form

$$\varphi(x, t) = \sum_{j=1}^{n} [x - x_j(t)]^{-1} r_j(t) , \tag{2.1.4}$$

so that $\varphi(x, t)$ is a rational function of x, the quantities $x_j(t)$ and $r_j(t)$ being its poles and residues.

The following formulae are immediate consequences of (2.1.4):

$$\varphi_x(x, t) = -\sum_{j=1}^{n} [x - x_j(t)]^{-2} r_j(t) , \tag{2.1.5}$$

$$\varphi_t(x, t) = \sum_{j=1}^{n} \{[x - x_j(t)]^{-2} r_j(t) \dot{x}_j(t) + [x - x_j(t)]^{-1} \dot{r}_j(t)\} , \tag{2.1.6}$$

$$\varphi^2(x, t) = \sum_{j=1}^{n} \{[x - x_j(t)]^{-2} r_j^2(t) + [x - x_j(t)]^{-1} 2r_j(t) \sum_{k=1}^{n}{}' r_k(t)/[x_j(t) - x_k(t)]\} . \tag{2.1.7}$$

Thus (2.1.4) satisfies (2.1.1) if and only if the following equations, that obtain from the requirement that the coefficients of the poles of first and second order vanish, hold:

$$r_j(t) = 1 - \dot{x}_j(t) , \qquad j = 1, 2, \ldots, n, \tag{2.1.8}$$

$$\dot{r}_j(t) = -\alpha r_j(t) - 2r_j(t) \sum_{k=1}^{n}{}' r_k(t)/[x_j(t) - x_k(t)] , \qquad j = 1, 2, \ldots, n. \tag{2.1.9}$$

They imply

$$\ddot{x}_j(t) = \alpha[1 - \dot{x}_j(t)] + 2[1 - \dot{x}_j(t)] \sum_{k=1}^{n}{}' [1 - \dot{x}_k(t)]/[x_j(t) - x_k(t)] , \qquad j = 1, 2, \ldots, n. \tag{2.1.10}$$

Note that (2.1.10) in its turn implies, for the « centre-of-mass » co-ordinate,

$$X(t) = n^{-1} \sum_{j=1}^{n} x_j(t) , \tag{2.1.11}$$

the simple equation

$$\ddot{X}(t) = \alpha[1 - \dot{X}(t)] , \tag{2.1.12}$$

that can be immediately integrated to yield

$$X(t) = X(0) + t + [\dot{X}(0) - 1][1 - \exp[-\alpha t]]/\alpha . \tag{2.1.13}$$

From (2.1.2)-(2.1.4) we conclude that the co-ordinates $x_j(t)$ are the n

solutions of the algebraic equation in x

$$\sum_{j=1}^{n}[1-\dot{x}_j(0)]/[x-t-x_j(0)]=-\alpha/[1-\exp[-\alpha t]]\,. \tag{2.1.14}$$

Thus, given $x_j(0)$ and $\dot{x}_j(0)$, the solution of the « equations of motion » (2.1.10) is reduced to the determination of the zeros of an explicitly given polynomial of degree n (15).

Let us now interpret (2.1.10) as the equations of motion defining a many-body problem.

Consider first the 2-body case with $\alpha=0$. Then, setting $x=x_1-x_2$ and noting that (2.1.13) with $\alpha=0$ yields $X(t)=X(0)+\dot{X}(0)t$, we get

$$\ddot{x}(t)=\{4[1-\dot{X}(0)]^2-[\dot{x}(t)]^2\}/x(t)\,, \tag{2.1.15}$$

that can be immediately integrated to yield

$$\begin{cases}[\dot{x}(t)]^2+K[x(t)]^{-2}=4[1-\dot{X}(0)]^2\,,\\ K=[x(0)]^2\{4[1-\dot{X}(0)]^2-[\dot{x}(0)]^2\}\,.\end{cases} \tag{2.1.16}$$

Thus in this case the time evolution of the relative co-ordinate x is just the same as for the two-body problem with the inverse-square potential! Explicit integration yields

$$x(t)=\{4t^2[1-\dot{X}(0)]^2+2tx(0)\dot{x}(0)+x^2(0)\}^{\frac{1}{2}}\,. \tag{2.1.17}$$

A condition necessary and sufficient to exclude vanishing of $x(t)$ for any t is

$$2|1-\dot{X}(0)|>|\dot{x}(0)|\,, \tag{2.1.18a}$$

or equivalently

$$[1-\dot{x}_1(0)]\,[1-\dot{x}_2(0)]>0\,. \tag{2.1.18b}$$

Its significance is clear; see also below.

Let us now discuss the n-body problem (2.1.10) for arbitrary n. It is translation invariant, but neither Galilei invariant, nor (evidently) Hamiltonian. It becomes approximately Galilei invariant and (evidently) Hamiltonian to

(15) Let us recall in this connection that OLSHANETSKY and PERELOMOV have also reduced the solution of the one-dimensional n-body problem with pair potential proportional to the inverse square of the interparticle distance to the search of the zeros of a polynomial of degree n, given explicitly in terms of t and of the initial data; see their papers listed in ref. (6).

the extent that the conditions

$$|\dot{x}_j(t)| \ll 1 , \qquad j = 1, 2, \ldots, n, \tag{2.1.19}$$

hold, in which case the velocity-dependent terms on the r.h.s. of (2.1.10) can be dropped. But, as we now show, this condition cannot remain valid for all time.

To further discuss the many-body problem (2.1.10), it is convenient to go over to the variables

$$y_j(t) = x_j(t) - t , \qquad j = 1, 2, \ldots, n; \tag{2.1.20}$$

let us assume hereafter that these are the variables that represent the particle positions. This change of variables corresponds of course to a (Galilei) transformation to a frame moving with unit speed.

In the new variables the equations of motion read

$$\ddot{y}_j(t) = - \alpha\dot{y}_j(t) + 2\dot{y}_j(t) \sum_{k=1}^{n}{}' \dot{y}_k(t)/[y_j(t) - y_k(t)] , \qquad j = 1, 2, \ldots, n. \tag{2.1.21}$$

These equations of motion imply that the speeds $\dot{y}_j(t)$ cannot change sign throughout the motion. Moreover, the second term on the r.h.s. represents a two-body interaction that is singular at zero interparticle separation and that is attractive or repulsive depending on whether the two particles have speeds of opposite or equal signs. Thus, if initially the speeds do not all have the same sign, adjacent particles with speeds of different signs approach each other and may eventually collide, producing at a finite time a singularity (« collapse »); the natural continuation of the solutions beyond the time of encounter yields complex conjugate values for the corresponding particle co-ordinates. If instead the particles all have velocities of the same sign, no collapse occurs and of course the ordering of the particles on the line does not change throughout the motion. Since this latter case allows a more straightforward physical interpretation, we begin the following discussion from this case.

Assume, therefore, that, say [16],

$$\dot{y}_j > 0 , \qquad j = 1, 2, \ldots, n, \tag{2.1.22}$$

and, moreover, for definiteness, that

$$y_j < y_{j+1} , \qquad j = 1, 2, \ldots, n-1 \tag{2.1.23}$$

[16] A particle with vanishing velocity has no interaction; thus it remains seated at his place and it can simply be ignored.

(note that what was just written implies that both these conditions, if valid at $t = 0$, remain valid at all (finite) times).

The previous analysis implies that, for given initial conditions $y_j(0)$, $\dot{y}_j(0)$, the solutions $y_j(t)$ of the equations of motion (2.1.21) are the n solutions of the equation in y

$$\sum_{j=1}^{n} \dot{y}_j(0)/[y - y_j(0)] = \alpha/[1 - \exp[-\alpha t]]\,. \tag{2.1.24}$$

A convenient and illuminating point of view to discuss this equation is based on the graphical representation of the l.h.s. as a function of y (note that this function has poles, with positive residues, for $y = y_j(0)$, and that it decreases everywhere). It is then easy to read off graphically the time evolution of the co-ordinates $y_j(t)$; and in particular one gets the following asymptotic behaviours:

$$\lim_{t\to\infty} [y_j(t)] = b_j(\alpha)\,, \qquad j = 1, 2, \ldots, n, \tag{2.1.25}$$

$$\lim_{t\to-\infty} [y_j(t)] = a_{j-1}\,, \qquad j = 2, 3, \ldots, n, \tag{2.1.26a}$$

$$y_1(t) = -\alpha^{-1} \exp[-\alpha t] \left[\sum_{j=1}^{n} \dot{y}_j(0)\right] \{1 + O[\exp[\alpha t]]\}\,, \qquad t \to -\infty, \tag{2.1.26b}$$

where the quantities $b_j(\alpha)$ are the n solutions, ordered so that $b_{j+1}(\alpha) > b_j(\alpha)$, of the algebraic equation in b

$$\sum_{j=1}^{n} \dot{y}_j(0)/[b - y_j(0)] = \alpha\,, \tag{2.1.27}$$

while the quantities a_j are the $n - 1$ (finite) solutions, ordered so that $a_{j+1} > a_j$, of the same equation with $\alpha = 0$, namely of the algebraic equation in a

$$\sum_{j=1}^{n} \dot{y}_j(0)/[a - y_j(0)] = 0\,. \tag{2.1.28}$$

These results hold for $\alpha > 0$; if instead $\alpha < 0$, the same results obtain, except for the exchange of t with $-t$.

For $\alpha = 0$ one gets instead, for $t \to -\infty$,

$$y_1(t) = vt + a_0 + O(|t|^{-1})\,, \tag{2.1.29a}$$

$$y_j(t) = a_{j-1} + O(|t|^{-1})\,, \qquad j = 2, 3, \ldots, n, \tag{2.1.29b}$$

and, for $t \to +\infty$,

$$y_j(t) = a_j + O(t^{-1})\,, \qquad j = 1, 2, \ldots, n-1, \tag{2.1.30a}$$

$$y_n(t) = vt + a_0 + O(t^{-1})\,, \tag{2.1.30b}$$

where

$$v = \sum_{j=1}^{n} \dot{y}_j(0)\,, \tag{2.1.31}$$

$$a_0 = \sum_{j=1}^{n} \dot{y}_j(0)\, y_j(0)/v\,, \tag{2.1.32}$$

and the a_j, $j = 1, 2, \dots, n-1$, are defined as above, namely as the roots, ordered in increasing order, of eq. (2.1.28).

This latter example with $\alpha = 0$ is particularly amusing; it corresponds to a many-body problem with only interparticle forces (see (2.1.21)), whose « centre of mass »

$$Y(t) = n^{-1} \sum_{j=1}^{n} y_j(t) \tag{2.1.33}$$

moves freely with speed $\dot{Y} = v/n$:

$$Y(t) = (v/n)\, t + Y(0)\,; \tag{2.1.34}$$

in the remote past, it sees $n-1$ particles (almost) [16] at rest at the positions a_j and one particle coming in (say, from the far left) with velocity v; at any intermediate time, it sees all the particles moving towards the right; in the remote future, it sees again $n-1$ particles (almost) at rest exactly in the same positions as in the remote past except for the fact that each particle has moved one place to the right, the first particle settling down in the first location a_1, while the last is escaping to the right along the same trajectory that the particle coming initially from the left would have followed had it been free to move through the others (as it would have been the case if the other $n-1$ particles had been *exactly* at rest initially).

Let us now discuss tersely the case in which the (initial) condition (2.1.22) does not hold. The time evolution of the co-ordinates $y_j(t)$ is still determined by (2.1.24), and a graphical representation of the l.h.s. of this equation as a function of y is still the best approach for the analysis. Given the initial conditions $y_j(0)$, $\dot{y}_j(0)$, now there are two possibilities for the future evolution of the system: either no collapse occurs, and in such a case the asymptotic behaviour of the system is the same as that described above by eqs. (2.1.25), (2.1.27), (2.1.28), (2.1.30), (2.1.31) and (2.1.32); or, at a finite time, two particles collide, disappearing after that into the complex plane (whence they might even re-emerge, at a different location, at a later time!). Which one of these two possibilities prevails, and, in the second case, when and where the collapse occurs is immediately evident from the graph of the l.h.s. of (2.1.24) and from the (very simple) function of time that appears on the r.h.s. of this equation; thus we shall not elaborate this point any further. Clearly, for given initial conditions $y_j(0)$, $\dot{y}_j(0)$, it is always possible to chose a sufficiently large (positive) value of α to exclude the occurrence of collapse; since the presence of the α-term (with $\alpha > 0$) has the effect to slow down every particle, this fact has a very clear physical meaning. In a similar manner it is possible to analyse the past behaviour of the system, and/or the case with $\alpha \leqslant 0$.

Let us emphasize that the many-body problems we have discussed, peculiar as they are due to the presence of velocity-dependent forces and trivial as they are due to the simplicity of their solution, do not involve any constraint on the positions or velocities at the initial time $t = 0$ (other than some inequalities, such as (2.1.22), to exclude collapse).

Note finally that introduction of novel co-ordinates $z_j(t)$ through the position

$$y_j(t) = \exp[z_j(t)]\,, \qquad j = 1, 2, \ldots, n, \tag{2.1.35}$$

transforms (2.1.21) into

$$\ddot{z}_j(t) = -\dot{z}_j(t)[\dot{z}_j(t) + \alpha] + 2\dot{z}_j(t) \sum_{k=1}^{n}{}' \dot{z}_k(t)/\{\exp[z_j(t) - z_k(t)] - 1\}\,, \qquad j = 1, 2, \ldots, n. \tag{2.1.36}$$

Thus the many-body model characterized by these (translation invariant) equations of motion is also solvable (for a more general version of this model, and a more detailed discussion, see subsect. **3**·4).

2·2. *An extension: non-translation-invariant problems.* – More general solvable many-body models can be generated by noting that (2.1.4) implies, besides (2.1.5)-(2.1.7), the equations

$$x\varphi = \sum_{j=1}^{n} \{[x - x_j]^{-1} x_j r_j + r_j\}\,, \tag{2.2.1}$$

$$x\varphi_x = -\sum_{j=1}^{n} \{[x - x_j]^{-2} x_j r_j + [x - x_j]^{-1} r_j\}\,, \tag{2.2.2}$$

$$x^2\varphi_x = -\sum_{j=1}^{n} \{[x - x_j]^{-2} x_j^2 r_j + [x - x_j]^{-1} 2x_j r_j + r_j\}\,, \tag{2.2.3}$$

$$x\varphi_t = \sum_{j=1}^{n} \{[x - x_j]^{-2} \dot{x}_j x_j r_j + [x - x_j]^{-1}(\dot{x}_j r_j + x_j \dot{r}_j) + \dot{r}_j\}\,, \tag{2.2.4}$$

$$x\varphi^2 = \sum_{j=1}^{n} \Big\{[x - x_j]^{-2} x_j r_j^2 + [x - x_j]^{-1}\Big[r_j^2 + 2x_j r_j \sum_{k=1}^{n}{}' r_k/(x_j - x_k)\Big]\Big\}, \tag{2.2.5}$$

$$x^2\varphi^2 = \sum_{j=1}^{n} \Big\{[x - x_j]^{-2} x_j^2 r_j^2 + [x - x_j]^{-1}\Big[2x_j r_j^2 + 2x_j^2 r_j \sum_{k=1}^{n}{}' r_k/(x_j - x_k)\Big]\Big\} + \Big(\sum_{j=1}^{n} r_j\Big)^2. \tag{2.2.6}$$

Thus to the nonlinear partial differential equation

$$(2.2.7)\qquad (A_0 + A_1 x)\varphi_t + (B_0 + B_1 x)\varphi + \\ + (C_0 + C_1 x + C_2 x^2)\varphi_x + (D_0 + D_1 x + D_2 x^2)\varphi^2 = 0$$

there corresponds for the poles $x_j(t)$ and residues $r_j(t)$ the $2n+1$ equations

$$(2.2.8)\qquad r_j = [C_0 + C_1 x_j + C_2 x_j^2 - \dot{x}_j(A_0 + A_1 x_j)]/(D_0 + D_1 x_j + D_2 x_j^2)\,, \\ j = 1, 2, \dots, n\,,$$

$$(2.2.9)\qquad (A_0 + A_1 x_j)\dot{r}_j + A_1 \dot{x}_j r_j + [B_0 - C_1 + (B_1 - 2C_2)x_j] r_j + \\ + (D_1 - 2D_2 x_j) r_j^2 + 2(D_0 + D_1 x_j + D_2 x_j^2) r_j \sum_{k=1}^{n}{}' r_k/(x_j - x_k) = 0\,, \quad j = 1, 2, \dots, n\,,$$

$$(2.2.10)\qquad A_1 \sum_{j=1}^{n} \dot{r}_j + \Big[B_1 - C_2 + D_2 \sum_{k=1}^{n} r_k\Big] \sum_{j=1}^{n} r_j = 0\,.$$

We have omitted for notational simplicity to indicate explicitly the time dependence; note that there is no *a priori* need to exclude that also the coefficients A_s, B_s, C_s and D_s be time dependent.

Substituting (2.2.8) in (2.2.9), one gets n « equations of motion » for the n quantities x_j, that are, however, generally not translation invariant (and also not very appealing); while care can be taken of the constraint (2.2.10) by an appropriate (if need be, time dependent) choice of A_1, B_1, C_2 and D_2, the simpler possibility is of course the choice $A_1 = D_2 = 0$, $B_1 = C_2$.

On the other hand, the nonlinear partial differential equation (2.2.7) can be reduced, by the simple change of dependent variable

$$(2.2.11)\qquad \varphi(x, t) = 1/\psi(x, t)\,,$$

to the linear equation

$$(2.2.12)\qquad (A_0 + A_1 x)\psi_t - (B_0 + B_1 x)\psi + \\ + (C_0 + C_1 x + C_2 x^2)\psi_x = D_0 + D_1 x + D_2 x\,.$$

The corresponding initial condition is clearly

$$(2.2.13)\qquad \psi(x, 0) = 1\Big/\sum_{j=0}^{n} [x - x_j(0)]^{-1} r_j(0)\,.$$

Thus the function ψ is completely determined, at the initial time $t = 0$, by the initial positions $x_j(0)$ and velocities $\dot{x}_j(0)$ (through (2.2.13) and (2.2.8));

its subsequent time evolution is provided by the linear first-order partial differential equation (2.2.12); and the positions of its zeros $x_j(t)$ coincide, as implied by (2.2.11) and (2.1.4), with the solutions $x_j(t)$ of the « many-body problem » that obtains from (2.2.8) and (2.2.9).

Note finally that the same approach could also be extended to equations for φ analogous to (2.2.7), but involving higher powers of x; and transformations of the « particle variables » analogous to (2.1.20) and/or (2.1.35) could also be used to enlarge the scope of many-body problems solvable by this technique.

2'3. *Another extension: periodic and hyperbolic forces.* – In this section we apply first of all an extension technique that is often used in the context of this type of problems, and in this manner we obtain novel solvable many-body problems with periodic and hyperbolic potentials. But these results, obtained by a cavalier procedure involving infinite series, are in fact incorrect. They are however suggestive; and we then proceed to a more careful derivation, thereby obtaining the correct version of these results.

The starting point of the more cavalier analysis is the many-body problem (2.1.21), where we now assume that there exist an infinite number of particles, arranged according to the following configuration: to every one of the n co-ordinates y_j an infinity of other particles is associated, whose co-ordinates z_{js} are related to y_j by the formula

$$z_{js} = y_j + \pi s/\beta\,, \qquad s = \pm 1,\, \pm 2, \dots \tag{2.3.1}$$

It is reasonable to assume, for symmetry reasons, that such a configuration is maintained throughout the time evolution, provided the initial velocities of all these particles coincide, namely

$$\dot{z}_{js}(0) = \dot{y}_j(0)\,, \qquad j = 1, 2, \dots, n,\ \ s = \pm 1,\, \pm 2, \dots \tag{2.3.2}$$

We therefore focus in the following only on the time evolution of the y_j's, interpreting these quantities as the co-ordinates of n particles.

By using the well-known formula

$$\sum_{s=-\infty}^{+\infty} (\beta - s)^{-1} \equiv \beta^{-1} + 2\beta \sum_{s=1}^{\infty} (\beta^2 - s^2)^{-1} = \pi \operatorname{ctg} \pi\beta\,, \tag{2.3.3}$$

and by ignoring any convergence problem, it is then easy to derive from (2.1.21) the equations of motion

$$\ddot{y}_j(t) = -\alpha \dot{y}_j(t) + 2\beta \dot{y}_j(t) \sum_{k=1}^{n}{}' \dot{y}_k(t) \operatorname{ctg} \beta[y_j(t) - y_k(t)]\,, \qquad j = 1, 2, \dots, n, \tag{2.3.4}$$

that now characterize a new n-body model, similar to (2.2.1) except for the replacement of the two-body force $\dot{y}_j\dot{y}_k/(y_j - y_k)$ by the periodic interaction $\beta\dot{y}_j\dot{y}_k \operatorname{ctg}[\beta(y_j - y_k)]$.

The same approach applied to (2.1.24) would imply that the solutions $y_j(t)$ of the equations of motion (2.3.4) coincide with the solutions of the equation in y

$$\beta \sum_{j=1}^{n} \dot{y}_j(0) \operatorname{ctg} \beta[y - y_j(0)] = \alpha/[1 - \exp[-\alpha t]] \,. \tag{2.3.5}$$

But, as we have indicated at the beginning of this section, these results are in fact incorrect. Rather than pinpointing the source of the error, now we provide a derivation that by-passes any handling of infinite series. This we do, taking as starting point a new ansatz for $\varphi(x, t)$ to replace (2.1.4), namely

$$\varphi(x, t) = \beta\varrho(t) + \beta \sum_{j=1}^{n} r_j(t) \operatorname{ctg} \beta[x - x_j(t)] \,. \tag{2.3.6}$$

This is of course suggested by the previous considerations, see in particular (2.1.4) and (2.3.3); note, however, the additional presence of the term proportional to $\varrho(t)$.

Let us now parallel the treatment of subsect. **2·1**. In place of (2.1.5), (2.1.6) and (2.1.7) we now have

$$\varphi_x(x, t) = -\beta^2 \sum_{j=1}^{n} r_j(t)/\sin^2 \beta[x - x_j(t)] \,, \tag{2.3.7}$$

$$\varphi_t(x, t) = \beta\dot{\varrho}(t) + \beta \sum_{j=1}^{n} [\beta r_j(t)\dot{x}_j(t)/\sin^2 \beta[x - x_j(t)] + \dot{r}_j(t) \operatorname{ctg} \beta[x - x_j(t)]] \,, \tag{2.3.8}$$

$$\varphi^2(x, t) = \beta^2\Big[\varrho^2(t) - R^2(t) + \sum_{j=1}^{n} r_j^2(t)/\sin^2 \beta[x - x_j(t)] + 2 \sum_{j=1}^{n} r_j(t) \operatorname{ctg} \beta[x - x_j(t)]\Big(\varrho(t) + {\sum_{k=1}^{n}}' r_k(t) \operatorname{ctg} \beta[x_j(t) - x_k(t)]\Big)\Big] \,. \tag{2.3.9}$$

To write the last formula we have taken advantage of the trigonometric identity

$$\operatorname{ctg} A \operatorname{ctg} B = -1 - (\operatorname{ctg} A - \operatorname{ctg} B) \operatorname{ctg} (A - B) \,, \tag{2.3.10}$$

and we have introduced the quantity

$$R(t) = \sum_{j=1}^{n} r_j(t) \,. \tag{2.3.11}$$

It is thus seen that now the requirement that $\varphi(x, t)$ satisfies (2.1.1) yields the following $2n+1$ equations:

$$\dot{\varrho} + \alpha\varrho + \beta(\varrho^2 - R^2) = 0 , \tag{2.3.12}$$

$$r_j = 1 - \dot{x}_j , \qquad j = 1, 2, \ldots, n, \tag{2.3.13}$$

$$\dot{r}_j + \alpha r_j + 2\beta r_j\Big\{\varrho + \sum_{k=1}^{n}{}' r_k \operatorname{ctg}[\beta(x_j - x_k)]\Big\} = 0 , \qquad j = 1, 2, \ldots, n. \tag{2.3.14}$$

Here, and occasionally below, we omit to indicate explicitly the t-dependence.

Summing (2.3.14) over j, we also get

$$\dot{R} + \alpha R + 2\beta R\varrho = 0 . \tag{2.3.15}$$

We then note that (2.3.12) and (2.3.15) imply that

$$(\varrho \pm iR)_t + \alpha(\varrho \pm iR) + \beta(\varrho \pm iR)^2 = 0 , \tag{2.3.16}$$

which is easily solved, yielding

$$\varrho(t) = -\frac{1}{2}\alpha\{[\beta + \gamma_+ \exp[\alpha t]]^{-1} + [\beta + \gamma_- \exp[\alpha t]]^{-1}\} , \tag{2.3.17}$$

$$R(t) = \frac{i}{2}\alpha\{[\beta + \gamma_+ \exp[\alpha t]]^{-1} - [\beta + \gamma_- \exp[\alpha t]]^{-1}\} . \tag{2.3.18}$$

The constants $\gamma_\pm$ that appear in this formula could be easily determined in terms of $\varrho(0)$ and $R(0)$.

On the other hand, from (2.3.13) and (2.3.14) we get the « equations of motion »

$$\begin{aligned} \ddot{x}_j(t) &= \alpha[1 - \dot{x}_j(t)] + \\ &+ 2\beta[1 - \dot{x}_j(t)]\Big[\varrho(t) + \sum_{k=1}^{n}{}' [1 - \dot{x}_k(t)] \operatorname{ctg}\beta[x_j(t) - x_k(t)]\Big] , \qquad j = 1, 2, \ldots, n. \end{aligned} \tag{2.3.19}$$

Before discussing this « many-body problem » and its solutions, we go over, as in subsect. **2˙1**, to the variables

$$y_j(t) = x_j(t) - t , \qquad j = 1, 2, \ldots, n, \tag{2.3.20}$$

that are hereafter interpreted as particle co-ordinates. Thus now the many-body problem is characterized by the equations of motion

$$\ddot{y}_j(t) = -\alpha\dot{y}_j(t) + 2\beta\dot{y}_j(t)\Big[-\varrho(t) + \sum_{k=1}^{n}{}' \dot{y}_k(t) \operatorname{ctg}\beta[y_j(t) - y_k(t)]\Big] , \qquad j = 1, 2, \ldots, n, \tag{2.3.21}$$

with $\varrho(t)$ given by (2.3.17) and the two constants γ_+ and γ_- constrained, in terms of the initial conditions, by the requirement

$$\alpha[(\beta+\gamma_+)^{-1}-(\beta+\gamma_-)^{-1}]=2in\dot{Y}(0)\,, \tag{2.3.22}$$

where we have again introduced the centre-of-mass co-ordinate

$$Y(t)=n^{-1}\sum_{j=1}^{n}y_j(t)\,, \tag{2.3.23}$$

whose time evolution is given by the explicit formula

$$\begin{aligned} Y(t)=\;&Y(0)+\frac{i}{2n\beta}\cdot\\ &\cdot\Big[\ln\{[\gamma_++\beta\exp[-\alpha t]]/(\gamma_++\beta)\}-\ln\{[\gamma_-+\beta\exp[-\alpha t]]/(\gamma_-+\beta)\}\Big]\,. \end{aligned} \tag{2.3.24}$$

The solutions $y_j(t)$ of this many-body problem are the roots of the explicit equation in y

$$\beta\sum_{j=1}^{n}\dot{y}_j(0)\operatorname{ctg}\beta[y-y_j(0)]=\beta\varrho(0)+\alpha/[1-\exp[-\alpha t]]\,, \tag{2.3.25}$$

as implied by the ansatz (2.3.6), by the position (2.3.20) and by the explicit form (2.1.2) of the solution of (2.1.1).

Of course for $\beta=0$ one merely recovers the results of subsect. **2·1**; thus we hereafter assume $\beta\neq 0$, keeping open the option to choose β real or imaginary.

The many-body problem characterized by the equations of motion (2.3.21) is marred by the presence of the explicitly time-dependent term involving $\varrho(t)$, and by the constraint (2.3.22). We, therefore, focus attention on special choices of the constants γ_+ and γ_- that eliminate (totally or partially) these shortcomings.

The first choice we consider is $\gamma_+=\gamma_-=0$; note that (2.3.17) then implies $\varrho=-\alpha/\beta$. Thus the many-body problem corresponding to this choice is characterized by the equations of motion

$$\ddot{y}_j(t)=\alpha\dot{y}_j(t)+2\beta\dot{y}_j(t)\sum_{k=1}^{n}{}'\,\dot{y}_k(t)\operatorname{ctg}\beta[y_j(t)-y_k(t)]\,,\qquad j=1,2,\ldots,n, \tag{2.3.26}$$

with the constraint on the initial conditions

$$\dot{Y}(0)=0\,, \tag{2.3.27}$$

that also implies in this case

$$\dot{Y}(t) = 0\,, \qquad Y(t) = Y(0)\,. \tag{2.3.28}$$

Its solutions are the roots of the explicit equation in y

$$f(y) \equiv \beta \sum_{j=1}^{n} \dot{y}_j(0) \operatorname{ctg} \beta[y - y_j(0)] = \alpha/[\exp[\alpha t] - 1]\,. \tag{2.3.29}$$

Note that now, for $\beta = 0$, one would recover the system discussed in subsect. **2'1**, but with the sign of α reversed.

To discuss the time evolution of the solutions $y_j(t)$ it is again very convenient to display graphically the l.h.s. of (2.3.39), $f(y)$, as a function of y. For real β, namely in the periodic case, it is sufficient to focus attention only on an interval of length π/β, since any particle can be transferred inside such an interval shifting its co-ordinates by an integral multiple of π/β, such shifts having no dynamical effect. For imaginary β, $\beta = i\gamma$, that still yields a real problem since

$$i\gamma \operatorname{ctg} i\gamma z = \gamma \operatorname{ctgh} \gamma z\,, \tag{2.3.30}$$

one should instead consider $f(y)$ in the whole interval $(-\infty, +\infty)$. Note that the condition (2.3.27) implies that not all the residues of the poles that occur for $y = y_j(0)$ can have the same sign; moreover, for imaginary β, this condition also implies

$$f(\pm\infty) = 0\,. \tag{2.3.31}$$

The phenomenology that results from the assignment of different initial positions $y_j(0)$ and velocities $\dot{y}_j(0)$, for the different possible choices of the values of α and β, is rather rich; yet in all cases it is very easily obtainable by the technique just described. A detailed description need, therefore, not be given here, since the reader may provide it easily by himself if he is interested. We merely mention that generally, for real β (periodic case), either collapse occurs, or as $t \to \pm\infty$ the particles tend to an equilibrium position, whose configuration is of course always provided by the solutions of (2.3.29) with $-\alpha$ or 0 on the r.h.s. depending on whether the product αt diverges to $-\infty$ or to $+\infty$. For imaginary β, the same asymptotic behaviour occurs for $t \to +\infty$ if $\dot{y}_1(0) > 0$ and $\dot{y}_n(0) < 0$ (we are assuming as usual the particle co-ordinates to be ordered so that $y_j < y_{j+1}$); if instead $\dot{y}_1(0) < 0$ and/or $\dot{y}_n(0) > 0$, the extremal particles escape towards infinity, but they go all the way only if $\alpha \geqslant 0$ (the physical reason is clear from (2.3.26)). We also note that, for any given initial data $y_j(0)$, $\dot{y}_j(0)$, $j = 1, 2, \ldots, n$, there always is a (possibly negative) value $\bar{\alpha}$ such that, for any $\alpha < \bar{\alpha}$, no collapse occurs for

$t > 0$; again a fact of obvious significance, since the presence of a negative α-term in the equations of motion (2.3.26) acts as a brake that damps the motion of each particle (indeed (2.3.29) implies that, in the limit $\alpha \to -\infty$, $y_j(t) \approx y_j(0)$ for $t > 0$).

A second possible choice of the constants γ_+ and γ_- that eliminates the explicit t-dependence in (2.3.21) is $\gamma_+ = \gamma_- = \infty$; but it is easily seen that this choice reproduces the case we have just discussed, except for the replacement of α by $-\alpha$.

A third possible choice of the constants γ_+ and γ_-, that also eliminates the explicit t-dependence in (2.3.21) and, moreover, yields a many-body problem with only interparticle forces and no constraint on the initial centre-of-mass velocity, is $\gamma_+ = \infty$, $\gamma_- = 0$ (the complementary choice, $\gamma_+ = 0$, $\gamma_- = \infty$, yields the same model and, therefore, need not be considered). Let us thus proceed to the analysis of this very interesting case.

As implied by (2.3.17) and (2.3.18), in this case $\varrho = -\frac{1}{2}\alpha/\beta$, $R = -n\dot{Y} = i\varrho$, so that the equations of motion (2.3.21) become

$$\ddot{y}_j(t) = 2\beta\dot{y}_j(t)\sum_{k=1}^{n}{}' \dot{y}_k(t)\operatorname{ctg}\beta[y_j(t) - y_k(t)]\,, \qquad j = 1, 2, \dots, n, \tag{2.3.32}$$

while the constraint (2.3.18) yields

$$\alpha = -2i\beta n\dot{Y}\,. \tag{2.3.33}$$

Thus in this case the constant α has disappeared from the equations of motion, and there is no constraint on the centre-of-mass velocity $\dot{Y}$, since now (2.3.33) can be viewed as the definition of α. Note that in this case, as implied by the equations of motion (2.3.32), the velocity of the centre-of-mass remains constant even if it is not zero,

$$\dot{Y}(t) = \dot{Y}(0) \equiv v/n\,. \tag{2.3.34}$$

The solutions of the equations of motion (2.3.32) are the roots of the equation in y

$$\sum_{j=1}^{n}\dot{y}_j(0)\operatorname{ctg}\beta[y - y_j(0)] = v\operatorname{ctg}\beta vt\,, \tag{2.3.35}$$

that obtains from (3.2.25) for $\varrho = -\frac{1}{2}\alpha/\beta$, α given by (2.3.33) and v defined by (2.3.34).

Let us re-emphasize that this formula provides the solution of the many-body problem characterized by the equations of motion (2.3.32), for any set of initial data $y_j(0)$, $\dot{y}_j(0)$, $j = 1, 2, \dots, n$; the (initial) centre-of-mass speed $\dot{Y}(0)$ is of course related to the initial speeds $\dot{y}_j(0)$ as implied by (2.3.23).

As repeatedly mentioned above, the behaviour of the solutions of the system (2.3.32) is better discussed by a graphical analysis of (2.3.35). We discuss here only the case characterized by the initial conditions

$$\dot{y}_j(0) > 0\,, \tag{2.3.36}$$

when no collapse can occur. In this case of course the centre of mass of the system moves towards the right with the constant positive speed $\dot{Y}(0) = v/n$.

Let us consider first the periodic case, with real β. We then assume of course that all n particles are initially within an interval of length π/β,

$$y_n(0) - y_1(0) < \pi/\beta \qquad (y_{j+1} > y_j)\,. \tag{2.3.37}$$

It is then easily seen ([17]) that, as time proceeds, all particles move to the right; at the time $t_1 = \delta$, where

$$\delta = \pi/\beta v = \pi/\beta n \dot{Y}(0)\,, \tag{2.3.38}$$

$$y_n(t_1) = y_1(0) + \pi/\beta\,, \tag{2.3.39a}$$

$$y_j(t_1) = y_{j+1}(0)\,, \qquad j = 1, 2, \dots, n-1\,; \tag{2.3.39b}$$

at the time $t_2 = 2\delta$,

$$y_n(t_2) = y_2(0) + \pi/\beta\,, \tag{2.3.40a}$$

$$y_{n-1}(t_2) = y_1(0) + \pi/\beta\,, \tag{2.3.40b}$$

$$y_j(t_2) = y_{j+2}(0)\,, \qquad j = 1, 2, \dots, n-2\,, \tag{2.3.40c}$$

and so on. At the time $t_n = n\delta$,

$$y_j(t_n) = y_j(0) + \pi/\beta\,, \qquad j = 1, 2, \dots, n\,, \tag{2.3.41}$$

namely the system has recovered exactly the initial structure, having moved collectively towards the right a distance π/β. Thereafter the process is repeated. Thus the system has an internal structure that oscillates periodically, with period

$$T = n\delta = \pi/\beta \dot{Y}(0)\,, \tag{2.3.42}$$

while it travels collectively with the constant speed $\dot{Y}(0)$ of its centre of mass (so that indeed in the time T it moves the distance π/β). Of course for special

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([17]) The l.h.s. of (2.3.35) is in this case an everywhere decreasing function of y, having n poles at $y = y_j(0)$ and being periodic with period π/β.

initial conditions the period of the internal motion might be a fraction of T; for instance, if initially the particles are equally spaced with mesh $\pi/n\beta$, clearly they acquire the same spatial configuration at the times t_1, t_2, etc.; and if, moreover, their initial speeds are all equal, then they continue to move with constant speed, namely in this special case there is no internal motion at all (see below).

Clearly to discuss this system it is convenient to go over to the variables

$$z_j(t) = y_j(t) - Vt\,, \tag{2.3.43}$$

where we set

$$V = \dot{Y}(0)\,. \tag{2.3.44}$$

Then the equations of motion read

$$\ddot{z}_j(t) = 2\beta[V + \dot{z}_j(t)] \sum_{k=1}^{n}{}' \, [V + \dot{z}_k(t)] \operatorname{ctg} \beta[z_j(t) - z_k(t)]\,, \qquad j = 1, 2, \ldots, n, \tag{2.3.45}$$

and, for these variables, the centre of mass

$$Z(t) = n^{-1} \sum_{j=1}^{n} z_j(t) \tag{2.3.46}$$

must be chosen at rest:

$$\dot{Z}(0) = 0\,, \qquad Z(t) = Z(0)\,. \tag{2.3.47}$$

The solutions $z_j(t)$ are the roots of the equation in z

$$\sum_{j=1}^{n} [V + \dot{z}_j(0)] \operatorname{ctg} \beta[z + Vt - z_j(0)] = nV \operatorname{ctg} \beta n V t\,; \tag{2.3.48}$$

as implied by the previous analysis, they are all real, provided that

$$|\dot{z}_j(0)| < V\,, \tag{2.3.49}$$

and they oscillate with period T, see (2.3.42), around the equilibrium positions

$$\bar{z}_j = z_0 + j\pi/\beta n\,, \qquad j = 1, 2, \ldots, n, \tag{2.3.50}$$

where z_0 is arbitrary (corresponding to the translation-invariant nature of the model).

Note that the many-body system (2.3.31) approximates, provided that

$$|\dot{z}_j(t)| \ll V\,, \tag{2.3.51}$$

the Hamiltonian system of n unit-mass particles interacting via the two-body periodic potential

$$W(z) = -\,2V^2 \ln|\sin\beta z|\,, \tag{2.3.52}$$

namely the n-body system characterized by the Hamiltonian

$$H(p, q) = \tfrac{1}{2}\sum_{j=1}^{n} p_j^2 - 2V^2 \sum_{j=1}^{n}\ \sum_{k=1}^{j-1} \ln|\sin\beta(q_j - q_k)|\,. \tag{2.3.53}$$

Condition (2.3.51) can of course be satisfied, provided the initial configuration of the system is sufficiently close to the equilibrium configuration.

Let us finally consider the behaviour of the system (2.3.32) for imaginary $\beta = i\gamma$, γ real. Now the equations of motion read

$$\ddot{y}_j(t) = 2\gamma\dot{y}_j(t) \sum_{k=1}^{n}{}' \dot{y}_k(t)\ \mathrm{ctgh}\,\gamma[y_j(t) - y_k(t)]\,, \qquad j = 1, 2, \ldots, n, \tag{2.3.54}$$

and the solutions $y_j(t)$ are the roots of the equation in y

$$\sum_{j=1}^{n} \dot{y}_j(0)\ \mathrm{ctgh}\,\gamma[y - y_j(0)] = nY(0)\ \mathrm{ctgh}\,[\gamma n\dot{Y}(0)\,t]\,. \tag{2.3.55}$$

Let us recall that the centre of mass Y moves with the constant velocity $\dot{Y}(0)$ and that there is no constraint on the initial positions $y_j(0)$ or velocities $\dot{y}_j(0)$, although we restrict for simplicity the analysis to the case characterized by the inequalities (2.3.36), that are sufficient to exclude the occurrence of collapse.

The usual analysis ([18]) implies then the following asymptotic behaviour for this system:

$$y_1(t) = vt + a_0^{(-)} + O[\exp[2\gamma t]]\,, \qquad t \to -\infty, \tag{2.3.56a}$$

$$\lim_{t\to-\infty} [y_j(t)] = a_{j-1}^{(-)}\,, \qquad j = 2, 3, \ldots, n, \tag{2.3.56b}$$

$$\lim_{t\to+\infty} [y_j(t)] = a_j^{(+)}\,, \qquad j = 1, 2, \ldots, n-1, \tag{2.3.57a}$$

$$y_n(t) = vt + a_0^{(+)} + O[\exp[-2\gamma t]]\,, \qquad t \to +\infty. \tag{2.3.57b}$$

([18]) The l.h.s. of (2.3.55) is in this case an everywhere decreasing function of y, with poles at $y = y_j(0)$, and with the asymptotic values $\varepsilon n\dot{Y}(0)$ as $\varepsilon t \to +\infty$, where $\varepsilon = \pm 1$ (for $\gamma > 0$).

Here

$$v \quad = n\dot{Y}(0) = \sum_{j=1}^{n} \dot{y}_j(0)\,, \tag{2.3.58}$$

$$a_0^{(\pm)} \quad = \pm\,(2\gamma)^{-1}\ln\Big\{\Big(\sum_{j=1}^{n}\dot{y}_j(0)\exp\,[\pm\,2\gamma y_j(0)]\Big)/v\Big\}, \tag{2.3.59}$$

and $a_j^{(\varepsilon)}$ are the $n-1$ real solutions, ordered so that $a_j^{(\varepsilon)} < a_{j+1}^{(\varepsilon)}$, of the equation in $a^{(\varepsilon)}$

$$\sum_{j=1}^{n}\dot{y}_j(0)\,\operatorname{ctgh}\gamma[a^{(\varepsilon)} - y_j(0)] = \varepsilon v\,, \tag{2.3.60}$$

where ε stands for $+1$ or -1. In this analysis we have assumed $\gamma > 0$; for $\gamma < 0$, the behaviour as $t \to +\infty$ is exchanged with that for $t \to -\infty$, and *vice versa*. Thus the behaviour of the many-body system characterized by the equations of motion (2.3.54) can be described as follows: in the remote past, one particle comes in from the extreme left with the (positive) velocity v, and the other $n-1$ particles are almost [16] at rest at the positions $a_j^{(-)}$; at intermediate times, all particles move to the right; in the extreme future, $n-1$ particles are (almost) at rest at the positions $a_j^{(+)}$, while the rightmost particle escapes to infinity with speed v. To assess the overall effect of the interaction note that the analysis outlined above implies

$$a_j^{(+)} < a_j^{(-)} < a_{j+1}^{(+)} \tag{2.3.61}$$

and, moreover,

$$a_0^{(+)} - a_0^{(-)} = (2\gamma)^{-1}\ln\Big\{\Big[\sum_{j=1}^{n}\dot{y}_j(0)\sum_{k=1}^{n}\dot{y}_k(0)\cosh 2\gamma[y_j(0) - y_k(0)]\Big]/v^2\Big\}, \tag{2.3.62}$$

so that $a_0^{(+)} - a_0^{(-)}$ is clearly translation invariant and positive (we are always assuming $\gamma > 0$).

In subsect. **3·6** we show that models similar to those discussed here, but considerably more general, can also be solved.

2·4. *Miscellaneous extensions: particles of two different types, symmetrical configurations, two-dimensional models.* – In this subsection we discuss tersely various extensions of the models discussed above, that can be obtained by using certain tricks that were previously used in the literature in analogous contexts.

The first trick leads to the introduction of two types of particles and is performed by a simple shift of some of the particle co-ordinates [19]. We apply

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[19] This trick was introduced in the first paper of ref. [6].

it only to the last models considered in the previous subsect. 2·3; its application to the other many-body problems discussed in the preceding section is left as an exercise for the interested reader.

Consider first the n-body model characterized by the equations of motion (2.3.32). Let us divide the n co-ordinates y_j into two groups, shifting by the amount $\frac{1}{2}\pi/\beta$ the co-ordinates of the second group. This we do replacing the co-ordinates y_j, for $j = n_1 + 1, n_1 + 2, \ldots, n = n_1 + n_2$, by the shifted co-ordinates u_j, defined as follows:

$$u_j = y_{n_1+j} + \tfrac{1}{2}\pi/\beta\,, \qquad j = 1, 2, \ldots, n_2 = n - n_1\,. \tag{2.4.1}$$

In the new co-ordinates the equations of motion read

$$\ddot{y}_j = 2\beta\dot{y}_j\Big\{\sum_{k=1}^{n_1}\dot{y}_k \operatorname{ctg}\beta(y_j - y_k) - \sum_{k=1}^{n_2}\dot{u}_k \operatorname{tg}\beta(y_j - u_k)\Big\}, \qquad j = 1, 2, \ldots, n_1, \tag{2.4.2a}$$

$$\ddot{u}_j = 2\beta\dot{u}_j\Big\{\sum_{k=1}^{n_2}\dot{u}_k \operatorname{ctg}\beta(u_j - u_k) - \sum_{k=1}^{n_1}\dot{y}_k \operatorname{tg}\beta(u_j - y_k)\Big\}, \qquad j = 1, 2, \ldots, n_2. \tag{2.4.2b}$$

Here and often below we omit for simplicity to indicate explicitly the time dependence.

If now we consider the y_j's, $j = 1, 2, \ldots, n_1$, and the u_j's, $j = 1, 2, \ldots, n_2$, as particle co-ordinates, we must interpret the equations of motion (2.4.2) as describing a many-body system composed of particles of two kinds labelled respectively by the co-ordinates y_j and u_j, with the force $2\beta\dot{z}_j\dot{z}_k \operatorname{ctg}\beta(z_j - z_k)$ acting between equal particles and the force $-2\beta\dot{z}_j\dot{z}_k \operatorname{tg}\beta(z_j - z_k)$ acting between different particles (here z_j stands for y_j or u_j, whichever the case may be). Note that the interaction between different particles is nonsingular, indeed it vanishes at zero separation, so that different particles can go through each other.

The solution of this many-body problem need not be discussed here, since it is trivially related to the solution of the many-body problem (2.3.32) by the change of variable (2.4.1).

There is, however, an amusing observation that is worth reporting. Since the force acting between unequal particles vanishes at zero interparticle separation, there clearly exists an equilibrium configuration of the two-body problem with two different particles located exactly at the same position and moving exactly with the same speed. Let us call such a two-body configuration a « molecule ». It is then also possible to consider the n-molecule problem, since such a configuration is compatible with the equations of motion, namely, if given initially, it is maintained throughout the motion. It appears thus that one is generating in this manner a novel solvable many-body problem.

But an elementary computation and the use of the trigonometric identity

$$\operatorname{ctg} A - \operatorname{tg} A = 2 \operatorname{ctg} 2A \tag{2.4.3}$$

imply that this novel model coincides exactly with the original one, except for a scaling of the particle variables by a factor of two! ([20])

Let us also note that, although, as we just pointed out, the many-body problem (2.4.2) is trivially related to the model (2.3.2), the remarkable richness of its dynamical evolution suggests an ample scope for applications. Notable in this connection is the existence of configurations containing « quasi-molecules »; indeed the form of the interaction indicates that two different particles moving with almost equal speeds and located close to one another form a rather stable compound. Of course one also has the possibility to perform an additional change of variables analogous to (2.3.43); indeed this may yield the most interesting models for applications in solid-state physics.

The trick described above to generate a model with two kinds of particles can of course be applied also in the hyperbolic case, namely for $\beta = i\gamma$, γ real. Then in place of (2.3.54) one has the equations of motion

$$\ddot{y}_j = 2\gamma \dot{y}_j \Big\{ \sum_{k=1}^{n_1}{}' \dot{y}_k \operatorname{ctgh} \gamma(y_j - y_k) + \sum_{k=1}^{n_2} \dot{u}_k \operatorname{tgh} \gamma(y_j - u_k) \Big\}, \qquad j = 1, 2, \ldots, n_1, \tag{2.4.4a}$$

$$\ddot{u}_j = 2\gamma \dot{u}_j \Big\{ \sum_{k=1}^{n_2}{}' \dot{u}_k \operatorname{ctgh} \gamma(u_j - u_k) + \sum_{k=1}^{n_1} \dot{y}_k \operatorname{tgh} \gamma(u_j - y_k) \Big\}, \qquad j = 1, 2, \ldots, n_2, \tag{2.4.4b}$$

also to be interpreted as describing a system composed of n_1 particles of one kind and n_2 particles of another with the (singular) force $2\gamma \dot{z}_j \dot{z}_k \operatorname{ctgh} \gamma(z_j - z_k)$ acting between equal particles and the (nonsingular) force $2\gamma \dot{z}_j \dot{z}_k \operatorname{tgh} \gamma(z_j - z_k)$ acting between different particles, where again z_j stands here for y_j or u_j, whichever the case may be.

The solutions of this system are now related to the solutions of (2.3.54) less trivially than in the previous case, since now an *imaginary* shift of n_2 co-ordinates has been performed. The prescription to find these solutions is thus the following one: the co-ordinates y_j, $j = 1, 2, \ldots, n_1$, of the particles of the first kind are the solutions of the equation in y

$$\sum_{j=1}^{n_1} \dot{y}_j(0) \operatorname{ctgh} \gamma[y - y_j(0)] + \sum_{j=1}^{n_2} \dot{u}_j(0) \operatorname{tgh} \gamma[y - u_j(0)] = v \operatorname{ctgh} \gamma v t, \tag{2.4.5a}$$

([20]) A similar phenomenon occurs in other recently discovered exactly solvable many-body problems; see the third paper of ref. ([6]).

while the co-ordinates u_j, $j = 1, 2, \ldots, n_2$, of the particles of the second kind are the solution of the equation in u

$$\sum_{j=1}^{n_2} \dot{u}_j(0) \operatorname{ctgh} \gamma[u - u_j(0)] + \sum_{j=1}^{n_1} \dot{y}_j(0) \operatorname{tgh} \gamma[u - y_j(0)] = v \operatorname{ctgh} \gamma v t \,. \tag{2.4.5b}$$

Here of course $y_j(0)$, $\dot{y}_j(0)$, $j = 1, 2, \ldots, n_1$, respectively $u_j(0)$, $\dot{u}_j(0)$, $j = 1, 2, \ldots, n_2$, are the initial positions and velocities of the particles of the first, respectively, second kind, while v/n is the (constant) velocity of the centre of mass of the whole system:

$$v = \sum_{j=1}^{n_1} \dot{y}_j(0) + \sum_{j=1}^{n_2} \dot{u}_j(0) = \sum_{j=1}^{n_1} \dot{y}_j(t) + \sum_{j=1}^{n_2} \dot{u}_j(t) \,. \tag{2.4.6}$$

As usual, if the number of real solutions of these equations is smaller than the corresponding number of particles, collapse has occurred. A sufficient condition to exclude this possibility is that all the particles of the same kind have initially (and, therefore, throughout the motion) speeds of the same sign, say

$$\dot{y}_j(0) > 0 \,, \qquad j = 1, 2, \ldots, n_1, \tag{2.4.7a}$$

and either

$$\dot{u}_j(0) > 0 \,, \qquad j = 1, 2, \ldots, n_2, \tag{2.4.7b}$$

or

$$\dot{u}_j(0) < 0 \,, \qquad j = 1, 2, \ldots, n_2. \tag{2.4.7c}$$

Hereafter we limit, for simplicity, our consideration to these cases. Note that, while the ordering of particles of the same kind cannot change throughout the motion (so that we assume hereafter, for definiteness, $y_{j+1} > y_j$ and $u_{j+1} > u_j$), particles of different kind can go through each other.

A discussion of the solution of (2.4.5) (and, therefore, also (2.4.4)) can be easily performed by the usual graphical technique. We report here only the results for the asymptotic behaviour of the (extremal) particles that escape, in the remote past or future. They have been obtained by using the well-known asymptotic expansions

$$\operatorname{tgh} x = \varepsilon\{1 - 2[\exp[-2\varepsilon x] - \exp[-4\varepsilon x] + \exp[-6\varepsilon x] + \ldots]\} \,, \tag{2.4.8a}$$

$$\operatorname{ctgh} x = \varepsilon\{1 + 2[\exp[-2\varepsilon x] + \exp[-4\varepsilon x] + \exp[-6\varepsilon x] + \ldots]\} \,, \tag{2.4.8b}$$

valid as $\varepsilon x \to +\infty$, $\varepsilon = \pm 1$. They imply an essential dependence on the sign of v and on the relative magnitude of the quantities

$$Y_m(\gamma) = \sum_{j=1}^{n_1} \dot{y}_j(0) \exp[2m\gamma y_j(0)], \qquad m = 0, 1, 2, \ldots, \tag{2.4.9a}$$

$$U_m(\gamma) = \sum_{j=1}^{n_2} \dot{u}_j(0) \exp[2m\gamma u_j(0)], \qquad m = 0, 1, 2, \ldots, \tag{2.4.9b}$$

as indicated by the following equations. Note that these definitions of $Y_m(\gamma)$ and $U_m(\gamma)$ imply that Y_0 and U_0 are independent of γ and that

$$Y_0 + U_0 = v = \sum_{j=1}^{n_1} \dot{y}_j(0) + \sum_{j=1}^{n_2} \dot{u}_j(0). \tag{2.4.10}$$

We assume throughout $\gamma > 0$.

For $t \to -\infty$

$$y_1 \approx vt - (2\gamma)^{-1} \ln\{[Y_1(-\gamma) - U_1(-\gamma)]/v\}, \qquad \text{if } v > 0 \text{ and } Y_1(-\gamma) > U_1(-\gamma), \tag{2.4.11a}$$

$$u_1 \approx vt - (2\gamma)^{-1} \ln\{[(U_1(-\gamma) - Y_1(-\gamma)]/v\}, \qquad \text{if } v > 0 \text{ and } U_1(-\gamma) > Y_1(-\gamma), \tag{2.4.11b}$$

$$y_1 \approx u_1 \approx \tfrac{1}{2} vt - (4\gamma)^{-1} \ln\{[Y_2(-\gamma) + U_2(-\gamma)]/v\}, \qquad \text{if } v > 0 \text{ and } Y_1(-\gamma) = U_1(-\gamma), \tag{2.4.11c}$$

$$u_{n_2} \approx vt + (2\gamma)^{-1} \ln\{[Y_1(\gamma) - U_1(\gamma)]/|v|\}, \qquad \text{if } v < 0, \tag{2.4.11d}$$

$$\begin{cases} y_1 \approx -(2\gamma)^{-1} \ln\{2\gamma|t|[Y_1(-\gamma) - U_1(-\gamma)]\}, & \text{if } v = 0, \\ u_{n_2} \approx (2\gamma)^{-1} \ln\{2\gamma|t|[Y_1(\gamma) - U_1(\gamma)]\}, & \text{if } v = 0. \end{cases} \tag{2.4.11e}$$

For $t \to +\infty$

$$y_{n_1} \approx vt + (2\gamma)^{-1} \ln\{[Y_1(\gamma) - U_1(\gamma)]/v\}, \qquad \text{if } v > 0 \text{ and } Y_1(\gamma) > U_1(\gamma), \tag{2.4.12a}$$

$$u_{n_2} \approx vt + (2\gamma)^{-1} \ln\{[U_1(\gamma) - Y_1(\gamma)]/v\}, \qquad \text{if } v > 0 \text{ and } U_1(\gamma) > Y_1(\gamma), \tag{2.4.12b}$$

$$y_{n_1} \approx u_{n_2} \approx \tfrac{1}{2} vt + (4\gamma)^{-1} \ln\{[Y_2(\gamma) + U_2(\gamma)]/v\}, \qquad \text{if } v > 0 \text{ and } Y_1(\gamma) = U_1(\gamma), \tag{2.4.12c}$$

$$(2.4.12d)\qquad u_1 \approx vt - (2\gamma)^{-1}\ln\{Y_1(-\gamma) - U_1(-\gamma)]/|v|\}\,, \qquad \text{if } v<0,$$

$$(2.4.12e)\qquad \begin{cases} y_{n_1} \approx (2\gamma)^{-1}\ln\{2\gamma t[Y_1(\gamma) - U_1(\gamma)]\}\,, & \text{if } v=0,\\ u_1 \approx -(2\gamma)^{-1}\ln\{2\gamma t[Y_1(-\gamma) - U_1(-\gamma)]\}\,, & \text{if } v=0. \end{cases}$$

These cases cover all possibilities (recall that the validity of the inequalities (2.7) is assumed). We emphasize that in each case all the particles whose asymptotic behaviour is not given explicitly by these formulae have asymptotically vanishing velocities; their asymptotic positions are given by the solutions of the equations obtained from (2.4.5) by replacing the term on the r.h.s. by its asymptotic value, namely $|v|$ for $t\to+\infty$ and $-|v|$ for $t\to-\infty$.

Attention should be called on the rather intriguing nature of the asymptotic results explicitly displayed by eqs. (2.4.11) and (2.4.12). Note in particular the possibility that a «molecule» emerge (see (2.4.12c)).

Let us terminate here the discussion of models involving two kinds of particles. Of course the technique we have discussed could be used in other cases besides the examples treated above.

Let us now return to the many-body model of subsect. 2·1, fixing our attention on the equations of motion (2.1.21), but considering a special symmetrical configuration that is maintained throughout the motion. The only such configuration occurs for even n, $n=2m$ [21], and is of the following type:

$$(2.4.13)\qquad y_j(t) = u_0 + u_j(t)\,,\qquad y_{j+m}(t) = u_0 - u_j(t)\,,\qquad j=1,2,\dots,m,$$

where u_0 is an arbitrary constant. Clearly, if this configuration of the $2m$ coordinates y_j is given initially, namely if (2.4.13) holds at $t=0$ and if, moreover,

$$(2.4.14)\qquad \dot y_j(0) = -\dot y_{j+m}(0)\,,\qquad j=1,2,\dots,m,$$

the configuration (2.4.13) is maintained at all times. One can then consider only the time evolution of the co-ordinates $u_j(t)$, $j=1,2,\dots,m$; and interpreting these quantities as particle co-ordinates, one obtains thereby a novel many-body problem, characterized by the equations of motion [22]

$$(2.4.15)\qquad \ddot u_j = -\alpha\dot u_j + 2\dot u_j\sum_{k=1}^{m}{}'\,\dot u_k/(u_j-u_k) - 2\dot u_j\sum_{k=1}^{m}\dot u_k/(u_j+u_k)\,,\quad j=1,2,\dots,m.$$

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(21) The symmetrical configuration with an odd number of particles has the central particle at rest; but then such a particle has no interaction and can be, therefore, ignored.

(22) This procedure, applied to the integrable many-body models of ref. [4-6], yields just those integrable Hamiltonian systems that have been introduced by OLSHANETSKY and PERELOMOV in connection with semi-simple Lie algebras; see, in particular, the second and the third of their papers listed in ref. [6].

This appears as a novel solvable many-body problem, whose physical interpretation is, however, marred by the non–translation-invariant character of the last term on the r.h.s. But as we show, it can be reduced back to the original model. Indeed the position

$$z_j = u_j^2 \tag{2.4.16}$$

implies, after some trivial algebra, the following equations of motion for z_j:

$$\ddot{z}_j = -\alpha\dot{z}_j + 2\dot{z}_j \sum_{k=1}^{m}{}' \dot{z}_k/(z_j - z_k)\,, \qquad j = 1, 2, \dots, m; \tag{2.4.17}$$

and these equations are identical, except for the replacement of n by m, to (2.1.21).

This intriguing result implies that the co-ordinates of the $2m$-body problem (2.1.21), characterized by the initial conditions that correspond to the special configuration (2.4.13), coincide exactly with the square roots of the co-ordinates of a corresponding m-body problem, obtained by eliminating one-half of the particles!

The same kind of trick can be applied to the models of subsect. **2·3**; and in this case again a rather extraordinary event occurs. Let us focus attention for simplicity just on the specific model characterized by the equations of motion (2.3.26), that we prefer to write here with the sign of α reversed ([23]):

$$\ddot{y}_j = -\alpha\dot{y}_j + 2\beta\dot{y}_j \sum_{k=1}^{n}{}' \dot{y}_k \operatorname{ctg}\beta(y_j - y_k)\,, \qquad j = 1, 2, \dots, n. \tag{2.4.18}$$

Let again $n = 2m$ and consider the symmetrical configuration (2.4.13) of this system. Then for the m co-ordinates u_j the equations of motion read

$$\ddot{u}_j = -\alpha\dot{u}_j + 2\beta\dot{u}_j\left\{\sum_{k=1}^{m}{}' \dot{u}_k \operatorname{ctg}\beta(u_j - u_k) - \sum_{k=1}^{m} \dot{u}_k \operatorname{ctg}\beta(u_j + u_k)\right\}, \qquad j = 1, 2, \dots, m. \tag{2.4.19}$$

Now set

$$z_j = \cos 2\beta u_j\,. \tag{2.4.20}$$

Then a little algebra reproduces for the co-ordinates z_j exactly the equations of motion (2.4.17).

We may, therefore, assert that the solutions of the $2m$-body problem (2.4.18) corresponding to the symmetrical configuration (2.4.13) are given, via (2.4.20), by the solutions of the m-body problem (2.4.17)!

([23]) Note that the requirement (2.3.27) that the centre of mass does not move is consistent with the symmetrical configuration considered below.

These results are of course valid for real, as well as for imaginary, β; and they are immediately extendible also to the other models of subsect. **2**'3.

The last topic that we take up in this subsection is the possibility of generating two-dimensional models by complexification [24]. We merely report the form that takes such a model, obtained by writing (2.1.21) with y_j replaced by z_j and then by setting $z_j = x_j + iy_j$:

$$\text{(2.4.21}a) \qquad \ddot{x}_j = -\alpha\dot{x}_j + 2\sum_{k=1}^{n}{}' [x_{jk}(\dot{x}_j\dot{x}_k - \dot{y}_j\dot{y}_k) + y_{jk}(\dot{x}_j\dot{y}_k + \dot{y}_j\dot{x}_k)]/r_{jk}^2, \qquad j = 1, 2, \dots, n,$$

$$\text{(2.4.21}b) \qquad \ddot{y}_j = -\alpha\dot{y}_j + 2\sum_{k=1}^{n}{}' [x_{jk}(\dot{x}_j\dot{y}_k + \dot{y}_j\dot{x}_k) + y_{jk}(\dot{y}_j\dot{y}_k - \dot{x}_j\dot{x}_k)]/r_{jk}^2, \qquad j = 1, 2, \dots, n.$$

Here we have used the synthetic notation

$$\text{(2.4.22)} \qquad x_{jk} \equiv x_j - x_k, \qquad y_{jk} \equiv y_j - y_k, \qquad r_{jk}^2 = x_{jk}^2 + y_{jk}^2.$$

The equations of motion (2.4.21) can be interpreted as describing a two-dimensional n-body problem, although one with non-spherically-symmetrical forces [24]. We shall not discuss here the detailed behaviour of the solutions of such a system, that are of course obtainable by its simple relation to the many-body problem (2.1.21), whose solution has been discussed in subsect. **2**'1. We merely mention that, in spite of the simplicity of this connection, the behaviour of the solutions of the system (2.4.21) is considerably richer in complexity than the behaviour of the solutions of (2.1.21), as implied by the availability of an extra dimension, and by the correspondingly more complicated phenomenology resulting from the consideration of the r.h.s. of (2.1.24) as a function of a complex, rather than a real, variable.

2'5. *More general partial differential equations.* – We have discussed in subsect. **2**'1 the solvable many-body problems that correspond to the motion of the poles of rational solutions, of type (2.1.4), of the very simple nonlinear partial differential equation (2.1.1), or of its variant (2.2.7); and in the subsequent subsect. **2**'2-**2**'4 we have considered various extensions that give rise to other solvable many-body problems. A natural question suggested by these results, as well as by the original findings of AMM and CC, is: can the same approach be applied to other nonlinear partial differential equations, and in particular to nonlinear partial differential equations, that can be solved by some analytical technique?

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[24] See the fourth paper of ref. [6].

To discuss this question it is convenient to rewrite here the ansatz (2.1.4),

$$\varphi(x,t)=\sum_{j=1}^{n}[x-x_j(t)]^{-1}r_j(t)\,, \tag{2.5.1}$$

and to construct from it table I, that displays the contributions R_{sj} that appear in the formula

$$F[\varphi,\varphi_x,\varphi_t,\varphi_{xx},\varphi_{xt},\varphi_{tt}]=\sum_{j=1}^{n}\sum_{s=1}^{3}[x-x_j(t)]^{-s}R_{sj}(t) \tag{2.5.2}$$

for all the choices of the function F that are compatible with (2.5.1) and (2.5.2).

TABLE I. – *Coefficients of eq.* (2.5.2).

F	R_{1j}	R_{2j}	R_{3j}
φ	r_j	0	0
φ_x	0	$-r_j$	0
φ_t	$\dot r_j$	$\dot x_j r_j$	0
φ_{xx}	0	0	$2r_j$
φ_{xt}	0	$-\dot r_j$	$-2\dot x_j r_j$
φ_{tt}	$\ddot r_j$	$2\dot x_j\dot r_j+\ddot x_j r_j$	$\dot x_j^2 r_j$
φ^2	$2r_j\sum_{k=1}^{n}{}' r_k/(x_j-x_k)$	r_j^2	0
$\varphi_x\varphi$	0	$-r_j\sum_{k=1}^{n}{}' r_k/(x_j-x_k)$	$-r_j^2$
$\varphi_t\varphi$	$\left[r_j\sum_{k=1}^{n}{}' r_k/(x_j-x_k)\right]_t$	$\dot r_j r_j+\dot x_j r_j\sum_{k=1}^{n}{}' r_k/(x_j-x_k)$	$\dot x_j r_j^2$
φ^3	$-3r_j^2\sum_{k=1}^{n}{}' r_k/(x_j-x_k)^2+$ $+3r_j\sum_{k=1}^{n}{}' r_k\sum_{l=1}^{n}{}' r_l/[(x_j-x_k)(x_j-x_l)]$	$3r_j^2\sum_{k=1}^{n}{}' r_k/(x_j-x_k)$	r_j^3

From table I one can immediately read the relationship between many-body problems and nonlinear partial differential equations, that are induced by the ansatz (2.5.1); for instance, one sees that to the Burgers-Hopf (BH) equation

$$\varphi_t + \varphi_{xx} + \varphi_x\varphi = 0 \tag{2.5.3}$$

there corresponds the $3n$ equations

$$\dot{r}_j = 0\,, \qquad j = 1, 2, \ldots, n, \tag{2.5.4}$$

$$\dot{x}_j = \sum_{k=1}^{n}{}' r_k/(x_j - x_k)\,, \qquad j = 1, 2, \ldots, n, \tag{2.5.5}$$

$$r_j = 2\,, \qquad j = 1, 2, \ldots, n, \tag{2.5.6}$$

or, equivalently (7),

$$\dot{x}_j = 2\sum_{k=1}^{n}{}' (x_j - x_k)^{-1}\,, \qquad j = 1, 2, \ldots, n. \tag{2.5.7}$$

An important remark is that generally to a nonlinear partial differential equation for $\varphi(x, t)$ obtained equating to zero a linear combination of the terms appearing in the first column of table I there correspond $3n$ equations, to be satisfied by the $2n$ quantities $x_j(t)$ and $r_j(t)$. Thus, generally, after the r_j's are eliminated, there obtain for the x_j's n constraints in addition to n « equations of motion ».

The only possibilities of avoiding the presence of constraints in addition to the equations of motion is either to take advantage of the occurrence of vanishing entries in table I to construct nonlinear partial differential equations for φ that give rise only to $2n$ equations for the x_j's and r_j's, or to combine the terms appearing in the first column of table I, so that of the corresponding $3n$ equations, that must be satisfied by the x_j's and r_j's, n equations are automatically implied by the remaining $2n$ equations. But, as is immediately seen, the first possibility corresponds only to the nonlinear partial differential equation (2.1.1), that has been discussed in subsect. 2·1; while, as regards the second possibility, the only nontrivial instance is the BH case mentioned above and treated in detail by CC.

One may, therefore, conclude that the only many-body problems without constraints that can be treated on the basis of the ansatz (2.5.1) are those described in the previous sections. This conclusion is not modified by the consideration of equations involving higher derivatives, or higher powers, of φ than those reported in the first column of table I, and a correspondingly more general ansatz for the function $\varphi(x, t)$, namely

$$\varphi(x, t) = \sum_{s=1}^{S}\,\sum_{j=1}^{n}[x - x_j(t)]^{-s} r_{sj}(t)\,, \tag{2.5.8}$$

that constitutes the natural extension of (2.5.1). We omit to report here a formal proof of this assertion; every reader can easily convince himself of its validity by the consideration of simple examples and by the recognition that the number of equations to be satisfied generally increases faster than the number of variables to be determined. Thus, even though an effect such as that exemplified above in the BH case by the fact that (2.5.4) are implied by (2.5.6) can occur in other instances, it is not enough to eliminate all constraints: for instance in the KdV case the ansatz (2.5.8) with $S = 2$ introduces the $3n$ variables x_j, r_{1j} and r_{2j}, $j = 1, 2, \ldots, n$, but poles of up to 5th order appear, so that the number of equations obtained by imposing that their coefficients vanish is $5n$; thus, although n equations can be eliminated, in analogy to the BH case, since they turn out not to be independent of the others, there still remain the n constraints (1.2), in addition to the n first-order equations of motion (1.3) (see AMM and CC).

It should of course also be noted that a first-order equation for the x_j's, even when it yields second-order equations interpretable as a many-body model with one- and two-body forces, implies that the (initial) velocities are determined by the (initial) positions, namely in these cases there always are constraints. This remark is of course related to our previous assertion, in the introduction, stating that the n-body problem associated to the BH equation is restricted by n constraints, and that associated with the KdV equation by $2n$ constraints.

It is also clear that the above considerations are essentially unmodified by extensions such as that discussed in subsect. **2**.2, involving nonautonomous partial differential equations and leading to non–translation-invariant models. Such an extension does, however, yield certain nontrivial results, such as the association of the nonlinear partial differential equation

$$\varphi_t + \varphi_{xx} + \varphi^2 + \omega(x\varphi_x + \varphi) = 0 \tag{2.5.9}$$

to the first-order equations of motion

$$\dot{x}_j = \omega x_j + 2 \sum_{k=1}^{n}{}' (x_j - x_k)^{-1}, \qquad j = 1, 2, \ldots, n, \tag{2.5.10}$$

that also imply [7] the second-order equations of motion

$$\ddot{x}_j = \omega^2 x_j - 4 \sum_{k=1}^{n}{}' (x_j - x_k)^{-3}, \qquad j = 1, 2, \ldots, n. \tag{2.5.11}$$

But, since these results can also be obtained by the (simpler) technique of the following section, we do not elaborate on them here any further (see subsect. **3**.2).

The extent to which the extensions discussed in subsect. 2'3 and 2'4 are also applicable in the context discussed in this subsection should also be sufficiently self-evident not to require any additional elaboration.

3. – Motion of zeros of linear evolution equations and related many-body problems.

Much of the discussion of the previous section was based on the analysis of the motion of the poles of special solutions of a (very simple) nonlinear partial differential equation. The simplicity of this equation was related to the possibility of transforming it by a simple change of dependent variable into a linear equation; the poles of solutions of the nonlinear equation coincide then with the zeros of solutions of the linear equation. Thus the above analysis could have been just as well based on the study of the zeros of (special) solutions of linear partial differential equations, although the special solutions discussed in the previous section were more naturally suggested, and more easily handled, in the nonlinear framework.

It is, therefore, natural to proceed to a direct study of the motion of the zeros of solutions of linear partial differential equations. This we do in the 6 subsections of this section. We consider mainly special solutions of polynomial type, or natural generalizations of such solutions; they are clearly suggested by our aim to generate by this approach solvable many-body problems. Several such models are indeed exhibited, and some of them are discussed; they are analogous to, and more general than, those of the previous section; indeed the two approaches, although not identical, are quite similar.

3'1. *Basic ansatz and formulae.* – Let us consider a polynomial of degree n in x, with n zeros at the positions $x_j(t)$:

$$\psi(x, t) = \prod_{j=1}^{n} [x - x_j(t)] \, . \tag{3.1.1}$$

This representation immediately implies the following formulae:

$$\psi_x = \psi \sum_{j=1}^{n} (x - x_j)^{-1} , \tag{3.1.2}$$

$$\psi_t = -\psi \sum_{j=1}^{n} (x - x_j)^{-1} \dot{x}_j , \tag{3.1.3}$$

$$\psi_{xx} = 2\psi \sum_{j=1}^{n} (x - x_j)^{-1} \sum_{k=1}^{n}{}' (x_j - x_k)^{-1} , \tag{3.1.4}$$

$$(3.1.5)\qquad \psi_{xt} = -\,\psi \sum_{j=1}^{n} (x - x_j)^{-1} {\sum_{k=1}^{n}}' (\dot x_j + \dot x_k)/(x_j - x_k)\,,$$

$$(3.1.6)\qquad \psi_{tt} = \psi \sum_{j=1}^{n} (x - x_j)^{-1}\Big[-\ddot x_j + 2\dot x_j {\sum_{k=1}^{n}}' \dot x_k/(x_j - x_k)\Big]\,,$$

$$(3.1.7)\qquad x\psi_x - n\psi = \psi \sum_{j=1}^{n} (x - x_j)^{-1} x_j\,,$$

$$(3.1.8)\qquad x\psi_{xx} = 2\psi \sum_{j=1}^{n} (x - x_j)^{-1} x_j {\sum_{k=1}^{n}}' (x_j - x_k)^{-1}\,,$$

$$(3.1.9)\qquad x\psi_{xt} = -\,\psi \sum_{j=1}^{n} (x - x_j)^{-1} x_j {\sum_{k=1}^{n}}' (\dot x_j + \dot x_k)/(x_j - x_k)\,,$$

$$(3.1.10)\qquad x^2\psi_{xx} - n(n-1)\,\psi = 2\psi \sum_{j=1}^{n} (x - x_j)^{-1} x_j^2 {\sum_{k=1}^{n}}' (x_j - x_k)^{-1}\,,$$

$$(3.1.11)\qquad x[x^2\psi_{xx} - 2(n-1)x\psi_x + n(n-1)\,\psi] = 2\psi \sum_{j=1}^{n} (x - x_j)^{-1} x_j^2 {\sum_{k=1}^{n}}' x_k/(x_j - x_k)\,,$$

$$(3.1.12)\qquad x[x\psi_{xt} - (n-1)\,\psi_t] = -\,\psi \sum_{j=1}^{n} (x - x_j)^{-1} x_j {\sum_{k=1}^{n}}' (\dot x_j x_k + \dot x_k x_j)/(x_j - x_k)\,.$$

In all these equations of course $\psi \equiv \psi(x, t)$ and $x_j \equiv x_j(t)$.

Now the procedure to relate many-body problems to linear partial differential equations is quite straightforward; indeed the assumption that a linear combination of the left-hand sides of eqs. (3.1.2)-(3.1.12) vanishes corresponds to the requirement that the ψ satisfies a linear partial differential equation (whose consistency with the original ansatz (3.1.1) must be ascertained); while the requirement that the same linear combination of the right-hand sides of these equations vanishes yields generally a set of «equations of motion» for the quantities $x_j(t)$, *i.e.* a many-body problem if these quantities (or others simply related to them; see below) are interpreted as particle co-ordinates. The most general many-body problem obtainable in this manner is clearly

$$(3.1.13)\qquad C\ddot x_j + E\dot x_j = B_0 + B_1 x_j + {\sum_{k=1}^{n}}' (x_j - x_k)^{-1}\cdot$$
$$\cdot[2(A_0 + A_1 x_j + A_2 x_j^2 + A_3 x_j^2 x_k) + 2C\dot x_j \dot x_k -$$
$$-(\dot x_j + \dot x_k)(D_0 + D_1 x_j) - D_2 x_j(\dot x_j x_k + \dot x_k x_j)]\,,\qquad j = 1, 2, \ldots, n,$$

corresponding to the linear partial differential equation

$$(3.1.14)\qquad [A_0 + A_1 x + A_2 x^2 + A_3 x^3]\,\psi_{xx} +$$
$$+ [B_0 + B_1 x - 2(n-1)A_3 x^2]\,\psi_x + C\psi_{tt} + [E - (n-1)D_2 x]\,\psi_t +$$
$$+ [D_0 + D_1 x + D_2 x^2]\,\psi_{xt} - [n(n-1)(A_2 - A_3 x) + nB_1]\,\psi = 0\,.$$

It is easy to verify that a polynomial solution of type (3.1.1) is indeed consistent with this differential equation.

The initial conditions for the equations of motion (3.1.13) prescribe the initial values $x_j(0)$, $\dot{x}_j(0)$, $j = 1, 2, \ldots, n$, for the « co-ordinates » $x_j(t)$ and the « velocities » $\dot{x}_j(t)$. These data also specify the initial conditions for the linear differential equation (3.1.14), since (3.1.1) and (3.1.3) imply

$$\psi(x, 0) = \prod_{j=1}^{n} [x - x_j(0)] , \tag{3.1.15a}$$

$$\psi_t(x, 0) = - \psi(x, 0) \sum_{j=1}^{n} [x - x_j(0)]^{-1} \dot{x}_j(0) . \tag{3.1.15b}$$

It is thus seen that the solutions $x_j(t)$ of the n-body problem characterized by the equations of motion (3.1.13) and by the initial conditions $x_j(0)$, $\dot{x}_j(0)$, $j = 1, 2, \ldots, n$, coincide with the n zeros of the solution of the linear partial differential equation (3.1.14) with initial conditions (3.1.15), a solution that has exactly n zeros, since it is indeed just a polynomial of degree n in x. Note that there is no constraint on the initial data (3.1.15), except possibly some inequality to guarantee that the zeros of the polynomial $\psi(x, t)$ remain always real, or equivalently to avoid the occurrence of collapse in the many-body problem (3.1.13) (see below).

The many-body problem (3.1.13), and the differential equation (3.1.14) related to it, are too general to yield a transparent physical picture; more interesting cases obtain by special choices of the quantities A_s, B_s, C, D_s and E, as discussed in subsequent subsections. Yet it is even possible to obtain more general models. Indeed the following formulae are also implied by the ansatz (3.1.1):

$$x[x\psi_x - n\psi] = \psi\Big[nX + \sum_{j=1}^{n} (x - x_j)^{-1} x_j^2\Big] , \tag{3.1.16}$$

$$x\psi_t = - \psi\Big[n\dot{X} + \sum_{j=1}^{n} (x - x_j)^{-1} \dot{x}_j x_j\Big] , \tag{3.1.17}$$

$$x\psi_{tt} = - \psi\Big\{n\ddot{X} + \sum_{j=1}^{n} (x - x_j)^{-1} \Big[\ddot{x}_j x_j - 2\dot{x}_j x_j \sum_{k=1}^{n}{}' \dot{x}_k/(x_j - x_k)\Big]\Big\}, \tag{3.1.18}$$

$$\begin{aligned} x^2[x^2\psi_{xx} - 2(n-1)\, x\psi_x + n(n-1)\, \psi] &= \\ &= \psi\Big[(nX)^2 - nX_2 + 2 \sum_{j=1}^{n} (x - x_j)^{-1} x_j^3 \sum_{k=1}^{n}{}' x_k/(x_j - x_k)\Big] , \end{aligned} \tag{3.1.19}$$

$$\begin{aligned} x^2[x\psi_{xt} - (n-1)\, \psi_t] = - \psi\Big[n^2\dot{X}X &+ \tfrac{1}{2} n\dot{X}_2 + \\ &+ \sum_{j=1}^{n} (x - x_j)^{-1} x_j^2 \sum_{k=1}^{n}{}' (\dot{x}_j x_k + \dot{x}_k x_j)/(x_j - x_k)\Big] ; \end{aligned} \tag{3.1.20}$$

they involve the collective co-ordinates

$$X(t) = n^{-1}\sum_{j=1}^{n} x_j(t)\,, \tag{3.1.21}$$

$$X_2(t) = n^{-1}\sum_{j=1}^{n} x_j^2(t)\,. \tag{3.1.22}$$

Thus the procedure indicated above can be applied by using formulae (3.1.16)-(3.1.20) in addition to (3.1.1)-(3.1.12). In this manner, one relates a linear partial differential equation more general than (3.1.14) to a many-body problem characterized by equations of motion more general than (3.1.13) and by one additional equation involving the collective co-ordinates X, X_2, whose consistency with the equations of motion for the co-ordinates x_j is a necessary requirement for the validity of the scheme. This condition must be verified in each case and, when it can be satisfied, it generally implies some constraint on the initial values of the collective co-ordinates and/or on their initial velocities. Merely to prove that such models do exist, we treat here tersely one such example. It is characterized by the equations of motion

$$\ddot{x}_j = \alpha\dot{x}_j x_j + 2\dot{x}_j \sum_{k=1}^{n}{}' \dot{x}_k/(x_j - x_k) \tag{3.1.23}$$

and by the constraint

$$\dot{X}(t) = 0\,. \tag{3.1.24}$$

This constraint is compatible with the equations of motion (3.1.23), provided the initial data are such that

$$\dot{X}(0) = 0\,, \qquad \dot{X}_2(0) = 0\,. \tag{3.1.25}$$

In fact, it is easily seen that (3.1.23) imply

$$\ddot{X} = \tfrac{1}{2}\alpha\dot{X}_2\,, \tag{3.1.26}$$

$$\ddot{X}_2 = \alpha\dot{X}_2\,. \tag{3.1.27}$$

The first of these two equations follows immediately by summing (3.1.23) over j; the second obtains with a little algebra by multiplying (3.1.23) by x_j and then by summing over j. Clearly, together with (3.1.25), they imply (3.1.24).

Thus, provided the initial data are constrained by (3.1.25), the many-body model characterized by the equations of motion (3.1.23) is solvable via the linear partial differential equation

$$\psi_{tt} + \alpha x\psi_t = 0\,, \tag{3.1.28}$$

that clearly corresponds through (3.1.6) and (3.1.17) to (3.1.23) and (3.1.24). In fact, using the explicit solution of this equation

$$\psi(x, t) = \psi(x, 0) + \psi_t(x, 0)[1 - \exp[-\alpha x t]]/\alpha x \tag{3.1.29}$$

and (3.1.3), one immediately concludes that the solutions $x_j(t)$ of (3.1.23) corresponding to the initial data $x_j(0)$, $\dot{x}_j(0)$ (such that (3.1.25) hold) coincide with the roots of the transcendental equation in x

$$\sum_{j=1}^{n} [x - x_j(0)]^{-1} \dot{x}_j(0) = \alpha x/[1 - \exp[-\alpha x t]] . \tag{3.1.30}$$

In this manner the many-body problem has been reduced to the solution of a single nondifferential equation. But it is not here the place to pursue the analysis of this model.

Yet other formulae may be obtained by multiplying by x those given above; for instance, from (3.1.16)-(3.1.18) it follows that

$$x^2[x\psi_x - n\psi] = \psi\Big[xnX + nX_2 + \sum_{j=1}^{n} (x - x_j)^{-1} x_j^3\Big] , \tag{3.1.31}$$

$$x^2 \psi_t = - \psi\Big[xn\dot{X} + \tfrac{1}{2} n\dot{X}_2 + \sum_{j=1}^{n} (x - x_j)^{-1} \dot{x}_j x_j^2\Big] , \tag{3.1.32}$$

$$x^2 \psi_{tt} = - \psi\Big\{xn\ddot{X} - (n\dot{X})^2 + \tfrac{1}{2} n\ddot{X}_2 + \\ + \sum_{j=1}^{n} (x - x_j)^{-1}\Big[\ddot{x}_j x_j^2 - 2\dot{x}_j x_j^2 \sum_{k=1}^{n}{}' \dot{x}_k/(x_j - x_k)\Big]\Big\} . \tag{3.1.33}$$

Using these formulae one may obtain still more general models, but the number of constraints to be satisfied also tends to increase.

Another possible extension of the approach is via the consideration of higher derivatives of ψ, for instance

$$\psi_{xxx} = 2\psi \sum_{j=1}^{n} (x - x_j)^{-1} \sum_{k=1}^{n}{}' (x_j - x_k)^{-1} \sum_{l=1}^{n}{}' [(x_j - x_l)^{-1} + (x_k - x_l)^{-1}] . \tag{3.1.34}$$

But the presence of a triple sum on the r.h.s. of this equation yields, through the procedure indicated above, equations of motion that, when interpreted as representing a many-body problem, involve the presence of three-body forces; and clearly the inclusion of a derivative of ψ of order m leads to m-body forces. Since we want to limit our consideration to many-body models with external potentials (one-body forces) and two-body forces, we do not consider these cases in the following.

It should be emphasized that there is no *a priori* requirement that the quantities A_s, B_s, C, D_s or E be time independent, although in most of the following this shall for simplicity be assumed.

We finally mention an extension of the ansatz (3.1.1) that leads to a further enlargement of the class of many-body problems solvable by these techniques. This is the position

$$\psi(x, t) = \exp\left[\int_{t_0}^{t} \mathrm{d}t' f(t')\right] \prod_{j=1}^{n} [x - x_j(t)] , \tag{3.1.35}$$

implying the formulae

$$\psi_x = \psi \sum_{j=1}^{n} (x - x_j)^{-1} , \tag{3.1.36}$$

$$\psi_t = \psi\left[f - \sum_{j=1}^{n} (x - x_j)^{-1} \dot{x}_j\right] , \tag{3.1.37}$$

$$\psi_{xx} = 2\psi \sum_{j=1}^{n} (x - x_j)^{-1} {\sum_{k=1}^{n}}' (x_j - x_k)^{-1} , \tag{3.1.38}$$

$$\psi_{xt} = \psi \sum_{j=1}^{n} (x - x_j)^{-1}\left[f - {\sum_{k=1}^{n}}' (\dot{x}_j + \dot{x}_k)/(x_j - x_k)\right] , \tag{3.1.39}$$

$$\psi_{tt} = \psi\left\{\dot{f} + f^2 + \sum_{j=1}^{n} (x - x_j)^{-1}\left[- 2f - \ddot{x}_j + 2\dot{x}_j {\sum_{k=1}^{n}}' \dot{x}_k/(x_j - x_k)\right]\right\}, \tag{3.1.40}$$

$$x\psi_x - n\psi = \psi \sum_{j=1}^{n} (x - x_j)^{-1} x_j , \tag{3.1.41}$$

$$x\psi_{xx} = 2\psi \sum_{j=1}^{n} (x - x_j)^{-1} x_j {\sum_{k=1}^{n}}' (x_j - x_k)^{-1} , \tag{3.1.42}$$

$$x\psi_{xt} = \psi\left\{nf + \sum_{j=1}^{n} (x - x_j)^{-1}\left[x_j f - x_j {\sum_{k=1}^{n}}' (\dot{x}_j + \dot{x}_k)/(x_j - x_k)\right]\right\} \tag{3.1.43}$$

and so on. Because the results flowing from the (simpler) ansatz (3.1.1) are already sufficiently rich (also in the light of the remark reported just above), we shall not exploit in the following the possibilities implied by the more general ansatz (3.1.35); there should be no difficulty for the interested reader to derive such results by using the techniques of this subsection and of those that follow.

3·2. *Equations of motion of first order.* – In this subsection we discuss tersely some models that obtain from (3.1.13) when $C = 0$, so that the equations (3.1.13) become of first order. By differentiating these equations and then using them to eliminate the first derivatives, it is of course generally possible to obtain also equations of second order that are, therefore, again similar to the equations of motion of an ordinary many-body problem. These equations of motion contain, however, generally also three-body forces, unless a cancellation occurs. We focus below just on two cases in which such cancellation does occur.

As already pointed out above, all the models that come under the heading of this subsection, when re-interpreted as many-body problems characterized by second-order equations, suffer of one drawback: while the initial positions are arbitrary, the initial velocities are not (they are given in terms of the initial co-ordinates by the first-order equations). Thus these models are characterized by the presence of constraints that determine the initial velocities in terms of the initial positions.

The first model obtains from (3.1.13) when $C = D_0 = D_1 = D_2 = A_1 = = A_2 = A_3 = 0$. We also set, for notational convenience, $B_0 = 0$, $B_1 = \omega$, $A_0 = \frac{1}{2} g$, $E = 1$. The equations of motion derived from (3.1.13) then read

$$\ddot{x}_j = \omega^2 x_j - 2g^2 \sum_{k=1}^{n}{}' (x_j - x_k)^{-3}, \qquad j = 1, 2, \ldots, n, \tag{3.2.1}$$

and the constraints on the initial conditions read

$$\dot{x}_j(0) = \omega x_j(0) + g \sum_{k=1}^{n}{}' (x_j - x_k)^{-1}, \qquad j = 1, 2, \ldots, n. \tag{3.2.2}$$

The corresponding partial differential equation reads

$$g\psi_{xx} + 2\omega x \psi_x + 2\psi_t - 2n\omega\psi = 0. \tag{3.2.3}$$

The equations of motion (3.2.1) are of course derivable from the Hamiltonian

$$H = \tfrac{1}{2} \sum_{j=1}^{n} (p_j^2 - \omega^2 x_j^2) - g^2 \sum_{j=2}^{n} \sum_{k=1}^{j-1} (x_j - x_k)^{-2}. \tag{3.2.4}$$

Note, however, that the potentials have the « wrong » sign; the singular two-body potential is attractive; the oscillator single-particle potential is repulsive. To change their signs, one should assume ω and g to be imaginary; but this is forbidden by the constraint (3.2.2) (assuming that the x_j's are real).

For $\omega = 0$, these results coincide with previous findings of CC.

The explicit solution of this many-body model could be easily discussed on the basis of (3.2.3), by using techniques such as those discussed below. But, since the solution of the many-body model (3.2.1) (even in the general case without the constraint (3.2.21)) has been extensively analysed in the literature [5,6], we skip here any further discussion of this model.

The second model that we discuss in this section obtains by setting $C = D_0 = D_1 = D_2 = A_0 = A_1 = A_3 = B_0 = 0$. We, moreover, set for notational convenience $E = 1$, $B_1 = 2B$, $A_2 = g$, and we perform the change of variables

$$x_j(t) = \exp[2z_j(t)]\,. \tag{3.2.5}$$

Then (3.1.31) yield

$$\dot{z}_j = B + g \sum_{k=1}^{n}{}' \exp[z_j - z_k]/\sinh(z_j - z_k)\,, \qquad j = 1, 2, \ldots, n, \tag{3.2.6}$$

and it is easily seen that these equations imply

$$\ddot{z}_j = -g^2 \sum_{j=1}^{n}{}' \sinh^{-3}(z_j - z_k)\cosh(z_j - z_k)\,, \qquad j = 1, 2, \ldots, n. \tag{3.2.7}$$

These are just the equations of motion derivable from the Hamiltonian

$$H = \tfrac{1}{2}\sum_{j=1}^{n} p_j^2 - 2g^2 \sum_{j=1}^{n}\sum_{k=1}^{j-1} \sinh^{-2}(z_j - z_k)\,, \tag{3.2.8}$$

that has been the subject of much recent study. Note, however, that with the present technique we obtain a model characterized by singular attractive interactions; and of course this technique is applicable only provided the initial velocities $\dot{z}_j(0)$ are related to the (arbitrary) initial positions by the constraints (3.2.6) (with B being an arbitrary constant, simply related to the (constant) speed of the centre of mass of the system; see below).

The partial differential equation connected to this many-body problem reads

$$gx^2\psi_{xx} + 2Bx\psi_x + \psi_t - [n(n-1)g + 2nB]\,\psi = 0\,. \tag{3.2.9}$$

It is immediately seen that the solution of this equation can be explicitly represented as follows:

$$\psi(x, t) = \sum_{m=0}^{n} c_m(t)\, x^m \tag{3.2.10}$$

with

$$c_m(t) = c_m(0)\exp\big[(n-m)[(n+m-1)g + 2B]\,t\big]\,, \qquad m = 0, 1, 2, \ldots, n. \tag{3.2.11}$$

Here $c_n(0) = 1$ (and, therefore, also $c_n(t) = 1$), while the n constants $c_m(0)$, $m = 0, 1, \ldots, n-1$, are fixed by the requirement that, at $t = 0$, the polynomial (3.2.10) has the n positive zeros $x_j(0) = \exp[2z_j(0)]$. On the other hand, the solutions $z_j(t)$ of the many-body problem (3.2.7) (with the constraints (3.2.6)) are given, through (3.2.5), by the time evolution of the zeros $x_j(t)$ of the explicit polynomial (3.2.10).

If we assume that the initial conditions are such to exclude the future occurrence of collapse, it is easy to find, from (3.2.10), the asymptotic behaviour of the particles. We find

$$z_j(t) \approx vt + a_j\,, \qquad j = 1, 2, \dots, n,\ (t \to +\infty), \tag{3.2.12}$$

with

$$v_j = B + (j-1)\,g\,, \tag{3.2.13}$$

$$a_j = \tfrac{1}{2}\ln[-c_{j-1}(0)/c_j(0)]\,. \tag{3.2.14}$$

These results obtain by noting that, in searching for the zeros of ψ as $t \to +\infty$, only two terms can be important on the r.h.s. of (3.2.10), and they must cancel exactly. The same analysis cannot be done for $t \to -\infty$, since necessarily collapse occurs in such an extrapolation if the initial data are such to exclude its occurrence for $t > 0$. Note that the above analysis requires $g \geqslant 0$ and implies that the centre of mass of the system moves with the speed $B + \frac{1}{2}(n-1)\,g$, a finding that is consistent with the constraints (3.2.6).

We finally note that the treatment given above applies even if B is time dependent, say

$$B = f(t)\,. \tag{3.2.15}$$

Then in place of (3.2.7) one has the equation of motion

$$\ddot{z}_j = \dot{f}(t) - g^2 \sum_{k=1}^{n}{}' \sinh^{-3}(z_j - z_k)\cosh(z_j - z_k)\,, \tag{3.2.16}$$

containing the (arbitrary) forcing term $\dot{f}(t)$, while (3.2.11) is replaced by the formula

$$c_m(t) = c_m(0)\exp\Big[(n-m)\Big[(n+m-1)\,gt + 2\int_0^t \mathrm{d}t' f(t')\Big]\Big]\,, \qquad m = 0, 1, 2, \dots, n. \tag{3.2.17}$$

3'3. *Some translation-invariant many-body models.* – In this subsection we consider the models characterized by the equations of motion (3.1.13) with $C \neq 0$ (so that these equations of motion are second order in time, and necessarily contain «velocity-dependent» forces), but with $A_1 = A_2 = A_3 = B_1 = D_1 = D_2 = 0$ (so that these equations of motion are translation invariant, namely such that, if $x_j(t)$, $j = 1, 2, \dots, n$, is a solution, $\tilde{x}_j(t) = x_j(t) + a$ is also a solution). Setting for notational convenience $A_0 = \lambda$, $B_0 = g$, $C = 1$, $D_0 = 2\mu$, $E = \alpha$ in (3.1.13), we thus get the equations of motion

$$\ddot{x}_j = g - \alpha\dot{x}_j + 2\sum_{k=1}^{n}{}' [\lambda - \mu(\dot{x}_j + \dot{x}_k) + \dot{x}_j\dot{x}_k]/(x_j - x_k)\,, \quad j = 1, 2, \dots, n, \tag{3.3.1}$$

where of course $x_j \equiv x_j(t)$. Note that now there is no constraint, neither on the initial positions nor on the initial velocities.

The many-body problem characterized by these equations of motion is similar to, but more general than, the problems discussed in subsect. **2'1**. Indeed for $g = \alpha$, $\lambda = \mu = 1$, the equations of motion (2.1.10) are reproduced; and for $g = \lambda = \mu = 0$, $x_j(t) = y_j(t)$, one gets instead the equations of motion (2.1.21) (this case is treated at the end of this subsection). By performing a scale transformation of the dependent variables x_j and of the independent variable t, in addition to a « Galilei » transformation to a frame moving with speed u, one re-obtains the eqs. (3.3.1), but with constants g, α, λ and μ replaced by

(3.3.2a) $$g' = (g - \alpha u)\,\varrho\sigma\,,$$

(3.3.2b) $$\alpha' = \sigma\alpha\,,$$

(3.3.2c) $$\lambda' = (\lambda - 2u\mu + u^2)\,\varrho^2\,,$$

(3.3.2d) $$\mu' = (\mu - u)\,\varrho\,,$$

where ϱ and σ are two nonvanishing constants. Note that there is no choice of ϱ, σ and u that reduces, via this transformation (3.3.21), the equations of motion (3.3.1) to (2.1.10), unless $\mu = g/\alpha$ and $\lambda = \mu^2$. Thus, in general, (3.3.1) cannot be trivially reduced to (2.1.10). In the following we keep all four constants g, α, λ and μ, although of course some of them could be replaced by unity, or made vanish, by the transformation (3.3.2).

Before proceeding with the discussion, it is worth re-emphasizing that, in spite of the similarity of the many-body models discussed in subsect. **2'1** and in this subsection and of the fact that in both cases the solution is achieved through the consideration of an associated partial differential equation, the two approaches are not identical; indeed, although in both cases the motion of the particles $x_j(t)$ coincides with the time evolution of the zeros of a function $\psi(x, t)$ satisfying a linear partial differential equation, in the case discussed in this subsection (and the two that precede it, as well as the two that follow it) ψ is a polynomial, while in the case of subsect. **2'1** it is a rational function.

The partial differential equation corresponding to the many-body model (3.3.1) reads as follows:

(3.3.3) $$\lambda\psi_{xx} + g\psi_x + \psi_{tt} + 2\mu\psi_{xt} + \alpha\psi_t = 0\,.$$

The correspondence is of course through the representation (3.1.1) of a (properly normalized) polynomial solution of this equation.

If n is small (« few-body problem »), the most convenient technique to solve (3.3.1) is by an explicit analysis of the time evolution of the zeros of the polynomial

$$\psi(x,t) = n!\sum_{m=0}^{n} c_m(t)\, x^m/m! \tag{3.3.4}$$

with the quantities $c_m(t)$ characterized by the following equations:

$$n!\sum_{m=0}^{n} c_m(0)\, x^m/m! = \prod_{j=1}^{n}[x - x_j(0)] = \psi(x,0)\,, \tag{3.3.5a}$$

$$n!\sum_{m=0}^{n-1} \dot{c}_m(0)\, x^m/m! = -\,\psi(x,0)\sum_{j=1}^{n}[x - x_j(0)]^{-1}\dot{x}_j(0)\,, \tag{3.3.5b}$$

$$\ddot{c}_m(t) + \alpha\dot{c}_m(t) = -\,\lambda c_{m+2}(t) - g c_{m+1}(t) - 2\mu\dot{c}_{m+1}(t)\,, \qquad m = 0, 1, 2, \ldots, n-2, \tag{3.3.6a}$$

$$\ddot{c}_{n-1}(t) + \alpha\dot{c}_{n-1}(t) = -\,g\,, \tag{3.3.6b}$$

$$c_n(t) = 1\,. \tag{3.3.6c}$$

Conditions (3.3.5) determine the initial values of $c_m(0)$ and $\dot{c}_m(0)$, $m = 0, 1, 2, \ldots, n-1$, in terms of the initial positions $x_j(0)$ and velocities $\dot{x}_j(0)$; they correspond to the requirement that (3.3.4) satisfies, at $t = 0$, (3.1.1) and (3.1.3). Equations (3.3.6), that can be solved by recursion starting from (3.3.6c) and (3.3.6b) and by proceeding then to obtain sequentially, through (3.3.6a), $c_m(t)$ for $m = n-2, n-3, \ldots, 0$, are of course equivalent, through (3.3.4), to the requirement that $\psi(x,t)$ satisfies (3.3.3).

Note, incidentally, that (3.3.4) and (3.1.1) imply the relationship

$$c_{n-1}(t) = -\,X(t) = -\,n^{-1}\sum_{j=1}^{n} x_j(t)\,, \tag{3.3.7}$$

and it is easily seen that (3.3.6b) is consistent with the equation for the centre-of-mass co-ordinate $X(t)$ that obtains by summing the n equations of motion (3.3.1).

Of course, as discussed in previous subsections, the zeros $x_j(t)$ of $\psi(x,t)$ need not remain real; they may collide (« collapse ») and then move into the complex plane. That this need not happen is implied by the discussion of subsect. 2·1. However, as now we show, there is a large class of problems in which, by an explicit solution of the differential equation (3.3.3), it is possible to conclude that collapse must necessarily occur.

Consider in particular problem (3.3.1) with $\alpha = g = 0$:

$$\ddot{x}_j = 2\sum_{k=1}^{n}{}'\,[\lambda - \mu(\dot{x}_j + \dot{x}_k) + \dot{x}_j\dot{x}_k]/(x_j - x_k)\,, \qquad j = 1, 2, \ldots, n. \tag{3.3.8}$$

If $\lambda = \mu^2$, the Galilei transformation $x_j = y_j + \mu t$ yields for the y_j's just the equations of motion (2.1.21), that have been extensively discussed in subsect. **2·1**; thus we do not discuss this case here (except in a special subcase, see below; but let us recall that in this case, $\lambda = \mu^2$, the results of subsect. **2·1** imply that there is a large class of initial conditions that exclude the occurrence of collapse). If instead $\lambda \neq \mu^2$, we now show that collapse is almost the rule; indeed, for $n > 2$, it certainly occurs if the centre-of-mass speed (that is clearly a constant of the motion for the system (3.3.8)) does not coincide with v_+ or v_-, defined by the formula

$$v_\pm = \mu \pm (\mu^2 - \lambda)^{\frac{1}{2}}; \tag{3.3.9}$$

such a coincidence is of course possible only if $\mu^2 > \lambda$, so that v_+ and v_- are real.

This result is not evident from the structure of the equations of motion (3.3.8). We prove it using the explicit solution of (3.3.3); indeed this equation for $g = \alpha = 0$ reads

$$\lambda \psi_{xx} + \psi_{tt} + 2\mu \psi_{xt} = 0 \tag{3.3.10}$$

and it admits, therefore, the general solution

$$\psi(x, t) = f_+(x - v_+ t) + f_-(x - v_- t), \tag{3.3.11}$$

where $f_+(z)$ and $f_-(z)$ are arbitrary functions and v_+, v_- are given by (3.3.9).

In our case the functions f_+ and f_- are determined by the requirement that $\psi(x, 0)$ and $\psi_t(x, 0)$ be given, in terms of the initial positions $x_j(0)$ and $\dot{x}_j(0)$, by the expressions (3.1.1) and (3.1.3). It is actually convenient to use for ψ the representation (3.3.4), which, used in conjunction with (3.3.11), yields, after a little algebra, the explicit formula

$$\begin{aligned}\psi(x, t) = c_0(0) + \tfrac{1}{2}(\mu^2 - \lambda)^{-\frac{1}{2}} n!\,\cdot \\ \cdot \sum_{m=1}^{n} \{[c_m(0)\, v_+ + \dot{c}_{m-1}(0)](x - v_- t)^m - [c_m(0)\, v_- + \dot{c}_{m-1}(0)](x - v_+ t)^m\}/m!,\end{aligned} \tag{3.3.12}$$

where of course $c_m(0)$ and $\dot{c}_m(0)$, $m = 0, 1, 2, \ldots, n-1$, are determined by the initial positions $x_j(0)$ and velocities $\dot{x}_j(0)$ through (3.3.5), while $c_n(0) = 1$, $\dot{c}_n(0) = 0$.

Thus the solutions $x_j(t)$ of (3.3.7) are the zeros of the explicit polynomial (of degree n in x) (3.3.12); note that, for real x, this polynomial is real, since the coefficients $c_m(0)$ and $\dot{c}_m(0)$ are real, while v_+ and v_- are real if $\mu^2 > \lambda$, complex conjugate if $\mu^2 < \lambda$.

For $t \to \pm \infty$, clearly

$$x_j(t) \approx v_j(\pm \infty)\, t + a_j(\pm \infty), \qquad j = 1, 2, \ldots, n, \tag{3.3.13}$$

where the sets $\{v_j(+\infty)\}$ and $\{v_j(-\infty)\}$ coincide with the set $\{v_j\}$ of the n solutions of the algebraic equation of degree n in v

$$[v_+ + \dot{c}_{n-1}(0)](v - v_-)^n = [v_- + \dot{c}_{n-1}(0)](v - v_+)^n . \tag{3.3.14a}$$

Note that this does not imply $v_j(+\infty) = v_j(-\infty)$, although it implies that the set $\{v_j(+\infty)\}$ coincides with the set $\{v_j(-\infty)\}$: the set of the particle speeds in the remote past coincides with the set of the particle speeds in the remote future, although each particle need not move in the extreme future with the same speed it had in the remote past ([25]).

Using (3.3.7), one may rewrite (3.3.14a) in the more elegant form

$$[(v - v_-)/(v - v_+)]^n = (V - v_-)/(V - v_+) , \tag{3.3.14b}$$

where

$$V \equiv \dot{X}(0) = \dot{X}(t) = n^{-1} \sum_{j=1}^{n} \dot{x}_j(t) = - \dot{c}_{n-1}(0) \tag{3.3.15}$$

is the (constant) speed of the centre of mass of the system.

Now it is clear that, if $V = v_-$ or $V = v_+$, (3.3.14) yields n equal solutions $v = V$; thus, in the special cases with initial conditions such that $V = v_-$ or $V = v_+$ (possible, as noted before, only if $\mu^2 > \lambda$), all particles move in the remote past and in the remote future (almost) ([26]) with the same speed V. If instead V does not coincide either with v_- or with v_+, the asymptotic velocities are given by the formula

$$v_j = (v_- - \eta_j v_+)/(1 - \eta_j) , \qquad j = 1, 2, \dots, n, \tag{3.3.16}$$

where the quantities

$$\eta_j = |(V - v_-)/(V - v_+)|^{1/n} \exp [i(\theta + 2\pi j)/n] \tag{3.3.17}$$

are the n-th roots of ([27])

$$(V - v_-)/(V - v_+) \equiv |(V - v_-)/(V - v_+)| \exp [i\theta] . \tag{3.3.18}$$

([25]) This is a familiar phenomenon for the solvable many-body problems recently considered, that correspond to integrable dynamical systems; see the papers of ref. ([4-6]).

([26]) Note that, if the speeds of all particles were to coincide, at any finite time, with v_+ or with v_-, this would imply that they remain constant throughout the motion, so that in such a case the whole system would move as a solid body, without any internal motion. This follows by inspection from the equations of motion (3.3.8).

([27]) So that, if v_+ and v_- are real, $\theta = 0$, while, if v_+ and v_- are complex conjugate, $|(V - v_+)/(V - v_-)| = 1$.

Clearly, if $n > 2$, the v_j's cannot be all real; thus collapse must have occurred.

The possibility to obtain explicitly the asymptotic speeds is nevertheless remarkable, and suggests that it may be of interest to consider the model obtained from (3.3.8) by complexification, according to the procedure described at the end of subsect. **2**'4. For the two-dimensional many-body model thus obtained, the occurrence of collapse would be the exception rather than the rule; yet all the analytic results described above would remain valid, including the property that the two sets of asymptotic (as $t \to \pm \infty$) speeds coincide ([28]).

We end this subsection showing how simply the results of subsect. **2**'1 can be re-obtained in the present framework. Let $g = \lambda = \mu = 0$, so that, except for the renaming of the variables $x_j(t)$ as $y_j(t)$, (3.3.1) coincide with (2.1.21). Corresponding to this choice of the parameters, the partial differential equation (3.3.3) takes the very simple form

$$\psi_{tt} + \alpha\psi_t = 0 \tag{3.3.19}$$

and is, therefore, immediately solved by the explicit formula

$$\psi(x, t) = \psi(x, 0) + [1 - \exp[-\alpha t]]\, \psi_t(x, 0)/\alpha \,. \tag{3.3.20}$$

Thus, using (3.1.3), one immediately concludes that the zeros of $\psi(x, t)$ are the solutions of the algebraic equation in x

$$\sum_{j=1}^{n} [x - x_j(0)]^{-1} \dot{x}_j(0) = \alpha/[1 - \exp[-\alpha t]] \,, \tag{3.3.21}$$

which coincides with (2.1.24) (as indeed it should).

3'4. *Other translation-invariant models.* – Another class of translation-invariant models obtains from (3.1.13) by setting $B_0 = A_0 = A_1 = A_3 = D_0 = D_2 = 0$ (so that the equations of motion for the x_j's are homogeneous), and then by using the same change of dependent variables already used at the end of subsect. **2**'1, namely

$$x_j(t) = \exp[z_j(t)] \,, \qquad j = 1, 2, \dots, n \,. \tag{3.4.1}$$

Essentially this same procedure was used in the second part of subsect. **3**'2.

Setting for notational convenience $C = 1$, $E = \alpha$, $B_1 = g$, $A_2 = \lambda$, $D_1 = 2\mu$,

([28]) It is easily seen that the same property also holds for the quantities $a_j(\pm \infty)$ of eq. (3.3.13), whose explicit expression can be easily obtained.

we thus get for the co-ordinates $x_j(t)$ the equations of motion

$$\ddot{x}_j = gx_j - \alpha\dot{x}_j + 2\sum_{k=1}^{n}{}' [\lambda x_j^2 - \mu x_j(\dot{x}_j + \dot{x}_k) + \dot{x}_j\dot{x}_k]/(x_j - x_k)\,, \qquad j = 1, 2, \ldots, n, \tag{3.4.2}$$

and these imply for the co-ordinates $z_j(t)$ the translation-invariant equations of motion

$$\ddot{z}_j = -\dot{z}_j^2(t) + G - A\dot{z}_j + 2\sum_{k=1}^{n}{}' [\lambda - \mu(\dot{z}_j + \dot{z}_k) + \dot{z}_j\dot{z}_k]/[\exp[z_j - z_k] - 1]\,, \qquad j = 1, 2, \ldots, n, \tag{3.4.3}$$

where

$$G = g + 2(n-1)\lambda\,, \tag{3.4.4}$$

$$A = \alpha + 2(n-1)\mu\,. \tag{3.4.5}$$

Clearly the study of the equations of motion (3.4.2) is essentially equivalent to the study of (3.4.3). The version (3.4.3) of the equations of motion may, however, be more appealing as a many-body model because of its translation-invariant nature; but it features a two-body force that is not an odd function of the interparticle distance. Both models are characterized by the possible occurrence of collapse, whenever the co-ordinates of two particles coincide. The co-ordinates z_j may, moreover, exist at $-\infty$ at a finite time, corresponding to the vanishing of x_j, as implied by (3.4.1); in the framework of the many-body model (3.4.3) the « cause » of such a possible divergence should be attributed to the presence of a quadratic velocity-dependent force (the first term on the r.h.s. of (3.4.3)).

In the following we investigate tersely the time evolution of these many-body models starting from initial conditions such to exclude the (future) occurrence of any divergence. Clearly such initial conditions do exist (see below).

The partial differential equation associated to the equations of motion (3.4.2) reads as follows:

$$\lambda x^2\psi_{xx} + gx\psi_x + \psi_{tt} + 2\mu x\psi_{xt} + \alpha\psi_t - [n(n-1)\lambda + ng]\psi = 0\,. \tag{3.4.6}$$

The most convenient way to study the time evolution of a solution of this equation that is a polynomial of degree n in x is through the ansatz

$$\psi(x, t) = \sum_{m=0}^{n} c_m(t)\, x^m\,, \tag{3.4.7}$$

since the time evolution of the co-efficients $c_m(t)$ is then determined by the decoupled equations

$$(3.4.8)\quad \ddot{c}_m(t) + (\alpha + 2m\mu)\,\dot{c}_m(t) - \\ - \{[n(n-1) - m(m-1)]\,\lambda + (n-m)\,g\}\,c_m(t) = 0\,, \qquad m = 0, 1, \dots, n\,.$$

The explicit solution of these equations is, of course,

$$(3.4.9)\quad c_m(t) = [b_m^{(+)} - b_m^{(-)}]^{-1}\{[\dot{c}_m(0) - b_m^{(-)}\,c_m(0)]\exp[b_m^{(+)}t] - \\ - [\dot{c}_m(0) - b_m^{(+)}\,c_m(0)]\exp[b_m^{(-)}t]\}\,, \qquad m = 0, 1, 2, \dots, n\,,$$

where $b_m^{(\pm)}$ are the two roots (assumed, for simplicity, different) of the second-degree equation in b

$$(3.4.10)\quad b^2 + (\alpha + 2m\mu)\,b - [n(n-1) - m(m-1)]\,\lambda - (n-m)\,g = 0\,.$$

The initial values $c_m(0)$, $\dot{c}_m(0)$, $m = 0, 1, \dots, n-1$, are determined by the initial positions and velocities of the particles through the two equations

$$(3.4.11)\qquad x^n + \sum_{m=0}^{n-1} c_m(0)\,x^m = \prod_{j=1}^{n}[x - x_j(0)] = \psi(x, 0)\,,$$

$$(3.4.12)\qquad \sum_{m=0}^{n-1} \dot{c}_m(0)\,x^m = -\,\psi(x, 0)\sum_{j=1}^{n}[x - x_j(0)]\,\dot{x}_j(0)\,,$$

while, of course,

$$(3.4.13)\qquad c_n(t) = c_n(0) = 1\,, \qquad \dot{c}_n(t) = 0$$

(note the consistency of these last formulae with (3.4.8)).

These equations characterize quite explicitly the time evolution of the polynomial $\psi(x, t)$ (of degree n in x; see (3.4.7)). On the other hand, the n zeros $x_j(t)$ of this polynomial, namely the values such that $\psi(x_j(t), t) = 0$, coincide with the solutions of (3.4.2) characterized by the initial data $x_j(0)$, $\dot{x}_j(0)$, $j = 1, 2, \dots, n$, and of course yield through (3.4.1) the corresponding solutions of the equations of motion (3.4.3) (subject to the condition that all the x_j's be positive).

The quantities $c_m(t)$, $m = 0, 1, \dots, n-1$, can be interpreted as convenient « collective variables » for the description of the many-body system; their relationship to the particle variables $x_j(t)$, $j = 1, 2, \dots, n$, is provided by the simultaneous validity of the representations (3.1.1) and (3.4.7). The advantage of these variables $c_m(t)$ over the particle co-ordinates $x_j(t)$ is of course that their time evolution is much simpler, being in fact given by the explicit formula (3.4.9).

The dynamics of these many-body models is quite rich, and a thorough discussion would require a separate paper. We report here only the asymptotic ($t \to +\infty$) analysis for the model (3.4.3) with $\alpha = \mu = 0$ and $g > 0$, $\lambda > 0$. It is then easily seen that

$$z_j(t) \approx v_j t + a_j\,, \qquad j = 1, 2, \ldots, n, \ (t \to +\infty), \tag{3.4.14}$$

where

$$v_j = \eta_j - \eta_{j-1}\,, \qquad j = 1, 2, \ldots, n, \tag{3.4.15}$$

$$a_j = \ln\{-(\eta_j/\eta_{j-1})[\dot c_{j-1}(0) + \eta_{j-1} c_{j-1}(0)]/[\dot c_j(0) + \eta_j c_j(0)]\}\,, \qquad j = 1, 2, \ldots, n, \tag{3.4.16}$$

$$\eta_m = \{[n(n-1) - m(m-1)]\,\lambda + (n-m)\,g\}^{\frac{1}{2}}\,, \qquad m = 0, 1, 2, \ldots, n. \tag{3.4.17}$$

These results obtain from (3.4.7), (3.4.9), (3.4.10) and (3.4.1), and from the remark that, for $t \to +\infty$, the zeros of ψ correspond to an exact cancellation between the two leading contributions in (3.4.7). Note that these findings imply that all the particles escape to the right with velocities that are independent of the initial conditions; since these asymptotic velocities are generally all different, the particles become asymptotically more and more separated. Of course these asymptotic results can also be inferred directly from the equations of motion (3.4.3), once the asymptotic separation of the particles is ascertained.

Let us finally mention that, by the trick of shifting part of the variables z_j by $i\pi$, one can generate a model with two kinds of particles; as is apparent from (3.4.3), the two-body interaction between different particles would then be nonsingular, so that in such a many-body problem different particles can cross each other.

3'5. *Some non–translation-invariant models*; *related properties of the zeros of the classical polynomials.* – In this subsection we discuss some non–translation-invariant many-body problems characterized by equations of motion that are special cases of (3.1.13). These models are interesting because their dynamical behaviour is rather rich; moreover, their dynamics is so closely related to properties of the zeros of the classical polynomials to allow the display of some remarkable, and we believe novel, properties satisfied by these quantities.

Let us consider first of all the subclass of the results of subsect. **3'1** that obtains for $A_3 = D_0 = D_1 = D_2 = E = 0$, $C = 1$, so that (3.1.13) become

$$\ddot x_j = B_0 + B_1 x_j + 2 \sum_{k=1}^{n}{}' (A_0 + A_1 x_j + A_2 x_j^2 + \dot x_j \dot x_k)/(x_j - x_k)\,, \qquad j = 1, 2, \ldots, n, \tag{3.5.1}$$

while (3.1.4) becomes

$$\psi_{tt} + (A_0 + A_1 x + A_2 x^2)\,\psi_{xx} + (B_0 + B_1 x)\,\psi_x - [n(n-1)A_2 + nB_1]\,\psi = 0 , \tag{3.5.2}$$

where of course $x_j \equiv x_j(t)$, $\psi \equiv \psi(x, t)$.

The many-body model characterized by the equations of motion (3.5.1), and by the corresponding partial differential equation (3.5.2), are still too general to allow a transparent analysis. Below we consider 3 special cases, that are particularly significant because of their close relationship with the classical polynomials. In each case, we take advantage of the possibility of performing trivial changes of variables, such as translations and scale transformations, to present the results in a canonical form that makes such relationship more evident.

The simpler and perhaps more interesting case obtains if $A_1 = A_2 = 0$. The canonical form is then displayed by setting $A_0 = \frac{1}{2}$, $B_0 = 0$, $B_1 = -1$, so that (3.5.1) become

$$\ddot{x}_j = -x_j + \sum_{k=1}^{n}{}' (1 + 2\dot{x}_j \dot{x}_k)/(x_j - x_k) , \qquad j = 1, 2, \ldots, n . \tag{3.5.3}$$

The interpretation of these equations of motion is clearly in terms of a n-body model with an external Hooke force (of unit strength in these dimensionless units) acting on each particle, and the velocity-dependent two-body force $(1 + 2\dot{x}_j \dot{x}_k)/(x_j - x_k)$ acting among the j-th and k-th particles. Such a system is clearly always confined, due to the presence of the Hooke force. It has one equilibrium configuration (see below); and it may, but it need not, give rise to collapse (see below). Particularly interesting is the (oscillatory) motion close to the equilibrium configuration.

The explicit dynamical evolution of this system can be studied by considering the associated partial differential equation, that obtains from (3.5.2) with the determination of the coefficients indicated above, reading, therefore,

$$\psi_{tt} + \tfrac{1}{2}\,\psi_{xx} - x\psi_x + n\psi = 0 . \tag{3.5.4}$$

The most convenient way to study the time evolution of the solutions of this equation is through the ansatz

$$\psi(x, t) = 2^{-n} \sum_{m=0}^{n} c_m(t)\, H_m(x) , \tag{3.5.5}$$

where $H_m(x)$ is the Hermite polynomial of order m [29]; for it is then easily

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[29] For the classical polynomials we use the notation of A. ERDÉLYI, Editor: *Higher Transcendental Functions*, Vol. **2** (New York, N. Y., 1953).

seen that (3.5.4) implies that the coefficients $c_m(t)$ satisfy the simple equations

$$\ddot{c}_m + (n-m)\,c_m = 0\,, \qquad m = 0, 1, \dots, n. \tag{3.5.6}$$

Thus their time evolution is given by the simple formula

$$c_m(t) = c_m(0)\cos(\omega_m t) + [\dot{c}_m(0)/\omega_m]\sin\omega_m t \tag{3.5.7}$$

with

$$\omega_m = (n-m)^{\frac{1}{2}}\,, \qquad m = 0, 1, \dots, n. \tag{3.5.8}$$

The initial data $c_m(0)$, $\dot{c}_m(0)$ are related, as usual, to the initial positions $x_j(0)$ and velocities $\dot{x}_j(0)$ of the many-body problem (3.5.3), through (3.1.1), (3.1.3) and (3.5.5):

$$2^{-n}\sum_{m=0}^{n} c_m(0)\,H_m(x) = \psi(x, 0) = \prod_{j=1}^{n}[x - x_j(0)]\,, \tag{3.5.9a}$$

$$2^{-n}\sum_{m=0}^{n-1} \dot{c}_m(0)\,H_m(x) = \psi_t(x, 0) = -\,\psi(x, 0)\sum_{j=1}^{n}[x - x_j(0)]^{-1}\dot{x}_j(0)\,. \tag{3.5.9b}$$

These equations imply of course ([29])

$$c_n(0) = c_n(t) = 1\,, \tag{3.5.10a}$$

$$c_m(0) = (\pi^{-\frac{1}{2}}2^{m-m}/m\,!)\int_{-\infty}^{+\infty}\mathrm{d}x\exp[-x^2]\,H_m(x)\prod_{j=1}^{n}[x - x_j(0)]\,, \qquad m = 0, 1, 2, \dots, n-1, \tag{3.5.10b}$$

$$\dot{c}_m(0) = -\,(\pi^{-\frac{1}{2}}2^{n-m}/m\,!)\int_{-\infty}^{+\infty}\mathrm{d}x\exp[-x^2]\cdot H_m(x)\prod_{j=1}^{n}[x - x_j(0)]\sum_{k=1}^{n}[x - x_k(0)]^{-1}\dot{x}_k(0)\,, \qquad m = 0, 1, \dots, n-1. \tag{3.5.10c}$$

The n zeros of the polynomial of degree n in x given by the explicit formulae (3.5.5), (3.5.7), (3.5.8) and (3.5.10) provide the solutions of the n-body model characterized by the equations of motion (3.5.3). Let us now discuss some special cases.

Consider first of all the configuration of the system characterized by the initial data $x_j(0) = \bar{x}_j$, $\dot{x}_j(0) = 0$, $j = 1, 2, \dots, n$, where $\bar{x}_j$ are the n zeros of the Hermite polynomial of degree n

$$H_n(\bar{x}_j) = 0\,, \qquad j = 1, 2, \ldots, n. \tag{3.5.11}$$

Then clearly $c_m(0) = \dot{c}_m(0) = 0$, $m = 1, 2, \ldots, n-1$, so that $\psi(x, t) = \psi(x, 0) = 2^{-n} H_n(x)$; the n-body problem is at equilibrium. We have thus proved that the n zeros of the Hermite polynomial of degree n provide the equilibrium configuration of the n-body system (3.5.3). And, since at equilibrium the velocity-dependent part of the two-body force can be of course ignored, we may also assert that the n zeros of the Hermite polynomial of degree n coincide with the equilibrium positions of n particles on the line whose dynamics is characterized by the Hamiltonian

$$H(p, q) = \tfrac{1}{2}\sum_{j=1}^{n}(p_j^2 + q_j^2) - \sum_{j=2}^{n}\sum_{k=1}^{j-1}\ln(q_j - q_k)\,, \tag{3.5.12}$$

since the equations of motion that obtain from this Hamiltonian are just the eqs. (3.5.3) with the velocity-dependent term on the r.h.s. omitted.

The connection of the solvable model of eqs. (3.5.3) with the Hamiltonian model of eq. (3.5.12) is very interesting, as well as the very direct relationship of its equilibrium configuration to the zeros of Hermite polynomials; the latter result is, however, not new, having in fact been discovered almost a century ago by STIELTJES [30].

The equivalence between the Hamiltonian model of eq. (3.5.12) and the solvable n-body model (3.5.3) is of course valid for any motion in which the velocities remain small; this is clearly the case if the system oscillates around its equilibrium configuration, without ever getting too far from it. Indeed, since the difference between the equations of motion (3.5.3) and those that correspond to the Hamiltonian (3.5.12) is quadratic in the velocities, the two models are identical not only as regards their equilibrium configurations, but also (to the first order) in accounting for the small oscillations around the equilibrium configuration. But for the model (3.5.3) it is clear that these small oscillations are characterized by the n frequencies ω_m, $m = 0, 1, 2, \ldots, n-1$, of eq. (3.5.8); see (3.5.7). On the other hand, the standard theory for the small oscillations of a dynamical system around its equilibrium configuration, applied to the system characterized by the Hamiltonian (3.5.12) and by the (clearly unique) equilibrium configuration $q_j = \bar{x}_j$, $j = 1, 2, \ldots, n$, described

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([30]) See subsect. **6**.7 of the classical textbook of G. SZEGÖ: *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., **23**, 1939. A discussion of these results is given in F. CALOGERO: *The zeros of the classical polynomials coincide with the equilibrium positions of simple one-dimensional many-body problems*, Nota Interna No. 682, Istituto di Fisica, Roma, April 1977.

above, implies after a little algebra that the frequencies $\tilde{\omega}_m$, $m = 0, 1, \ldots, n-1$, of the normal modes are just the square roots of the eigenvalues of the matrix A of rank n, whose matrix elements are given in terms of the n quantities $\bar{x}_j$ by the simple formula

$$A_{ml} = \delta_{ml} \sum_{k=1}^{n}{}' (\bar{x}_m - \bar{x}_k)^{-2} - (1 - \delta_{lm})(\bar{x}_m - \bar{x}_l)^{-2}, \tag{3.5.13}$$

where of course δ_{ml} is the Kronecker symbol, $\delta_{ml} = 1$ if $m = l$, $\delta_{ml} = 0$ if $m \neq l$. Since these frequencies $\tilde{\omega}_m$ must coincide with the frequencies ω_m of eq. (3.5.8), while the quantities $\bar{x}_j$ coincide with the zeros of $H_n(x)$, the following *theorem* obtains for the zeros of Hermite polynomials: *Let* $x_m^{(n)}$, $m = 1, 2, \ldots, n$, *indicate the* n *zeros of the Hermite polynomial* $H_n(x)$, $H_n(x_m^{(n)}) = 0$; *let the matrix* A *of rank* n *be defined in terms of these zeros by the simple expression*

$$A_{lm} = \delta_{lm} \sum_{k=1}^{n}{}' [x_m^{(n)} - x_k^{(n)}]^{-2} - (1 - \delta_{lm})[x_l^{(n)} - x_m^{(n)}]^{-2}; \tag{3.5.14}$$

then the n *eigenvalues* a_s *of* A *are the natural numbers from* 0 *to* $n-1$:

$$a_s = s - 1, \qquad s = 1, 2, \ldots, n. \tag{3.5.15}$$

This result is presumably new; we have reported it elsewhere together with a terse discussion of its implications ([31]).

Let us now return to discuss the solutions of the n-body model (3.5.3). There are clearly special solutions that oscillate periodically with one of the frequencies (3.5.8); some of these lend themselves to a very illuminating graphical display. Consider, for instance, the solution characterized by initial conditions such that $c_m(0) = \dot{c}_m(0) = 0$ for $m = 1, 2, \ldots, n-1$ (see (3.5.9) and (3.5.10)). Then the solutions $x_j(t)$ of (3.5.3) are given by the n roots of the equation in x

$$H_n(x) = -C \cos[n^{\frac{1}{2}} t + \varphi], \tag{3.5.16}$$

where

$$C = \{[c_0(0)]^2 + [\dot{c}_0(0)]^2/n\}^{\frac{1}{2}}, \tag{3.5.17a}$$

$$\varphi = \operatorname{arctg}[n^{\frac{1}{2}} c_0(0)/\dot{c}_0(0)]. \tag{3.5.17b}$$

A convenient way of visualizing the corresponding behaviour of the particles characterized by the co-ordinates $x_j(t)$ is to draw a graph of the Hermite polynomial $H_n(x)$ and to consider the intersection of this graph with a straight

([31]) F. CALOGERO: *Lett. Nuovo Cimento*, **19**, 505 (1977).

line parallel to the x-axis that oscillates periodically as described by the r.h.s. of (3.5.16). Note that in this manner one discovers that adjacent particles have generally opposite speeds; moreover, one finds that the condition necessary and sufficient to esclude the occurrence of collapse in this case is the requirement that the amplitude C of the oscillations of the r.h.s. of (3.5.16) be less than the moduli of all the (local) extrema of $H_n(x)$.

Another solution that also lends itself to a very explicit graphical display is that corresponding to initial conditions such that $c_m(0) = \dot{c}_m(0) = 0$, $m = 0, 2, 3, \ldots, n-1$, in which case (3.5.16) is replaced by

$$H_n(x) = -xC' \cos[(n-1)^{\frac{1}{2}}t + \varphi'] , \tag{3.5.18}$$

$$C' = 2\{[c_1(0)]^2 + [\dot{c}_1(0)]^2/(n-1)\}^{\frac{1}{2}} , \tag{3.5.19a}$$

$$\varphi' = \operatorname{arctg}[(n-1)^{\frac{1}{2}} c_1(0)/\dot{c}_1(0)] . \tag{3.5.19b}$$

In this case the co-ordinates $x_j(t)$ are given by the intersections of the graph of the Hermite polynomial $H_n(x)$ with a straight line that rotates in an oscillatory way around the origin, representing the r.h.s. of (3.5.18).

We conclude this discussion emphasizing an interesting feature of this n-body model, that may make it relevant for several applications: it is the explicit role played by the transition from the n particle variables $x_j(t)$ to the n quantities $c_m(t)$, $m = 0, 1, 2, \ldots, n-1$. The connection is of course provided by the formula

$$\sum_{m=0}^{n} c_m(t) H_m(x) = 2^n \prod_{j=1}^{n} [x - x_j(t)] . \tag{3.5.20}$$

The quantities $c_m(t)$, $m = 0, 1, \ldots, n-1$, are characterized by their simple time evolution, see (3.5.7); they may be considered the « normal co-ordinates » for the system under consideration (although their relationship (3.5.20) to the particle co-ordinates is not linear). They clearly constitute « collective co-ordinates » for the description of the many-body system, particularly suited to the description of its time development. Note incidentally that the last of these co-ordinates is essentially just the centre of mass X of the system:

$$c_{n-1}(t) = -2nX(t) = -2\sum_{j=1}^{n} x_j(t) . \tag{3.5.21}$$

This last formula follows from (3.5.20); note its compatibility with (3.5.6) and with the equation of motion $\ddot{X} + X = 0$, that follows directly from (the sum of all) the eqs. (3.5.3).

The second model to be discussed in this subsection corresponds to (3.5.1) and (3.5.2) with $A_2 = 0$, but $A_1 \neq 0$. Its canonical form obtains by setting

$A_0 = 0$, $A_1 = 1$, $B_0 = 1 + \alpha$, $B_1 = -1$; we assume hereafter $\alpha > -1$ (the motivation for this assumption shall be clear in the following, as well as the fact that many results remain valid even if this assumption does not hold).

With this choice of coefficients the equations of motion (5.3.1) become

$$\ddot{x}_j = 1 + \alpha - x_j + 2 \sum_{k=1}^{n}{}' (x_j + \dot{x}_j \dot{x}_k)/(x_j - x_k)\,, \qquad j = 1, 2, \dots, n, \tag{3.5.20a}$$

or, equivalently,

$$\ddot{x}_j = n + \alpha - x_j + \sum_{k=1}^{n}{}' (x_j + x_k + 2\dot{x}_j \dot{x}_k)/(x_j - x_k)\,, \qquad j = 1, 2, \dots, n. \tag{3.5.20b}$$

Note incidentally that these last equations imply for the centre-of-mass coordinate $X = n^{-1} \sum_{j=1}^{n} x_j$ the equation

$$\ddot{X} = n + \alpha - X\,, \tag{3.5.21a}$$

namely the time evolution

$$X(t) = n + \alpha + [X(0) - (n + \alpha)] \cos t + \dot{X}(0) \sin t\,. \tag{3.5.21b}$$

Equations (3.5.20) have a less appealing interpretation as equations of motion of a many-body problem than those given above in this subsection, due to the non–translation-invariant character of the two-body force. Nevertheless, this model is interesting because of its relationship with properties of Laguerre polynomials. Indeed, because the results are analogous, except for the replacement of Hermite polynomials by Laguerre polynomials, to those described above, we present them below very tersely.

The partial differential equation that corresponds to the equations of motion (3.5.20) reads

$$\psi_{tt} + x\psi_{xx} + (1 + \alpha - x)\, \psi_x + n\psi = 0\,. \tag{3.5.22}$$

The convenient ansatz to discuss the time evolution of a polynomial solution is

$$\psi(x, t) = (-)^n n\,! \sum_{m=0}^{n} c_m(t)\, L_m^{\alpha}(x)\,, \tag{3.5.23}$$

where $L_m^{\alpha}(x)$ is the Laguerre polynomial [29] of degree m, since (3.5.22) then implies

$$\ddot{c}_m + (n - m)\, c_m = 0\,, \qquad m = 0, 1, \dots, n, \tag{3.5.24}$$

namely

$$c_m(t) = c_m(0) \cos \omega_m t + [\dot{c}_m(0)/\omega_m] \sin \omega_m t\,, \qquad m = 0, 1, \dots, n, \tag{3.5.25}$$

where again

$$\omega_m = (n - m)^{\frac{1}{2}}\,, \qquad m = 0, 1, \ldots, n. \tag{3.5.26}$$

As for the initial data $c_m(0)$, $\dot{c}_m(0)$, they are related to the initial positions $x_j(0)$ and velocities $\dot{x}_j(0)$ by

$$(-)^n n\,!\sum_{m=0}^{n} c_m(0)\, L_m^\alpha(x) = \psi(x, 0) = \prod_{j=1}^{n} [x - x_j(0)]\,, \tag{3.5.27a}$$

$$(-)^n n\,!\sum_{m=0}^{n-1} \dot{c}_m(0)\, L_m^\alpha(x) = -\, \psi(x, 0) \sum_{j=1}^{n} [x - x_j(0)]^{-1} \dot{x}_j(0)\,. \tag{3.5.27b}$$

These equations determine the $2n$ quantities $c_m(0)$ and $\dot{c}_m(0)$, $m = 0, 1, \ldots, n-1$, in terms of the $2n$ initial data $x_j(0)$, $\dot{x}_j(0)$, while

$$c_n(t) = 1\,. \tag{3.5.28}$$

The relationship between the n particle co-ordinates $x_j(t)$, $j = 1, 2, \ldots, n-1$, and the n « collective co-ordinates » $c_m(t)$, $m = 0, 1, \ldots, n-1$, is given by

$$(-)^n n\,!\sum_{m=0}^{n} c_m(t)\, L_m^\alpha(x) = \prod_{j=1}^{n} [x - x_j(t)]\,. \tag{3.5.29}$$

In particular, $c_{n-1}(t)$ is simply related to the centre-of-mass co-ordinate $X(t)$:

$$X(t) = n + \alpha + c_{n-1}(t)\,. \tag{3.5.30}$$

The n zeros $\bar{x}_j$ of the Laguerre polynomial $L_n^\alpha(x)$, $L_n^\alpha(\bar{x}_j) = 0$, yield the (unique) equilibrium configuration of the system characterized by the equations of motion (3.5.20). This equilibrium configuration corresponds of course to a solution of the equations that obtains by equating to zero the r.h.s. of (3.5.20), with moreover $\dot{x}_j = 0$, $j = 1, 2, \ldots, n$:

$$\sum_{k=1}^{n}{}' (\bar{x}_j - \bar{x}_k)^{-1} = \tfrac{1}{2}[1 - (1 + \alpha)/x_j]\,, \qquad j = 1, 2, \ldots, n. \tag{3.5.31}$$

Note that, to write these equations, we have divided by $2x_j$ the r.h.s. of (3.5.20a). This is convenient, since it is then immediately recognized that the zeros $\bar{x}_m$ of the Laguerre polynomial $L_n^\alpha(x)$ coincide with the unique equilibrium configuration of the (Hamiltonian) system characterized by the Hamiltonian

$$H(p, q) = \tfrac{1}{2}\sum_{j=1}^{n} p_j^2 + \tfrac{1}{2}\sum_{j=1}^{n} [q_j - (1 + \alpha) \ln q_j] - \sum_{j=2}^{n} \sum_{k=1}^{j-1} \ln (q_j - q_k)\,, \tag{3.5.32}$$

a result that is again not new, going in fact back to STIELTJES [30]. Note that the single-particle potential constrains the particles on the *positive* real axis.

The time evolution of the system (3.5.20) is clearly characterized by the n frequencies ω_m of eq. (3.5.26), with $m = 0, 1, \ldots, n-1$; these frequencies must, therefore, coincide with the eigenfrequencies of the small oscillations of this system around its equilibrium configuration. But these eigenfrequencies can be computed by the standard procedure of linearization of (3.5.20) around the equilibrium configuration. In this manner, in analogy to the previous case, one proves, with a little algebra, the following *theorem*: *Let* $x_m(n, \alpha)$, $m = 1, 2, \ldots, n$, *indicate the n zeros of the Laguerre polynomial* $L_n^\alpha(x)$, $L_n^\alpha[x_m(n, \alpha)] = 0$; *let the matrix B of rank n be defined in terms of these zeros by the simple expression*

$$B_{lm} = \delta_{lm} \sum_{k=1}^{n}{}' [x_m(n, \alpha) - x_k(n, \alpha)]^{-2} x_k(n, \alpha) - \\ - (1 - \delta_{lm})[x_l(n, \alpha) - x_m(n, \alpha)]^{-2} x_m(n, \alpha); \tag{3.5.33}$$

then the n eigenvalues b_s *of B are one-half of the natural numbers from zero to* $n-1$:

$$b_s = \tfrac{1}{2}(s-1), \qquad s = 1, 2, \ldots, n. \tag{3.5.34}$$

We have reported, and discussed, also this result elsewhere [31].

A very explicit display of the motion of the particles obeying (3.5.20) can be given in some special cases, the simpler of these being of course that characterized by $c_0(t) \neq 0$, $c_m(t) = 0$ for $1, 2, \ldots, n-1$; we do not elaborate this analysis here, since the interested reader will have no difficulty to duplicate the treatment given above in analogous cases.

The last model of this subsection corresponds to (3.5.1) and (3.5.2) with $A_2 \neq 0$. The canonical form obtains then by setting $A_0 = -1$, $A_1 = 0$, $A_2 = 1$, $B_0 = \alpha - \beta$, $B_1 = \alpha + \beta + 2$; we assume hereafter $\alpha > -1$, $\beta > -1$, for reasons that shall be clear presently (but many of the following results do not require these restrictions). Since the treatment to be given now is essentially a repetition of those just given, except for the fact that now the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [29] are taking the place of the Hermite and Laguerre polynomials, we merely report the main equations and results, without any elaboration.

The equations of motion read

$$\ddot{x}_j = \alpha - \beta + (\alpha + \beta + 2) x_j + 2 \sum_{k=1}^{n}{}' (x_j^2 - 1 + \dot{x}_j \dot{x}_k)/(x_j - x_k), \\ j = 1, 2, \ldots, n, \tag{3.5.35a}$$

or, equivalently,

$$\ddot{x}_j = \alpha - \beta + (\alpha + \beta + n)\, x_j + nX + \sum_{k=1}^{n}{}' (x_j^2 + x_k^2 - 2 + 2\dot{x}_j \dot{x}_k)/(x_j - x_k)\,, \qquad j = 1, 2, \ldots, n, \tag{3.5.35b}$$

where of course X is the centre-of-mass co-ordinate.

The corresponding partial differential equation reads

$$\psi_{tt} + (x^2 - 1)\,\psi_{xx} + [\alpha - \beta + (\alpha + \beta + 2)\, x]\,\psi_x - (n + \alpha + \beta + 1)\,\psi = 0\,. \tag{3.5.36}$$

The convenient ansatz for its solution reads

$$\psi(x, t) = [2^n n\,!(n + \alpha + \beta)\,!/(2n + \alpha + \beta)\,!] \sum_{m=0}^{n} c_m(t)\, P_m^{(\alpha,\beta)}(x)\,, \tag{3.5.37}$$

since then

$$\ddot{c}_m + [n(n + \alpha + \beta + 1) - m(m + \alpha + \beta + 1)]\, c_m = 0\,, \qquad m = 0, 1, \ldots, n, \tag{3.5.38}$$

implying of course

$$c_m(t) = c_m(0) \cos \omega_m t + [\dot{c}_m(0)/\omega_m]/\sin \omega_m t\,, \tag{3.5.39}$$

where, however, now

$$\omega_m = [n(n + \alpha + \beta + 1) - m(m + \alpha + \beta + 1)]^{\frac{1}{2}}\,, \qquad m = 0, 1, 2, \ldots, n. \tag{3.5.40}$$

The relationship between the particle co-ordinates $x_j(t)$ and the collective co-ordinates $c_m(t)$ is given by

$$[2^n n\,!(n + \alpha + \beta)\,!/(2n + \alpha + \beta)\,!] \sum_{m=0}^{n} c_m(t)\, P_m^{(\alpha,\beta)}(x) = \prod_{j=1}^{n} [x - x_j(t)]\,, \tag{3.5.41}$$

which also implies

$$c_n(t) = 1\,. \tag{3.5.42}$$

Clearly the initial data are correspondingly related by

$$[2^n n\,!(n + \alpha + \beta)\,!/(2n + \alpha + \beta)\,!] \sum_{m=0}^{n} c_m(0)\, P_n^{(\alpha,\beta)}(x) = \psi(x, 0) = \prod_{j=1}^{n} [x - x_j(0)]\,, \tag{3.5.43a}$$

$$(3.5.43b)\qquad [2^n n!(n+\alpha+\beta)!/(2n+\alpha+\beta)!]\sum_{m=0}^{n}\dot{c}_m(0)P_m^{(\alpha,\beta)}(x) = -\psi(x,0)\sum_{j=1}^{n}[x-x_j(0)]^{-1}\dot{x}_j(0)\,.$$

The equilibrium configuration for the system (3.5.35) is given by $x_j = \bar{x}_j$, the n quantities $\bar{x}_j$ coinciding with the zeros $x_j(n, \alpha, \beta)$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, $P_n^{(\alpha,\beta)}[x_j(n, \alpha, \beta)] = 0$, $j = 1, 2, \ldots, n$. This same configuration corresponds also to equilibrium for the (Hamiltonian) system characterized by the Hamiltonian

$$(3.5.44)\qquad H(p,q) = \tfrac{1}{2}\sum_{j=1}^{n}p_j^2 - \tfrac{1}{2}\sum_{j=1}^{n}[(1+\alpha)\ln(1-q_j) + (1+\beta)\ln(1+q_j)] - \sum_{j=2}^{n}\sum_{k=1}^{j-1}\ln(q_j-q_k)\,,$$

again a result going back to STIELTJES [30]. Now the single-particle potential confines all the particles in the interval $(-1, 1)$.

A comparison of the approximate (linearized) treatment of the small oscillations of the system (3.5.35) around its equilibrium configuration with the exact treatment given above leads to the following *theorem* [31]: *Let $x_m(n, \alpha, \beta)$ be the n zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$; let the matrix C of rank n be defined in terms of these zeros by the formula*

$$(3.5.45)\qquad C_{lm} = \delta_{lm}\sum_{k=1}^{n}{}'(1-x_k^2)(x_m-x_k)^{-2} - (1-\delta_{lm})(1-x_m^2)(x_m-x_l)^{-2}\,;$$

then the n eigenvalues c_s of C are given by the formula

$$(3.5.46)\qquad c_s = \tfrac{1}{2}(s-1)(2n-s+\alpha+\beta)\,,\qquad s = 1, 2, \ldots, n\,.$$

Of course analogous results can be given for Gegenbauer and Legendre polynomials, since these are just special cases of the Jacobi polynomials, corresponding respectively to $\alpha = \beta$ and to $\alpha = \beta = 0$ [29].

3·6. *Periodic and hyperbolic forces.* – In analogy to the treatment given in the previous section, see in particular subsect. **2·3**, we extend here the results of the previous subsections to models involving circular and hyperbolic functions. As will be presently seen, the solvable many-body models thus obtained encompass all those discussed in subsect. **2·3**. Here, however, we shall forsake a detailed discussion of each model, limiting our treatment to the identification and display of these many-body models and of the technique by which they can be analysed. We, moreover, restrict our consideration to translation-invariant problems.

Now the basic ansatz is

$$(3.6.1)\qquad \psi(x, t) = \beta^{-n}\prod_{j=1}^{n}\sin\beta[x - x_j(t)]\,,$$

that reduces of course to (3.1.1) for $\beta = 0$. It implies the following formulae:

$$(3.6.2)\qquad \psi_x = \psi\beta\sum_{j=1}^{n}\operatorname{ctg}\beta(x - x_j)\,,$$

$$(3.6.3)\qquad \psi_t = -\psi\beta\sum_{j=1}^{n}\dot{x}_j\operatorname{ctg}\beta(x - x_j)\,,$$

$$(3.6.4)\qquad \psi_{xx} = \psi\Big\{-\beta^2 n^2 + 2\beta^2\sum_{j=1}^{n}\operatorname{ctg}[\beta(x - x_j)]\sum_{k=1}^{n}{}'\operatorname{ctg}\beta(x_j - x_k)\Big\},$$

$$(3.6.5)\qquad \psi_{xt} = \psi\Big\{\beta^2 n^2\dot{X} - \beta^2\sum_{j=1}^{n}\operatorname{ctg}[\beta(x - x_j)]\sum_{k=1}^{n}{}'(\dot{x}_j + \dot{x}_k)\operatorname{ctg}\beta(x_j - x_k)\Big\}\,,$$

$$(3.6.6)\qquad \psi_{tt} = \psi\Big\{-\beta^2 n^2\dot{X}^2 + \beta\sum_{j=1}^{n}\operatorname{ctg}\beta(x - x_j)\cdot$$
$$\cdot\Big(-\ddot{x}_j + 2\beta\dot{x}_j\sum_{k=1}^{n}{}'\dot{x}_k\operatorname{ctg}\beta(x_j - x_k)\Big)\Big\}.$$

In all these equations of course $\psi \equiv \psi(x, t)$, $x_j \equiv x_j(t)$, $X \equiv X(t)$. Note that, to obtain the last 3 equations, we have used the trigonometric identity (2.3.10); and we have of course introduced the centre-of-mass co-ordinate

$$(3.6.7)\qquad X(t) = n^{-1}\sum_{j=1}^{n}x_j(t)\,.$$

Assume now that the function $\psi(x, t)$ satisfies the linear partial differential equation

$$(3.6.8)\qquad A\psi_{xx} + B\psi_x + C\psi_{tt} + D\psi_{xt} + E\psi_t + F\psi = 0\,,$$

where the quantities A, B, C, D, E and F are independent of x, but might depend on t (see below). There clearly correspond for the co-ordinates $x_j(t)$ the equations of motion

$$(3.6.9)\qquad C\ddot{x}_j + E\dot{x}_j = B + \beta\sum_{k=1}^{n}{}'[2A - D(\dot{x}_j + \dot{x}_k) + 2C\dot{x}_j\dot{x}_k]\operatorname{ctg}\beta(x_j - x_k)\,,$$
$$j = 1, 2, \ldots, n,$$

with the additional equation

$$F = \beta^2 n^2 (A - D\dot{X} + C\dot{X}^2)\,. \tag{3.6.10}$$

The solutions of the equations of motion (3.6.9) with no constraint on the initial data are in this manner related to the linear partial differential equation (3.6.8), where now F should be considered as given by eq. (3.6.10). Clearly the centre-of-mass co-ordinate that enters in this last equation satisfies the equation

$$C\ddot{X} + E\dot{X} = B\,, \tag{3.6.11}$$

that follows immediately from (3.6.9).

We assume hereafter, for simplicity, A, B, C, D and E to be constant (although the possibility to solve the many-body problem (3.6.9) even when these quantities depend arbitrarily on t should be emphasized; and note that the techniques described below apply also in this more general case). It is then immediately seen that (3.6.11) implies

$$X(t) = X(0) + (B/E)\,t + (C/E)[\dot{X}(0) - (B/E)]\,[1 - \exp[-Et/C]]\,, \tag{3.6.12}$$

and, inserting the corresponding formula for $\dot{X}(t)$ in (3.6.10), one gets the explicit expression of $F(t)$ in terms of A, B, C, D, E and of the initial position $X(0)$ and velocity $\dot{X}(0)$ of the centre of mass of the system.

The many-body model characterized by the equations of motion (3.6.9) is clearly translation invariant; it is not Galilei invariant, although the change of variables $x_j(t) = x'_j(t) + ut$ (corresponding to a description of the same system as seen in a frame of reference moving with constant speed u) reproduces the same model, but with the new coefficients $A' = A - Du + Cu^2$, $B' = B - Eu$, $C' = C$, $D' = D - 2Cu$, $E' = E$, and with F' given in terms A', C', D' and X' by the same eq. (3.6.10) that gives F in terms of A, C, D and $X = X' + ut$.

The linear partial differential equation (3.6.8) is most conveniently solved through the ansatz

$$\psi(x, t) = \beta^{-n}(2i)^{-n} \sum_{m=-n}^{n} c_m(t) \exp[i\beta m x]\,, \tag{3.6.13}$$

that is clearly consistent with (3.6.1); for the time evolution of the « collective co-ordinates » $c_m(t)$ is then given by the simple equations

$$C\ddot{c}_m(t) + (E + i\beta m D)\,\dot{c}_m(t) + (F + i\beta m B - \beta^2 m^2 A)\,c_m(t) = 0\,, \qquad m = 0, \pm 1, \pm 2, \dots, \pm n. \tag{3.6.14}$$

Of course the explicit relationship between the particle co-ordinates $x_j(t)$ and the collective co-ordinates is univocally determined by (3.6.1) and (3.6.13), namely by the formula

$$\sum_{m=-n}^{n} c_m(t) \exp[i\beta m x] = (2i)^n \prod_{j=1}^{n} \sin\beta[x - x_j(t)]\,, \tag{3.6.15}$$

that of course implies (when all the x_j's are real)

$$c_m(t) = (-)^n c^*_{-m}(t)\,, \qquad m = 0, 1, \dots, n, \tag{3.6.16}$$

a property that is clearly consistent with (3.6.14).

The explicit solution of (3.6.14) is generally easy. Moreover, if the centre of mass of the system moves with constant speed,

$$\dot{X}(t) = V\,, \tag{3.6.17}$$

as is the case if $B = E = 0$ or, in the general case, if $\dot{X}(0) = B/E$, then F, as given by (3.6.10), is also time independent, so that all the coefficients in (3.6.14) become constant. Then clearly

$$c_m(t) = c_m^{(+)} \exp[\alpha_m^{(+)} t] + c_m^{(-)} \exp[\alpha_m^{(-)} t]\,, \qquad m = 0, \pm 1, \pm 2, \dots, \pm n, \tag{3.6.18}$$

where $\alpha_m^{(+)}$ and $\alpha_m^{(-)}$ are the two roots of the second-degree equation in α

$$C\alpha^2 + (E + i\beta m D)\alpha + (F + i\beta m B - \beta^2 m^2 A) = 0\,, \qquad m = 0, \pm 1, \pm 2, \dots, \pm n, \tag{3.6.19}$$

and the constants $c_m^{(+)}$ and $c_m^{(-)}$ are related to the initial data $c_m(0)$ and $\dot{c}_m(0)$ by the formula

$$c_m^{(\pm)} = \pm [\alpha_m^{(+)} - \alpha_m^{(-)}]^{-1} [\dot{c}_m(0) - \alpha_m^{(\mp)} c_m(0)]\,, \qquad m = 0, \pm 1, \dots, \pm n. \tag{3.6.20}$$

As for the initial data $c_m(0)$ and $\dot{c}_m(0)$, they are of course related to the initial positions $x_j(0)$ and velocities $\dot{x}_j(0)$ through (3.6.15); note incidentally that, for small n, this equation is easily solved in explicit form, for instance for $n = 2$

$$c_1(t) = 0\,, \qquad c_0(t) = -2\cos\beta[x_1(t) - x_2(t)]\,, \qquad c_2(t) = \exp[-2i\beta X(t)]\,, \tag{3.6.21}$$

while for $n = 3$

$$\begin{cases} c_0(t) = c_2(t) = 0\,, \\ c_1(t) = -\exp\big[i\beta[x_1(t) - x_2(t) - x_3(t)]\big] - \\ \qquad - \exp\big[i\beta[x_2(t) - x_3(t) - x_1(t)]\big] - \exp\big[i\beta[x_3(t) - x_1(t) - x_2(t)]\big]\,, \\ c_3(t) = \exp[-3i\beta X(t)]\,. \end{cases} \tag{3.6.22}$$

As suggested by these formulae, and clearly implied by (3.6.15), it is generally true that $c_m(t) = 0$ if m and n have different parities, and that $c_n(t)$ is simply related to the centre-of-mass co-ordinate,

$$c_n(t) = \exp[-i\beta nX(t)]\,. \tag{3.6.23}$$

It is of course always possible to work with real quantities, by introducing as « collective variables » the real and imaginary parts of $c_m(t)$, in place of $c_m(t)$ and its complex conjugate (up to a sign; see (3.6.16)) $c_{-m}(t)$.

These results open the possibility of a detailed discussion of the (rather general) many-body problem characterized by eqs. (3.6.9), including a determination of whether and when collapse occurs; but, as indicated at the beginning of this subsection, space limitations prevent us from elaborating this matter any further here.

We end this subsection with two remarks. First of all we note that, although in our discussion above we implicitly assumed β to be real, an interesting many-body model obtains also for imaginary $\beta = i\gamma$. In such a model the circular functions are replaced by hyperbolic functions, so that the physical behaviour becomes of course qualitatively different. On the other hand, the analytical treatment, displaying the possibility to achieve a quite complete and rather explicit solution of this many-body problem, remains applicable, with obvious modifications, also in the case with $\beta = i\gamma$, γ real (indeed it remains formally valid, although hardly interesting, even for complex β).

Secondly we remark that it is very easy, although certainly nontrivial as regards the changes induced in the qualitative physical picture, to modify these models so that they involve two kinds of particles, following the procedure of subsect. 2˙4. Indeed also the other extensions discussed in that subsection are clearly easily applicable in the context of the results presented in this subsection, and more generally in this section.

4. – Summary and outlook.

A rich harvest of many-body models has been introduced in this paper. The main focus has been on models involving no constraints on the initial velocities or positions of the particles. The models considered are all amenable to analytic treatment, the main technique of solution being through the identification of the motion of the particles with that of the zeros of specially simple solutions of linear partial differential equations. Because these latter equations are generally solvable in rather explicit form, the motion of the particles described by these many-body problems can be studied in great detail, and can often be visualized quite explicitly. In many cases, « collective co-ordinates » that are particularly appropriate to describe the dynamical evolution of these systems are quite naturally introduced through the process of solution.

Typical prototypes of the models discussed in this paper are those characterized by equations of motion such as

$$\ddot{x}_j = a + b\dot{x}_j + cx_j + \sum_{k=1}^{n}{}' [A + B(\dot{x}_j + \dot{x}_k) + C(x_j + x_k) + 2\dot{x}_j\dot{x}_k]/(x_j - x_k)\,, \quad j = 1, 2, \ldots, n, \tag{4.1}$$

$$\ddot{x}_j = a + b\dot{x}_j + \beta \sum_{k=1}^{n}{}' [A + B(\dot{x}_j + \dot{x}_k) + 2\dot{x}_j\dot{x}_k] \operatorname{ctg} \beta(x_j - x_k)\,, \quad j = 1, 2, \ldots, n, \tag{4.2}$$

$$\ddot{x}_j = a + b\dot{x}_j + \gamma \sum_{k=1}^{n}{}' [A + B(\dot{x}_j + \dot{x}_k) + 2\dot{x}_j\dot{x}_k] \operatorname{ctgh} \gamma(x_j - x_k)\,, \quad j = 1, 2, \ldots, n; \tag{4.3}$$

although models considerably more general than those reported here are also considered. The coefficients a, b, c, A, B, C appearing in these equations could be time dependent, although in the cases discussed in more explicit detail their constancy was generally assumed.

These models give rise to an ample variety of motions, including cases in which the particles can escape to infinity and cases in which they are confined in a finite region of configuration space; cases in which the motion is periodic, or quasi-periodic, or not periodic; cases in which collapse, whose possibility is of course implied by the singular character of the two-body force appearing on the r.h.s. of these equations of motion, does or does not occur. Some of these models are translation invariant, some are not; they are generally characterized by the presence of velocity-dependent forces; but, when the system admits an equilibrium configuration around which it oscillates, there are regimes in which the velocity-dependent components are small, so that in these cases the solvable system provides a good approximation to a system without velocity-dependent forces, that is then evidently of Hamiltonian type.

A detailed analysis of the dynamical behaviour has been reported only for a few cases; the techniques used provide tools that allow an easy treatment of the many other models contained in this general framework.

Applications have not been mentioned; but the variety of behaviours that have been displayed suggest an ample scope.

There are several directions of research suggested by the findings of this paper. Since all the results obtained follow from the choice of a simple ansatz, one may wonder whether additional results could be attained by modifications or extensions of these; indeed the basis of some developments of this kind have been given, but perhaps more radical modifications may produce further progress. But, even without drastic departures from the present framework, there are two important directions in which an extension of the results of this paper appear particularly interesting: an investigation of models involving an infinite number of particles ([32]), and an extension of the approach to more than one space dimension.

Two other topics should also be mentioned, in which the research suggested by the findings reported in this paper appears called for and promising: the connection with the recent results on integrable dynamical systems ([5,6]), and the extension to the quantal case.

Finally it should be noted that, while our main interest has been here on the discussion of solvable many-body problems, some of the results obtained (in particular those concerning the zeros of the classical polynomials) are of purely mathematical nature.

* * *

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([32]) *Note added in proofs.* – Recent results on the zeros of Bessel functions are steps in this direction. F. CALOGERO: *Lett. Nuovo Cimento*, **20**, 254 (1977).

● RIASSUNTO

Si mostra come il moto degli zeri e dei poli di soluzioni particolari di alcune equazioni alle derivate parziali lineari e non lineari possa essere interpretato come un problema a molti corpi. Si introducono in tal modo numerosi esempi di problemi a molti corpi risolubili. L'analisi è limitata a modelli con un numero finito di particelle che si muovono in una dimensione.

Движение полюсов и нулей частных решений нелинейных и линейных дифференциальных уравнений в частных производных и родственные «решаемые» проблемы многих тел.

Резюме (*). — Показывается, что движение полюсов и нулей частных решений некоторых нелинейных и линейных дифференциальных уравнений в частных производных может быть интерпретировано в терминах эквивалентных проблем многих тел. Обсуждаются некоторые решаемые модели многих тел. Рассмотрение ограничивается проблемами, включающими конечное число частиц, движущихся в одномерном пространстве.

(*) *Переведено редакцией.*