Theoretical and Mathematical Physics, **151**(3): 851–862 (2007)

JACOBI'S LAST MULTIPLIER, LIE SYMMETRIES, AND HIDDEN LINEARITY: "GOLDFISHES" GALORE

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In addition to the reduction method, we present a novel application of Jacobi's last multiplier for finding Lie symmetries of ordinary differential equations algorithmically. These methods and Lie symmetries allow unveiling the hidden linearity of certain nonlinear equations that are relevant in physics. We consider the Einstein–Yang–Mills equations and Calogero's many-body problem in the plane as examples.

Keywords: Lie group analysis, first integral, Jacobi's last multiplier

1. Introduction

Lie group analysis is the most powerful tool for finding the general solution of ordinary differential equations (ODEs). But it is useless when applied to systems of n first-order equations because they admit an infinite number of symmetries, and there is no systematic way to find even a one-dimensional Lie symmetry algebra other than trivial groups such as the translations in time admitted by autonomous systems. Deriving an admitted n-dimensional solvable Lie symmetry algebra can be attempted by making an ansatz on the form of its generators, but when successful (rarely), this is just the result of a lucky guess. But we noted that any system of n first-order equations can be transformed into an equivalent system where at least one of the equations is of the second order, the admitted Lie symmetry algebra is then no longer infinite-dimensional, and nontrivial symmetries of the original system can be retrieved [1]. This idea was successfully applied in several instances [1]–[9]. In [10], we also showed that first integrals can be obtained by Lie group analysis even if the system under study does not come from a variational problem, i.e., without using Noether's theorem [11].

If we consider a system of first-order equations and derive an equivalent system with one equation being of the second order by eliminating one of the dependent variables, then applying Lie group analysis to that equivalent system yields the first integral(s) of the original system, and the eliminated dependent variable is absent from the integral(s). Of course, this requires that such first integrals exist. To find all such first integrals, the procedure should be repeated as many times as there are dependent variables.

The first integrals correspond to the characteristic curves of determining equations of parabolic type constructed by the Lie group analysis method. We used this method to find first integrals in [4], [5]. Unfortunately, the reduction method is not the ultimate method for finding symmetries. Therefore, in [12], we devised another method using Jacobi's last multiplier [13]–[20] to transform any system of n first-order equations into an equivalent system of n equations where one of the equations is of the second order, namely, the order of the system is raised by one. Among other examples in [12], the method was successfully applied to the second-order equation (see Chap. 6 in [21])

$$y'' = \frac{y'^2}{y} + f'(x)y^{p+1} + pf(x)y'y^p, \tag{1}$$

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Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 151, No. 3, pp. 495–509, June, 2007.

where $p \neq 0$ is a real constant and $f \neq 0$ is an arbitrary function of the independent variable x. This equation does not have Lie point symmetries for general $f(x)$ and yet is trivially integrable [22]. Our method leads to an equivalent system of two equations (a first-order and a second-order equation) that admits sufficient Lie symmetries to be integrable by quadratures.

We exemplify all the above methods in this paper. In Sec. 2, we derive the two well-known first integrals of the static spherically symmetric Einstein–Yang–Mills equations [23] by the reduction method. In Sec. 3, we find several properties (integration, linearization, reduction of order) of a many-body problem in the plane derived by Calogero [24]. Throughout the paper, we use our own interactive REDUCE programs to calculate Lie symmetries [25].

2. First integrals of Einstein–Yang–Mills equations

In [23], the static spherically symmetric Einstein–Yang–Mills equations were given:[1]

$$\begin{aligned} &\dot{w}_1 = w_1 w_3, \qquad \dot{w}_2 = w_1 w_5, \qquad \dot{w}_3 = (w_4 - w_3) w_3 - 2w_5^2, \\ &\dot{w}_4 = s(1 - 2aw_1) + 2w_5^2 - w_4^2, \qquad \dot{w}_5 = s w_2 w_6 + (w_3 - w_4) w_5, \\ &\dot{w}_6 = 2w_2 w_5 - w_3 w_6, \end{aligned} \tag{2}$$

where a is the cosmological constant, $s \in \{-1; 1\}$, and the dot over a variable denotes the derivative with respect to the space–time variable t. It is known that this system admits two first integrals, i.e.,

$$I_1 = w_1 w_6 - w_2^2, \tag{3}$$

$$I_2 = w_1^2(-4asw_1 - 3sw_6^2 + 3s + 3w_3^2 - 6w_3w_4 + 6w_5^2). \tag{4}$$

We show how these two first integrals can be obtained using the reduction method and Lie group analysis (more details on the method can be found in [10]).

2.1. Obtaining I_1. If we derive w_5 from the second equation[2] in system (2), i.e., $w_5 = \dot{w}_2/w_1$, then we obtain a system of five equations, one of the second order and four of the first order:

$$\begin{aligned} &\dot{w}_1 = w_1 w_3, \qquad \ddot{w}_2 = (2w_3 - w_4)\dot{w}_2 + s w_1 w_2 w_6, \\ &\dot{w}_3 = (w_4 - w_3) w_3 - 2\left(\frac{\dot{w}_2}{w_1}\right)^2, \qquad \dot{w}_4 = s(1 - 2aw_1) + 2\left(\frac{\dot{w}_2}{w_1}\right)^2 - w_4^2, \\ &\dot{w}_6 = 2w_2 \frac{\dot{w}_2}{w_1} - w_3 w_6. \end{aligned} \tag{5}$$

Applying Lie group analysis to system (5),[3] we obtain a determining equation of parabolic type for V in two independent variables. Its characteristic curve is $w_6 - w_2^2/w_1$, which suggests the transformation $w_6 = r_6 + w_2^2/w_1$, where $r_6(t)$ is a new unknown function of t. System (5) then transforms into[4]

$$\begin{aligned} &\ddot{u}_1 = su_1^3 + su_1u_2u_5 + 2\dot{u}_1u_3 - \dot{u}_1u_4, \qquad \dot{u}_2 = u_2u_3, \\ &\dot{u}_3 = (u_4 - u_3)u_3 - 2\left(\frac{\dot{u}_1}{u_2}\right)^2, \\ &\dot{u}_4 = s(1 - 2au_2) + 2\left(\frac{\dot{u}_1}{u_2}\right)^2 - u_4^2, \qquad \dot{u}_5 = -u_3u_5, \end{aligned} \tag{6}$$

[1] We note that the original notation is changed here, i.e., $w_1 = r$, $w_2 = W$, $w_3 = N$, $w_4 = k$, $w_5 = U$, and $w_6 = T$.

[2] It does not matter which equation is used to derive w_5. We could have derived w_5 from the sixth equation and obtained I_1 similarly.

[3] The Lie operator is $\Gamma = V\partial_t + \sum G_k \partial_{w_k}$.

[4] We note that the transformation suggested by the characteristic curve eliminated $\dot{w}_2$ from the fifth equation in system (5).

where

$$u_1 = w_2, \qquad u_2 = w_1, \qquad u_3 = w_3, \qquad u_4 = w_4, \qquad u_5 = r_6. \tag{7}$$

Applying Lie group analysis to system (6), we obtain a determining equation of parabolic type for V in two independent variables. Its characteristic curve is $u_4 + u_3$, which suggests the transformation $u_4 = r_4 - u_3$, where $r_4(t)$ is a new unknown function of t. System (6) then transforms into

$$\begin{aligned}
&\ddot{u}_1 = su_1^3 + su_1u_2u_5 + 3\dot{u}_1u_3 - \dot{u}_1u_4, \qquad \dot{u}_2 = u_2u_3,\\
&\dot{u}_3 = (u_4 - 2u_3)u_3 - 2\left(\frac{\dot{u}_1}{u_2}\right)^2,\\
&\dot{u}_4 = s(1 - 2au_2) + 3(u_4 - u_3)u_3 - u_4^2, \qquad \dot{u}_5 = -u_3u_5,
\end{aligned} \tag{8}$$

where

$$u_1 = w_2, \qquad u_2 = w_1, \qquad u_3 = w_3, \qquad u_4 = r_4, \qquad u_5 = r_6. \tag{9}$$

Applying Lie group analysis to system (8), we obtain a determining equation of parabolic type for V in two independent variables. Its characteristic curve is $\xi = u_2u_5$. We then have $V = \psi(\xi)$, where ψ is an arbitrary function of ξ, and the operator

$$\Gamma_1 = \psi(\xi)\,\partial_t \tag{10}$$

is consequently a generator of a Lie point symmetry for system (8). Transforming ξ into the original unknown functions yields $\xi = w_1w_6 - w_2^2$, which is exactly the first integral I_1 in (3). We note that if we had derived w_4 from the third equation in system (2), i.e., $w_4 = (\dot{w}_3 + w_3^2 + 2w_5^2)/w_3$, then we would have obtained I_1 analogously. The same would have happened if we had derived w_3 from the first equation in system (2). In fact, the expression for I_1 does not contain any of the dependent variables w_3, w_4, or w_5.

2.2. Obtaining I_2. If we derive w_2 from the sixth equation in system (2), i.e., $w_2=(\dot{w}_6+w_3w_6)/(2w_5)$, then we obtain a system of five equations, one of the second order and four of the first order. Applying Lie group analysis to this system, we obtain a determining equation of parabolic type for V in two independent variables. Its characteristic curve is $2w_5^2 - sw_6^2$, which suggests the transformation $w_5 = \sqrt{(sw_6^2 + 2r_5)/2}$, where $r_5(t)$ is a new unknown function of t. System (8) then transforms into

$$\begin{aligned}
&\ddot{u}_1 = -\big[2\big(\dot{u}_1u_4 - 2u_2u_5 - 2(u_3^2 - u_3u_4 + u_5)u_1\big)u_5 - (u_1 + u_2)s^2u_1^4 - su_1\dot{u}_1^2 -\\
&\qquad - \big((3u_3^2 - 2u_3u_4 + 4u_5)u_1^2 + ((2u_3 - u_4)\dot{u}_1 + 4u_2u_5)u_1\big)su_1\big]/(su_1^2 + 2u_5),\\
&\dot{u}_2 = u_2u_3, \qquad \dot{u}_3 = -\big(su_1^2 + 2u_5 + (u_3 - u_4)u_3\big),\\
&\dot{u}_4 = -\big(u_4^2 - 2u_5 + 2asu_2 - (u_1^2 + 1)s\big), \qquad \dot{u}_5 = (2u_3 - u_4)su_1^2 + 2(u_3 - u_4)u_5,
\end{aligned} \tag{11}$$

where

$$u_1 = w_6, \qquad u_2 = w_1, \qquad u_3 = w_3, \qquad u_4 = w_4, \qquad u_5 = r_5. \tag{12}$$

Applying Lie group analysis to system (11), we obtain a determining equation of parabolic type for V in four independent variables. Its characteristic curve is

$$\eta = u_2^2u_5 - \frac{2}{3}asu_2^3 + \frac{1}{2}u_2^2s - u_2^2u_3u_4 + \frac{1}{2}u_3^2u_2^2. \tag{13}$$

We then have $V = \phi(\eta)$, where ϕ is an arbitrary function of η, and the operator

$$\Gamma_2 = \phi(\eta)\partial_t \tag{14}$$

is consequently a generator of a Lie point symmetry for system (11). Transforming η into the original unknown functions yields the first integral I_2 as in (4).

3. Calogero's N-body problem in the plane

In [26], Calogero derived a solvable many-body problem of the form

$$\ddot{z}_n = 2 \sum_{\substack{m=1,\\ m\neq n}}^{N} \frac{\dot{z}_n \dot{z}_m}{z_n - z_m}, \quad n = 1, \ldots, N, \tag{15}$$

by considering the solvable nonlinear partial differential equation

$$\varphi_t + \varphi_x + \varphi^2 = 0, \qquad \varphi \equiv \varphi(x,t),$$

and examining the behavior of the poles of its solution. In [27], the same system (15) was presented, its properties were further studied, and its solution was given in terms of the roots of the algebraic equation in z

$$\sum_{m=1}^{N} \frac{\dot{z}_m(0)}{z - z_m(0)} = \frac{1}{t}. \tag{16}$$

Calogero called system (15) "a goldfish" following a statement by Zakharov [28]. In [6], we used the reduction method and Lie group analysis as introduced in [1] to show that Calogero's "goldfish" (15) is linearizable and can be transformed into the trivial system of $N-1$ second-order equations

$$\frac{d^2 \tilde{u}_j}{d\tilde{y}^2} = 0, \quad j = 1, \ldots, N-1, \tag{17}$$

a school of free particles indeed. Calogero also introduced a generalization of system (15),

$$\ddot{z}_n = 2 \sum_{\substack{m=1,\\ m\neq n}}^{N} a_{nm} \frac{\dot{z}_n \dot{z}_m}{z_n - z_m}, \quad n = 1, \ldots, N, \tag{18}$$

where the coupling constants satisfy the symmetry requirement $a_{nm} = a_{mn}$ and are otherwise arbitrary. Calogero and his collaborators have studied system (18) in numerous papers (see [29] and the references therein). For example, it was shown in [30] that if $N = 3$ then the general solution of system (18) can be obtained in closed form with special choices of the coupling constants a_{nm}. In [31], the Painlevé analysis was applied to system (18) in the cases $N = 2, 3, 4$.

Here, we apply the reduction method and show that for any $N > 2$ and any value of a_{nm}, system (18) can be reduced to one first-order ODE and a separate system of $N-1$ second-order equations that admits a two-dimensional Lie symmetry algebra and is therefore reducible to a system of $2N-4$ first-order ODEs. In particular, we exemplify the case $N = 3$, which we reduce to a system of two first-order ODEs, and the case $N = 4$, which we reduce to a system of four first-order ODEs. If $N = 2$, then we can always integrate system (18) by quadratures; in fact, the reduction method reduces system (18) to a single first-order ODE and a separate second-order ODE that always admits a two-dimensional Lie symmetry algebra L_2 and is therefore integrable by quadratures. As a consequence, Jacobi's last multiplier is obtained for this single second-order ODE by using L_2, and a Lagrangian is thus obtained [20]. Finally, we show that the special choice $a_{12} = -1/2$ yields a three-dimensional Lie symmetry algebra $sl(2, \mathbb{R})$, and that increasing the order via Jacobi's last multiplier yields a system of two equations (one of the first order and one of the second order) that admits a ten-dimensional Lie symmetry algebra and can therefore be linearized. A similar occurrence was found in [8] and also in [32].

3.1. Case $N = 2$. In the case $N = 2$, system (18) reduces to

$$\ddot{z}_1 = 2a_{12}\frac{\dot{z}_1\dot{z}_2}{z_1 - z_2}, \qquad \ddot{z}_2 = 2a_{12}\frac{\dot{z}_2\dot{z}_1}{z_2 - z_1}. \tag{19}$$

If we introduce four new dependent variables w_1, w_2, w_3, and w_4 such that

$$z_1 = w_1, \qquad z_2 = w_2, \qquad \dot{z}_1 = w_3, \qquad \dot{z}_2 = w_4, \tag{20}$$

then system (19) becomes an autonomous system of four first-order equations, i.e.,

$$\dot{w}_1 = w_3, \qquad \dot{w}_2 = w_4, \qquad \dot{w}_3 = 2a_{12}\frac{w_3 w_4}{w_1 - w_2}, \qquad \dot{w}_4 = 2a_{12}\frac{w_4 w_3}{w_2 - w_1}. \tag{21}$$

If we use the reduction method and introduce a new independent variable $y = w_1$, for example,[5] then system (21) reduces to the nonautonomous system of three first-order equations

$$w_2' = \frac{w_4}{w_3}, \qquad w_3' = 2a_{12}\frac{w_3 w_4}{w_3(y - w_2)}, \qquad w_4' = 2a_{12}\frac{w_4 w_3}{w_3(w_2 - y)}, \tag{22}$$

where the prime denotes the derivative with respect to y. Expressing w_4 from the first equation in system (22), i.e., $w_4 = w_2' w_3$, we obtain a system of two equations, the first-order equation

$$w_3' = -2a_{12}\frac{w_2' w_3}{w_2 - y} \tag{23}$$

and the second-order equation

$$w_2'' = 2a_{12}\frac{w_2'(w_2' + 1)}{w_2 - y}. \tag{24}$$

If we apply Lie group analysis to this system, then the parabolic equations and their characteristic curves suggest the transformations of dependent variables

$$w_3 = -\frac{r_3}{(w_2 - y)^{2a_{12}}}, \tag{25}$$

where r_3 is a new variable y, which transforms (23) into

$$r_3' = -2a_{12}\frac{r_3}{w_2 - y}, \tag{26}$$

and

$$w_2 = u + y, \tag{27}$$

where u is a new variable dependent on y, which transforms (24) into

$$u'' = 2a_{12}\frac{(u' + 2)(u' + 1)}{u}. \tag{28}$$

We note that this equation is independent of r_3 and that Eq. (26) can be easily integrated as soon as the general solution of (28) is determined. Applying Lie group analysis to (28), we obtain a non-Abelian, transitive, two-dimensional Lie symmetry algebra[6] spanned by the operators

$$\Gamma_1 = \partial_y, \qquad \Gamma_2 = y\partial_y + u\partial_u, \tag{29}$$

[5]We could choose any other dependent variable as the new independent variable.

[6]We showed in [6] that if $a_{12} = 1$, then this equation admits an eight-dimensional Lie symmetry algebra and is therefore linearizable [18].

and the general solution of (28) can therefore be easily obtained in implicit form by two quadratures, i.e.,

$$-2ua_{12} - u\,\mathrm{LerchPhi}\left(2u^{2a_{12}}c_1, 1, \frac{1}{2a_{12}}\right) - 4ya_{12} = c_2, \tag{30}$$

where c_1 and c_2 arbitrary constants and $\mathrm{LerchPhi}(z, a, v) = \sum_{n=0}^{\infty} z^n/(v+n)^a$ as defined in MAPLE 9.

Lie discovered the link between symmetries and Jacobi's last multiplier [18], [19], namely, that Jacobi's last multiplier can be obtained as the inverse of a certain matrix that contains the admitted symmetries.[7] In fact, Jacobi's last multiplier M of Eq. (28) is equal to the inverse of the determinant

$$\Delta = \det \begin{pmatrix} 1 & u' & 2a_{12}\dfrac{(u'+2)(u'+1)}{u} \\ y & u & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{31}$$

i.e.,

$$M = \frac{1}{\Delta} = -\frac{1}{2a_{12}(u'+2)(u'+1)}. \tag{32}$$

The well-known relation between Jacobi's last multiplier M and the Lagrangian $L(u, u', y)$ for any second-order equation [20]

$$M = \frac{\partial^2 L}{\partial u'^2} \tag{33}$$

then yields the Lagrangian for (28)

$$L = -\frac{2\log(u)a_{12} + 1 + (u'+1)\log(u'+1) - (u'+2)\log(u'+2)}{2a_{12}}. \tag{34}$$

Finally, we note that if $a_{12} = -1/2$, then Eq. (28), i.e.,

$$u'' = -\frac{(u'+2)(u'+1)}{u} \Rightarrow (u_1 = u, u_2 = u') \Rightarrow \begin{array}{l} u_1' = u_2, \\ u_2' = -\dfrac{(u_2+2)(u_2+1)}{u_1}, \end{array} \tag{35}$$

admits a three-dimensional Lie symmetry algebra $sl(2, \mathbb{R})$, spanned by the operators

$$\Gamma_1 = \partial_y, \qquad \Gamma_2 = y\partial_y + u\partial_u, \qquad \Gamma_3 = (u^2 - 4y^2)\partial_y - 4u(u+2y)\partial_u. \tag{36}$$

This means that we can obtain three Jacobi last multipliers (and consequently three different Lagrangians), namely, M_1 as given in (32), i.e.,

$$M_1 = \frac{1}{(u_2+2)(u_2+1)}, \tag{37}$$

M_2 via the determinant

$$\Delta_2 = \det \begin{pmatrix} 1 & u_2 & -\dfrac{(u_2+2)(u_2+1)}{u_1} \\ 1 & 0 & 0 \\ u_1^2 - 4y^2 & u_1(-4u_1 - 8y) & 2u_1(-u_2^2 - 4u_2 - 4) \end{pmatrix}, \tag{38}$$

[7] A review of the properties of Jacobi's last multiplier can be found in [12].

i.e.,

$$M_2 = \frac{1}{\Delta_2} = \frac{1}{2(u_2+2)(u_2^2u_1 + 4u_2y + 4u_2u_1 + 4y + 2u_1)}, \tag{39}$$

and M_3 via the determinant

$$\Delta_3 = \det \begin{pmatrix} 1 & u_2 & -\dfrac{(u_2+2)(u_2+1)}{u_1} \\ y & u_1 & 0 \\ u_1^2 - 4y^2 & u_1(-4u_1 - 8y) & 2u_1(-u_2^2 - 4u_2 - 4) \end{pmatrix}, \tag{40}$$

i.e.,

$$M_3 = \frac{1}{\Delta_3} = \frac{1}{(u_2+2)(2u_2y + 2y - u_1)(u_2u_1 + 3u_1 + 2y)}. \tag{41}$$

An important property of Jacobi's last multiplier is that the ratio of two multipliers is a first integral. Therefore, the three ratios

$$\begin{aligned} I_1 &= \frac{M_1}{M_2} = 2\frac{u_2^2u_1 + 4u_2y + 4u_2u_1 + 4y + 2u_1}{u_2+1}, \\ I_2 &= \frac{M_1}{M_3} = \frac{(2u_2y + 2y - u_1)(u_2u_1 + 3u_1 + 2y)}{u_2+1}, \\ I_3 &= \frac{M_3}{M_2} = 2\frac{u_2^2u_1 + 4u_2y + 4u_2u_1 + 4y + 2u_1}{(2u_2y + 2y - u_1)(u_2u_1 + 3u_1 + 2y)} \end{aligned} \tag{42}$$

are first integrals[8] of equation/system (35). Other applications of this property can be found in [12], [33]–[35].

We now use the method developed in [12] to show that (35) hides linearity. Jacobi's last multiplier J of (35) satisfies the equation

$$\frac{d\log J}{dy} + \frac{\partial u_2}{\partial u_1} + \frac{\partial}{\partial u_2}\left(\frac{-(u_2+2)(u_2+1)}{u_1}\right) = \frac{d\log J}{dy} - \frac{2u_2+3}{u_1} = 0. \tag{43}$$

This suggests increasing the order of system (35) by introducing a new variable r dependent on y such that

$$u_2 = \frac{r'}{2r}u_1 - \frac{3}{2}. \tag{44}$$

System (35) then becomes the system of two equations (one of the first order and one of the second order)

$$u_1' = \frac{r'u_1 - 3r}{2r}, \qquad r'' = \frac{3r'u_1 + r}{2u_1^2}. \tag{45}$$

If we apply Lie group analysis to this system, then the parabolic equation and its characteristic curve suggest the transformation of the dependent variable $u_1 = \sqrt{r}s$, where s is a new variable dependent on y, which transforms system (45) into

$$s' = -\frac{3\sqrt{r}}{2r}, \qquad r'' = \frac{3\sqrt{r}r's + r}{2rs^2}. \tag{46}$$

[8]Obviously, only two of the integrals are linearly independent.

This system admits a ten-dimensional Lie symmetry algebra spanned by the operators

$$
\begin{aligned}
X_1 &= \frac{1}{s^{2/3}}(s^2 r\partial_y - 3(s^2 r^{1/2} + sy)\partial_s + 2ry\partial_r),\\
X_2 &= \frac{1}{s^{1/3}}(sr\partial_y - 3sr^{1/2}\partial_s + 2r^{3/2}\partial_r),\\
X_3 &= (s^2 r - 4y^2)\partial_y - (3s^2 r^{1/2} + 6sy)\partial_s - (2r^{3/2}s + 4ry)\partial_r,\\
X_4 &= -\frac{1}{3s^{1/3}}(-4sy\partial_y - 3s^2\partial_s + (8r^{1/2}y + 2sr)\partial_r),\\
X_5 &= -\frac{1}{3s^{2/3}}(-3s\partial_s + 2r\partial_r),\\
X_6 &= s\partial_s - 2r\partial_r, \qquad X_7 = y\partial_y + 2r\partial_r, \qquad X_8 = s^{4/3}\partial_y - 4r^{1/2}s^{1/3}\partial_r,\\
X_9 &= \frac{1}{s^{1/3}}(s\partial_y - 2r^{1/2}\partial_r), \qquad X_{10} = \partial_y
\end{aligned}
\tag{47}
$$

and is therefore linearizable. To find the linearizing transformation, we seek a three-dimensional subalgebra $A_1 \oplus A_2$ and transform it into its canonical form, i.e., to the form

$$
\partial_{\tilde{y}}, \qquad \partial_{\tilde{u}_1}, \qquad \tilde{u}_1\partial_{\tilde{u}_1} + \tilde{u}_2\partial_{\tilde{u}_2}. \tag{48}
$$

We find that one such algebra is generated by the operators X_{10}, X_5, and $3X_6/2$. It is then easy to find that the canonical variables are

$$
\tilde{y} = y, \qquad \tilde{u}_1 = \frac{3}{2}s^{2/3}, \qquad \tilde{u}_2 = \frac{1}{s^{1/3}\sqrt{r}}, \tag{49}
$$

and system (46) becomes

$$
\frac{d\tilde{u}_1}{d\tilde{y}} = -\frac{3}{2}\tilde{u}_2, \qquad \frac{d^2\tilde{u}_2}{d\tilde{y}^2} = \frac{3}{\tilde{u}_2}\left(\frac{d\tilde{u}_2}{d\tilde{y}}\right)^2. \tag{50}
$$

Expressing $\tilde{u}_2$ from the first equation, i.e., $\tilde{u}_2 = (2/3)\,d\tilde{u}_1/d\tilde{y}$, we obtain the third-order ODE

$$
\frac{d^3\tilde{u}_1}{d\tilde{y}^3}\frac{d\tilde{u}_1}{d\tilde{y}} = 3\left(\frac{d^2\tilde{u}_1}{d\tilde{y}^2}\right)^2, \tag{51}
$$

which admits a six-dimensional Lie symmetry algebra (in fact, ten contact symmetries) and is therefore linearizable. Equation (51) is the well-known Schwarzian differential equation, which can be transformed into the Riccati equation

$$
\frac{dv}{d\tilde{y}} = 2v^2, \qquad v = \frac{d^2\tilde{u}_1}{d\tilde{y}^2}\bigg/\frac{d\tilde{u}_1}{d\tilde{y}}.
$$

We note that the second equation in (50) admits an eight-dimensional Lie symmetry and is therefore linearizable [18], i.e.,

$$
\frac{d^2U}{dx^2} = 0, \qquad \text{with } U = \frac{1}{2\tilde{y}\tilde{u}_2}, \quad \frac{1}{\tilde{y}}.
$$

3.2. Case $N = 3$. If we introduce six new dependent variables w_1, w_2, w_3, w_4, w_5, and w_6 such that

$$z_1 = w_1, \qquad z_2 = w_2, \qquad z_3 = w_3, \qquad \dot{z}_1 = w_4, \qquad \dot{z}_2 = w_5, \qquad \dot{z}_3 = w_6, \tag{52}$$

then system (18) becomes the autonomous system of six first-order equations

$$\begin{aligned} &\dot{w}_1 = w_4, \qquad \dot{w}_2 = w_5, \qquad \dot{w}_3 = w_6, \qquad \dot{w}_4 = 2\left(a_{12}\frac{w_4 w_5}{w_1 - w_2} + a_{13}\frac{w_4 w_6}{w_1 - w_3}\right), \\ &\dot{w}_5 = 2\left(a_{12}\frac{w_5 w_4}{w_2 - w_1} + a_{23}\frac{w_5 w_6}{w_2 - w_3}\right), \qquad \dot{w}_6 = 2\left(a_{13}\frac{w_6 w_4}{w_3 - w_1} + a_{23}\frac{w_6 w_5}{w_3 - w_2}\right). \end{aligned} \tag{53}$$

If we follow the reduction method and introduce a new independent variable $y = w_1$, for example,[9] then system (53) reduces to the nonautonomous system of five first-order equations

$$\begin{aligned} &w_2' = \frac{w_5}{w_4}, \qquad w_3' = \frac{w_6}{w_4}, \\ &w_4' = 2\left(a_{12}\frac{w_5}{y - w_2} + a_{13}\frac{w_6}{y - w_3}\right), \\ &w_5' = 2\left(a_{12}\frac{w_5}{w_2 - y} + a_{23}\frac{w_5 w_6}{w_4(w_2 - w_3)}\right), \\ &w_6' = 2\left(a_{13}\frac{w_6}{w_3 - y} + a_{23}\frac{w_6 w_5}{w_4(w_3 - w_2)}\right), \end{aligned} \tag{54}$$

where the prime denotes the derivative with respect to y. Expressing w_5 from the first equation and w_6 from the second equation in system (54), $w_5 = w_2' w_4$ and $w_6 = w_3' w_4$, we obtain a single first-order equation for w_4 and a separate system of two second-order equations for w_2 and w_3. If we apply Lie group analysis to this system, then the parabolic equations and their characteristic curves suggest the transformations of the dependent variables

$$w_4 = \frac{r_4}{(w_3 - y)^{2a_{13}}(w_2 - y)^{2a_{12}}}, \tag{55}$$

where r_4 is a new variable dependent on y, which transforms the single first-order equation for w_4 into the equation

$$r_4' = -2r_4\frac{a_{12}(w_3 - y) + a_{13}(w_2 - y)}{(w_2 - y)(w_3 - y)}, \tag{56}$$

and

$$w_2 = u_1 + y, \qquad w_3 = u_2 + y, \tag{57}$$

where u_1 and u_2 are new variables dependent on y, which transforms the system of two second-order equations for w_2 and w_3 into the system

$$\begin{aligned} u_1'' &= \frac{2\big[\big((u_1' + 2)a_{12}u_2 + (u_2' + 1)a_{13}u_1\big)(u_1 - u_2) + (u_2' + 1)a_{23}u_1 u_2\big](u_1' + 1)}{(u_1 - u_2)u_1 u_2}, \\ u_2'' &= \frac{2\big[\big((u_1' + 1)a_{12}u_2 + (u_2' + 2)a_{13}u_1\big)(u_1 - u_2) - (u_1' + 1)a_{23}u_1 u_2\big](u_2' + 1)}{(u_1 - u_2)u_1 u_2}. \end{aligned} \tag{58}$$

[9] We could choose any other dependent variable as the new independent variable.

Applying Lie group analysis only to system (58), we obtain a two-dimensional Lie symmetry algebra[10] spanned by the operators

$$\Gamma_1 = \partial_y, \qquad \Gamma_2 = y\partial_y + \sum_{n=1}^{2} u_n \partial_{u_n}. \tag{59}$$

Its differential invariants of order ≤ 1 are

$$\tilde{y} = \frac{u_2}{u_1}, \qquad \tilde{u}_1 = u_1', \qquad \tilde{u}_2 = u_2', \tag{60}$$

which reduce system (58) to the system of two first-order equations

$$\begin{aligned} \frac{d\tilde{u}_1}{d\tilde{y}} &= -\frac{2\big[\big((\tilde{y}-1)a_{13} - a_{23}\tilde{y}\big)(\tilde{u}_2+1) + (\tilde{u}_1+2)(\tilde{y}-1)a_{12}\tilde{y}\big](\tilde{u}_1+1)}{(\tilde{u}_1\tilde{y} - \tilde{u}_2)(\tilde{y}-1)\tilde{y}}, \\ \frac{d\tilde{u}_2}{d\tilde{y}} &= -\frac{2\big[\big((\tilde{y}-1)a_{12} + a_{23}\big)(\tilde{u}_1+1)\tilde{y} + (\tilde{u}_2+2)(\tilde{y}-1)a_{13}\big](\tilde{u}_2+1)}{(\tilde{u}_1\tilde{y} - \tilde{u}_2)(\tilde{y}-1)\tilde{y}}. \end{aligned} \tag{61}$$

3.3. Case $N = 4$. If we introduce eight new dependent variables w_1, w_2, w_3, w_4, w_5, w_6, w_7, and w_8 such that

$$\begin{aligned} &z_1 = w_1, \qquad z_2 = w_2, \qquad z_3 = w_3, \qquad z_4 = w_4, \\ &\dot{z}_1 = w_5, \qquad \dot{z}_2 = w_6, \qquad \dot{z}_3 = w_7, \qquad \dot{z}_4 = w_8, \end{aligned} \tag{62}$$

then system (18) becomes an autonomous system of eight first-order equations. Following the reduction method and introducing a new independent variable $y = w_1$, for example, we obtain a nonautonomous system of seven first-order equations. Expressing w_6 from the first equation of that system, w_7 from the second equation, and w_8 from the third equation,

$$w_6 = w_2' w_5, \qquad w_7 = w_3' w_5, \qquad w_8 = w_4' w_5, \tag{63}$$

we obtain the single first-order equation

$$w_5' = \frac{-2w_5\big[\big((w_3-y)w_4'a_{14} + (w_4-y)w_3'a_{13}\big)(w_2-y) + (w_3-y)(w_4-y)w_2'a_{12}\big]}{(w_2-y)(w_3-y)(w_4-y)} \tag{64}$$

and a system of three second-order equations. If we apply Lie group analysis to this system, then the parabolic equations and their characteristic curves suggest the transformations of the dependent variables

$$w_5 = \frac{r_5}{(w_4-y)^{2a_{14}}(w_3-y)^{2a_{13}}(w_2-y)^{2a_{12}}}, \tag{65}$$

where r_5 is a new variable dependent on y, which transforms (64) into the equation

$$r_5' = -2r_5 \frac{\big(a_{13}(w_4-y) + a_{14}(w_3-y)\big)(w_2-y) + a_{12}(w_4-y)(w_3-y)}{(w_2-y)(w_3-y)(w_4-y)}, \tag{66}$$

and

$$w_2 = u_2 + y, \qquad w_3 = u_3 + y, \qquad w_4 = u_1 + y, \tag{67}$$

[10] We showed in [6] that if $a_{nm} = 1$, then this system admits a 15-dimensional Lie symmetry algebra and is therefore linearizable.

where u_1, u_2, and u_3 are new variables dependent on y, which transforms the system of three second-order equations in w_4, w_2, and w_3 into a system of three second-order equations in u_1, u_2, and u_3 that admits a two-dimensional Lie symmetry algebra[11] spanned by the operators

$$\Gamma_1 = \partial_y, \qquad \Gamma_2 = y\partial_y + \sum_{n=1}^{3} u_n \partial_{u_n}. \tag{68}$$

Its differential invariants of order ≤ 1 are

$$\tilde{y} = \frac{u_2}{u_1}, \qquad \tilde{u}_1 = u_1', \qquad \tilde{u}_2 = u_2', \qquad \tilde{u}_3 = u_3', \qquad \tilde{u}_4 = \frac{u_4}{u_1}, \tag{69}$$

which reduce system (58) to the system of four first-order equations

$$\begin{aligned}
\frac{d\tilde{u}_1}{d\tilde{y}} &= \bigl[-2\bigl(((\tilde{u}_4 - 1)a_{13} - a_{34}\tilde{u}_4)(\tilde{u}_3 + 1)(\tilde{y} - 1)\tilde{y} + \\
&\quad + ((\tilde{y} - 1)a_{12} - a_{24}\tilde{y})(\tilde{u}_2 + 1)(\tilde{u}_4 - 1)\tilde{u}_4 + \\
&\quad + (\tilde{u}_1 + 2)(\tilde{u}_4 - 1)(\tilde{y} - 1)a_{14}\tilde{u}_4\tilde{y}\bigr)(\tilde{u}_1 + 1)\bigr] \times \\
&\quad \times \bigl[(\tilde{u}_1\tilde{y} - \tilde{u}_2)(\tilde{u}_4 - 1)(\tilde{y} - 1)\tilde{u}_4\tilde{y}\bigr]^{-1}, \\
\frac{d\tilde{u}_2}{d\tilde{y}} &= \bigl[-2\bigl(\bigl(((\tilde{u}_4 - \tilde{y})a_{13} - a_{23}\tilde{u}_4)(\tilde{u}_3 + 1)\tilde{y} + \\
&\quad + (\tilde{u}_2 + 2)(\tilde{u}_4 - \tilde{y})a_{12}\tilde{u}_4\bigr)(\tilde{y} - 1) + \\
&\quad + ((\tilde{y} - 1)a_{14} + a_{24})(\tilde{u}_1 + 1)(\tilde{u}_4 - \tilde{y})\tilde{u}_4\tilde{y}\bigr)(\tilde{u}_2 + 1)\bigr] \times \\
&\quad \times \bigl[(\tilde{u}_1\tilde{y} - \tilde{u}_2)(\tilde{u}_4 - \tilde{y})(\tilde{y} - 1)\tilde{u}_4\tilde{y}\bigr]^{-1}, \\
\frac{d\tilde{u}_3}{d\tilde{y}} &= \bigl[-2\bigl(\bigl(((\tilde{u}_4 - \tilde{y})a_{12} + a_{23}\tilde{y})(\tilde{u}_2 + 1)\tilde{u}_4 + \\
&\quad + (\tilde{u}_3 + 2)(\tilde{u}_4 - \tilde{y})a_{13}\tilde{y}\bigr)(\tilde{u}_4 - 1) + \\
&\quad + ((\tilde{u}_4 - 1)a_{14} + a_{34})(\tilde{u}_1 + 1)(\tilde{u}_4 - \tilde{y})\tilde{u}_4\tilde{y}\bigr)(\tilde{u}_3 + 1)\bigr] \times \\
&\quad \times \bigl[(\tilde{u}_1\tilde{y} - \tilde{u}_2)(\tilde{u}_4 - \tilde{y})(\tilde{u}_4 - 1)\tilde{u}_4\tilde{y}\bigr]^{-1}, \\
\frac{d\tilde{u}_4}{d\tilde{y}} &= \frac{\tilde{u}_1\tilde{u}_4 - \tilde{u}_3}{\tilde{u}_1\tilde{y} - \tilde{u}_2}.
\end{aligned} \tag{70}$$

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