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A model for linear dragging

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Abstract

We try to carry over, as closely as possible, the well-known results for rotational dragging (Thirring, Brill and Cohen) to dragging due to linearly accelerated masses. To this end, a spherical, charged mass shell is linearly accelerated by a (weak) external, axisymmetric and dipolar charge distribution. It is shown that the interior of this (Reissner–Nordström-like) shell stays flat. The dragging of neutral test particles inside the shell, defined by their acceleration, scaled by the overall acceleration of the (rigid) shell, is calculated for the weak field case for a highly massive but weakly charged shell and for the general strong field case. The results compare favourably with the corresponding results for rotational dragging.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The idea that in a relativistic gravitation theory accelerated masses may induce a dragging effect on test bodies—in analogy with electromagnetic induction—was first formulated by Einstein in 1912 within a preliminary relativistic scalar gravitation theory [1]. In this paper, he also introduced for the first time the extremely useful model of an infinitely thin spherical mass shell. One year later Einstein showed, now in the tensorial Entwurf theory, that a rotating mass shell induces a Coriolis-type dragging force on the inertial frames in its interior [2]. (The dragging by a linearly accelerated mass shell within the Entwurf theory is calculated in the Einstein–Besso manuscript (June 1913) on the motion of the perihelion of Mercury [3].) As is well known, in final general relativity the Coriolis-type dragging force was confirmed by Thirring in a paper [4] whose essential results took shape only after Einstein, in correspondence [5] with Thirring, provided decisive hints and corrections [6].

(The corresponding Lense–Thirring dragging effect in the exterior of a rotating body is now—86 years after its theoretical prediction—for the first time directly experimentally confirmed within 10% by the LAGEOS satellites [7], and a confirmation with better than 1% is expected from the recently launched Gravity Probe B satellite [8].) For a mass shell with mass M, radius R and angular velocity ω, Thirring derived in the first orders of M/R and ω a Coriolis-type force with dragging factor $4M/3R$. In second order of ω, an additional force showed up which was treated by Thirring as a centrifugal force although it had also an axial component and could not be made zero in the same rotating frame in which the Coriolis-type force vanishes. In 1966 Brill and Cohen succeeded in extending Thirring's calculations to arbitrary values of M/R, but still only in first order of ω [9]. Their main result was that in the collapse limit of the mass shell the dragging factor attains the value 1, i.e. inertial systems inside the mass shell are dragged along with the full angular velocity ω of the shell. This result confirms (at least partly) that for physically reasonable models within general relativity the Machian postulate of relativity of rotation is fulfilled. An extension of these calculations to higher orders of ω was performed in [10–12]. By allowing for a nonspherical form of the rotating mass shell, for a nonspherical mass distribution on it and for differential rotation, it was (for the first time) possible to derive a correct centrifugal force in its interior, and to realize flat geometry inside the shell in all orders of ω. The question of whether and how rotating masses influence properties other than inertial ones, especially of how they influence electromagnetic phenomena, was taken up in the papers [13–16]. For instance, it was shown that a rotating mass shell induces (in first order of ω) to a charge in its interior a dipolar magnetic field. The claim in [15] that some of these effects are 'Mach-negative or, at best, Mach-neutral' was corrected in [17].

The question addressed in the present paper is whether and how all these dragging effects derived for rotating mass shells carry over from a rotational acceleration to a linear acceleration, as initiated by Einstein in [1]. A quite general and severe problem with linearly accelerated bodies is that they need—in contrast to rotating bodies—a perpetual supply of energy in order to maintain the acceleration. And since in general relativity the equations of motion of bodies are already contained in the field equations, the energy source (or the motor of the accelerated system) has to be included in the considered system in order to deal with a self-consistent problem. This difficulty may be the reason why, besides the historical Einstein paper [1], only a few articles [18–21] treat (or claim to treat) dragging effects due to linearly accelerated masses. And even these papers compare only quite poorly to the rotating systems considered in [4] and [9–17], because they treat only the weak field case or contain special relations between mass M and charge q of the shell. Furthermore, in some of these papers the source of acceleration is not really fixed, or is removed to infinity, with the consequence that the equations of motion are in danger of being violated. And in no model considered so far is it guaranteed that the geometry inside the shell is flat so that pretend dragging effects cannot be clearly distinguished from local gravitational effects due to curvature.

In this paper we start (in section 2) with a spherical Reissner–Nordström (RN) shell of nearly arbitrary mass M, charge q and radius R, and we discuss the (weak and dominant) energy conditions for the shell material. In the main section 3 we consider a (first-order) translational acceleration of this RN shell and study the resulting dragging effects. In order to include the source of acceleration into the system but to disturb electrically neutral test particles inside the shell as little as possible, it seems appropriate to take as the source of acceleration a charge distribution outside the shell. This has a further advantage because (at least in the electro-vacuum regions) the Einstein–Maxwell equations are not much more complicated than the pure Einstein equations. (We expect, however, that other acceleration mechanisms lead to similar dragging results.) The simplest nontrivial model obviously is an axisymmetric and dipolar charge distribution $\lambda\sigma(r)\cos\vartheta$, treated in first order of the dimensionless smallness parameter λ. We require that the system is time-symmetric around $t = 0$, and we treat the system up to terms of order t^2, in order to be able to calculate the acceleration of the RN shell. (For later times t the system may change drastically because part of the charges $\sigma(r)$ will hit the RN shell.) In the case of an appropriate asymptotic fall-off behaviour of the charge function $\sigma(r)$ the system is asymptotically flat, like the systems in [4] and [9–17]. For this model we give (in section 3.1) the complete Einstein–Maxwell equations in first order of λ. We choose the solutions of the Maxwell equations (on the RN background) such that there are no magnetic fields up to order t^2, and therefore also no electromagnetic waves (which could violate the time symmetry of the system). We prove that the interior of the RN shell stays flat in this order, and we give the geodesic equation for neutral test particles in this region (in section 3.2).

In section 3.3 we discuss the Einstein–Maxwell equations in the exterior region, and we state (in the 'invariant' Israel formalism [22]) the junction conditions between exterior and interior solutions. In order to fix our model system completely, i.e. to fix all integration constants of the Einstein–Maxwell equations, we have to specify the geometrical, mechanical and electrical properties of the accelerated RN shell. We do this in as complete analogy to the rotating mass shell (in [4, 9]) as possible. Since a mass shell rotating in first order of the angular velocity ω stays automatically spherical and does not in this order receive corrections to its mass density and pressure, we demand that our RN shell is (in first order of λ) rigidly accelerated, and we set the correction terms to energy density and pressure equal to zero. Furthermore, the shell material is chosen as electrically isolating. With these conditions the linear acceleration g of neutral test particles inside the RN shell (as measured at infinity), and also the acceleration b of the RN shell are uniquely determined, and herewith also the 'dragging factor' $d = g/b$. In analogy with the historical development of dragging effects for a rotating shell in [4, 9, 17], we divide the detailed discussion of our results into three steps: we begin in section 3.3.1 with the weak field case $M/R \ll 1$, and $q/R \ll 1$, and we find that for the simplest power law charge distribution $\sigma(r) \sim r^{-5}$, having a finite dipole moment, the dragging factor coincides with Thirring's value $4M/3R$. We calculate and discuss the factor d also for other charge distributions. In section 3.3.2 we consider a shell with arbitrary mass but small charge, and we find that for $\sigma(r) \sim r^{-5}$ the dragging factor d has a similar dependence on M/R as for the rotating mass shell in [9]. Especially, in the collapse limit $2M/R \to 1$ we have $d \to 1$, i.e. total dragging, for arbitrary charge distributions $\sigma(r)$. In section 3.3.3 we analyse the general strong field case. Since we did not succeed in separating the system of differential equations, we turn to a numerical integration procedure. We then compare (graphically) the linear dragging factor, in its dependence on M/R and q/R, with the results for rotational acceleration in [23].

In total it is now clear that for (first-order) linear and rotational accelerations of a mass shell, the interior of this shell can be kept flat, and that the dragging effects in this shell exactly mimic the corresponding well-known 'inertial forces' in accelerated reference systems in Newtonian physics. Since general accelerations can (in principle) be constructed from appropriate linear and circular accelerations, we have now in hand very good arguments for the validity of a 'quasiglobal equivalence principle' [10] in general relativity: 'If some finite laboratory (a flat region in spacetime) is in arbitrary (weak) accelerated motion relative to the fixed stars, then all motions of free particles and all physical laws, measured from laboratory axes, are modified by inertial forces. It is argued that exactly the same modified motions and laws can be induced (at least for some time) at all places of a laboratory at rest relative to the fixed stars, by suitable and suitably moving masses outside the laboratory (e.g. in a mass shell).' In this connection it may be remarked that already in the dawn of general relativity, in the years

1912–1913, similar ideas arose in discussions of Einstein with Ehrenfest [24] and Mie [2]. But at that time the participants were quite sceptical about such a 'macro-equivalence'.

2. The Reissner–Nordström shell

Our zero-order model is a spherical and static Reissner–Nordström (RN) shell. In the standard RN coordinates $(x^0, x^1, x^2, x^3) = (t, r, \vartheta, \varphi)$ this is given by the metric

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu = -F(r)\,dt^2 + F(r)^{-1}\,dr^2 + r^2(\,d\vartheta^2 + \sin^2\vartheta\,d\varphi^2), \qquad (1)$$

with $F(r) = 1 - 2M/r + q^2/r^2$ for $r > R$, and $F(r) = 1$ for $r < R$. The Ricci tensor of this metric is of course identically zero for $r < R$, and has components $R^0{}_0 = R^1{}_1 = -R^2{}_2 = -R^3{}_3 = q^2/r^4$ (and therefore Ricci scalar zero) for $r > R$. The electromagnetic field tensor $F_{\mu\nu}$ is identically zero for $r < R$, and has only one non-zero component $F_{10} = -F_{01} = q/r^2$ for $r > R$. For the calculation of the energy–momentum tensor of the RN shell at $r = R$ we use the Israel formalism [22] which expresses this tensor in a purely geometrical way by the extrinsic curvatures of the embeddings of this shell Σ in the different spacetimes V^+ for $r > R$, and V^- for $r < R$, and does not ask for a continuous metric across the shell. (For an extension of the Israel formalism to charged shells, see [25].) In contrast to Israel and Kuchař, we define the signs of the Riemann tensor and of the extrinsic curvature according to the convention in modern textbooks. Let $(\tau, \vartheta, \varphi)$ be the intrinsic coordinates in the shell, with $\tau = t^- = \sqrt{F(R)}t^+$. (The RN time t is discontinuous at $r = R$!) As basis vectors in Σ we choose $e^{+\mu}_{\tau} = (F(R)^{-1/2}, 0, 0, 0)$, $e^{-\mu}_{\tau} = (1, 0, 0, 0)$, $e^{+\mu}_{\vartheta} = e^{-\mu}_{\vartheta} = (0, 0, 1, 0)$, $e^{+\mu}_{\varphi} = e^{-\mu}_{\varphi} = (0, 0, 0, 1)$. Then the unit normal vectors to Σ in V^+ and V^- are $n^{+\mu} = (0, F(R)^{1/2}, 0, 0)$ and $n^{-\mu} = (0, 1, 0, 0)$. The symmetric extrinsic curvature 3-tensors (in V^+ and V^-) are then given by

$$K_{ab} = n_\mu e^\mu{}_{a;b}, \qquad (2)$$

with $a, b \in (\tau, \vartheta, \varphi)$, and where the covariant derivatives are calculated from the 3-metric $g_{ab} = \mathrm{diag}(-1, R^2, R^2\sin^2\vartheta)$ in Σ. The resulting non-zero components of K_{ab} are

$$K^+_{\tau\tau} = F'(R)/2\sqrt{F(R)}, \qquad (3a)$$

$$K^+_{\vartheta\vartheta} = K^+_{\varphi\varphi}/\sin^2\vartheta = -R\sqrt{F(R)}, \qquad (3b)$$

$$K^-_{\vartheta\vartheta} = K^-_{\varphi\varphi}/\sin^2\vartheta = -R, \qquad (3c)$$

with $F'(R) = \frac{d}{dr}F(r)\big|_{r=R} = 2M/R^2 - 2q^2/R^3$. With the tensor $\gamma_{ab} = K^+_{ab} - K^-_{ab}$, the energy–momentum tensor S_{ab} of the shell Σ is, according to Israel [22], given by $8\pi S_{ab} = \gamma_{ab} - g_{ab}\gamma^c_c$, with the results

$$8\pi S^\tau{}_\tau = \frac{2}{R}(\sqrt{F(R)} - 1) = \frac{2}{R}(\sqrt{1 - \alpha + \beta} - 1), \qquad (4a)$$

$$8\pi S^\vartheta{}_\vartheta = 8\pi S^\varphi{}_\varphi = \frac{1}{R\sqrt{F(R)}}\left(\frac{R}{2}F'(R) + F(R) - \sqrt{F(R)}\right)$$
$$= \frac{1}{R\sqrt{1 - \alpha + \beta}}\left(1 - \frac{\alpha}{2} - \sqrt{1 - \alpha + \beta}\right), \qquad (4b)$$

with the useful, dimensionless abbreviations $\alpha = 2M/R$ and $\beta = q^2/R^2$. Obviously, the stresses in the shell vanish in the extreme RN case $\beta = \alpha^2/4$, and they diverge in the collapse limit $\beta = \alpha - 1$.

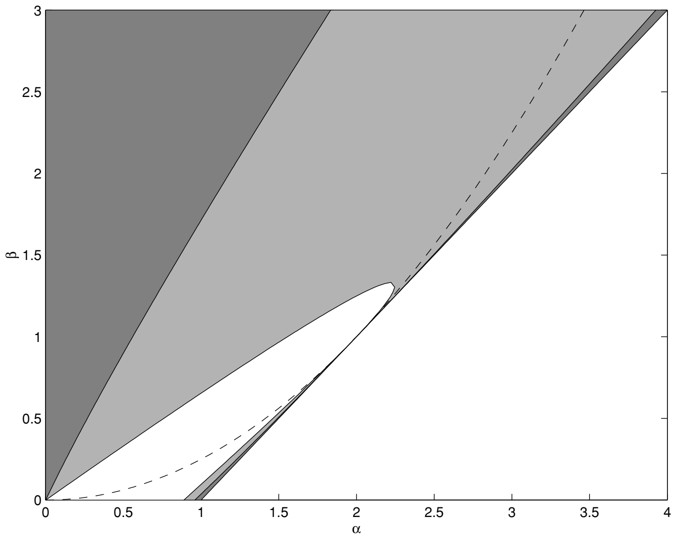

Figure 1. In the variables $\alpha = 2M/R$ and $\beta = q^2/R^2$ are shown: (*a*) in white: the region where both the weak and the dominant energy conditions are fulfilled, (*b*) in light grey shading: the region where the weak energy condition is violated, (*c*) in dark grey shading: the region where the dominant energy condition is violated. The dashed parabola $\beta = \alpha^2/4$ represents the extreme RN case.

In addition to the conditions $\alpha \geqslant 0, \beta \geqslant 0$ and $\beta \geqslant \alpha - 1$, one may wish that the charged shell material has 'reasonable physical properties', and that therefore the dragging induced on neutral test particles in the interior of an accelerated shell (in section 3) fulfils reasonable physical expectations. Standard conditions for a 'reasonable shell material' are the fulfilment of the weak energy condition and of the dominant energy condition [26]. (For rotational acceleration, it was shown in [16, 23] that at least in part of the parameter region (α, β) where these energy conditions are violated, one has counterintuitive effects such as anti-dragging. Therefore, we would like to discuss similar questions also for our linearly accelerated RN shell in section 3.3.3.) The weak energy condition $S_{ab}u^a u^b \geqslant 0$ for all timelike vectors u^a (in Σ) consists, in our case, of the two inequalities $S^{\tau}{}_{\tau} \leqslant 0$, and $S^{\tau}{}_{\tau} - S^{\vartheta}{}_{\vartheta} \leqslant 0$. These conditions are fulfilled 'inside' the (tilted) parabola $\beta = \frac{1}{8}(6\alpha - 3 \pm \sqrt{9 - 4\alpha})$ in figure 1. The energy-violating region is shown in light grey shading. The dominant energy condition reads $\left|S^{\tau}{}_{\tau}\right| \geqslant \left|S^{\vartheta}{}_{\vartheta}\right|$, and it is fulfilled 'inside' the (tilted) parabola $\beta = \frac{1}{8}(10\alpha - 3 \pm 3\sqrt{1 + 4\alpha})$. The region where this condition is violated (shown in dark grey shading in figure 1) consists, in our case, of a small strip near the collapse limit $\beta = \alpha - 1$, and a triangle-shaped region in the upper left part of figure 1.

(A similar analysis of the energy conditions for the RN shell, but in a different notation based on the isotropic radial coordinate instead of the RN coordinate, has already been performed in [23, 27]. However, in the figures of these papers the limits of the dominant energy condition were not explicitly marked.)

3. The model of a linearly accelerated shell

A first-order perturbation of the RN shell from section 2 shall be caused by a small, axisymmetric and dipolar charge distribution $\lambda\sigma(t, r)\cos\vartheta$ in the region $r \geqslant R$, which furthermore is assumed to be momentarily static at $t = 0 : \frac{\partial}{\partial t}\sigma(0, r) = 0$. For the perturbed metric we write

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}, \tag{5}$$

where $g_{\mu\nu}$ is the RN metric of equation (1), and $\delta g_{\mu\nu}$ is of order λ. A detailed analysis of the perturbations of the RN metric, including the dipolar case, has been performed by Bičák [28]. According to this work, for our dipolar perturbation a (Regge–Wheeler-type) gauge can be chosen such that only $\delta g_{00} = F(r)H_0(t,r)\cos\vartheta$, $\delta g_{01} = H_1(t,r)\cos\vartheta$ and $\delta g_{11} = F(r)^{-1}H_2(t,r)\cos\vartheta$ are non-zero. (Herewith the gauge is fixed up to an arbitrary function $f(t)$. In the following calculations nowhere appears a function depending only on t, and not on r. Therefore the above gauge freedom $f(t)$ is irrelevant for us. Furthermore, according to a theorem by Stewart and Walker [29], for a first-order perturbation of the RN metric, perturbation quantities—such as our acceleration g of test particles, the acceleration b of the RN shell and the dragging factor $d = g/b$—are gauge invariant if the corresponding unperturbed quantities vanish.) In the perturbed electromagnetic field tensor $\tilde{F}_{\mu\nu} = F_{\mu\nu} + \delta F_{\mu\nu}$ the perturbations reduce then to the non-zero elements $\delta F_{02} = a_0(t,r)\sin\vartheta$, $\delta F_{12} = a_1(t,r)\sin\vartheta$ and $\delta F_{01} = e(t,r)\cos\vartheta$, with $e = \dot{a}_1 - a_0'$, where $\cdot$ denotes the t derivative, and $'$ the r derivative, and there is no electromagnetic gauge freedom left. We assume that the RN shell consists of isolating material, and that therefore no additional (induced or mirror) charges, besides the charge q of the RN shell and the exterior charge distribution $\lambda\sigma\cos\vartheta$, have to be taken into account. The energy–momentum tensor for our perturbed system has (in the exterior region $r > R$) the form

$$\tilde{T}_{\mu\nu} = \frac{1}{4\pi}\left(\tilde{F}_{\mu\xi}\tilde{F}_{\nu}{}^{\xi} - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{F}_{\xi\chi}\tilde{F}^{\xi\chi}\right) + \varepsilon v_\mu v_\nu, \tag{6}$$

where ε is the mass density of the charged particles constituting the charge distribution σ, and v_μ is the 4-velocity of these particles which at $t = 0$ has the form $v_\mu(t = 0) = \sqrt{F(r)}(-1, 0, 0, 0)$. We assume that the effects of the mass density ε are generally negligible as compared to the effects of the charge distribution σ. However, if we would take ε to be exactly zero, the charged shell at $r = R$ would induce an infinite acceleration on the charged particles at $r > R$. Mathematically, this is encoded in the equation of motion under Lorentz force $\varepsilon v^\nu v^\mu{}_{;\nu} = \lambda\sigma F^\mu_\nu v^\nu\cos\vartheta$ for $\mu = 1$ which reads in our approximation

$$\frac{\partial}{\partial t}(\varepsilon v^1 v^0) = \lambda\sigma F^1{}_0 v^0\cos\vartheta. \tag{7}$$

3.1. The perturbation equations

The first order (in λ) perturbation terms of the Einstein equations $\tilde{G}_{\mu\nu} = 8\pi\tilde{T}_{\mu\nu}$ for $r \neq R$ are nontrivial for the index pairs $(\mu,\nu) = (0,0), (0,1), (0,2), (1,1), (1,2)$ and (2,2), and read in this order:

$$rFH_2' + 2H_2 - \frac{q^2}{r^2}H_0 = -2qe, \tag{8}$$

$$\dot{H}_2 + \frac{1}{r}H_1 = -\frac{8\pi r}{\cos\vartheta}\varepsilon v^1 v^0, \tag{9}$$

$$-\dot{H}_2 + FH_1' + F'H_1 = -\frac{4Fq}{r^2}a_1, \tag{10}$$

$$\dot{H}_1 - \frac{F}{2}H_0' - \frac{1}{2r}H_2 + \frac{1}{2r}\left(1 + \frac{q^2}{r^2}\right)H_0 = \frac{q}{r}e, \tag{11}$$

$$-\dot{H}_1 + FH_0' + \left(\frac{F}{r} + \frac{F'}{2}\right) H_2 - \left(\frac{F}{r} - \frac{F'}{2}\right) H_0 = -\frac{4q}{r^2} a_0, \tag{12}$$

$$\ddot{H}_2 + F^2 H_0'' - 2F\dot{H}_1' - \left(F' + \frac{2F}{r}\right)\dot{H}_1 + \frac{F^2}{r}(H_0' + H_2')$$
$$+ \frac{FF'}{2}(3H_0' + H_2') + \frac{F}{r^2}(H_2 - H_0) + \frac{2Fq^2}{r^4} H_0 = \frac{4Fq}{r^2} e. \tag{13}$$

The complicated equation (13) can be substituted by the much simpler equation

$$\ddot{H}_2 + \frac{1}{r}\dot{H}_1 = -8\pi\lambda q\sigma \frac{F}{r} v^0, \tag{14}$$

which follows from the time derivative of equation (9), together with equation (7). Equations (8)–(14) contain no wave equation because for dipole perturbations of the RN metric there exist no gravitational wave degrees of freedom. This is a decisive advantage of our dipolar model, because gravitational waves would in any case also penetrate into the interior of the shell and destroy the flatness there.

The first-order perturbation terms of the inhomogeneous Maxwell equations $\tilde{F}^{\mu\nu}{}_{;\nu} = 4\pi \tilde{J}^\mu$ for $r \neq R$, with $\tilde{J}^\mu = \lambda\sigma v^\mu \cos\vartheta$, are nontrivial for $\mu = 0, 1, 2$, and read in this order:

$$(r^2 e)' + \frac{2}{F} a_0 - \frac{q}{2}(H_0' - H_2') = -4\pi r^2 \lambda\sigma v^0, \tag{15}$$

$$r^2 \dot{e} + 2Fa_1 - \frac{q}{2}(\dot{H}_0 - \dot{H}_2) = 0, \tag{16}$$

$$\frac{1}{F}\dot{a}_0 - (Fa_1)' = 0. \tag{17}$$

The homogeneous Maxwell equations reduce to the already mentioned relation

$$e = \dot{a}_1 - a_0'. \tag{18}$$

We are mainly interested in the (momentarily static) situation at $t \approx 0$, where we have $H_1(0, r) = a_1(0, r) = 0$, and $\dot{H}_0(0, r) = \dot{H}_2(0, r) = \dot{a}_0(0, r) = \dot{e}(0, r) = 0$, while $\dot{H}_1(0, r)$ and $\dot{a}_1(0, r)$ can in principle be non-zero. We analyse equations (8)–(18) near $t = 0$, and separate them into time-independent equations and dynamical equations. Obviously, equations (8) and (15) are time independent. Further time-independent equations are obtained by adding equations (9) and (10),

$$rFH_1' + \left(1 + \frac{2M}{r} - \frac{2q^2}{r^2}\right) H_1 = -\frac{4Fq}{r} a_1, \tag{19}$$

and by adding equations (11) and (12),

$$rFH_0' - \left(1 - \frac{6M}{r} + \frac{3q^2}{r^2}\right) H_0 + \left(1 - \frac{2M}{r}\right) H_2 = -\frac{8q}{r} a_0 + 2qe. \tag{20}$$

The remaining Einstein and Maxwell equations are dynamical equations. At least in the weak field case $M/R \ll 1$, and $q/R \ll 1$, where $F(r) \approx 1$, and the terms $q(H_0 - H_2)$ can be neglected, the equations (15)–(18) constitute a linear, inhomogeneous system of differential equations for the electromagnetic fields $\delta F_{\mu\nu}$. Their general solution is the sum of a special solution, e.g. with $\dot{a}_1(0, r) = 0$, and the general homogeneous solution. But the asymptotically decaying homogeneous solutions are of wave type (also in the strong field case, because $F(r) \to 1$, and $q(H_0 - H_2) \to 0$, asymptotically). We fix now the electromagnetic part of our model system by requiring that there are no electromagnetic (dipole) waves at $t \approx 0$, which would anyhow endanger the desired time symmetry of our system. Then the magnetic field component $\dot{a}_1$ is zero in the considered time interval around $t = 0$, and we have, according to equation (18),

$$e = -a_0'. \tag{21}$$

With $\dot{a}_1(0, r) = 0$ the time derivative of equation (19) reduces to a homogeneous differential equation for $\dot{H}_1(0, r)$. Its solution behaves in the interior and asymptotically like r^{-1}, and has therefore to be set equal to zero, in order to guarantee regularity at the origin, and asymptotic flatness. Therefore, for our model in the considered approximation also the perturbed metric is diagonal. In the rest of the paper we usually omit the argument $t = 0$ in the metric and electric functions.

3.2. Analysis in the interior of the shell

For our shell system the perturbed Einstein–Maxwell equations can be considered separately for the interior region $r < R$, and the exterior region $r > R$. At the end, interior and exterior solutions have to be joined according to the Israel conditions [22].

In the interior region we have flat spacetime (and therefore $F \equiv 1$), and also $\sigma = 0$. The electric fields in the interior (continuing the exterior electric fields), are not relevant for the electrically neutral test particles inside the shell, in whose acceleration we are interested. In the interior, equation (8) reduces to a homogeneous differential equation for H_2 whose solution $H_2 \sim r^{-2}$ has to be set equal to zero in order to guarantee regularity at the origin. In contrast, equation (20), with $H_2 = 0$, has for $r < R$ the admissible solution

$$H_0(r) = Hr, \tag{22}$$

with an arbitrary constant H.

It can easily be proved that the resulting perturbed interior metric with the non-zero elements $\tilde{g}_{00} = -(1 - Hr\cos\vartheta)$, $\tilde{g}_{11} = 1$, $\tilde{g}_{22} = r^2$ and $\tilde{g}_{33} = r^2 \sin^2\vartheta$ can be transformed to the Minkowski metric in Cartesian coordinates $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ with $\bar{t} = t - \frac{1}{2}Htr\cos\vartheta$, $\bar{z} = r\cos\vartheta - \frac{1}{4}Ht^2\cos\vartheta$, $\bar{x} = r\sin\vartheta\cos\varphi$ and $\bar{y} = r\sin\vartheta\sin\varphi$, i.e. the system $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ has acceleration $\frac{1}{2}H$ in $\bar{z}$-direction with respect to the original system. Therefore, the perturbed metric inside the shell is flat, and any acceleration of neutral test particles within this shell is caused by the global dragging effect, and not by local curvature effects.

The acceleration of test particles (with 4-velocity u^μ) is of course ruled by the geodesic equation $\mathrm{d}u^\mu/\mathrm{d}s = -\tilde{\Gamma}^\mu_{\xi\chi} u^\xi u^\chi$, with s denoting proper time. Since inside the unperturbed RN shell there exists no acceleration of test particles whatsoever, this equation reduces to $\mathrm{d}u^\mu/\mathrm{d}s = -\delta\Gamma^\mu_{\xi\chi} u^\xi u^\chi$, and only the components with $\mu = 1, 2, 3$ are relevant for the spatial acceleration. For our interior metric perturbation with the only non-zero component $\delta g_{00} = Hr\cos\vartheta$, the relevant non-zero components $\delta\Gamma^\mu_{\xi\chi}$ are, due to $\dot{H}_2(0, r) = -(1/r)H_1(0, r) = 0$: $\delta\Gamma^1_{00} = -\frac{1}{2}H\cos\vartheta$, $\delta\Gamma^2_{00} = (1/2r)H\sin\vartheta$. Transforming to Cartesian coordinates and to coordinate time t, and specializing to static test particles, the only non-zero component of the acceleration is

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{1}{2}H. \tag{23}$$

Therefore the single quantity ruling the dragging of interior test particles by the accelerated RN shell is the constant H. This constant (in its dependence on M, q and R) has to be determined in section 3.3 by the Israel conditions, together with an appropriate fixation of the properties of the shell material.

3.3. Analysis in the exterior of the shell, and junction conditions at the shell

We have to analyse the system of equations (8), (20), (15) and (21) for the functions H_2, H_0, e and a_0 at $t = 0$. We first see that insertion of (21) into (15) results in the Gauss law of electrostatics

$$(r^2 a_0')' - \frac{2}{F} a_0 + \frac{q}{2}(H_0' - H_2') = 4\pi r^2 \lambda \sigma v^0, \tag{24}$$

modified by the 'curvature terms' F^{-1} and $\frac{1}{2}q(H_0' - H_2')$ which, in general, prevent a direct and explicit calculation of a_0 and of $e = -a_0'$ from a given charge distribution $\sigma(r)$. However, equation (24) suggests performing one integration

$$e(r) = \frac{q}{2r^2}(H_0 - H_2) - \frac{1}{2q} k(r), \tag{25}$$

with

$$k(r) = \frac{2q}{r^2} \int_\infty^r \mathrm{d}r' \left[4\pi r'^2 \lambda \sigma(r') v^0 + \frac{2}{F(r')} a_0(r') \right].$$

Insertion of (25) into (8) results in the (partly) separated equation

$$rFH_2' + \left(2 - \frac{q^2}{r^2}\right) H_2 = k. \tag{26}$$

Insertion of (25) into (20) does not immediately lead to a separated equation for H_0. However, for $H_3 := 3H_0 - H_2$ we get, in combination with (26), the (partly) separated equation

$$rFH_3' - \left(1 - \frac{6M}{r} + \frac{4q^2}{r^2}\right) H_3 = -4k - \frac{24q}{r} a_0, \tag{27}$$

with

$$a_0(r) = \frac{F(r)}{2} \left\{ \frac{[r^2 k(r)]'}{2q} - 4\pi r^2 \lambda \sigma(r) v^0 \right\}.$$

In general the 'inhomogeneities' $k(r)$ and $a_0(r)$ in equations (26) and (27) still depend, due to equation (24), on the metric potentials H_0 and H_2. However, asymptotically for $r \to \infty$ this dependence drops out. Therefore, it is generally true that the homogeneous solution $H_2^{(\mathrm{h})}$ of (26) behaves asymptotically like $c_2 r^{-2}$ and is therefore physically admissible, whereas the homogeneous solution $H_3^{(\mathrm{h})}$ of (27) behaves like r and has to be set equal to zero. The constant c_2 will, in the following, be fixed by appropriate physical conditions for the shell material.

As announced in the introduction, we now analyse the junction conditions between interior and exterior solutions at the shell Σ, and we fix the geometrical, mechanical and electrical properties of the accelerated RN shell. Again we do this in the Israel formalism [22]. Since, in analogy with a first-order rotation of the shells in [4, 9] and [13–17], the RN shell should (at least in first order of λ) be rigidly accelerated in the z-direction, the position of this shell is given by $r^+ = r^- = R + \frac{1}{2} b\tau^2 \cos\vartheta$, with the acceleration 'constant' b depending only on M/R and q^2/R^2. (The identity $r^+ = r^-$ follows from the Israel condition that the interior and exterior 4-metrics (5) induce the same 3-metric in Σ.) In the generalization of the choices in section 2, appropriate basis vectors in Σ are $e^{+\mu}_{\tau} = (F(R)^{-1/2}, b\tau\cos\vartheta, 0, 0)$, $e^{-\mu}_{\tau} = (1, b\tau\cos\vartheta, 0, 0)$, $e^{+\mu}_{\vartheta} = e^{-\mu}_{\vartheta} = (0, -\frac{1}{2}b\tau^2\sin\vartheta, 1, 0)$, $e^{+\mu}_{\varphi} = e^{-\mu}_{\varphi} = (0, 0, 0, 1)$. Then the unit normal vectors to Σ in V^+ and V^- are

$$n^{+\mu} = \left(\frac{b\tau}{F(R)} \cos\vartheta, \sqrt{F(R)} \left(1 - \frac{1}{2} H_2(R) \cos\vartheta\right), \frac{b\tau^2}{2R^2\sqrt{F(R)}} \sin\vartheta, 0 \right),$$

and

$$n^{-\mu} = \left(b\tau\cos\vartheta, 1, \frac{b\tau^2}{2R^2}\sin\vartheta, 0\right).$$

The Israel condition $\tilde{g}^+_{\tau\tau} = \tilde{g}^-_{\tau\tau}$ in Σ leads to the continuity condition $H_0^+(0,R) = H_0^-(0,R) = HR$, according to equation (22). But the Israel conditions lead to no continuity condition for $H_2(0,r)$ at Σ. The 3-metric in Σ reads then

$$\tilde{g}_{ab} = \operatorname{diag}\left(-1 + RH\cos\vartheta, R^2\left(1 + \frac{b\tau^2}{R}\cos\vartheta\right), R^2\left(1 + \frac{b\tau^2}{R}\cos\vartheta\right)\sin^2\vartheta\right).$$

Herewith, the nontrivial components of the extrinsic curvature tensors of Σ in V^+ and V^- are

$$K^+_{\tau\tau} = \frac{1}{2\sqrt{F(R)}}\left\{F'(R) - \left[F(R)H_0'(R) - 2\dot{H}_1(R) + F'(R)\left(H_0(R) + \frac{1}{2}H_2(R)\right) - 2b\right]\cos\vartheta\right\}, \tag{28a}$$

$$K^+_{\tau\vartheta} = -\frac{b\tau}{\sqrt{F(R)}}\sin\vartheta, \tag{28b}$$

$$K^+_{\vartheta\vartheta} = K^+_{\varphi\varphi}/\sin^2\vartheta = -R\sqrt{F(R)}\left(1 - \frac{1}{2}H_2(R)\cos\vartheta\right), \tag{28c}$$

$$K^-_{\tau\tau} = \left(b - \frac{1}{2}H\right)\cos\vartheta, \tag{28d}$$

$$K^-_{\tau\vartheta} = -b\tau\sin\vartheta, \tag{28e}$$

$$K^-_{\vartheta\vartheta} = K^-_{\varphi\varphi}/\sin^2\vartheta = -R, \tag{28f}$$

where terms of order τ^2 have been omitted, because we consider our system only near $\tau = 0$. With the quantities in equations (28*a*)–(28*f*), and inserting $FH_0' - 2\dot{H}_1$ from equation (11), we get the following order λ-corrections to the energy–momentum tensor $S^a_{\ b}$ of the RN shell (in equations (4*a*), (4*b*)):

$$8\pi\delta S^\tau_{\ \tau} = -\frac{\sqrt{F(R)}}{R}H_2(R)\cos\vartheta \tag{29a}$$

$$8\pi\delta S^\vartheta_{\ \vartheta} = 8\pi\delta S^\varphi_{\ \varphi} = \frac{1}{2R\sqrt{F(R)}}\left[\left(\sqrt{F(R)} - 1 - \frac{q^2}{R^2}\right)HR + \frac{M}{R}H_2(R) + 2qe(R) + 2Rb(1 - \sqrt{F(R)})\right]\cos\vartheta. \tag{29b}$$

In order to fix our system completely, we now have to define what mechanical properties the shell material shall have in the accelerated state. (That, in addition to the fulfilment of the Einstein–Maxwell equations in V^+ and V^-, the Israel junction conditions on Σ and initial conditions, there has to be given 'a suitable description of the matter on Σ', is especially clearly expressed in [30].) As already stated in the introduction, our guiding principle for fixing the material quantities $\delta S^\tau_{\ \tau}$ and $\delta S^\vartheta_{\ \vartheta}$ in equations (29*a*) and (29*b*) is the analogy to the corresponding rotating shells in [4, 9], and [13–17]. (Otherwise, the final comparison of our translational dragging effects with the rotational dragging results in these papers would not make much sense.) Now, for a rotating mass shell it is trivial from symmetry considerations that corrections to the energy density $S^\tau_{\ \tau}$ and to the isotropic pressure $S^\vartheta_{\ \vartheta}$ can only appear in even orders of the angular velocity ω. Therefore, the analogy to the rotating shells is realized in an optimal way if we set the first order (in λ)-correction terms $\delta S^{\tau}_{\ \tau}$ and $\delta S^{\vartheta}_{\ \vartheta}$ equal to zero. (In principle, and in an extremely unphysical way, it would be possible to arrange the changes from the static to the accelerated state of the shell in such a way that $b = 0$, i.e. that the shell is not accelerated at all, but that the 'force' exerted on it by the charge distribution σ is completely compensated by an inner tension of the shell material.) From $\delta S^{\tau}_{\ \tau} = 0$, and equation (29*a*) follows $H_2(R) = 0$, i.e. also the metric function $H_2(R)$ is continuous at $r = R$. $\delta S^{\vartheta}_{\ \vartheta} = 0$ results then, together with equation (29*b*), in

$$b = \frac{\left(1 - \sqrt{F(R)} + \frac{q^2}{R^2}\right)HR - 2qe(R)}{2R(1 - \sqrt{F(R)})}. \tag{30}$$

Therefore, with equation (25), and after solving equations (26) and (27) with the appropriate boundary conditions at $r = R$, we get a unique expression for the acceleration b of the RN shell. With the boundary conditions for the metric potentials now fixed, the constant H in equation (23) is also fixed, and herewith the dragging factor $d = g/b$ due to the linear acceleration of the RN shell.

In the general (strong field) case we presently see no way to separate equations (8), (20), (15) and (21), or the reduced system (24)–(27) completely. However, in special and still physically interesting cases, considerable simplifications happen: for weak gravitational and electric fields ($M \ll R, q \ll R$) equation (24) reduces to the standard Gauss law of electrostatics, i.e. $a_0(r)$ and $k(r)$ can be explicitly calculated from $\sigma(r)$. Then also equations (26) and (27) for the metric potentials H_2 and H_3 can be explicitly integrated, at least for some simple and characteristic charge distributions $\sigma(r)$. For strong gravitational fields, but weak electric fields, the term $\frac{1}{2}q(H_0' - H_2')$ in equation (24), being of order q^2, is still negligible, and the system (24), (26), (27) separates. However, the Schwarzschild factor $F(r) = 1 - 2M/r$ makes the integration of equation (24), and of (26) and (27) more complicated. The division of the following analysis into the three subsections 3.3.1 (weak field case), 3.3.2 (small charge q) and 3.3.3 (strong field case) is, however, appropriate not only for these mathematical reasons but also in analogy with the development and understanding of the (simpler) dragging phenomena due to a rotating shell, which happened historically in the three decisive steps, 'weak field', 'strong gravitational field' and 'strong gravitational and electric field'.

3.3.1. The weak field case. In this case equation (24) reduces to the standard Gauss law

$$(r^2 a_0')' - 2a_0 = 4\pi r^2 \lambda\sigma. \tag{31}$$

In order that the charge distribution $\sigma(r)$ have a finite dipole moment $\mathrm{d}_z = \int \mathrm{d}^3 r (r\cos\vartheta)\lambda\sigma(r)\cos\vartheta$, $\sigma(r)$ has to fall off asymptotically faster than r^{-4}, and as a first representative example we choose $4\pi\sigma_1(r) = \delta r^{-5}$, with a (dimensional) constant δ. Herewith equation (31) has in the exterior $r > R$ a special inhomogeneous solution $a_0^{(\mathrm{inh})} = \lambda\delta/4r^3$, and the admissible homogeneous solution $a_0^{(\mathrm{h1})} \sim r^{-2}$. In the interior $r < R$ we have $\sigma(r) = 0$, and the admissible homogeneous solution $a_0^{(\mathrm{h2})} \sim r$. The total solution

$$a_0\,(r > R) = \frac{\lambda\delta}{4r^3} + \frac{c_1}{r^2} \qquad \text{and} \qquad a_0\,(r < R) = Cr, \tag{32}$$

with constants c_1 and C, has to be continuous at $r = R$, and, due to $a_0'(r) = -e(r)$ and the isolator property of the shell material, it has to have a continuous derivative $a_0'(R)$. These boundary conditions result in $c_1 = -\lambda\delta/3R$ (coinciding with the negative dipole moment of

$\sigma_1(r)$), and $C = -\lambda\delta/12R^4$. Therefore, the complete exterior solution of equation (31) for this model reads

$$a_0(r) = -\frac{\lambda\delta}{3Rr^2} + \frac{\lambda\delta}{4r^3}. \tag{33}$$

The fact that a_0 and $e = -a_0'$ decrease asymptotically faster than r^{-1} confirms that we have no electromagnetic radiation. From equation (33) we get

$$k(r) = \frac{4q\lambda\delta}{3R}\left(\frac{1}{r^3} - \frac{9R}{8r^4}\right),$$

and from equation (27): $H_3(r) = -2q\lambda\delta/3Rr^3$. According to our determination of the material properties of the shell, also the function $H_2(r)$, and with it the function $H_3(r)$, is continuous at the shell position $r = R$. Therefore, together with equations (22) and (23), the acceleration (in z-direction) of the interior test particles is given by

$$g_1 = \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{H_3(R)}{6R} = -\frac{q\lambda\delta}{9R^5}. \tag{34}$$

In order to derive the dragging effect, this acceleration has to be scaled by the acceleration b_1 (in z-direction) which the charge distribution $\sigma_1(r)$ induces on the charged shell as a whole. In the weak field case we have $1 - \sqrt{F(R)} \approx M/R$, and therefore in the numerator of equation (30) the first term is negligible in comparison to the second. We get $b = -qe(R)/M$, and with equations (21) and (33):

$$b_1 = -\frac{q\lambda\delta}{12MR^4}. \tag{35}$$

This result coincides with a calculation of b_1 from the Coulomb law of classical electrostatics (interaction of $\sigma_1(r)$ with the RN shell which can here be substituted by a point charge at the origin),

$$b_1 = -\frac{1}{M}\int \mathrm{d}^3 r \frac{1}{r^2} q\cos\vartheta\,\lambda\sigma_1(r)\cos\vartheta,$$

which constitutes (in the weak field case) another justification for setting $\delta S^\tau_{\ \tau} = \delta S^\vartheta_{\ \vartheta} = 0$. With equation (35), the dragging factor for the charge distribution $\sigma_1(r)$ is

$$d_1 = \frac{g_1}{b_1} = \frac{4M}{3R}. \tag{36}$$

By accident this result coincides exactly with the Thirring result [4] for a slowly rotating and weakly massive shell. It is, of course, to be expected that the number in front of the ratio M/R in equation (36) changes by going to other charge distributions. Therefore, we have calculated (along the same procedure as above) the dragging factor d also for some alternative charge distributions $4\pi\sigma_2(r) = 6R\delta(r^{-6} - Rr^{-7}), 4\pi\sigma_3(r) = 3\delta(r^{-5} - Rr^{-6})$ and $4\pi\sigma_4(r) = 6\delta(r^{-5} - 2Rr^{-6} + R^2r^{-7})$, for all of which $\sigma(r)$ vanishes at the shell position $r = R$, which all have the same 'total charge' $\lambda\delta/8R^2$ in the upper hemisphere $0 \leqslant \vartheta \leqslant \pi/2$, but which, according to figure 2, have increasingly higher values of the 'centre of charge'.

The resulting dragging factors $d_2 = 37M/21R \approx 1.76M/R, d_3 = 2M/R$ and $d_4 = 52M/21R \approx 2.47M/R$ then increase 'accordingly'. In order to analyse in more detail the dependence of d on the 'position' of the charge, we have also calculated the dragging factor d for the δ-type (respectively shell type) charge distribution $4\pi\sigma(r) = \delta/[2R^2(\zeta R)^2]\delta(r - \zeta R)$ for arbitrary $\zeta > 1$, with the result

$$d = \frac{M}{R}\left(\frac{16}{3}\zeta - 5\right). \tag{37}$$

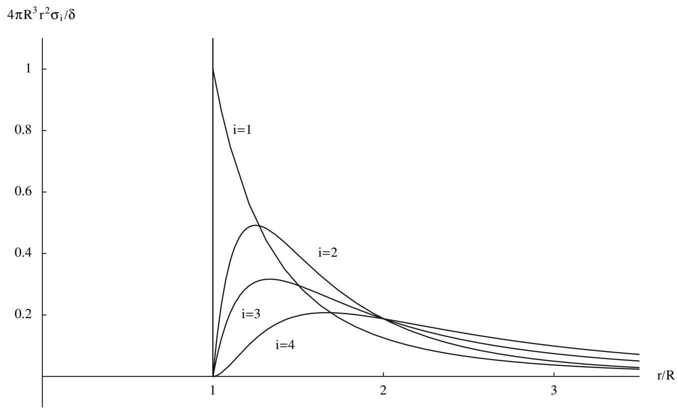

Figure 2. A comparison of the functions $r^2\sigma_i(r)$ which integrate to the total charge (in a hemisphere), for the different charge distributions $\sigma_i(r)$ $(i = 1, 2, 3, 4)$ given above.

The qualitative structure of these results for the dragging factor d follows (disregarding numerical factors) already from general and dimensional arguments: if, in a first step, we do not care for the detailed structure of the charge distribution $\sigma(r)$, the only model parameters on which d can depend, are R, M and q, where in the weak field case M and q can occur only linearly. However, it is clear that in the ratio $d = g/b$ the parameter q cancels. Therefore, the dimensionless quantity d has to be proportional to M/R. Furthermore, since the acceleration b is essentially determined by the 'total charge' of the distribution $\sigma(r)$ whereas the acceleration g is dominated by the dipole moment d_z of this distribution, the dominating part of the ratio d has to contain a factor ζ, if $\sigma(r)$ is centred around $r = \zeta R$.

3.3.2. A shell with arbitrary mass but small charge. As shown in the first part of section 3.3, in this case the Einstein–Maxwell equations (24), (26), (27) again separate. Therefore, we can first integrate equation (24) for the function $a_0(r)$ which reads in the present case

$$\left(r^2 a_0'\right)' - \frac{2r}{r-2M} a_0 = 4\pi r^2 \lambda \sigma v^0. \tag{38}$$

We would first like to consider the rhs of equation (38) for the charge function $\sigma_1(r) \sim r^{-5}$: if we demand that the 'total charge' in a hemisphere stays independent of M/R, the rhs of equation (38) has to read $\lambda\delta r^{-3}$, as in the weak field case, and without additional 'Schwarzschild factors' $(1 - 2M/r)$. (Corresponding expressions result for the charge functions $\sigma_2(r)$, $\sigma_3(r)$ and $\sigma_4(r)$ of section 3.3.1.) One homogeneous solution of equation (38) reads $a_0^{(\mathrm{h2})} \sim r - 2M$ which, however, diverges for $r \to \infty$. The other homogeneous solution, which is physically acceptable in the exterior, is constructed from here by d'Alembert's reduction procedure:

$$a_0^{(\mathrm{h1})}(y) \sim \frac{1-y}{y} \log(1-y) + 1 - \frac{y}{2}, \tag{39}$$

where $y = 2M/r$. This solution is also known from the literature [31]. For charge functions $\sigma_i(r)$ corresponding to the simple powers of r^{-1} or sums of such terms in section 3.3.1, also the inhomogeneous solution of equation (38), decreasing asymptotically at least like r^{-3} can be found in analogy with equation (39). For the rhs $\lambda\delta r^{-3}$ we get

$$a_0^{(\mathrm{inh})}(y) = \frac{\lambda\delta}{(2M)^3} \left[\frac{1-y}{y} \log(1-y) + 1 - \frac{y}{2} - \frac{y^2}{6}(1-y) \right]. \tag{40}$$

(For the charge functions $\sigma_2(r), \sigma_3(r)$ and $\sigma_4(r)$, the solution looks similar but additional powers $\sim y^4, y^5$ have to be 'subtracted' from the log term.) The exterior total solution

$$a_0(y) = a_0^{(\mathrm{inh})}(y) + c_1 a_0^{(\mathrm{h1})}(y) \tag{41}$$

now has to join continuously, and with continuous derivative, to the interior solution Cr of equation (38) with $M = 0$ and $\sigma(r) = 0$. Thereby, the constant c_1 is fixed to

$$c_1 = -\frac{\lambda\delta}{(RY)^3}\left\{1 + \frac{Y^2(3-4Y)}{6[\log(1-Y)+Y]}\right\}, \tag{42}$$

with $Y = 2M/R$. (The asymptotic behaviour of $a_0(y)$ is now no longer ruled by the dipole moment of $\sigma_1(r)$ in the Schwarzschild background.) The constant C attains a similar but less interesting value. With $e(r) = -a_0'(r)$ we derive from equation (41)

$$e(y) = \frac{\lambda\delta}{6(RY)^4}\left\{\frac{Y^2(3-4Y)}{\log(1-Y)+Y}\left[\log(1-y)+y+\frac{y^2}{2}\right] - 2y^3 + 3y^4\right\}. \tag{43}$$

In order to determine the function $H_3(r)$ from equation (27), we need also to know the auxiliary function $k(r)$ which reads in the variable y

$$k(y) = -\frac{4q}{RY}y^2\int_0^y \mathrm{d}y'\left[\frac{\lambda\delta}{2(RY)^3}y' + \frac{a_0(y')}{y'^2(1-y')}\right],$$

and integrates to

$$k(y) = -\frac{q\lambda\delta}{3(RY)^4}\left\{\frac{Y^2(3-4Y)}{\log(1-Y)+Y}\left[\log(1-y)+y+\frac{y^2}{2}\right] - 2y^3 + 3y^4\right\} = -2qe(y), \tag{44}$$

where the last identity is already evident from equation (25). With $k(y)$ and $a_0(y)$ known, the function $H_3(y)$ can be explicitly calculated from equation (27), by the ansatz $H_3(y) = c(y)H_3^{(\mathrm{h})}(y) = c(y)/y(1-y)^2$:

$$H_3(y) = \frac{2q\lambda\delta}{3(RY)^4}\left\{\frac{Y^2(3-4Y)}{\log(1-Y)+Y}\left[\frac{3-2y}{y}\log(1-y) + 3 - \frac{y}{2}\right] - \frac{y^3(5-4y)}{10(1-y)^2}\right\}. \tag{45}$$

Also for more general, physically reasonable charge distributions $\sigma(r)$, $H_3(y)$ diverges in the collapse limit $y \to 1$ like $(1-y)^{-2}$, whereas $e(y)$ goes to zero in this limit. With equation (34), the acceleration (in the z-direction) of the interior test particles is given by

$$g = \frac{H_3(Y)}{6R}. \tag{46}$$

Omitting terms of order q^2, equation (30) for the acceleration b of the RN shell reads

$$b = \frac{H_3(Y)}{6R} - \frac{qe(Y)}{R(1-\sqrt{1-Y})}, \tag{47}$$

and therefore

$$d_{\mathrm{linear}} = \frac{g}{b} = \frac{1}{1-\kappa}, \tag{48}$$

with

$$\kappa = \frac{6qe(Y)}{(1-\sqrt{1-Y})H_3(Y)} = \frac{Y(1-Y)^2}{2(1-\sqrt{1-Y})} \times \frac{(1-Y)^2\log(1-Y) + Y\left(1-\frac{3}{2}Y+\frac{1}{3}Y^2\right)}{\left[(1-Y)^4 - \frac{Y^2}{6}\left(1-\frac{8}{5}Y+\frac{2}{3}Y^2\right)\right]\log(1-Y) + Y(1-Y)^2\left(1-\frac{3}{2}Y+\frac{2}{9}Y^2\right) - \frac{Y^3}{18}\left(1-\frac{4}{5}Y\right)}. \tag{49}$$

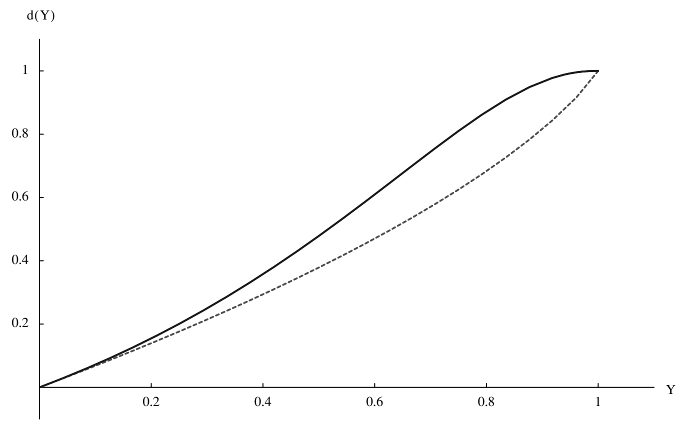

Figure 3. A comparison of the linear dragging coefficient d_{linear} from equation (48) (solid line), with the dragging coefficient d_{BC} inside a rotating mass shell (dotted line), in dependence of $Y = 2M/R$.

Obviously we have $\kappa \to 0$ in the collapse limit $Y \to 1$ (and this for all reasonable charge distributions $\sigma(r)$). Therefore, the dragging factor d reaches the limit of total dragging, $d \to 1$, and this with correction term $\sim(1-Y)^{-2}$, i.e. with horizontal tangent. In figure 3, our result from equation (48) is compared to the dragging factor d_{BC} for a rotating mass shell, according to [9], which reads in our notation (with the Schwarzschild radial coordinate): $d_{\text{BC}} = 2 - 2\sqrt{1-Y} - Y/(1+2\sqrt{1-Y})$.

We see that for $Y < \frac{1}{4}$ $d_{\text{linear}}(Y)$ exceeds d_{BC} by at most 10%. For higher Y-values $d_{\text{linear}}(Y)$ exceeds d_{BC} by up to 30% because d_{linear} reaches the limiting value $d(Y=1) = 1$ with horizontal tangent whereas d_{BC} has slope $\frac{\mathrm{d}}{\mathrm{d}Y} d_{\text{BC}}\big|_{Y=1} = 3$. For the charge distributions $\sigma_2(r), \sigma_3(r)$ and $\sigma_4(r)$ from section 3.3.1, d_{linear} exceeds d_{BC} of course in the whole region $0 < Y < 1$, and coincides with d_{BC} only at the limiting points $Y = 0$ and $Y = 1$.

3.3.3. The general strong field case. As already mentioned at the end of section 3.3, in this case the differential equations (8), (15), (20) and (21) for the functions $H_0(r), H_2(r), a_0(r)$ and $e(r)$ could not be separated, and therefore cannot be solved analytically. We therefore turn to a numerical solution of these equations. We use the dimensionless variable $y = 2M/r$, and introduce the 'reduced' functions $u_1(y) = (3H_0(y) - H_2(y))/qy^3$, $u_2(y) = (H_0(y) + H_2(y))/qy^2$, $u_3(y) = 2e(y)/y^3$ and $u_4(y) = 4a_0(y)/My^2$, all of which approach asymptotically (for $y \to 0$) non-zero constants. With $p = q^2/4M^2$, and with $F(y) = 1 - y + py^2$, the differential equations for $u_1(y), \ldots, u_4(y)$ then read:

$$yF(y)\frac{\mathrm{d}u_1}{\mathrm{d}y} + (4 - 6y + 5py^2)u_1 + 2pyu_2 + 4u_3 - 3u_4 = 0, \tag{50a}$$

$$yF(y)\frac{\mathrm{d}u_2}{\mathrm{d}y} + yF(y)u_1 - y(2 - 3py)u_2 - yu_4 = 0, \tag{50b}$$

$$\begin{aligned} yF(y)\frac{\mathrm{d}u_3}{\mathrm{d}y} &- py^3\left(1 - \frac{p}{2}y\right)u_1 + py\left(1 + \frac{p}{2}y^2\right)u_2 + (1 - y + 3py^2)u_3 \\ &- \left(\frac{1}{2} + py^2\right)u_4 = \frac{2\lambda\delta}{(RY)^4}yF(y), \end{aligned} \tag{50c}$$

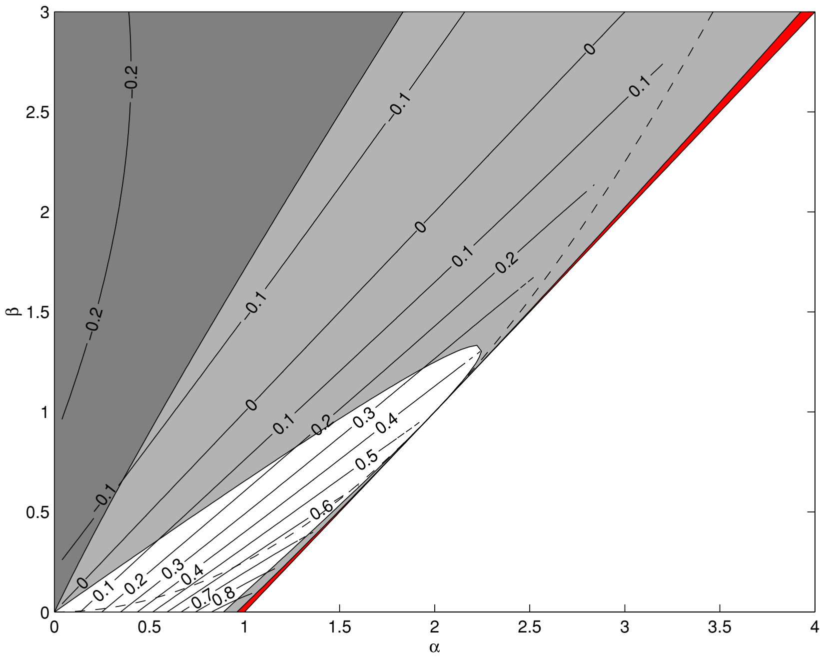

Figure 4. Level lines for representative values of d_{linear} in the physical region of the model parameters $\alpha = 2M/R$ and $\beta = q^2/R^2$ (compare figure 1).

$$y\frac{\mathrm{d}u_4}{\mathrm{d}y} - 4u_3 + 2u_4 = 0. \qquad (50d)$$

The boundary conditions at the shell $y = Y$ are

$$u_2(Y) = \frac{Y}{3}u_1(Y), \qquad (51a)$$

resulting from $H_2(R) = 0$, and

$$u_4(Y) = -4u_3(Y), \qquad (51b)$$

resulting from the continuous connection of $a_0(r)$ and $e(r)$ to the interior solutions $a_0(r < R) = Cr$, and $e(r < R) = -C$, with a constant C. Together with the regularity of $u_1(y), \ldots, u_4(y)$ at the boundary $y = 0$, the conditions (51*a*), (51*b*) lead to a unique solution of the system (50*a*)–(50*d*).

Since this system is a boundary value problem we choose to search for the solution by expanding the functions as a finite sum of (appropriately scaled) Chebyshev polynomials. This procedure ensures that the solution will be regular even though the equations are degenerate at $F(y) = 0$. Inserting these sums into the equations and the boundary conditions yields a linear system of equations for the expansion coefficients which can be solved uniquely.

In this way the solution of the differential equations can be found with machine accuracy. With $H_0(R)$ and $e(R)$ calculated in this way, the linear dragging coefficient results from $d_{\text{linear}} = g/b$, with $g = H_3(R)/6R$, and b from equation (30). Figure 4 shows characteristic lines $d_{\text{linear}} = \text{const}$ in the plane of the model parameters $\alpha = 2M/R$ and $\beta = q^2/R^2$ (compare figure 1). In the underextreme case $p < \frac{1}{4}$, the RN shell collapses in the limit $Y \to (1/2p)(1 - \sqrt{1-4p}) =: Y_0$, where the function $F(Y_0)$ vanishes. In this limit the

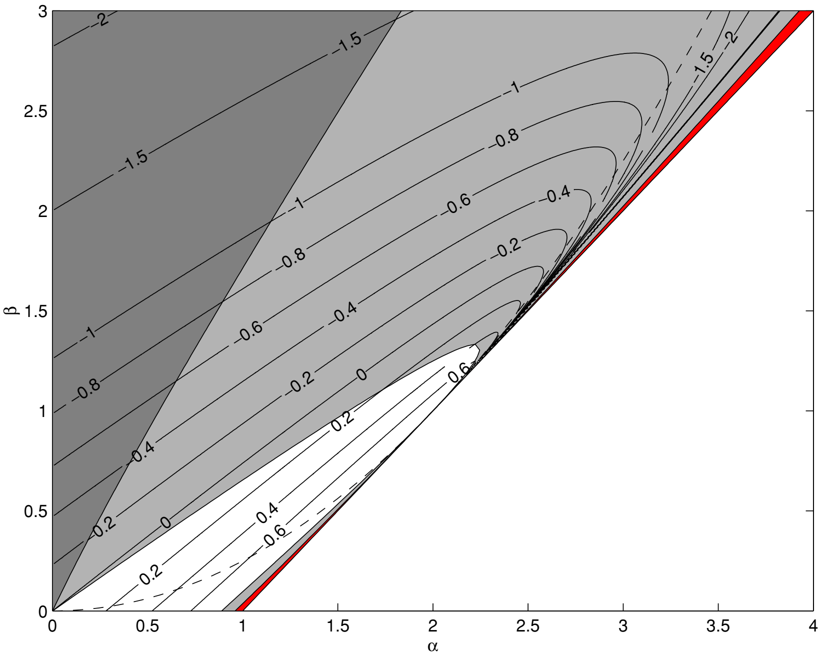

Figure 5. Level lines for representative values of d_{rot} in the physical region of the model parameters $\alpha = 2M/R$ and $\beta = q^2/R^2$ (compare figure 1).

functions $u_1(y \to Y_0)$ and $u_2(y \to Y_0)$ diverge, and the above numerical procedure may loose its reliability. However, this limit can again be treated analytically: as in the case of a small charge (in section 3.3.2), in the collapse limit the functions $u_3(y)$ and $u_4(y)$, and their derivatives, are negligible as compared to the functions $u_1(y)$ and $u_2(y)$. Herewith, equation (50*c*) reduces to $u_2(y) \approx y u_1(y)$, and both equations (50*a*) and (50*b*) attain the same form

$$yF(y)\frac{\mathrm{d}u_1}{\mathrm{d}y} - (3 - y)u_1 = 0. \tag{52}$$

An ansatz $u_1(y \to Y_0) \sim (y - Y_0)^{-j}$ leads then, together with $F(y \to Y_0) \approx -\sqrt{1-4p}(y - Y_0)$, to

$$j = \frac{6p - 1 + \sqrt{1-4p}}{4p - 1 + \sqrt{1-4p}}. \tag{53}$$

For $p = 0$, we have $j = 2$, in agreement with the divergence $H_3(y) \sim (1 - y)^{-2}$ of equation (45) in the collapse limit $y \to 1$. In the region $0 < p \leqslant \frac{1}{4}$, the exponent j increases monotonically until $j = \infty$ for $p = \frac{1}{4}$. Since in the collapse limit we can neglect $e(R)$ as compared to $H_3(R)$, and we have $F(R) \approx 0$, equation (30) reduces to $b = H_3(R)(1 + q^2/R^2)/6R$, and we get for the linear dragging coefficient in the collapse limit

$$d_{\mathrm{linear}}(Y_0) = \frac{1}{1 + q^2/R^2} = \frac{1}{1+\beta}. \tag{54}$$

These values coincide nicely with the limiting values of the numerical curves $d_{\mathrm{linear}} = \mathrm{const}$ in figure 4.

For comparison of $d_{\mathrm{linear}}(\alpha, \beta)$ with the dragging factor inside a rotating RN shell, we have transformed figure 1 of the paper [23] from the parameters used there to the parameters

α and β, and we show $d_{\mathrm{rot}}\,(\alpha, \beta)$ in figure 5. Whereas d_{rot} is equal to 1 (total dragging) on the whole line $\beta = \alpha - 1$ (collapse limit), d_{linear} attains the value 1 only at the point $\alpha = 1$, $\beta = 0$. Accordingly, the level lines with d between 1 and 0.3 are 'steeper' in the rotational case, as compared to the linear case. For $d \lesssim 0.3$ this behaviour changes, partly because all lines $d_{\mathrm{rot}} = \mathrm{const}$ have to 'turn around' in order to meet in the 'singular point' $\alpha = 2$, $\beta = 1$. (In the variables used in the paper [23], which are scaled by the shell radius in isotropic coordinates, this point is at infinity.) The level line $d_{\mathrm{linear}} = 0$ is exactly represented by the line $\beta = \alpha$ because then $F(R) = 1 - \alpha + \beta = 1$ and, according to equation (30), the acceleration b of the RN shell diverges. In the linear and in the rotational case, the dragging factor attains negative values ('antidragging') in part of the region (in light grey shading) where the weak energy condition is violated. However, whereas d_{rot} attains arbitrarily negative values in the region (in dark grey shadow) where also the dominant energy condition is violated (see figure 1 in [23]), d_{linear} seems to be limited by $d_{\mathrm{linear}} \gtrsim -0.22$.

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References

[1] Einstein A 1912 Vierteljahrschrift für gerichtliche Medizin und öffentliches Sanitätswesen **44** 37
See also Einstein A 1995 *The Collected Papers of Albert Einstein* vol 4 ed M J Klein *et al* (Princeton, NJ: Princeton University Press) pp 174–9
[2] Einstein A 1913 *Phys. Z.* **14** 1249
[3] Klein M J *et al* (ed) 1995 *The Collected Papers of Albert Einstein* vol 4 (Princeton, NJ: Princeton University Press) pp 436–7
[4] Thirring H 1918 *Phys. Z.* **19** 33
[5] Schulmann R *et al* (ed) 1998 *The Collected Papers of Albert Einstein* vol 8 (Princeton, NJ: Princeton University Press) documents 361, 369, 401 and 405
[6] Pfister H 2005 On the history of the so-called Lense–Thirring effect *Preprint* submitted
[7] Ciufolini I and Pavlis E C 2004 *Nature* **431** 958
[8] Everitt C W F *et al* 2001 Gravity Probe B: countdown to launch *Gyros, Clocks, Interferometers ...: Testing Relativistic Gravity in Space (Lecture Notes in Physics* vol 562*)* ed C Lämmerzahl *et al* (Berlin: Springer) pp 52–82
[9] Brill D R and Cohen J M 1966 *Phys. Rev.* **143** 1011
[10] Pfister H and Braun K H 1985 *Class. Quantum Grav.* **2** 909
[11] Pfister H and Braun K H 1986 *Class. Quantum Grav.* **3** 335
[12] Pfister H 1989 *Class. Quantum Grav.* **6** 487
[13] Hofmann K-D 1962 *Z. Phys.* **166** 567
[14] Cohen J M 1966 *Phys. Rev.* **148** 1264
[15] Ehlers J and Rindler W 1971 *Phys. Rev.* D **4** 3543
[16] Briggs C, Cohen J M, DeWoolfson G L and Kegeles L S 1981 *Phys. Rev.* D **23** 1235
[17] King M and Pfister H 2001 *Phys. Rev.* D **63** 104004
[18] Farhoosh H and Zimmermann R L 1980 *Phys. Rev.* D **21** 317
[19] Grøn Ø and Eriksen E 1989 *Gen. Rel. Grav.* **21** 105
[20] Lynden-Bell D, Bičák J and Katz J 1999 *Ann. Phys.* **271** 1
[21] Menshikov L I, Perevalova I A and Pinzul A N 2000 *Preprint* gr–qc/0008012
[22] Israel W 1966 *Nuovo Cimento* B **44** 1
Israel W 1966 *Nuovo Cimento* B **48** 463 (erratum)
[23] Pfister H and King M 2002 *Phys. Rev.* D **65** 084033
[24] Klein M J *et al* 1993 *The Collected Papers of Albert Einstein* vol 5 (Princeton, NJ: Princeton University Press) documents 409 and 411
[25] Kuchař K 1968 *Czech. J. Phys.* B **18** 435

[26] Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Space-Time* (Cambridge: Cambridge University Press) pp 89–91
[27] Pfister H and King M 2003 *Class. Quantum Grav.* **20** 205
[28] Bičák J 1979 *Czech. J. Phys.* B **29** 945
[29] Stewart J M and Walker M 1974 *Proc. R. Soc.* A **341** 49
[30] Ipser J and Sikivie P 1984 *Phys. Rev.* D **30** 712
[31] Ginzburg V L 1964 *Sov. Phys.—Dokl.* **9** 329
Israel W 1968 *Commun. Math. Phys.* **8** 245