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# Applications of Symmetry Methods to Partial Differential Equations

Springer

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# Contents

# Preface

This book is a sequel to *Symmetries and Integration Methods* (2002), by George W. Bluman and Stephen C. Anco. It includes a significant update of the material in the last three chapters of *Symmetries and Differential Equations* (1989; reprinted with corrections, 1996), by George W. Bluman and Sukeyuki Kumei. The emphasis in the present book is on how to find systematically symmetries (local and nonlocal) and conservation laws (local and nonlocal) of a given PDE system and how to use systematically symmetries and conservation laws for related applications. In particular, for a given PDE system, it is shown how systematically (1) to find higher-order and nonlocal symmetries of the system; (2) to construct by direct methods its conservation laws through finding sets of conservation law multipliers and formulas to obtain the fluxes of a conservation law from a known set of multipliers; (3) to determine whether it has a linearization by an invertible mapping and construct such a linearization when one exists from knowledge of its symmetries and/or conservation law multipliers, in the case when the given PDE system is nonlinear; (4) to use conservation laws to construct equivalent nonlocally related systems; (5) to use such nonlocally related systems to obtain nonlocal symmetries, nonlocal conservation laws and non-invertible mappings to linear systems; and (6) to construct specific solutions from reductions arising from its symmetries as well as from extensions of symmetry methods to find such reductions.

This book is aimed at applied mathematicians, scientists and engineers interested in finding solutions of partial differential equations and is written in the style of the above-mentioned 1989 book by Bluman and Kumei. There are numerous examples involving various well-known physical and engineering PDE systems.

The preceding book by Bluman and Anco includes comprehensive treatments of dimensional analysis, Lie groups of transformations, the discovery and use of symmetries to construct solutions of ordinary differential equations, and also shows how to construct conservation laws (first integrals) of ordinary differential equations through multipliers (integrating factors) as well as how to construct invariant solutions of partial differential equations from their point symmetries.

Chapter 1 reviews essential material from the Bluman and Anco book on one-parameter Lie groups of point transformations and how to find point symmetries of PDE systems and extends this material to the consideration of one-parameter higher-order local transformations and the finding of higher-order symmetries of PDE systems. This is followed by a comprehensive treatment on how to construct directly the local conservation laws essentially for any given PDE system. This treatment is based on first finding conservation law multipliers. It is shown how this treatment is related to and subsumes the classical Noether's theorem (which only holds for variational systems). In particular, multipliers are symmetries of a given PDE system only when the system is variational as written. There is a full discussion on connections between symmetries and conservation laws including the use of symmetries to find one or more additional conservation laws from a known conservation law.

Chapter 2 deals with the construction of local mappings relating a given PDE system to a target system of interest (or a member of a target class of PDE systems) from knowledge of the symmetries and/or conservation law multipliers of the given PDE system. In particular it is shown how to determine whether (1) a given nonlinear PDE system can be mapped invertibly to a linear PDE system and it is shown how to construct such a mapping when one exists; (2) a given linear PDE with variable coefficients can be mapped invertibly to a linear PDE with constant coefficients and it is shown how to construct such a mapping when one exists.

Chapter 3 considers perhaps the most important application of the material on conservation laws presented in Chapter 1. In particular, it is shown how to use local conservation laws and subsystems of a given PDE system to construct systematically a tree of equivalent nonlocally related systems. One of the many exhibited examples involves the planar gas dynamics equations, for which it is shown how the Euler and Lagrange systems are related systematically within such a tree of nonlocally related systems.

Chapter 4 considers the applications of such nonlocally related systems to find systematically nonlocal symmetries and nonlocal conservation laws of a given PDE system. In turn, it is shown how to use such nonlocal symmetries to construct nonlocal mappings of nonlinear PDE systems to equivalent linear PDE systems and to use conservation law multipliers of nonlocally related systems to construct nonlocal mappings of linear PDEs with variable coefficients to equivalent linear PDEs with constant coefficients.

The topics of Chapter 5 include how to use various kinds of symmetries to construct explicit solutions of PDEs, a discussion of the complexity involved in the construction of interesting nonlocally related systems in multi-dimensions, and a discussion of existing software to implement the procedures presented in this book.

If one is primarily interested in the material of Chapters 3–5, then Chapter 2 can be skipped. Chapter 1 is essential reading for all subsequent chapters.

Every topic is illustrated by examples. All sections have many exercises. It is essential to do some of the exercises to obtain a working knowledge of the presented material. Each chapter begins with a comprehensive Introduction section. The Discussion section at the end of each chapter discusses related work and puts the subject matter of the chapter in context for later chapters.

Within each section of a given chapter, definitions, theorems, corollaries, and remarks are numbered separately and consecutively. For example, Remark 4.2.1 refers to the first remark in Section 4.2. Exercises appear at the end of each section; Exercise 4.2.2 refers to the second problem of Exercises 4.2, i.e., the second problem at the end of Section 4.2.

There are separate Author and Subject indices as well as a References section. In addition there is a Theorem, Corollary and Lemma Index.

The authors are grateful to their many collaborators without whom this book would not have been possible. In particular we wish to thank Julian Cole (posthumously), Sukeyuki Kumei, Gregory Reid, Vladimir Shtelen, Zhenya Yan, Temuerchaolu, Oleg Bogoyavlenskij, Nataliya Ivanova, Dennis The, Sheng Liu, Thomas Wolf, and Juha Pohjanpelto.

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# Introduction

This book is concerned with some modern developments related to symmetries and conservation laws for partial differential equations (PDEs). It is a sequel to *Symmetry and Integration Methods for Differential Equations* (2002) by George W. Bluman and Stephen C. Anco (2002), which focused on Lie groups of transformations and their applications to solving ordinary differential equations (ODEs) and finding invariant solutions of PDEs. The present volume primarily concentrates on recent research of the authors and their collaborators. Most important, we attempt to put this work in a form accessible to graduate students and researchers in applied mathematics, the physical sciences and engineering. Most of the material in this book did not appear in *Symmetries and Differential Equations* [(1989); reprinted with corrections (1996)], by George W. Bluman and Sukeyuki Kumei, and includes a significant updating of the final three chapters.

In the latter part of the $19^{\text{th}}$ century, Sophus Lie initiated his studies on continuous groups (Lie groups) with the aim to put order to, and thereby extend systematically, the hodgepodge of heuristic techniques for solving ODEs. He showed that the problem of finding the Lie group of point transformations leaving invariant a DE (ordinary or partial), i.e., a point symmetry of a DE, reduced to solving related linear systems of determining equations for its infinitesimal generators. Lie also showed that a point symmetry of a DE leads, in the case of an ODE, to reducing the order of the DE (irrespective of any imposed initial conditions) and, in the case of a PDE, to finding special solutions called invariant (similarity) solutions of the DE. Moreover, he showed that a point symmetry of a DE generates a one-parameter family of solutions from any known solution of a DE that is not an invariant solution arising from the symmetry. Most importantly, Lie's work is applicable to nonlinear DEs. His work is discussed in the two above-mentioned books as well as many other excellent references therein. The direct applicability of Lie's work to PDEs, especially nonlinear PDEs, is rather limited, even when a given PDE has a point symmetry, since the resulting invariant solutions yield only a small subset of the solution set of the PDE and hence few posed boundary value problems can be solved.

The extensions of Lie's work to PDEs have focused on finding further applications of point symmetries to include linearization mappings and solutions of boundary value problems, extending the spaces of symmetries of a given PDE system to include local symmetries (higher-order symmetries) as well as nonlocal symmetries, extending the applications of symmetries to include variational symmetries that yield conservation laws for variational systems, extending variational symmetries to multipliers and resulting conservation laws for any given PDE system, finding further solutions that arise from the extension of Lie's method to the "nonclassical method" as well as other generalizations, and efficiently solving the (over-determined) linear system of symmetry and/or multiplier determining equations through the development of symbolic computation software as well as related calculations for solving the nonlinear system of determining equations for the nonclassical method.

A symmetry of a PDE system is any transformation of its solution manifold into itself, i.e., a symmetry transforms (maps) any solution of a PDE system to another solution of the same system. Consequently, continuous symmetries of PDE systems are defined topologically and hence are not restricted to just point symmetries. Thus, in principle, any nontrivial PDE system has symmetries. The problem is to find and use such symmetries. Practically, to find a symmetry of a PDE system, one must consider transformations, acting locally in some finite-dimensional space, whose variables include the dependent variables of the PDE system. However, as it will be seen, these transformation variables do not have to be restricted to the independent and dependent variables of a given PDE system.

One such extension is to consider *higher-order symmetries* (*local symmetries*) where the solutions of the linear determining equations for the components of infinitesimal generators of symmetries are allowed to depend on a finite number of derivatives of the given dependent variables of the PDE. [By comparison, components of infinitesimal generators of point symmetries allow dependence at most linearly on the first derivatives of the dependent variables whereas components of infinitesimal generators of contact symmetries allow arbitrary dependence on first derivatives of dependent variables.] In making this extension, it is essential to realize that the linear determining equations for local symmetries are the linearized system of the given PDE that holds for *all* of its solutions. *Globally*, point and contact symmetries act on finite-dimensional spaces whereas higher-order symmetries act on infinite-dimensional spaces consisting of the dependent and independent variables as well as *all* of their derivatives. Well-known integrable equations of mathematical physics such as the Korteweg–de Vries equation have an infinite number of higher-order local symmetries.

Another extension is to consider solutions of the determining equations that allow an ad-hoc dependence on nonlocal variables such as integrals of the dependent variables. Usually such symmetries are found formally through recursion operators that depend on inverse differentiation. Integrable equations such as the sine-Gordon and cubic Schrödinger equations have an infinite number of such nonlocal symmetries.

In her celebrated 1918 paper, Emmy Noether showed that if a system of DEs admits a variational principle, then any local transformation group leaving invariant the action integral for its Lagrangian density, i.e., an admitted *variational symmetry*, yields a local conservation law. Conversely, any local conservation law of a variational DE system arises from a variational symmetry, and hence there is a direct correspondence between local conservation laws and variational symmetries (Noether's theorem and its generalizations due to Bessel-Hagen (1921) and Boyer (1967)).

There are several limitations to Noether's theorem for finding the local conservation laws for a given DE system. First of all, it is restricted to variational systems. Consequently, for this theorem to be applicable to a given DE system *as written*, the system must have the same number of dependent variables as the number of equations in the given system, and have no dissipation. Moreover, if a given DE system consists of one scalar equation, it must be of even order. In particular, a given system of DEs, as written, is variational if and only if its linearized system is self-adjoint. There is also the difficulty of finding local symmetries of the action integral. In general, not all local symmetries of a variational DE system are variational symmetries. Moreover, the use of Noether's theorem to find local conservation laws is coordinate-dependent.

A conservation law of a given DE system is a divergence expression that vanishes on all solutions of the DE system. Conservation laws describe essential properties of the process modeled by a given DE system and are also used for existence, uniqueness and stability analysis and for the development of numerical methods. In general, all such divergences that yield local conservation laws arise from linear combinations of the DEs of a given system taken with sets of local multipliers in which each multiplier is an expression depending on the independent and dependent variables as well as derivatives (up to some finite order) of the dependent variables of a given DE system. It will be seen that a given DE system has a local conservation law if and only if there exists a set of local multipliers such that the corresponding linear combination of the DEs in the system is identically annihilated by the Euler operators associated with each of its dependent variables without restricting these variables to solutions of the DE system, i.e., the dependent variables are now treated as arbitrary functions. If a given DE system, as written, is variational then its local conservation law multipliers correspond to variational symmetries. In this case, it turns out that its local conservation law multipliers satisfy a system of determining equations that includes the linearizing system of the given DE system augmented by additional determining equations that taken together correspond to the action integral being invariant under the associated variational symmetry. More generally, for *any* given DE system, all local conservation law multipliers are the solutions of an easily found linear determining system that includes the adjoint system of the linearizing DE system. For any set of local conservation law multipliers, one can either directly find the fluxes and density of the corresponding local conservation law or, if this proves difficult, there is an integral formula that yields them without the need of a specific functional (Lagrangian) even in the case when the given DE system is variational.

Another important application of symmetries of PDEs is to determine whether a given PDE system can be mapped into an equivalent target PDE system of interest. This is especially significant if a target class of PDEs can be completely characterized in terms of its symmetries. Target classes with such complete characterizations include linear PDE systems and linear PDEs with constant coefficients. Consequently, from knowledge of the point or contact symmetries of a given PDE system, one can determine whether it can be mapped invertibly to a linear PDE system by a point or contact transformation and explicitly find such a mapping when one exists. Moreover, one can also see whether such a linearization is possible from knowledge of the local conservation law multipliers of a given PDE system. From knowledge of the point symmetries of a linear PDE with variable coefficients, one can determine whether it can be mapped by an invertible point transformation to a linear PDE with constant coefficients and find such an explicit mapping when one exists.

In order to effectively apply symmetry methods to PDE systems, one needs to work in some specific coordinate frame in order to perform calculations. A procedure to find symmetries that are nonlocal and yet are local in some related coordinate frame involves embedding a given PDE system in another PDE system obtained by adjoining nonlocal variables in such a way that the related PDE system is equivalent to the given system and the given system arises through projection. Consequently, any local symmetry of the related system yields a symmetry of the given system. If the local symmetry of the related system has an essential dependence on the nonlocal variables after projection, then it yields a nonlocal symmetry of the given PDE system.

A systematic way to find such an embedding is through local conservation laws of a given PDE system. For each local conservation law, one can introduce a potential variable(s). By adjoining the resulting potential equations to the given PDE system, one can construct an augmented system (*potential system*) of PDEs. By construction, such a potential system is nonlocally equivalent to the given PDE system since, through built in integrability conditions, any solution of the given PDE system yields a solution of the potential system and, conversely, through projection any solution of the potential system yields a solution of the given PDE system. But this relationship is nonlocal since there is no one-to-one correspondence between solutions of the given and potential systems. If a local symmetry of the potential system has an essential dependence on the potential variables when projected onto the space of variables of the given system, then it yields a nonlocal symmetry (*potential symmetry*) of the given PDE system. It turns out that many PDE systems have such potential symmetries. Moreover, one can find other nonlocal symmetries of a given PDE system through seeking local symmetries of an equivalent subsystem of the given system or one of its potential systems provided that such a subsystem is nonlocally related to the given PDE system. Invariant solutions of such potential systems and subsystems can yield further solutions of the given PDE system. A potential symmetry is a local symmetry of a potential system, thus it generates a one-parameter family of solutions from any known solution of the potential system that in turn yields a one-parameter family of solutions from a known solution of the given PDE system. Similarly, this will be the case for a nonlocal symmetry arising from a subsystem. Furthermore, local conservation laws of potential systems can yield nonlocal conservation laws of a given PDE system provided that their local conservation law multipliers have an essential dependence on the potential variables. Linearizations of such potential systems through local symmetry or local conservation law multiplier analysis can yield explicit nonlocal linearizations of a given PDE system. Moreover, through a potential system one can extend the mappings of linear systems with variable coefficients to linear systems with constant coefficients to include nonlocal mappings between such systems.

One can further extend embeddings through using local conservation laws to systematically construct trees of nonlocally related but equivalent systems of PDEs. If a given PDE system has $n$ local conservation laws, then each conservation law yields potentials and corresponding potential systems. Most importantly, from the $n$ local conservation laws, one can directly construct up to $2^n - 1$ independent nonlocally related systems of PDEs by considering the corresponding potential systems individually ($n$ singlets), in pairs ($n(n-1)/2$ couplets), ..., and taken all together (one $n$-plet). In turn, any one of these $2^n - 1$ systems could lead to the discovery of new nonlocal symmetries and/or nonlocal conservation laws of the given PDE system or any of the other nonlocally related systems. Moreover, such nonlocal conservation laws could yield further nonlocally related systems, etc. Furthermore, subsystems of such nonlocally related systems could yield further nonlocally related systems. Correspondingly, a tree of nonlocally related systems is constructed. Through such constructions, one can systematically relate Eulerian and Lagrangian coordinate descriptions of gas dynamics and nonlinear elasticity. In both cases, for a corresponding PDE system written in Eulerian coordinates, there exists a nonlocally related system that yields a corresponding PDE system written in Lagrangian coordinates.

For a given class of PDEs with classifying (constitutive) functions, it is of interest to classify its trees of nonlocally related systems and corresponding symmetries and conservation laws with respect to various forms of its constitutive functions. When a system is variational, i.e., its linearized system is self-adjoint, then of course the local conservation laws arise from a subset of its local symmetries and, in particular, the number of linearly independent conservation laws cannot exceed the number of corresponding higher-order symmetries. But from the above, one can see that, in general, this will not be the case when a system is not variational. Here a specific constitutive function could yield more local conservation laws than local symmetries as well as vice versa.

For any given PDE system, a transformation group (continuous or discrete) that leaves it invariant yields a formula that maps a conservation law to a conservation law of the same system, whether or not the given system is variational. If the group is continuous, then in terms of a parameter expansion a given conservation law could map into more than one additional conservation law for the given PDE system.

Another important extension relates to Lie's work on finding invariant solutions for PDE systems. As mentioned previously, a point symmetry of a PDE system maps each of its solutions into a one-parameter family of solutions. But some solutions map into themselves, i.e., they are themselves invariant. Such solutions satisfy the characteristic PDE given by the invariant surface condition yielding the invariants of the point symmetry. The invariant solutions arising from the point symmetry are the solutions of the given PDE system that satisfy the augmented system consisting of this characteristic PDE with known coefficients (obtained from the point symmetry) and the given PDE system itself. The invariant solutions arise as solutions of a reduced system with one less independent variable. This method ("classical method") of Lie to find invariant solutions of a given PDE is generalized by the *nonclassical method* introduced in Bluman's 1967 PhD thesis where one seeks solutions of an augmented system consisting of the given PDE system and the characteristic PDE with unknown coefficients as well as differential consequences of the augmented system. Here the unknown coefficients are determined by substituting the characteristic equation, and its differential consequences, into the determining system for point symmetries of the augmented system. The resulting over-determined system is nonlinear (even if the given PDE system is linear) in these unknown coefficients, but less over-determined than is the case when finding point symmetries of the given PDE system. Each solution of the determining system for point symmetries is a solution of the determining system for the unknown coefficients of the characteristic PDE. Solving for the unknown coefficients, one then proceeds to find the corresponding "nonclassical" solutions of the augmented system that, by construction, include the classical invariant solutions.

The solutions of a PDE that can be obtained by the nonclassical method include all of its solutions that satisfy a particular functional form (ansatz) of some generality that allows an arbitrary dependence on a similarity variable (depending on the independent and dependent variables of the PDE) and an arbitrary dependence on a function of a similarity variable and the independent variables of the PDE. The solutions obtained by the nonclassical method include all solutions obtained "directly" from such an ansatz by the *direct method* introduced by Clarkson and Kruskal in 1988.

For many PDE systems arising in applications, the linear determining equations for local symmetry components or local conservation law multipliers split into over-determined linear PDE systems that can contain hundreds of equations. To generate, simplify and solve such PDE systems, symbolic software is used. Modern symbolic packages include routines for the automatic generation of determining equations, their subsequent simplification and solution (including classification with respect to constitutive functions and/or parameters of a given DE system), to yield local symmetries and conservation laws of a given DE system.