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A new conservation theorem

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Abstract

A general theorem on conservation laws for arbitrary differential equations is proved. The theorem is valid also for any system of differential equations where the number of equations is equal to the number of dependent variables. The new theorem does not require existence of a Lagrangian and is based on a concept of an adjoint equation for non-linear equations suggested recently by the author. It is proved that the adjoint equation inherits all symmetries of the original equation. Accordingly, one can associate a conservation law with any group of Lie, Lie–Bäcklund or non-local symmetries and find conservation laws for differential equations without classical Lagrangians.

Keywords: Adjoint equation to non-linear equations; Lagrangian; Symmetry of adjoint equations; Conservation theorem; KdV equation

1. Introduction

Noether's theorem [5] establishes a connection between symmetries of differential equations and conservation laws, provided that the equations under consideration are obtained from the variational principle, i.e. they are Euler–Lagrange equations. However, Lagrangians exists only for very special types of differential equations. The restriction to Euler–Lagrange equations reduces applications of Noether's theorem significantly. For example, Noether's theorem is not applicable to evolution equations, to differential equations of an odd order, etc. Moreover, a symmetry of Euler–Lagrange equations should satisfy an additional property to leave invariant the variational integral. In spite of the fact that certain attempts have been made to overcome these

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restrictions and various generalizations of Noether's theorem have been discussed, I do not know in the literature a general result associating a conservation law with *every infinitesimal symmetry of an arbitrary differential equation.*

In a recent paper [3], I made a step toward the solution of this problem. Namely, I defined an adjoint equation for non-linear differential equations and constructed a Lagrangian for an arbitrary (linear and non-linear) equation considered together with its adjoint equation. The same construction furnishes us with a Lagrangian for any system of linear and non-linear differential equations considered together with the adjoint system, provided that the number of equations in the given system is equal to the number of dependent variables. However, the problem was not completely solved because it was not proved that the combined system comprising a given differential equation and the adjoint equation inherits all symmetries of the original equation. The present paper is aimed at filling this gap and completing the proof of a new general conservation theorem.

2. Preliminaries

This section contains a brief discussion of the space $\mathcal{A}$ of differential functions, the basic operators X, $\delta/\delta u^\alpha$ and N^i and the *fundamental identity* connecting these operators. They play a central role in studying symmetries and conservation laws of differential equations. Most of the concepts presented in this section have been introduced in [1, Chapters 4 and 5] (see also [2, Sections 8.4 and 9.7]).

2.1. Lie–Bäcklund operators

Let $x = (x^1, \ldots, x^n)$ be n independent variables, and $u = (u^1, \ldots, u^m)$ be m dependent variables with the partial derivatives $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\}, \ldots$ of the first, second, etc. orders, where $u_i^\alpha = \partial u^\alpha/\partial x^i$, $u_{ij}^\alpha = \partial^2 u^\alpha/\partial x^i \partial x^j$. Denoting

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \tag{2.1}$$

the total differentiation with respect to x^i, we have:

$$u_i^\alpha = D_i\big(u^\alpha\big), \quad u_{ij}^\alpha = D_i\big(u_j^\alpha\big) = D_i D_j\big(u^\alpha\big), \quad \ldots.$$

The variables u^α are also known as *differential variables.*

A function $f(x, u, u_{(1)}, \ldots)$ of a finite number of variables $x, u, u_{(1)}, u_{(2)}, \ldots$ is called a *differential function* if it is locally analytic, i.e., locally expandable in a Taylor series with respect to all arguments. The highest order of derivatives appearing in the differential function is called the order of this function. The set of all differential functions of all finite orders is denoted by $\mathcal{A}$. This set is a vector space with respect to the usual addition of functions and becomes an associative algebra if multiplication is defined by the usual multiplication of functions. The space $\mathcal{A}$ is closed under the total differentiations: if $f \in \mathcal{A}$ then $D_i(f) \in \mathcal{A}$.

Let $\xi^i, \eta^\alpha \in \mathcal{A}$ be differential functions depending on any finite number of variables $x, u, u_{(1)}, u_{(2)}, \ldots$. A first-order linear differential operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots, \tag{2.2}$$

where

$$\zeta_i^\alpha = D_i\bigl(\eta^\alpha - \xi^j u_j^\alpha\bigr) + \xi^j u_{ij}^\alpha,$$
$$\zeta_{i_1 i_2}^\alpha = D_{i_1} D_{i_2}\bigl(\eta^\alpha - \xi^j u_j^\alpha\bigr) + \xi^j u_{j i_1 i_2}^\alpha, \quad \ldots \tag{2.3}$$

is called a *Lie–Bäcklund operator* (see [2, Section 8.4], and the references therein). The Lie–Bäcklund operator (2.2) is often written in the abbreviated form

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \tag{2.4}$$

where the prolongation given by (2.2)–(2.3) is understood. The operator (2.2) is formally an infinite sum, but it truncates when acting on any differential function. Hence, *the action of Lie–Bäcklund operators is well defined on the space $\mathcal{A}$.*

The commutator $[X_1, X_2] = X_1 X_2 - X_2 X_1$ of any two Lie–Bäcklund operators,

$$X_\nu = \xi_\nu^i \frac{\partial}{\partial x^i} + \eta_\nu^\alpha \frac{\partial}{\partial u^\alpha} + \cdots \quad (\nu = 1, 2),$$

is identical with the Lie–Bäcklund operator given by

$$[X_1, X_2] = \bigl(X_1\bigl(\xi_2^i\bigr) - X_2\bigl(\xi_1^i\bigr)\bigr)\frac{\partial}{\partial x^i} + \bigl(X_1\bigl(\eta_2^\alpha\bigr) - X_2\bigl(\eta_1^\alpha\bigr)\bigr)\frac{\partial}{\partial u^\alpha} + \cdots, \tag{2.5}$$

where the terms denoted by dots are obtained by prolonging the coefficients of $\partial/\partial x^i$ and $\partial/\partial u^\alpha$ in accordance with Eqs. (2.3).

The set of all Lie–Bäcklund operators is an infinite-dimensional Lie algebra with respect to the commutator (2.5). It is called the *Lie–Bäcklund algebra* and denoted by $L_\mathcal{B}$. The algebra $L_\mathcal{B}$ is endowed with the following properties:

I. $D_i \in L_\mathcal{B}$. In other words, the total differentiation (2.1) is a Lie–Bäcklund operator. Furthermore,

$$X_* = \xi_*^i D_i \in L_\mathcal{B} \tag{2.6}$$

for any $\xi_*^i \in \mathcal{A}$.

II. Let L_* be the set of all Lie–Bäcklund operators of the form (2.6). Then L_* is an ideal of $L_\mathcal{B}$, i.e., $[X, X_*] \in L_*$ for any $X \in L_\mathcal{B}$. Indeed,

$$[X, X_*] = \bigl(X\bigl(\xi_*^i\bigr) - X_*\bigl(\xi^i\bigr)\bigr)D_i \in L_*.$$

III. In accordance with property II, two operators $X_1, X_2 \in L_\mathcal{B}$ are said to be *equivalent* (i.e. $X_1 \sim X_2$) if $X_1 - X_2 \in L_*$. In particular, every operator $X \in L_\mathcal{B}$ is equivalent to an operator (2.2) with $\xi^i = 0$, $i = 1, \ldots, n$. Namely, $X \sim \widetilde{X}$ where

$$\widetilde{X} = X - \xi^i D_i = \bigl(\eta^\alpha - \xi^i u_i^\alpha\bigr)\frac{\partial}{\partial u^\alpha} + \cdots. \tag{2.7}$$

The operators of the form

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \quad \eta^\alpha \in \mathcal{A}, \tag{2.8}$$

are called *canonical Lie–Bäcklund operators*. Hence, the property III means that any operator $X \in L_\mathcal{B}$ is equivalent to a canonical Lie–Bäcklund operator.

IV. Generators of Lie point transformation groups are operators (2.4) with the coefficients ξ^i and η^α depending only on x, u:

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{2.9}$$

The Lie–Bäcklund operator (2.2) is equivalent to a generator (2.9) of a point transformation group if and only if its coordinates have the form

$$\xi^i = \xi_1^i(x,u) + \xi_*^i, \qquad \eta^\alpha = \eta_1^\alpha(x,u) + \big(\xi_2^i(x,u) + \xi_*^i\big)u_i^\alpha,$$

where $\xi_*^i \in \mathcal{A}$ are arbitrary differential functions and $\xi_1^i, \xi_2^i, \eta_1^\alpha$ are arbitrary functions of x and u.

Example 2.1. Let t, x be the independent variables. The generator of the Galilean transformation and its canonical Lie–Bäcklund form (2.7) are written as:

$$X = \frac{\partial}{\partial u} - t\frac{\partial}{\partial x} \sim \widetilde{X} = (1 + tu_x)\frac{\partial}{\partial u} + \cdots.$$

Example 2.2. The generator of non-homogeneous dilations (see operator X_2 in Section 4.1) and its canonical Lie–Bäcklund representation (2.7) are written as:

$$X = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} \sim \widetilde{X} = (2u + 3tu_t + xu_x)\frac{\partial}{\partial u} + \cdots.$$

2.2. Fundamental identity

The *Euler–Lagrange operator* in $\mathcal{A}$ is defined by the formal sum

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1,\dots,m, \tag{2.10}$$

where, for every s, the summation is presupposed over the repeated indices $i_1\cdots i_s$ running from 1 to n.

In the case of one independent variable x and one dependent variable y the Euler–Lagrange operator reads:

$$\frac{\delta}{\delta y} = \sum_{s=0}^{\infty}(-1)^s D_x^s\frac{\partial}{\partial y^{(s)}} = \frac{\partial}{\partial y} - D_x\frac{\partial}{\partial y'} + D_x^2\frac{\partial}{\partial y''} - D_x^3\frac{\partial}{\partial y'''} + \cdots, \tag{2.11}$$

where D_x is the total differentiation with respect to x:

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots.$$

Let $X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha}$ be any Lie–Bäcklund operator (2.2). We associate with X the following n operators $\mathcal{N}^i$ $(i = 1,\dots,n)$ by the formal sums:

$$\mathcal{N}^i = \xi^i + W^\alpha\frac{\delta}{\delta u_i^\alpha} + \sum_{s=1}^{\infty}D_{i_1}\cdots D_{i_s}\big(W^\alpha\big)\frac{\delta}{\delta u^\alpha_{ii_1\cdots i_s}}, \tag{2.12}$$

where

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad \alpha = 1,\dots,m, \tag{2.13}$$

and the Euler–Lagrange operators with respect to derivatives of u^α are obtained from (2.10) by replacing u^α by the corresponding derivatives, e.g.

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{j_1}\cdots D_{j_s}\frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}}. \tag{2.14}$$

The Euler–Lagrange (2.10), Lie–Bäcklund (2.2) and the associated operators (2.14) are connected by the following *fundamental identity* (N.H. Ibragimov, 1979, see [2, Section 8.4.4]):

$$X + D_i\big(\xi^i\big) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i \mathcal{N}^i. \tag{2.15}$$

2.3. Noether's theorem

Let us begin with Euler–Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\right) = 0, \quad \alpha = 1, \ldots, m, \tag{2.16}$$

where $\mathcal{L}(x, u, u_{(1)})$ is a first-order Lagrangian, i.e. it involves, along with the independent variables $x = (x^1, \ldots, x^n)$ and the dependent variables $u = (u, \ldots, u^m)$, the first-order derivatives $u_{(1)} = \{u_i^\alpha\}$ only.

Noether's theorem states that if the variational integral with the Lagrangian $\mathcal{L}(x, u, u_{(1)})$ is invariant under a group G with a generator

$$X = \xi^i(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial u^\alpha} \tag{2.17}$$

then the vector field $C = (C^1, \ldots, C^n)$ defined by

$$C^i = \xi^i \mathcal{L} + \big(\eta^\alpha - \xi^j u_j^\alpha\big) \frac{\partial \mathcal{L}}{\partial u_i^\alpha}, \quad i = 1, \ldots, n, \tag{2.18}$$

provides a conservation law for the Euler–Lagrange equations (2.16), i.e. obeys the equation $\operatorname{div} C \equiv D_i(C^i) = 0$ for all solutions of (2.16), i.e.

$$D_i\big(C^i\big)\big|_{(2.16)} = 0. \tag{2.19}$$

Any vector field C^i satisfying (2.19) is called a *conserved vector* for Eq. (2.16).

Remark 2.1. It is manifest from Eq. (2.19) that any linear combination of conserved vectors is a conserved vector. Furthermore, any vector vanishing on the solutions of Eq. (2.16) is a conserved vector, a *trivial conserved vector,* for Eq. (2.16). In what follows, conserved vectors will be considered up to addition of trivial conserved vectors.

The invariance of the variational integral implies that the Euler–Lagrange equations (2.16) admit the group G. Therefore, in order to apply Noether's theorem, one has first of all to find the symmetries of Eq. (2.16). Then one should single out the symmetries leaving invariant the variational integral (2.16). This can be done by means of the following infinitesimal test for the invariance of the variational integral (see [1] or [2]):

$$X(\mathcal{L}) + \mathcal{L} D_i\big(\xi^i\big) = 0, \tag{2.20}$$

where the generator X is prolonged to the first derivatives $u_{(1)}$ by the formula

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \big[D_i\big(\eta^\alpha\big) - u_j^\alpha D_i\big(\xi^j\big)\big] \frac{\partial}{\partial u_i^\alpha}. \tag{2.21}$$

If Eq. (2.20) is satisfied, then the vector (2.18) provides a conservation law.

The invariance of the variational integral is sufficient, as said above, for the invariance of the Euler–Lagrange equations, but not necessary. Indeed, the following lemma shows that if one adds to a Lagrangian the divergence of any vector field, the Euler–Lagrange equations remain invariant.

Lemma 2.1. *A function $f(x,u,\dots,u_{(s)}) \in \mathcal{A}$ with several independent variables $x=(x^1,\dots,x^n)$ and several dependent variables $u=(u^1,\dots,u^m)$ is the divergence of a vector field $H=(h^1,\dots,h^n)$, $h^i \in \mathcal{A}$, i.e.*

$$f = \operatorname{div} H \equiv D_i\big(h^i\big), \tag{2.22}$$

if and only if the following equations hold identically in $x, u, u_{(1)}, \dots$:

$$\frac{\delta f}{\delta u^\alpha} = 0, \quad \alpha = 1,\dots,m. \tag{2.23}$$

Therefore, one can add to the Lagrangian $\mathcal{L}$ the divergence of an arbitrary vector field depending on the group parameter and replace the invariance condition (2.20) by the divergence condition

$$X(\mathcal{L}) + \mathcal{L} D_i\big(\xi^i\big) = D_i\big(B^i\big). \tag{2.24}$$

Then Eq. (2.16) is again invariant and has a conservation law $D_i(C^i)=0$, where (2.18) is replaced by

$$C^i = \xi^i \mathcal{L} + \big(\eta^\alpha - \xi^j u_j^\alpha\big)\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - B^i. \tag{2.25}$$

It follows from Eqs. (2.15) and (2.20) that if a variational integral $\int \mathcal{L}\,dx$ with a higher-order Lagrangian $\mathcal{L}(x,u,u_{(1)},u_{(2)},u_{(3)},\dots)$ is invariant under a group with a generator (2.21), then the vector

$$C^i = \mathcal{N}^i(\mathcal{L}) \tag{2.26}$$

provides a conservation law for the corresponding Euler–Lagrange equations. Dropping the differentiations of $\mathcal{L}$ with respect to higher-order derivative $u_{(4)},\dots$ and changing the summation indices, we obtain from (2.26) and (2.12):

$$\begin{aligned} C^i = \xi^i \mathcal{L} &+ W^\alpha\left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) - \cdots\right] \\ &+ D_j\big(W^\alpha\big)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) + \cdots\right] + D_j D_k\big(W^\alpha\big)\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots\right], \end{aligned} \tag{2.27}$$

where $\mathcal{N}^i$ is the operator (2.12) and $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$ is given by (2.13).

In the case of first-order Lagrangians, Eqs. (2.27) coincide with Eqs. (2.18).

In the case of second-order Lagrangians $L(x,u,u_{(1)},u_{(2)})$, the Euler–Lagrange equations (2.16) and the conserved vector (2.18) are replaced by

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\right) + D_i D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha}\right) = 0 \tag{2.28}$$

and

$$C^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} \right) \right] + D_k \left(W^\alpha \right) \frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha}, \tag{2.29}$$

respectively.

In the case of third-order Lagrangians $L(x, u, u_{(1)}, u_{(2)}, u_{(3)})$, the Euler–Lagrange equations are written as:

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i \left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha} \right) + D_i D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} \right) - D_i D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) = 0 \tag{2.30}$$

and the conserved vector (2.27) becomes

$$\begin{aligned} C^i = {} & \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) \right] \\ & + D_j \left(W^\alpha \right) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) \right] + D_j D_k \left(W^\alpha \right) \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}. \end{aligned} \tag{2.31}$$

3. Basic definitions and theorems

3.1. Adjoint equations

Recall that the *adjoint operator* to a linear differential operator L is usually defined as a linear operator L^* such that the equation

$$v L[u] - u L^*[v] = \operatorname{div} P(x) \tag{3.1}$$

holds for all functions u and v, where $P(x) = (p^1(x), \ldots, p^n(x))$ is any vector and $\operatorname{div} P = D_i(p^i)$. The equation $L^*[v] = 0$ is called the adjoint equation to $L[u] = 0$. The operator L and the equation $L[u] = 0$ are said to be *self-adjoint* if $L[u] = L^*[u]$ for any function $u(x)$. For example, if L is a linear second-order differential operator,

$$L[u] = a^{ij}(x) D_i D_j(u) + b^i(x) D_i(u) + c(x) u, \tag{3.2}$$

Eq. (3.1) yields the adjoint operator L^* given by

$$L^*[v] = D_i D_j \left(a^{ij}(x) v \right) - D_i \left(b^i(x) v \right) + c(x) v. \tag{3.3}$$

The operator (3.2) is self-adjoint provided that

$$b^i(x) = D_j \left(a^{ij} \right), \quad i = 1, \ldots, n. \tag{3.4}$$

The definitions of the adjoint operator and the adjoint equation are the same for systems of differential equations. For example, in the case of systems of second-order equations the adjoint operator is obtained by assuming that u is an m-dimensional vector-function and that the coefficients $a^{ij}(x)$, $b^i(x)$ and $c(x)$ of the operator (3.2) are $(m \times m)$-matrices. The following two second-order equations provide an example of a self-adjoint system:

$$\begin{aligned} & x^2 u_{xx} + u_{yy} + 2x u_x + w = 0, \\ & w_{xx} + y^2 w_{yy} + 2y w_y + u = 0. \end{aligned}$$

Linearity of equations is crucial for defining adjoint equations by means of Eq. (3.1). The following definition of an adjoint equation suggested in [3] is applicable to any system of linear and non-linear differential equations.

Definition 3.1. Consider a system of sth-order partial differential equations

$$F_\alpha(x, u, \ldots, u_{(s)}) = 0, \quad \alpha = 1, \ldots, m, \tag{3.5}$$

where $F_\alpha(x, u, \ldots, u_{(s)}) \in \mathcal{A}$ are differential functions with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$, $u = u(x)$. We introduce the differential functions

$$F^*_\alpha(x, u, v, \ldots, u_{(s)}, v_{(s)}) = \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha}, \quad \alpha = 1, \ldots, m, \tag{3.6}$$

where $v = (v^1, \ldots, v^m)$ are new dependent variables, $v = v(x)$, and define the system of *adjoint equations* to Eqs. (3.5) by

$$F^*_\alpha(x, u, v, \ldots, u_{(s)}, v_{(s)}) = 0, \quad \alpha = 1, \ldots, m. \tag{3.7}$$

In the case of linear equations, Definition 3.1 is equivalent to the classical definition of the adjoint equation. Namely, taking for the sake of simplicity scalar equations, we can formulate the statement as follows.

Theorem 3.1. *The operator L^* to a linear operator L defined by Eq.* (3.1) *is identical with the operator L^* given by*

$$L^*[v] = \frac{\delta(vL[u])}{\delta u}. \tag{3.8}$$

One can easily verify that if $L[u]$ is the second-order operator given by (3.2), then operator $L^*[v]$ defined by Eq. (3.8) coincides with the operator $L^*[v]$ defined by Eq. (3.3).

Remark 3.1. The adjoint equation to a linear equation $F(x, u, \ldots, u_{(s)}) = 0$ for $u(x)$ is a linear equation $F^*(x, v, \ldots, v_{(s)}) = 0$ for $v(x)$. If Eqs. (3.5) are non-linear, the adjoint equations are linear with respect to $v(x)$, but non-linear in the coupled variables u and v.

Definition 3.2. A system of equations (3.5) is said to be *self-adjoint* if the system obtained from the adjoint equations (3.7) by the substitution $v = u$:

$$F^*_\alpha(x, u, u, \ldots, u_{(s)}, u_{(s)}) = 0, \quad \alpha = 1, \ldots, m, \tag{3.9}$$

is identical with the original system (3.5).

Remark 3.2. Definition 3.2 does not mean that the left-hand sides of a self-adjoint system (3.5) and of Eqs. (3.9) coincide. So, in general, it may happen that $F^*_\alpha(x, u, u, \ldots, u_{(s)}, u_{(s)}) \neq F_\alpha(x, u, \ldots, u_{(s)})$ even though (3.5) is self-adjoint. See Example 3.3.

Example 3.1. For the heat equation $u_t - u_{xx} = 0$, Eq. (3.6) yields

$$F^* = \frac{\delta}{\delta u}\big[v(v_t - u_{xx})\big] = \left(-D_t\frac{\partial}{\partial u_t} + D_x^2\frac{\partial}{\partial u_{xx}}\right)\big[v(u_t - u_{xx})\big] = -D_t(v) - D_x^2(v).$$

Hence, the adjoint equation (3.7) to the heat equation is $v_t + v_{xx} = 0$. It is manifest that the heat equation is not self-adjoint.

3.2. Lagrangians

Theorem 3.2. *Any system of sth-order differential equations* (3.5) *considered together with its adjoint equation* (3.7) *has a Lagrangian. Namely, the Euler–Lagrange equations* (2.16) *with the Lagrangian*

$$\mathcal{L} = v^\beta F_\beta(x, u, \ldots, u_{(s)}) \tag{3.10}$$

provide the simultaneous system of equations (3.5)–(3.7) *with* $2m$ *dependent variables* $u = (u^1, \ldots, u^m)$ *and* $v = (v^1, \ldots, v^m)$.

Proof. The proof given in [3] is straightforward. Indeed, the variation of $\mathcal{L}$ given by (3.10) yields:

$$\frac{\delta \mathcal{L}}{\delta v^\alpha} = F_\alpha(x, u, \ldots, u_{(s)}), \tag{3.11}$$

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = F_\alpha^*(x, u, v, \ldots, u_{(s)}, v_{(s)}). \qquad \square \tag{3.12}$$

Example 3.2. According to Theorem 3.2, the heat equation $u_t - u_{xx} = 0$ together with its adjoint $v_t + v_{xx} = 0$ are Euler–Lagrange equations (2.28) with the second-order Lagrangian $\mathcal{L} = v(u_t - u_{xx})$. Using Lemma 2.1 and the identity $-vu_{xx} = (-vu_x)_x + u_x v_x$, one can take the first-order Lagrangian $\mathcal{L} = vu_t + u_x v_x$. The variational derivatives of both Lagrangians are as follows:

$$\frac{\delta \mathcal{L}}{\delta v} = u_t - u_{xx}, \qquad \frac{\delta \mathcal{L}}{\delta u} = -(v_t + v_{xx}).$$

Let us extend Example 3.2 to any linear second-order differential equation

$$L[u] \equiv a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = f(x). \tag{3.13}$$

The Lagrangian (3.10) is written as $\mathcal{L} = (a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u - f(x))v$. We rewrite it in the form

$$\mathcal{L} = D_j\big(va^{ij}u_i\big) - vu_i D_j\big(a^{ij}\big) - a^{ij}u_i v_j + vb^i u_i + cuv - f(x)v.$$

The first term at the right-hand side can be dropped by Lemma 2.1, and hence

$$\mathcal{L} = cuv + vb^i(x)u_i - vu_i D_j\big(a^{ij}\big) - a^{ij}u_i v_j - f(x)v. \tag{3.14}$$

The differential function (3.14) provides a Lagrangian for Eq. (3.13) considered together with the adjoint equation $D_i D_j(a^{ij}v) - D_i(b^i v) + cv = 0$. Namely,

$$\frac{\delta \mathcal{L}}{\delta v} = cu + b^i(x)u_i - u_i D_j\big(a^{ij}\big) + D_j\big(a^{ij}u_i\big) - f = a^{ij}u_{ij} + b^i u_i + cu - f$$

and

$$\frac{\delta \mathcal{L}}{\delta u} = cv - D_i\big(b^i v\big) + D_i\big(vD_j\big(a^{ij}v\big)\big) + D_i\big(a^{ij}v_j\big) = D_i D_j\big(a^{ij}v\big) - D_i\big(b^i v\big) + cv.$$

In particular, if the operator $L[u]$ is self-adjoint, then Eq. (3.13) is obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\big[c(x)u^2 - a^{ij}(x)u_i u_j\big]. \tag{3.15}$$

Indeed, the second and the third terms in the right-hand side of Eq. (3.14) annihilate each other by the condition (3.4). Now we set $v = u$, divide by two and arrive at the Lagrangian (3.15).

Example 3.3. Consider the Korteweg–de Vries (KdV) equation

$$u_t = u_{xxx} + uu_x. \tag{3.16}$$

Equation (3.6) is written as $F^*(t, x, u, v, \ldots, u_{(3)}, v_{(3)}) = -(v_t - v_{xxx} - uv_x)$. It follows that the adjoint equation to the KdV equation is

$$v_t = v_{xxx} + uv_x \tag{3.17}$$

and that $F^*(t, x, u, u, \ldots, u_{(3)}, u_{(3)}) = -F(t, x, u, \ldots, u_{(3)})$. Thus, the *KdV equation is self-adjoint* and provides an example to Remark 3.2. Using Eq. (3.10), we obtain the third-order Lagrangian for the KdV equation:

$$\mathcal{L} = v[u_t - uu_x - u_{xxx}]. \tag{3.18}$$

Since $-vu_{xxx} = (-vu_{xx})_x + v_x u_{xx}$, we can use Lemma 2.1 and take the second-order Lagrangian

$$\mathcal{L} = vu_t - vuu_x + v_x u_{xx}. \tag{3.19}$$

It is equivalent to the following second-order Lagrangian:

$$\mathcal{L} = v_x u_{xx} - uv_t + \frac{1}{2}u^2 v_x. \tag{3.20}$$

Any of the Lagrangians (3.18)–(3.20) yield the KdV equation (3.16) and its adjoint equation (3.17). Indeed:

$$\frac{\delta\mathcal{L}}{\delta v} = u_t - uu_x - u_{xxx}, \qquad \frac{\delta\mathcal{L}}{\delta u} = -v_t + v_{xxx} + uv_x.$$

3.3. Symmetry of adjoint equations

Let us show that the adjoint equations (3.7) inherit all Lie and Lie–Bäcklund symmetries of Eqs. (3.5). We will begin with scalar equations.

Theorem 3.3. *Consider an equation*

$$F(x, u, u_{(1)}, \ldots, u_{(s)}) = 0 \tag{3.21}$$

with n independent variables $x = (x^1, \ldots, x^n)$ and one dependent variable u. The adjoint equation

$$F^*(x, u, v, \ldots, u_{(s)}, v_{(s)}) \equiv \frac{\delta(vF)}{\delta u} = 0 \tag{3.22}$$

to Eq. (3.21) *inherits the symmetries of Eq.* (3.21). *Namely, if Eq.* (3.21) *admits an operator*

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}, \tag{3.23}$$

where X is either a generator of a point transformation group, i.e. $\xi^i = \xi^i(x, u)$, $\eta = \eta(x, u)$, or a Lie–Bäcklund operator, i.e. $\xi^i = \xi^i(x, u, u_{(1)}, \ldots, u_{(p)})$ and $\eta = \eta(x, u, u_{(1)}, \ldots, u_{(q)})$ are any differential functions, then Eq. (3.22) *admits the operator* (3.23) *extended to the variable v by the formula*

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \eta_* \frac{\partial}{\partial v} \tag{3.24}$$

with a certain function $\eta_* = \eta_*(x, u, v, u_{(1)}, \ldots)$.

Proof. Let the operator (3.23) be a Lie point symmetry of Eq. (3.21). Then

$$X(F) = \lambda F, \tag{3.25}$$

where $\lambda = \lambda(x, u, \ldots)$. In Eq. (3.25), the prolongation of X to all derivatives involved in Eq. (3.21) is understood. Furthermore, the simultaneous system (3.21), (3.22) has the Lagrangian (3.10):

$$\mathcal{L} = vF. \tag{3.26}$$

We take an extension of the operator (3.23) in the form (3.24) with an unknown coefficient η_* and require that the invariance condition (2.20) be satisfied:

$$Y(\mathcal{L}) + \mathcal{L}D_i\big(\xi^i\big) = 0. \tag{3.27}$$

We have:

$$\begin{aligned} Y(\mathcal{L}) + \mathcal{L}D_i\big(\xi^i\big) &= Y(v)F + vX(F) + vFD_i\big(\xi^i\big) = \eta_* F + v\lambda F + vFD_i\big(\xi^i\big) \\ &= \big[\eta_* + v\lambda + vD_i\big(\xi^i\big)\big]F. \end{aligned}$$

Hence, the requirement (3.27) leads to the equation

$$\eta_* = -\big[\lambda + D_i\big(\xi^i\big)\big]v \tag{3.28}$$

with λ defined by Eq. (3.25). Since Eq. (3.27) guarantees the invariance of the system (3.21), (3.22) we conclude that the adjoint equation (3.22) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} - \big[\lambda + D_i\big(\xi^i\big)\big]v \frac{\partial}{\partial v} \tag{3.29}$$

thus proving the theorem for Lie point symmetries.

Let us assume now that the symmetry (3.23) is a Lie–Bäcklund operator. Then Eq. (3.25) is replaced by (see [1])

$$X(F) = \lambda_0 F + \lambda_1^i D_i(F) + \lambda_2^{ij} D_i D_j(F) + \lambda_3^{ij} D_i D_j D_k(F) + \cdots, \tag{3.30}$$

where $\lambda_2^{ij} = \lambda_2^{ji}, \ldots$. Therefore, using the operator (3.24), we have:

$$\begin{aligned} Y(\mathcal{L}) + \mathcal{L}D_i\big(\xi^i\big) &= Y(v)F + vX(F) + vFD_i\big(\xi^i\big) \\ &= \big[\eta_* + v\lambda_0 + vD_i\big(\xi^i\big)\big]F + v\lambda_1^i D_i(F) + v\lambda_2^{ij} D_i D_j(F) \\ &\quad + v\lambda_3^{ijk} D_i D_j D_k(F) + \cdots. \end{aligned}$$

Now we use the identities

$$\begin{aligned} &v\lambda_1^i D_i(F) = D_i\big(v\lambda_1^i F\big) - FD_i\big(v\lambda_1^i\big), \\ &v\lambda_2^{ij} D_i D_j(F) = D_i\big[v\lambda_2^{ij} D_j(F) - FD_j\big(v\lambda_2^{ij}\big)\big] + FD_i D_j\big(v\lambda_2^{ij}\big), \\ &v\lambda_3^{ijk} D_i D_j D_k(F) = D_i[\cdots] - FD_i D_j D_k\big(v\lambda_3^{ijk}\big), \end{aligned}$$

etc., and obtain:

$$\begin{aligned} Y(\mathcal{L}) + \mathcal{L}D_i\big(\xi^i\big) &= D_i\big[v\lambda_1^i F + v\lambda_2^{ij} D_j(F) - FD_j\big(v\lambda_2^{ij}\big) + \cdots\big] \\ &\quad + \big[\eta_* + v\lambda_0 + vD_i\big(\xi^i\big) - D_i\big(v\lambda_1^i\big) + D_i D_j\big(v\lambda_2^{ij}\big) \\ &\quad - D_i D_j D_k\big(v\lambda_3^{ijk}\big) + \cdots\big]F. \end{aligned}$$

Finally, we complete the proof of the theorem by setting

$$\eta_* = -\big[\lambda_0 + D_i\big(\xi^i\big)\big]v + D_i\big(v\lambda_1^i\big) - D_i D_j\big(v\lambda_2^{ij}\big) + D_i D_j D_k\big(v\lambda_3^{ijk}\big) - \cdots \tag{3.31}$$

and arriving at Eq. (2.24),

$$X(\mathcal{L}) + \mathcal{L}D_i\big(\xi^i\big) = D_i\big(B^i\big), \tag{2.24}$$

with

$$B^i = -v\lambda_1^i F - v\lambda_2^{ij} D_j(F) + F D_j\big(v\lambda_2^{ij}\big) - \cdots. \qquad \square \tag{3.32}$$

Let us prove a similar statement on symmetries of adjoint equations for systems of m equations with m dependent variables. For the sake of simplicity we will prove the theorem only for Lie point symmetries. The proof can be extended to Lie–Bäcklund symmetries as it has been done in Theorem 3.3.

Theorem 3.4. *Consider a system of m equations*

$$F_\alpha(x, u, u_{(1)}, \ldots, u_{(s)}) = 0, \quad \alpha = 1, \ldots, m, \tag{3.33}$$

with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$. The adjoint system

$$F_\alpha^*(x, u, v, \ldots, u_{(s)}, v_{(s)}) \equiv \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{3.34}$$

inherits the symmetries of the system (3.33). *Namely, if the system* (3.33) *admits a point transformation group with a generator*

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{3.35}$$

then the adjoint system (3.34) *admits the operator* (3.35) *extended to the variables v^α by the formula*

$$Y = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \eta_*^\alpha\frac{\partial}{\partial v^\alpha} \tag{3.36}$$

with appropriately chosen coefficients $\eta_*^\alpha = \eta_*^\alpha(x, u, v, \ldots)$.

Proof. Now the invariance condition (3.25) is replaced by

$$X(F_\alpha) = \lambda_\alpha^\beta F_\beta, \quad \alpha = 1, \ldots, m, \tag{3.37}$$

where the prolongation of X to all derivatives involved in Eqs. (3.33) is understood. We know that the simultaneous system (3.33), (3.34) has the Lagrangian

$$\mathcal{L} = v^\alpha F_\alpha. \tag{3.38}$$

We take an extension of the operator (3.35) in the form (3.36) with undetermined coefficients η_*^α and require that the invariance condition (2.20) be satisfied:

$$Y(\mathcal{L}) + \mathcal{L}D_i\big(\xi^i\big) = 0. \tag{3.39}$$

We have:

$$\begin{aligned} Y(\mathcal{L}) + \mathcal{L} D_i\big(\xi^i\big) &= Y\big(v^\alpha\big) F_\alpha + v^\alpha X(F_\alpha) + v^\alpha F_\alpha D_i\big(\xi^i\big) \\ &= \eta_*^\alpha F_\alpha + \lambda_\alpha^\beta v^\alpha F_\beta + v^\alpha F_\alpha D_i\big(\xi^i\big) \\ &= \big[\eta_*^\alpha + \lambda_\beta^\alpha v^\beta + v^\alpha D_i\big(\xi^i\big)\big] F_\alpha. \end{aligned}$$

Therefore, the requirement (3.39) leads to the equations

$$\eta_*^\alpha = -\big[\lambda_\beta^\alpha v^\beta + v^\alpha D_i\big(\xi^i\big)\big], \quad \alpha = 1, \ldots, m, \tag{3.40}$$

with λ_β^α defined by Eqs. (3.37). Since Eqs. (3.39) guarantee the invariance of the system (3.33), (3.34), the adjoint system (3.34) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} - \big[\lambda_\beta^\alpha v^\beta + v^\alpha D_i\big(\xi^i\big)\big] \frac{\partial}{\partial v^\alpha}. \tag{3.41}$$

This proves the theorem. □

Remark 3.3. Theorems 3.3 and 3.4 are valid for non-local symmetries as well.

3.4. The main conservation theorem

Theorem 3.5. *Every Lie point, Lie–Bäcklund and non-local symmetry*

$$X = \xi^i(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial u^\alpha} \tag{3.42}$$

of differential equations

$$F_\alpha(x, u, \ldots, u_{(s)}) = 0, \quad \alpha = 1, \ldots, m, \tag{3.43}$$

provides a conservation law for the system of differential equations comprising Eqs. (3.43) *and the adjoint equations* (3.34).

Proof. To prove the statement, I present the procedure for computing the conserved vector associated with the symmetry (3.42). The construction follows from Theorem 3.3 for scalar equations and Theorem 3.4 for systems.

Let (3.43) be a scalar equation with one dependent variable u. Then the symmetry (3.42) has the form (3.23) and the corresponding conserved vector C^i is obtained by applying Eq. (2.27) to the Lagrangian (3.26) and to the operator (3.24) with the coefficient η_* given by (3.28) if (3.23) is a Lie point symmetry and by (3.31) if (3.23) is a Lie–Bäcklund symmetry. In the latter case one can ignore the presence of the term $D_i(B^i)$ in Eq. (2.24) since the vector B^i defined by (3.32) vanishes on the solutions of Eq. (3.43) (see Remark 2.1).

Let (3.43) be a system with $m > 1$ and let (3.35) be a point symmetry. The conserved vector C^i is obtained by applying Eq. (2.27) to the Lagrangian (3.38) and to the operator (3.41). We complete the proof by invoking Remark 3.3. □

3.5. An example on Theorem 3.5

The heat equation $u_t - u_{xx} = 0$ together with its adjoint equation $v_t + v_{xx} = 0$ have the Lagrangian $\mathcal{L} = v(u_t - u_{xx})$. We will apply here Theorem 3.5 to a Lie point symmetry and a Lie–Bäcklund symmetry.

Let us take the following generator of a point transformation group:

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} \tag{3.44}$$

admitted by the heat equation and extend it to the variable v. The prolongation of X to the derivatives involved in the heat equation has the form

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} - (xu_t + 2u_x)\frac{\partial}{\partial u_t} - (2u_x + xu_{xx})\frac{\partial}{\partial u_{xx}}.$$

The reckoning shows that Eq. (3.25) is written as $X(u_t - u_{xx}) = -x(u_t - u_{xx})$, hence $\lambda = -x$. Noting that in our case $D_i(\xi^i) = 0$ and using (3.28) we obtain $\eta_* = xv$. Hence, the extension (3.29) of the operator (3.44) to v has the form

$$Y = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} + xv\frac{\partial}{\partial v}. \tag{3.45}$$

One can readily verify that it is admitted by the system $u_t - u_{xx} = 0,\ v_t + v_{xx} = 0$.

Let us find the conservation law provided by the symmetry (3.44). Since we deal with a second-order Lagrangian, $\mathcal{L} = v(u_t - u_{xx})$, we compute the conserved vector by the formula (2.29). Denoting $t = x^1, x = x^2, u = u^1, v = u^2$, we have for the extended operator (3.45):

$$\xi^1 = 0, \qquad \xi^2 = 2t, \qquad \eta^1 = -xu, \qquad \eta^2 = xv, \qquad W = -(xu + 2tu_x).$$

In our case, (2.29) provides the conservation equation $D_t(C^1) + D_x(C^2) = 0$ for the vector $C = (C^1, C^2)$ with the following components:

$$C^1 = W\frac{\partial \mathcal{L}}{\partial u_t} = vW = -v(xu + 2tu_x),$$
$$C^2 = 2t\mathcal{L} + WD_x(v) - vD_x(W) = v(2tu_t + u + xu_x) - (xu + 2tu_x)v_x.$$

This vector involves an arbitrary solution v of the adjoint equation $v_t + v_{xx} = 0$, and hence provides an infinite number of conservation laws. Let us take, e.g. the solutions $v = -1$, $v = -x$ and $v = -\mathrm{e}^t \sin x$. In the first case, we have

$$C^1 = xu + 2tu_x, \qquad C^2 = -(2tu_t + u + xu_x).$$

Noting that $D_t(2tu_x) = D_t D_x(2tu) = D_x D_t(2tu) = D_x(2u + 2tu_t)$ we can transfer the term $2tu_x$ from C^1 to C^2 in the form $2u + 2tu_t$. Then the components of the conserved vector are written simply as

$$C^1 = xu, \qquad C^2 = u - xu_x.$$

In the second case, $v = -x$, we obtain the vector

$$C^1 = x^2u + 2txu_x, \qquad C^2 = \left(2t - x^2\right)u_x - 2txu_t,$$

and simplifying it as before arrive at

$$C^1 = \left(x^2 - 2t\right)u, \qquad C^2 = \left(2t - x^2\right)u_x + 2xu.$$

In the case $v = -\mathrm{e}^t \sin x$, we note that $2tu_x\mathrm{e}^t \sin x = D_x(2tu\mathrm{e}^t \sin x) - 2tu\mathrm{e}^t \cos x$ and simplifying as before obtain:

$$C^1 = \mathrm{e}^t(x \sin x - 2t \cos x)u, \qquad C^2 = (u + 2tu - xu_x)\mathrm{e}^t \sin x + (xu + 2tu_x)\mathrm{e}^t \cos x.$$

The heat equation has also Lie–Bäcklund symmetries. One of them is

$$X = (xu_{xx} + 2tu_{xxx})\frac{\partial}{\partial u}. \tag{3.46}$$

We prolong its action to u_t and u_{xx}, denote the prolonged operator again by X and obtain

$$X(u_t - u_{xx}) = xD_x^2(u_t - u_{xx}) + 2tD_x^3(u_t - u_{xx}).$$

It follows that Eq. (3.30) is satisfied and that the only non-vanishing coefficients in (3.30) are $\lambda_2^{22} = x$ and $\lambda_3^{222} = 2t$. Accordingly, Eq. (3.31) yields:

$$\eta_* = -D_x^2(xv) + D_x^3(2tv) = -2v_x - xv_{xx} + 2tv_{xxx},$$

and hence the extension (3.24) of the operator (3.46) to the variable v is

$$Y = \eta\frac{\partial}{\partial u} + \eta_*\frac{\partial}{\partial v} \equiv (xu_{xx} + 2tu_{xxx})\frac{\partial}{\partial u} + (2tv_{xxx} - 2v_x - xv_{xx})\frac{\partial}{\partial v}. \tag{3.47}$$

For the operator (3.47), we have $\xi^1 = \xi^2 = 0$ and $W = \eta = xu_{xx} + 2tu_{xxx}$. Therefore, (2.29) provides the conserved vector with the following components:

$$C^1 = W\frac{\partial \mathcal{L}}{\partial u_t} = \eta v = [xu_{xx} + 2tu_{xxx}]v,$$

$$C^2 = 2t\mathcal{L} - WD_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right) + D_x(W)\frac{\partial \mathcal{L}}{\partial u_{xx}} = \big[2t(u_t - u_{xx}) - D_x(\eta)\big]v + \eta D_x(v)$$

$$= \big[2t(u_t - u_{xx} - u_{xxxx}) - u_{xx} - xu_{xxx}\big]v + [u_{xx} + 2tu_{xxx}]v_x.$$

4. Applications

4.1. The Korteweg–de Vries equation

For the KdV equation (3.16), $u_t = u_{xxx} + uu_x$, let us take the Lagrangian (3.18),

$$\mathcal{L} = v[u_t - uu_x - u_{xxx}],$$

and apply Theorem 3.5 to the following two point symmetries (the generators of the Galilean transformation and a scaling transformation) of the KdV equation:

$$X_1 = \frac{\partial}{\partial u} - t\frac{\partial}{\partial x}, \qquad X_2 = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}.$$

The reckoning shows that the extension (3.24) of X_1 to the variable v coincides with X_1. Let us find the extension of X_2. Its prolongation to the derivatives involved in the KdV equation is written as

$$X_2 = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + 5u_t\frac{\partial}{\partial u_t} + 3u_x\frac{\partial}{\partial u_x} + 4u_{xx}\frac{\partial}{\partial u_{xx}} + 5u_{xxx}\frac{\partial}{\partial u_{xxx}}.$$

Consequently, $X_2(u_t - uu_x - u_{xxx}) = 5(u_t - uu_x - u_{xxx})$, and hence $\lambda = 5$. Since $D_i(\xi^i) = -4$, Eq. (3.28) yields $\eta_* = -v$. Thus, the extension (3.24) of X_2 is

$$Y_2 = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}.$$

The operator X_1 yields the conservation law $D_t(C^1) + D_x(C^2) = 0$, where the conserved vector $C = (C^1, C^2)$ is given by (2.31) and has the components

$$C^1 = (1 + tu_x)v, \qquad C^2 = t(v_x u_{xx} - u_x v_{xx} - vu_t) - uv - v_{xx}.$$

Since the KdV equation is self-adjoint (see Example 3.3), we let $v = u$, transfer the term $tuu_x = D_x(\frac{1}{2}tu^2)$ from C^1 to C^2 in the form $tuu_t + \frac{1}{2}u^2$ and obtain

$$C^1 = u, \qquad C^2 = -\frac{1}{2}u^2 - u_{xx}. \tag{4.1}$$

Let us make more detailed calculations for the operator X_2. For this operator, we have $W = (2u + 3tu_t + xu_x)$ and the vector (2.31) is written as:

$$\begin{aligned}
C^1 &= -3t\mathcal{L} + Wv = (3tu_{xxx} + 3tuu_x + xu_x + 2u)v,\\
C^2 &= -x\mathcal{L} - (uv + v_{xx})W + v_x D_x(W) - vD_x^2(W)\\
&= -\left(2u^2 + xu_t + 3tuu_t + 4u_{xx} + 3tu_{txx}\right)v + (3u_x + 3tu_{tx} + xu_{xx})v_x\\
&\quad - (2u + 3tu_t + xu_x)v_{xx}.
\end{aligned}$$

As before, we let $v = u$, simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from C^1 to C^2 and obtain

$$C^1 = u^2, \qquad C^2 = u_x^2 - 2uu_{xx} - \frac{2}{3}u^3. \tag{4.2}$$

The KdV equation has an infinite algebra of Lie–Bäcklund and non-local symmetries [4]. The Lie–Bäcklund symmetry of the lowest (fifth) order is

$$X_3 = f_5\frac{\partial}{\partial u} \quad \text{with } f_5 = u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_2u_2 + \frac{5}{6}u^2u_1,$$

where $u_1 = u_x,\ u_2 = u_{xx}, \ldots.$ The reckoning shows that the invariance condition (3.30) for $F = u_t - uu_x - u_{xxx}$ is satisfied in the following form:

$$X_3(F) = \left[\frac{5}{3}(u_3 + uu_1) + \frac{5}{6}\left(4u_2 + u^2\right)D_x + \frac{10}{3}u_1D_x^2 + \frac{5}{3}uD_x^3 + D_x^5\right](F).$$

The first component of the conserved vector (2.31) is $C^1 = vf_5$. Upon setting $v = u$, we have $uf_5 = D_x(uu_4 - u_1u_3 + \frac{1}{2}u_2^2 + \frac{5}{3}u^2u_2 + \frac{5}{24}u^4)$. Hence, the Lie–Bäcklund symmetry X_3 provides only a trivial conserved vector (2.31) with $C^1 = 0$.

Let us apply our technique to non-local symmetries (see Remark 3.3). The KdV equation has an infinite set of non-local symmetries, namely:

$$\mathcal{X}_{n+2} = g_{n+2}\frac{\partial}{\partial u}, \tag{4.3}$$

where g_{n+2} are given recurrently by ([4], see also [1, Eq. (18.36)])

$$g_1 = 1 + tu_1, \qquad g_{n+2} = \left(D_x^2 + \frac{2}{3}u + \frac{2}{3}D_x^{-1}\right)g_n, \quad n = 1, 3, \ldots. \tag{4.4}$$

The operator $\mathcal{X}_1 = (1 + tu_1)\frac{\partial}{\partial u}$ corresponding to g_1 is the canonical Lie–Bäcklund representation of the generator X_1 of the Galilean transformation (Example 2.1). Equation (4.4) yields $g_3 = (1/3)[2u + 3t(u_3 + uu_1) + xu_1] \equiv (1/3)(2u + 3tu_t + xu_x)$, hence $\mathcal{X}_3$ coincides, up to the constant factor $1/3$, with the canonical Lie–Bäcklund representation of the scaling generator X_2

(cf. Example 2.2). Continuing the recursion (4.4), we arrive at the following non-local symmetry of the KdV equation:

$$\mathcal{X}_5 = g_5 \frac{\partial}{\partial u} \quad \text{with } g_5 = tf_5 + \frac{x}{3}(u_3 + uu_1) + \frac{4}{3}u_2 + \frac{4}{9}u^2 + \frac{1}{9}u_1\varphi,$$

where f_5 is the coordinate of the Lie–Bäcklund operator X_3 used above and φ is a *non-local variable* defined by the following integrable system of equations:

$$\varphi_x = u, \qquad \varphi_t = u_{xx} + \frac{1}{2}u^2.$$

We will write down only the first component of the conserved vector (2.31). It is given by $C^1 = vg_5$. Setting $v = u$, transferring the terms of the form $D_x(\ldots)$ from C^1 to C^2, eliminating an immaterial constant factor and returning to the original notation $u_1 = u_x$, we arrive at a non-trivial conservation law with

$$C^1 = u^3 - 3u_x^2. \tag{4.5}$$

The non-local variable φ is involved in the component C^2 of the conserved vector. We can also take in $C^1 = vg_5$ the solution $v = 1$ of the adjoint equation (3.17), $v_t = v_{xxx} + uv_x$. Then we will arrive again to the conserved vector (4.2).

Dealing likewise with all non-local symmetries (4.3), we obtain an infinite set of non-trivial conservation laws. For example, $\mathcal{X}_5$ yields

$$C^1 = 29u^4 + 852uu_1^2 - 252u_2^2. \tag{4.6}$$

4.2. *The Black–Scholes equation*

For the Black–Scholes equation

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0, \tag{4.7}$$

the adjoint equation has the form

$$\frac{1}{2}A^2x^2v_{xx} + \left(2A^2 - B\right)xv_x - v_t + \left(A^2 - B - C\right)v = 0 \tag{4.8}$$

and the Lagrangian is

$$\mathcal{L} = \left(u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu\right)v. \tag{4.9}$$

Let us find the conservation law corresponding, e.g. to the time-translational invariance of Eq. (4.7), i.e. provided by the infinitesimal symmetry $X = \frac{\partial}{\partial t}$. For this operator, we have $W = -u_t$, and Eq. (2.29) written for the second-order Lagrangian (4.9) yields the conserved vector

$$C^1 = \mathcal{L} - u_t v = \left(\frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu\right)v,$$

$$C^2 = -\left[Bxv - D_x\left(\frac{1}{2}A^2x^2v\right)\right]u_t - \frac{1}{2}A^2x^2vu_{tx}$$

$$= \left[-Bxv + A^2xv + \frac{1}{2}A^2x^2v_x\right]u_t - \frac{1}{2}A^2x^2vu_{tx}.$$

We can substitute here any solution $v = v(t, x)$ of the adjoint equation (4.8). Let us take, e.g. the invariant solution obtained by using the operator $X = x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}$. The solution has the form

$$v = \frac{1}{x}\mathrm{e}^{-Ct}, \quad \text{where } C \text{ is from Eq. (4.7).}$$

Substituting it in the above C^1, C^2 and simplifying as before, we obtain

$$C^1 = \frac{u}{x}\mathrm{e}^{-Ct}, \qquad C^2 = \left(\frac{1}{2}A^2 x u_x + Bu - \frac{1}{2}A^2 u\right)\mathrm{e}^{-Ct}. \tag{4.10}$$

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