Report of the meeting of the British association for the advancement of science

at 501 feet. The difference between them agrees well with the generally accepted rate of 1° for 300 feet, and indicates about 48° as the surface temperature at small elevations, such as 30 feet. The pits in the East Manchester coal-field from which we have observations, namely, Astley Pit (Dukinfield), Ashton Moss, Bredbury, Denton, and Nook Pit, are all sunk in ground at elevations of between 300 and 350 feet. It would therefore appear that the assumption of a surface temperature of 49°, which we made in reducing these observations, is about 2° in excess of the truth.

A very elaborate paper on Underground Temperature has recently been communicated to the Royal Society by one of the members of the Committee—Professor Prestwich. It contains probably the fullest collection that has ever been made of observations of underground temperature, accompanied in most cases by critical remarks; and adduces arguments to show that most of the temperatures observed are too low, owing to the influence of the air in mines, and of convection currents in wells. Professor Prestwich is disposed to adopt 1° F. in 45 feet as the most probable value of the normal gradient.

Report on Electrical Theories.

By Professor J. J. THOMSON, *M.A.*, *F.R.S.*

IN this report I have confined myself exclusively to the consideration of those theories of electrical action which only profess to give mathematical expressions for the forces exerted by a system of currents, and which make no attempt to give any physical explanation of these forces; for it is evident that before we can test any theory of electrical action we must know what the actions are which it has to explain, and we cannot do this until we have a satisfactory mathematical theory. I have further limited myself to the consideration of the fundamental assumptions of each theory, and have not attempted to give any account of its mathematical developments, except in so far as they lead to results capable of distinguishing between the various theories.

I have divided the theories into the following classes:—

1. Theories in which the action between elements of current is deduced by geometrical considerations combined with assumptions which are not explicitly, at any rate, founded on the principle of the Conservation of Energy.

This class includes the theories of Ampère, Grassmann, Stefan, and Korteweg.

2. Theories which explain the action of currents by assuming that the forces between electrified bodies depend upon the velocities and accelerations of the bodies.

This class includes the theories of Gauss, Weber, Riemann, and Clausius.

3. Theories which are based upon dynamical considerations, but which neglect the action of the dielectric.

This class contains F. E. Neumann's potential theory and v. Helmholtz's extension of it.

4. C. Neumann's theory.

5. Theories which are based upon dynamical considerations, and which take into account the action of the dielectric.

This class includes the theories of Maxwell and v. Helmholtz.

We shall now proceed to the detailed consideration of these theories.

Theories in which the action between elements of current is deduced by geometrical considerations combined with certain assumptions which are not explicitly, at any rate, founded on the Principle of the Conservation of Energy.

The best known theory of this class is that of Ampère. Others, however, have been given by Grassmann, Stefan, and Korteweg, which we shall consider in order.

Ampère's Theory.

This theory was first published in 1820. In 1823 appeared his great paper, the 'Mémoire sur la Théorie Mathématique des Phénomènes Electro-dynamiques,' *Mémoires de l'Institut*, t. vi., which Maxwell describes as 'perfect in form and unassailable in accuracy,' and which at once brought the action between electric currents under the power of mathematics. Ampère founded his theory on certain postulates which he attempted to establish by experiment; inasmuch, however, as he always dealt with closed circuits in his experiments and elements of circuit in his postulates, the experimental evidence is not quite satisfactory. Ampère's experiments have been repeated by v. Ettingshausen [1] with much more delicate apparatus.

The postulates used by Ampère are as follows. The first four are given in the words of Professor Tait:—[2]

I. 'Equal and opposite currents in the same conductor produce equal and opposite effects on other conductors; whence it follows that an element of one current has no effect on an element of another which lies in the plane bisecting the former at right angles.'

II. 'The effect of a conductor bent or twisted in any manner is equivalent to that of a straight one, provided that the two are traversed by equal currents and the former nearly coincides with the latter.'

III. 'No closed circuit can set in motion an element of a circular conductor about an axis through the centre of the circle and perpendicular to its plane.'

IV. 'In similar systems traversed by equal currents the forces are equal.'

V. 'The action between two elements of current is a *force* along the straight line joining them, and proportional to the product of the lengths of the elements and the currents flowing through them.'

It follows from IV. that the force between two elements of current varies inversely as the square of the distance between them.

The assumption V. is one that can only be justified by the correctness of the results to which it leads. We have no right to assume *à priori* that the action is equivalent to a single force, and not to a force and a couple; and we have no more right to assume that the force is along the line joining the elements than we have to assume that the force between

[1] 'Ueber Ampère's elektrodynamische Fundamentalversuche,' *Wien. Ber.* (11), 77, p. 109, 1878.

[2] Tait's *Quaternions*, 2nd edit. p. 249.

two small magnets is along the line joining their centres, and in this case the assumption is untrue. It is in the nature of the assumption that Ampère's theory differs from others of this class. The second part of I. depends upon V. It is not true unless we assume that the force between two elements is along the line joining them.

Ampère deduces the force between two elements of current from these principles in the following way:—Suppose we have two elements of current of lengths ds_1, ds_2 traversed by currents of strengths i, j respectively. Let us take the line joining the centres of these currents as the axis of x; let the plane of ds_1 and x be taken as the plane of xy; let θ_1, θ_2 be the angles which ds_1, ds_2 respectively make with the axis of x, η the angle which the plane through ds_2 and r makes with the plane of xy.

By Ampère's second proposition the action of ds_1 on ds_2 will be the sum of the action of

$$\begin{cases} ds_1 \cos\theta_1 \text{ or } \alpha_1 \text{ along } x \\ ds_1 \sin\theta_1 \text{ or } \beta_1 \text{ along } y \end{cases}$$

on

$$\begin{cases} ds_2 \cos\theta_2 \text{ or } \alpha_2 \text{ along } x \\ ds_2 \sin\theta_2 \cos\eta \text{ or } \beta_2 \text{ along } y \\ ds_2 \sin\theta_2 \sin\eta \text{ or } \gamma_2 \text{ along } z. \end{cases}$$

Now by proposition I. α_1 cannot exert a force on either β_2 or γ_2, because it is in planes which bisect β_2 and γ_2 at right angles, so that the only component on which α_1 can exert a force is α_2. Let the force between these components be

$$\frac{a}{r^2}\alpha_1\alpha_2.$$

where r is the distance between the centres of the elementary currents.

In the same way we can show that the only component on which β_1 can exert any force is β_2. Let the force between these two elements be

$$\frac{b}{r^2}\beta_1\beta_2.$$

Thus the force between the two elements ds_1, ds_2 is

$$\frac{1}{r^2}\{a\alpha_1\alpha_2 + b\beta_1\beta_2\},$$

or, substituting for $\alpha_1\alpha_2$, $\beta_1\beta_2$ their values:

$$\frac{1}{r^2}\{a\cos\theta_1\cos\theta_2 + b\sin\theta_1\sin\theta_2\cos\eta\}\, ij\, ds_1\, ds_2.$$

The proposition III., that the action of a closed circuit on an element of current is always at right angles to the element, leads on integration to the condition

$$2a + b = 0,$$

so that the force between the two elements equals

$$\frac{a}{r^2}\{\cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2\cos\eta\}\, ij\, ds_1\, ds_2.$$

From this we are able to find the force between any two circuits or parts of circuits. To find the force on a magnetic system, Ampère used his principle that the magnetic action of an electric current was the same as that due to a magnetic shell bounded by the circuit and magnetised to the proper intensity. In this way Ampère gave a complete theory of the action of currents upon currents and upon magnets—in fact, a complete theory of all the effects produced by a current which were known when his paper was published.

It is difficult to overrate the service which Ampère's theory has rendered to the science of electrodynamics. Perhaps the best evidence of its value for practical purposes is the extreme difficulty of finding any experiment which proves that it is insufficient. In spite of this, however, as a dynamical theory it is very unsatisfactory. If, as we are led to do by Ampère, we attach physical importance to elements of current, and regard them as something more than mathematical helps for calculating the force between two closed circuits, then we are driven to ask, not only what is the law of force between the elements, but what is the energy possessed by a system consisting of two such elements. If we do this, and find this energy by calculating the amount of work required to pull the elements an infinite distance apart, we arrive at the conclusion that the energy must depend upon the angles which the elements make with each other and with the line joining them; but if this is so, then the force between the elements cannot be along the line joining them, and there must in addition to this force be couples acting on the elements. For these reasons we see that Ampère's theory cannot give the complete action between two elements of current. What it does—and this for practical purposes is an advantage and not a disadvantage—is to give in most cases, instead of the complete action between two elements, that part of it which really affects the case under consideration.

Before discussing cases, however, in which the terms which Ampère neglects might be expected to produce measurable effects, we shall, in order to compare the various theories more easily, proceed to consider other theories of the same class.

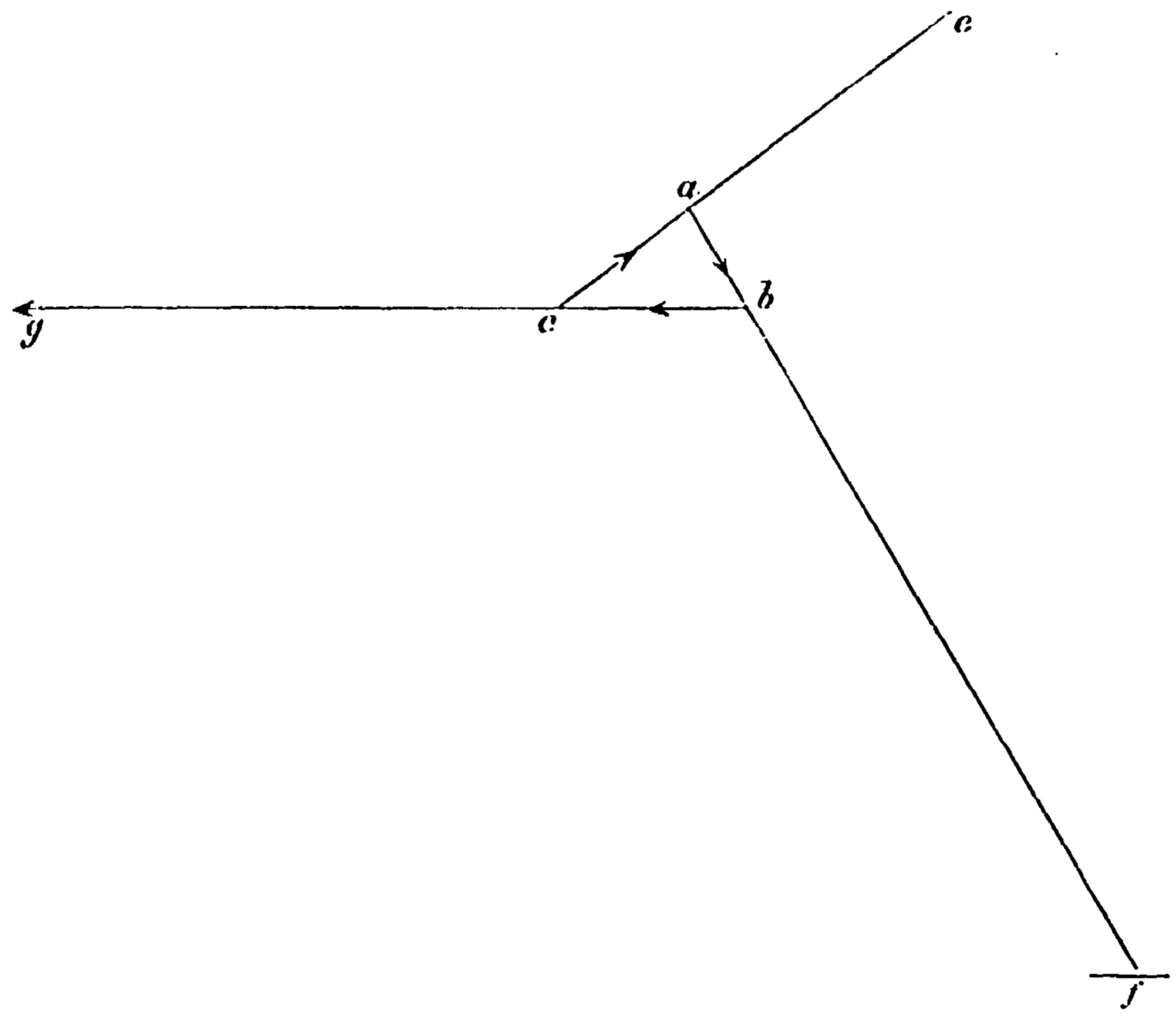
Grassmann's Theory.[1]

The method by which Grassmann obtains his theory is very remarkable. He objects to Ampère's formula for the force between two elements of current, because it makes the force between two parallel elements change from an attraction to a repulsion when the angle which the elements make with the line joining them passes through the value $\cos^{-1} 2/3$, and the object of his investigation is to get a law of force free from this peculiarity, and which, while giving the same result as Ampère's for closed circuits, shall yet be as simple as possible. He begins by regarding any circuit as built up of 'Winkelströme,' *i.e.*, currents flowing along the two infinite lines which form any angle. He points out that a circuit of any shape can be built up of such currents; the circuit *abc*, for example, may be regarded as built up of the 'Winkelströme' *eaf*, *fbg*, and *gce*.

Grassmann proceeds to calculate by Ampère's formula the action of a 'Winkelstrom' upon an element of current (a). Since the 'Winkelstrom' will have no action upon an element of current perpendicular to its plane, we see that it is only necessary to calculate its action upon the component (a') of a in its own plane. Grassmann does this by calculating the effect due to each arm of the 'Winkelstrom' separately. He

[1] Pogg. *Ann.* 64, p. 1, 1845; Crelle, 83, p. 57

finds expression for the forces along and perpendicular to a', due to an infinite rectilinear current starting from a definite point. The force of such a current along a' does not depend on the angle the current makes with the line from its end to a', so that the effects of two such currents starting from the same point and flowing in opposite directions, *i.e.* of a 'Winkelstrom,' will be to produce no force along a'; thus the effect of a 'Winkelstrom' on an element of current in its own plane will be a force at right angles to the element. The force at right angles to a' due to a rectilinear current will consist of two parts, one independent of the angle made by the current with the line joining its end to the element, the other depending upon this angle. The first part will vanish when we consider a 'Winkelstrom'; the second part only will produce any effect.



Now Grassmann says that it will much simplify the analysis, and obviously (since any closed circuit may be built up of 'Winkelströme') lead, for closed circuits, to the same result as Ampère's formula, if we suppose that the law of force between elements of currents is such that the only effects produced by a rectilinear current are those which do not vanish for a 'Winkelstrom,' and hence that a straight current exerts on an element of current a force at right angles to the projection of the element on the plane containing the centre of the element and the rectilinear current, and that the magnitude of this force is

$$\frac{ij\,ds'}{r}\cot\frac{a}{2},$$

where i is the strength of the rectilinear current, j the strength of the element of current, ds' its projection on the plane through its centre containing the straight current, r the distance of the element from the end of the straight current, and α the angle which the rectilinear current makes with the line joining its extremity to the elementary current. By taking the difference of two such rectilinear currents, Grassmann finds the action of an element (β) of current on another element (α) is a force at right angles to α', the component of α in the plane containing β and the middle point of α and equal to

$$ij \frac{d\sigma ds'}{r^2} \sin \theta,$$

where θ is the angle which β makes with r, $d\sigma$ the length of (β), and j the current flowing through it.

The direction of the force is along AB, where A is the centre of the element (α) and B the point where the normal to α' is cut by β produced *in the direction* of the current.

If we treat this theory in the same way as we did Ampère's on p. 99 by considering the action of the component α_1, β_1 of an element of current ds_1 on the components α_2, β_2, γ_2 of another element ds_2, we see that Grassmann's theory is equivalent to supposing that α_1 exerts no force on α_2, β_2, or γ_2; that β_1 exerts a force $A\beta_1\alpha_2$ on α_2 at right angles to α_2 in the plane of xy, and a force $A\beta_1\beta_2$ on β_2 at right angles to it, that is, along the line joining the element, and that it exerts no force on γ_2.

Thus we see that Grassmann's theory is equivalent to replacing Ampère's assumption, that the force between two elements of current acts along the line joining them, by the assumption that two elements of current in the same straight line exert no force on each other.

As a dynamical theory of electrodynamics, Grassmann's theory is open to the same objection as Ampère's, that it does not take into account the couples which may exist between the elements, and also to the additional objection that, according to it, the action of an element of current ds_1 on another element ds_2 is not equal and opposite to the action of ds_2 on ds_1, so that the momentum of the two elements ds_1 and ds_2 will not remain constant, and, as the theory does not take into account the surrounding ether, there is no way of explaining what has become of the momentum lost or gained by the elements. As a piece of geometrical analysis, however, the theory is very elegant and worthy of the author of the 'Ausdehnungslehre.'

From the way in which Grassmann's theory was developed we see that between closed circuits it must give the same forces as Ampère's; for unclosed circuits this is not the case, and Grassmann, at the end of the paper quoted above, mentions a case where the two theories would give opposite results, assuming that unclosed streams exist. Suppose we have a magnet ns and an unclosed current AB in the same plane as the magnet and passing through its middle point, then if Ampère's theory be true, the magnet will twist in one direction; if Grassmann's, it will twist in the opposite. This depends upon the change, according to Ampère's theory, of the force between two parallel elements from attraction to repulsion, when they make the angle with the line joining them at less than $\sin^{-1} 1/\sqrt{3}$, while according to Grassmann's theory, there is no such change.

Stefan's Theory.[1]

This resembles Ampère's theory very closely, except that Stefan does not make the assumption that the force between two elements of current is along the line joining them: this difference leads to the introduction of two forces which Ampère neglects.

We shall use the same notation as when we discussed Ampère's theory, and consider, as before, the action of an element of current ds_1 on another element ds_2. Stefan, like Ampère, assumes that we may replace an element of current by its component, so that we have to consider the action of the components (α_1, β_1) of ds_1 on the components $(\alpha_2, \beta_2, \gamma_2)$ of ds_1.

As in Ampère's theory, the component α_1 is supposed to exert a force

$$\frac{a\alpha_1\alpha_2}{r^2}$$

on α_2, this force by symmetry must be along the line joining the elements.

α_1 is supposed to exert a force on β_2 equal to

$$\frac{c\alpha_1\beta_2}{r^2}$$

along the axis of y. We can see that this force may exist, for it is conceivable that it should be in the same direction as β_2 when α_1 points from the middle of ds_1 to the middle of ds_2, and in the opposite direction to β_2 when α_1 points in the opposite direction. Stefan assumes that α_1 exerts no force on β_2 parallel to the axis of z, and no force at all on γ_2.

β_1 is supposed to exert a force on α_2 parallel to the axis of y and equal to

$$\frac{d}{r^2}\beta_1\alpha_2.$$

We may see, by the same reasoning as we used before for the force between β_1 and α_2, that it is conceivable that this force may exist. β_1 is supposed to exert no force on α_2 parallel to the axis of z.

As in Ampère's theory, β_1 is supposed to exert a force on β_2 equal to

$$\frac{b}{r^2}\beta_1\beta_2,$$

this force must by symmetry be along the line joining the elements; β_1 is supposed to exert no force on γ_2.

Thus the action of ds_1 on ds_2 consists of a force

$$\frac{1}{r^2}\left\{a\alpha_1\alpha_2 + b\beta_1\beta_2\right\}$$

along the line joining the elements, and a force

$$\frac{1}{r^2}\left\{c\alpha_1\beta_2 + d\beta_1\alpha_2\right\}$$

at right angles to this line in the plane containing ds_1 and r. If we take

[1] Stefan, *Wien. Sitzungsberichte*, 59, p. 693, 1869.

arbitrary coordinate axes and suppose that x, y, z are the coordinates of ds_1, x^1, y^1, z^1 those of ds_2, then the x component of the force on ds_2 due to ds_1 is shown by Stefan to be equal to

$$i\,j\,ds_1\,ds_2\left\{m\frac{d_2}{ds_1ds_2}\frac{(x^1-x)}{r}+n\frac{d}{ds_1}\frac{1}{r}\frac{dx^1}{ds_2}+p\frac{d}{ds_2}\frac{1}{r}\frac{dx}{ds_1}+q\frac{x^1-x}{r^3}\cos\epsilon\right\}$$

with similar expressions for the force parallel to the axes of y and z.

Here i, j are the currents through ds_1, ds_2 respectively, ϵ is the angle between the elements of current, and

$$m=-\tfrac{1}{3}\{a-b-c-d\}$$
$$n=\tfrac{1}{3}\{a-b-c+2d\}$$
$$p=-\tfrac{1}{3}\{a-b+2c-d\}$$
$$q=\tfrac{1}{3}\{a+2b-c-d\}.$$

We see from this expression for the force parallel to x that the last term is the only one which does not vanish when integrated round two closed circuits of which ds_1 and ds_2 are elements. So that the force will depend only upon q; the value of q will depend upon the units we adopt: in Stefan's work q is put equal to $-1/2$.

This is the only condition to be got by considering the translatory force between two circuits; we can get another by considering the couple acting on the closed circuit, supposed rigid, of which ds_2 forms a part. For the z component N of this couple Stefan finds the expression

$$N=ijq\iint\frac{y^1x-x^1y}{r^3}\cos\epsilon\,ds_1ds_2-ijp\iint\frac{1}{r}\left\{\frac{dx^1}{ds_2}\frac{dy}{ds_1}-\frac{dy^1}{ds_2}\frac{dx}{ds_1}\right\}ds_1ds_2.$$

But supposing the two circuits to have a potential

$$ij\iint\frac{\cos\epsilon}{r}\,ds_1\,ds_2,$$

we can easily see that the couple

$$=ij\iint\frac{y^1x-x^1y}{r^3}\cos\epsilon\,ds_1ds_2-ij\iint\frac{1}{r}\left\{\frac{dx^1}{ds_2}\frac{dy}{ds_1}-\frac{dy^1}{ds_2}\frac{dx}{ds_1}\right\}ds_1ds_2\,;$$

thus if two circuits have a potential

$$p=q,$$

or substituting for p and q their values,

$$2a+b+c-2d=0.$$

If $c=0$ and $d=0$, as in Ampère's theory, this relation becomes

$$2a+b=0,$$

which is the same relation as Ampère deduced by finding the condition that the force due to a closed circuit on an element of current should be at right angles to the element, and Stefan has proved that on his theory the same condition leads to the equation

$$p=q,$$

i.e., the same condition as the one which expresses that two closed circuits have a potential.

Stefan shows that, from the consideration of the action of closed circuits on elements of other circuits or of themselves, it is impossible to get any other relation between the quantities a, b, c, d, so that we have only two relations between the quantities a, b, c, d, and thus two of them must be indeterminate.

We may give any values we please to these quantities, provided they satisfy these two relations; if we put $c = 0$, $d = 0$ we get Ampère's theory; if $a = 0$, $c = 0$, Grassmann's; and we can get a number of other theories by giving different values to these quantities.

Stefan's theory is open to the same objection as Ampère's, since it does not take into account the couples which one element may produce on another. He also limits the generality of his theory by supposing that the force between two elements of currents in one plane is in that plane.

Korteweg's Theory.[1]

According to this theory, the forces between two elements of current are the same as in Stefan's theory; Korteweg, however, considers in addition the couples which one element may produce on another.

If we use the notation we adopted in discussing Stefan's theory, we have, considering the force on ds_2, a force

$$\frac{1}{r^2}\{a\alpha_1\alpha_2 + b\beta_1\beta_2\}$$

along the line joining the elements, and a force

$$\frac{1}{r^2}\{c\alpha_1\beta_2 + d\alpha_2\beta_1\}$$

parallel to the axis of y.

In addition to these forces, Korteweg supposes that from the action of α_1 on β_2 there is a couple whose axis is parallel to the axis of z equal to

$$f\alpha_1\beta_2,$$

and from the action of α_1 on γ_2 a couple on γ_2 whose axis is parallel to the axis of y and equal to

$$-f\alpha_1\gamma_2;$$

from the action of β_1 on α_2 there is a couple on α_2 whose axis is parallel to the axis of z and equal to

$$g\beta_1\alpha_2,$$

and from the action of β_1 on γ_2 there is a couple on γ_2 whose axis is parallel to the line joining the elements and equal to

$$h\beta_1\alpha_2.$$

If we now take arbitrary co-ordinate axes, the forces on the element ds_2 are the same as those given by Stefan's theory. The couples, however, are different. The component parallel to the axis of x of the couple on ds_2 is given by the equation

$$\mathrm{L} = \left[\frac{d}{r^2}\frac{dr}{ds_2}\left(y\frac{dz}{ds_1} - z\frac{dy}{ds_1}\right) - \frac{a-d-c}{r^3}\frac{dr}{ds_2}\frac{dr}{ds}(y'z - z'y)\right.$$

[1] Crelle, xc. p. 49, 1881.

$$- \frac{b}{r^2} \frac{d^2 r}{ds_1 ds_2} (y^1 z - z^1 y) - \frac{c}{r^2} \frac{dr}{ds_1} \left(z \frac{dy^1}{ds_2} - y \frac{dz_1}{ds_2} \right)$$

$$+ \frac{(h+g)}{r} \frac{dr}{ds_1} \left\{ (y^1 - y) \frac{dz_1}{ds_2} - (z^1 - z) \frac{dy^1}{ds_2} \right\}$$

$$+ \frac{h-f}{r} \frac{dr}{ds_2} \left\{ (y^1 - y) \frac{dz}{ds_1} - (z^1 - z) \frac{dy}{ds_1} \right\}$$

$$- h \left\{ \frac{dy^1}{ds_2} \frac{dz}{ds_1} - \frac{dy}{ds_1} \frac{dz}{ds_2} \right\} \Big] ii\, ds_1\, ds_2,$$

with similar expressions for the components of couple around the axes of y and z.

By making the force between two closed circuits have the same value as that given by Ampère's theory, Korteweg finds that

$$a + 2b - d - e = -3A^2,$$

where A is a constant quantity whose value depends upon the unit of current adopted.

By making the couples produced by one closed circuit on another have the same value as that given by Ampère and the potential theory, he finds that

$$\frac{d}{dr}(r^2 h) + (g - h) r - c + 2A^2 = 0.$$

Korteweg considers that the experiments of v. Ettingshausen, quoted above, prove (1) that the force on an element of circuit produced by a closed circuit is at right angles to the element, and (2) that the couple on an element due to a closed circuit has the value given by Ampère's theory. The first condition gives

$$c - b = 2A^2;$$

the second the two conditions

$$\frac{d}{dr}(rh) - f = 0$$

$$h + g = 0.$$

And he points out that we cannot get any more conditions by considering the action between two closed circuits, or the action of a closed circuit on an element of another.

It should be noticed that since, according to this theory, part of the action of one element of a circuit on another consists of a couple, the condition that the force due to a closed circuit on an element of another should be at right angles to the element is not, as in Stefan's theory, identical with the condition that the expression for the couple exerted by one closed circuit on another should be the same as that given by Ampère.

This theory is valuable because it is the most general one of the class we are considering which has been published. It is the only one which takes into account the couples, and by giving special values to the quantities a, b, c, d, f, g, h, we can get any of the other theories of this class.

On the theories which explain the action of currents by assuming that the forces between two electrified bodies depend upon the velocities and accelerations of the bodies.

According to these theories a body conveying an electric current contains equal quantities of positive and negative electricity, so that it will not exert any ordinary electrostatic effect: the positive electricity is supposed, however, to be moving differently from the negative. In some of the theories (Weber's, Gauss's, Riemann's) Fechner's hypothesis, that the electric current consists of positive electricity moving in one direction (the direction of the current), and an equal quantity of negative electricity moving at the same speed in the opposite direction, is assumed; in other theories (Clausius') only one of the electricities is supposed to move, the other remains at rest. We can see in a general way how the assumption that the forces between two electrified particles depend on the velocities and the accelerations of the particles can explain the effects produced by an electric current.

Let us take first the mechanical action between two circuits A and B, and let us consider the action of an element (a) of A on an element (b) of B. We shall consider first the action of the two electricities which are flowing through a on the positive electricity which is flowing through b. Since the motion of the positive electricity in a relative to that of the positive electricity in b is not the same as the motion of the negative electricity in a relative to that of the positive in b, the forces due to the positive and negative electricities in a will not counterbalance, so that there will be a resultant force on the positive electricity in b depending on the inequality between the motion of the positive and negative electricities in a relative to that of the positive in b. Similarly there will be a force on the negative electricity in b depending on the inequality between the velocities of the positive and negative electricities in a relative to that of the negative in b, and, except for special laws of force and special values of the velocities of the electricities in b, this force will not be equal and opposite to the force on the positive electricity in b, so that a mechanical force on b will be produced by the currents through a.

Let us now consider how inductive forces can be explained by this hypothesis: let us suppose that the element a is moving, and that the element b is at rest. The velocity of the electricity in a will be the resultant of the velocity with which the electricity flows through a and the velocity of translation of a itself, so that since the velocities of flow of the positive and negative electricities are different, the actual velocity of the positive electricity will differ in magnitude from the velocity of the negative (unless, assuming Fechner's hypothesis, the element a is moving at right angles to itself); thus the force due to the positive electricity in a on a unit of positive electricity at b will not be equal and opposite to that due to the negative electricity in a, and thus there will be an E.M.F. at b due to the motion of a. This explains induction due to the motion of the primary circuit.

Let us now consider induction due to the variation of the intensity of the current in the primary circuit. According to all the theories there is a force produced by a moving electrified body proportional to the first power of the acceleration of that body. Let us consider the elements a and b again, and suppose that a variable current is flowing through a and no current through b; then if we suppose that a variation in the intensity of a current is accompanied by an alteration in the velocity of flow, the acceleration of the positive electricity will, if we take Fechner's hypothesis, be equal and opposite to that of the negative; but since there is a part of the force due to the moving electrified body which changes sign both with the electrification and the acceleration, the force due to the acceleration of the positive electricity will be equal in all respects to that due to the acceleration of the negative, so that there will be a resultant force on a unit of positive electricity at b, and this force is the electromotive intensity at b due to the alteration of the intensity of the current in a. In this way we can explain the induction due to the variation of the current in the primary circuit.

Theories of this kind have been given by Gauss, Weber, Riemann, and Clausius, and these writers have given expressions for the force between two electrified particles moving in any way. We shall afterwards consider these expressions in detail, but we may remark in passing that the theories of Gauss, Weber, and Riemann have much in common; among other things they all lead to impossible results. In addition Clausius has shown that, unless we make Fechner's hypothesis about a current, viz. that it consists of equal quantities of positive and negative electricity moving with equal speeds in opposite directions, a current would on these theories exert a force on an electrified body at rest.

The question of the forces due to moving electrified bodies is interesting in connection with electrolysis. Taking the ordinary view that the current is carried by the ions, we know from Hittorf's researches that the anion and the cation move at different rates, so that the forces produced by these will be different; hence we should expect an electrolyte conveying a current to exert a force on a charged particle at rest.

We shall now go on to consider the various theories separately.

Gauss's Theory.[1]

Gauss assumes that the force between two particles separated by a distance r and charged with quantities of electricity e and e' is along the line joining the particles and equal to

$$\frac{ee'}{r^2}\left\{1+\frac{1}{c^2}\left\{u^2-\frac{3}{2}\left(\frac{dr}{dt}\right)^2\right\}\right\}$$

where u is the relative velocity of the two particles and c is a constant. This law will, if we make Fechner's hypothesis, explain the mechanical force between two circuits; but, since it contains no term depending on the acceleration, it cannot explain the E.M.F. produced by the variation of the strength of the current in the primary; it is also inconsistent with the principle of the Conservation of Energy, and so we need not consider it any further.

W. E. Weber's Theory.[2]

Weber assumes that the force between two charged particles, using the same notation as before, is

$$\frac{ee'}{r^2}\left\{1+\frac{1}{c^2}\left(r\frac{d^2r}{dt^2}-\frac{1}{2}\left(\frac{dr}{dt}\right)^2\right)\right\}$$

[1] Gauss's theory was published after his death in his collected works, Göttingen edition, vol. v. p. 616. See also Maxwell's *Electricity and Magnetism*, 2nd edit. vol. ii. p. 440.

[2] Weber's theory was published in 1846 in *Abhandlungen der Königlich-Säch-*

This formula is not inconsistent with the principle of the Conservation of Energy; making Fechner's hypothesis, it will explain the mechanical force between circuits conveying currents; it will also explain induction due both to the motion of the primary and the alteration in the strength of the current in the primary. We shall see, however, that it makes a body under certain circumstances behave as if its mass were negative; *i.e.* if it were acted on by a force in a direction opposite to that in which it is moving, its velocity would continually increase.

Riemann's Theory.

This is explained in his 'Schwere Electricität und Magnetismus,' edited by Hallendorff, p. 327. According to this theory the force between two electrified bodies is not altogether along the line joining them, but consists of the following parts:—

1. A force along the line joining the particles equal with the same notation as before to

$$\frac{ee'}{r^2}\left\{1+\frac{u^2}{c^2}\right\}$$

2. A force on the first particle parallel to its velocity relative to the second equal to

$$-\frac{2ee'}{c^2r^2}u\frac{dr}{dt}.$$

3. A force on the first particle parallel to its acceleration relative to the second equal to

$$\frac{2ee'}{c^2r}f,$$

where f is the relative acceleration of the particles.

There are of course similar forces acting on the second particles, and we see from the form of the expressions of the forces that the force on the first particle is equal and opposite to the force on the second. Riemann's law of force is not inconsistent with the principle of the conservation of energy, and it explains the mechanical force between two circuits; hence it must explain the induction of currents. We shall see, however, that it is open to the same objection as Weber's theory, viz. that it makes an electrified particle under certain circumstances behave as if its mass were negative.

Clausius' Theory.[1]

If x, y, z are the co-ordinates of the first electrified particle, x', y', z' those of the second, then according to this theory the x component of the force on the first particle is equal to

$$-ee'\left[\frac{d}{dx}\left\{(1-vv'\cos\epsilon/c^2)\frac{1}{r}\right\}-\frac{1}{c^2}\frac{d}{dt}\left(\frac{1}{r}\frac{dx'}{dt}\right)\right]$$

With similar expressions for the components parallel to y and z, here

sischen Gesellschaft der Wissenschaften, 1846, p. 211; it is reprinted in *Electrodynamische Maassbestimmungen*, 1871. A good account of the theory is given in Maxwell's *Electricity and Magnetism*, 2nd edit. vol. ii. chap. xxiii.

[1] This theory is given in Crelle, vol. 82, p. 85. There is also a full abstract in Wiedemann's *Beiblätter*, vol. i. p. 143.

v and v' are the velocities of the first and second particles respectively, and ϵ is the angle between their directions of motion. We may analyse these forces a little differently, and say that the force on the first particle consists of—

1. A force along the line joining the particles equal to

$$\frac{ee'}{r^2}\left\{1-vv'\cos\epsilon/c^2\right\}$$

2. A force parallel to the velocity of the *second* particle and equal to

$$\frac{ee'}{c^2r^2}\frac{dr}{dt}v'.$$

3. A force parallel to the acceleration of the *second* particle equal to

$$-\frac{ee'}{c^2r}\frac{dv'}{dt}.$$

We have, of course, corresponding expressions for the force on the second particle.

Clausius' formulæ differ from those of Gauss, Weber, and Riemann in two very important respects.

1. They make the forces between two electrified bodies depend on the absolute velocities and accelerations of the bodies, while the others make them depend only on the relative velocities and accelerations.

2. They do not make the forces between the bodies equal and opposite, so that the momentum of the system does not remain constant.

These results show that if this theory is true, we must take the ether surrounding the bodies into account. The first result can then be explained by supposing that the velocities which enter into the formulæ are the velocities of the bodies relatively to the ether at a considerable distance from the bodies, and the second result by supposing that the ether possesses a finite density, and that the momentum lost or gained by the bodies is added to or taken from the surrounding ether.

The case is analogous to the case of two spheres A and B moving in an incompressible fluid; in this case the forces on the sphere A depend on the velocities and accelerations of B relatively to the fluid at a great distance from the sphere, and are independent of the velocity and acceleration of A; the forces are not equal and opposite, and the momentum lost or gained by the system is added to or taken from the momentum of the fluid. At the end of this section we shall see that, if we assume that variations in what Maxwell calls the electric displacement produce effects analogous to those produced by ordinary conduction currents, we get the same forces between moving electrified bodies as are given by Clausius' theory.

Clausius' theory is not inconsistent with the principle of the conservation of energy, and we shall see that it does not lead to the same difficulty as the theories of Weber and Riemann, viz., that under special circumstances a body would behave as if its mass were negative.

Assuming that in an electric current we have equal quantities of positive and negative electricity moving with different velocities, Clausius has shown in the paper already cited that his theory gives Ampère's results for the mechanical force between two circuits, and the usual expression for the induction due to the motion of the primary circuit, or variation in the strength of the current passing through it.

Fröhlich[1] urges against Clausius' law that since, according to it, an electric current in motion exerts an electromotive force on a moving electrified particle, even though the particle is moving at the same rate as the circuit, every current on the earth's surface ought to exert an electromotive force on an electrified particle relatively at rest, since each is moving with the velocity of the earth. This force is one that can be derived from a potential, so that the integral of the force taken round a closed curve would vanish, and thus, even if this result were true, two circuits would not induce currents in each other if they were relatively at rest. Budde[2] points out, however, that the moving circuit would exert an electromotive force at each point of itself, and thus cause a separation of the electricity in the circuit, so that it would get coated with a distribution of electricity, the electrostatic action of which would balance that due to the action due to its motion on a point relatively at rest. The velocities which enter into Clausius' formulæ are velocities relative to the ether, so that if the ether moves with the earth, an electric current will, according even to this theory, exert no electromotive force on a point relatively at rest, and there will be no electrification on the surface of the circuit. The velocity c which occurs in all these theories is a velocity comparable with the velocity of light.

General Considerations on these Theories.[3]

We shall now go on to discuss a general way of treating theories of the kind we have been considering. Perhaps the best way of doing this is to consider not the forces between the electrified bodies, but the energy possessed by them. If the energy depends on the electrification there will be forces between two electrified bodies. Now the potential energy depends on the electrification, and this dependence produces the ordinary electrostatic forces between two electrified bodies at rest. If, however, the kinetic energy as well as the potential depends on the electrification, then the forces between two electrified bodies in motion will be different from the forces between the same bodies at rest. An easy way of seeing this is by means of Lagrange's equations.

If T be the kinetic energy, and x a co-ordinate of any kind, then we have, by Lagrange's equations,

$$\frac{d}{dt}\frac{dT}{d\dot{x}} - \frac{dT}{dx} = \text{external force of type } x.$$

Hence if we have any term T′ in the expression for the kinetic energy, we may, if we like, regard it as producing a force equal to

$$-\frac{d}{dt}\frac{dT'}{d\dot{x}} + \frac{dT'}{dx}.$$

A simple illustration of this is afforded by the centrifugal force. In

[1] Fröhlich, *Wied. Ann.*, ix. p. 277, 1880.
[2] *Wied. Ann.*, x. p. 553, 1880.
[3] See Clausius 'On the Employment of the Electrodynamic Potential for the Determination of the Ponderomotive and Electromotive Forces,' *Phil. Mag.*, 1880, v. 10, p. 255.

the expression for the kinetic energy of a moving particle there is the term

$$\tfrac{1}{2}mr^2\,\dot{\theta}^2,$$

where r is the distance of the particle from some fixed point, and θ the angle which the radius from this point to the particle makes with some fixed line; m is the mass of the particle. This term, by the above rule, will give rise to a force of type r, *i.e.*, along the radius vector equal to

$$mr\dot{\theta}^2,$$

and this is the ordinary centrifugal force.

Now let us consider a moving electrified body. If it is symmetrical, and moves in an isotropic dielectric, it is evident that the electrification, if it enters at all, can only enter as a factor of the total velocity q, and cannot affect the separate components of the velocity differently.

Let us suppose that the body is charged with a quantity of electricity denoted by e, then the kinetic energy, if it depends on the electrification, must be of the form

$$\tfrac{1}{2}mq^2 + f(e)q^2,$$

where $f(e)$ denotes some function of e. Now $f(e)$ must be always positive, for if it were negative we could make

$$\tfrac{1}{2}m + f(e)$$

negative, and then the electrified body would behave like one of negative mass. The simplest form satisfying this condition which we can take for $f(e)$ is ae^2, where a is some positive constant; so that the form of expression for the kinetic energy may be taken as

$$(\tfrac{1}{2}m + ae^2)q^2.$$

Now let us go on to the case where we have two electrified bodies present, with charges e and e' of electricity; let m and m' be their masses, q, q' their velocities, of which the components parallel to the axes of x, y, z are (u, v, w), (u', v', w') respectively, the co-ordinates of the particles being (x, y, z), (x', y', z').

If everything is symmetrical, the expression for the kinetic energy, if it only involves second powers of the charges of electricity, will be of the form

$$\tfrac{1}{2}mq^2 + \tfrac{1}{2}mq'^2 + ae^2q^2 + \beta e'^2\,q'^2 + ee'\,\kappa\,.\,f\,\{u, v, w, u', v', w'\}$$

where $f\,(u, v, w, u', v', w')$ is a quadratic function of u, v, w, u', v', w'.

By Lagrange's equations we see that the last term will give rise to a force parallel to the axis of x on the particle whose charge is e equal to

$$\kappa ee' \left\{ \frac{df}{dx} - \frac{d}{dt}\frac{df}{du} \right\},$$

with similar expressions for the forces parallel to y and z. We can see, by substituting in this expression, that we get Weber's law if we make

$$f = \frac{1}{r}\left\{ \frac{x-x'}{r}(u-u') + \frac{y-y'}{r}(v-v') + \frac{z-z'}{r}(w-w') \right\}^2;$$

Riemann's law, if we make

$$f=\frac{1}{r}\{(u-u')^2+(v-v')^2+(w-w')^2\};$$

Clausius' law, if we make

$$f=\frac{1}{r}\{uu'+vv'+ww'\};$$

and that we cannot get Gauss's law in this way; this is in accordance with the fact that Gauss's law does not satisfy the principle of the conservation of energy. This way of considering the theories enables us to see that neither Weber's nor Riemann's formulæ can be right, for if they were, an electrified body, when in presence of another, would, under certain circumstances, behave as if its mass were negative. Thus take Weber's law as an example: let us suppose that two electrified bodies are moving along the line joining them, which we may take as the axis of x; then the expression for the kinetic energy, putting in the value of f which corresponds to Weber's law, is

$$\tfrac{1}{2}mq^2+\tfrac{1}{2}mq'^2+\alpha e^2q^2+\beta e'^2q'^2+\frac{\kappa ee'}{r}\{q-q'\}^2,$$

so that if

$$\tfrac{1}{2}m+\alpha e^2+\frac{\kappa ee'}{r}$$

be negative, then the coefficient of q^2 in the kinetic energy will be negative, and the body will behave as if its mass were negative; and, by sufficiently increasing e' or diminishing r, we can make this expression negative, so that Weber's law leads to results which are inconsistent with experience. This result of Weber's law was first pointed out by Helmholtz.[1]

Exactly the same objection applies to Riemann's theory, and indeed we see that it will apply to any theory which makes the force between two electrified bodies depend on *relative* velocities and accelerations.

The same objection need not apply to Clausius' theory, for substituting the value of f belonging to his theory, the kinetic energy equals

$$(\tfrac{1}{2}m+\alpha e^2)q^2+(\tfrac{1}{2}m'+\beta e'^2)q'^2+\kappa ee'\,\frac{qq'\cos\epsilon}{r},$$

so that the kinetic energy will be always positive if

$$(\tfrac{1}{2}m+\alpha e')\ \ (\tfrac{1}{2}m'+\beta e'^2)>\frac{\kappa^2e^2e'^2\cos^2\epsilon}{4r^2}.$$

This condition will evidently be satisfied if

$$\alpha\beta>\frac{\kappa^2}{4r^2},$$

and this relation does not involve the electrification. We cannot assume that we can make r so small that this condition is not satisfied, for r has a minimum value depending upon the shape and size of the electrified bodies. For example, if these are spheres, r cannot be less than the sum of their radii. On the other hand, α and β may be functions of the

[1] *Ueber die Theorie der Elektrodynamik.* Crelle, vol. lxxv. p. 535; Collected Works, Bd. 1, S. 647.

sizes of the electrified bodies, and the geometrical relations may be such that the condition written above must be always satisfied.

Physical reasons why the force between two electrified bodies should depend on their velocities and accelerations.

If we assume Maxwell's hypothesis that a change in the electric polarisation produces the same effect as an electric current, then we see that the kinetic energy of an electrified body must be different from the kinetic energy of the same body moving at the same rate but not electrified. For let us suppose that we have an electrified body at rest, and consider the amount of work necessary to start it with a velocity q. It is evident that it will be greater than when it is not electrified, for when it is electrified and in motion the electric polarisation in the surrounding dielectric will be in changing, and so in addition to starting the body with a velocity q we have, if Maxwell's hypothesis be true, to establish what is equivalent to a field full of electric currents. The production of these currents of course requires work, so that more work is required to start the body with a velocity q when it is electrified than when it is not; in other words, the kinetic energy of a moving electrified body is greater than that of one not electrified, but under similar conditions as to mass and velocity. In fact in this case electricity behaves as if it possessed inertia.

In a paper published in the 'Philosophical Magazine,' April 1881, I have shown that the kinetic energy of a charged sphere of radius a and mass m moving at a velocity q

$$= \tfrac{1}{2}mq^2 + \tfrac{2}{15}\frac{e^2\mu}{a}q^2,$$

where μ is the magnetic permeability of the surrounding dielectric and e the charge on the sphere. If there are two spheres in the field, then I have shown in the same paper that the kinetic energy

$$= \tfrac{1}{2}mq^2 + \tfrac{2}{15}\frac{\mu e^2}{a}q'^2 + \tfrac{1}{2}m'q'^2 + \tfrac{2}{15}\frac{\mu e'^2}{a'}q'^2 + \tfrac{1}{3}\frac{\mu ee'}{R}qq'\cos\epsilon,$$

where corresponding quantities for the two spheres are denoted by plain and accented letters. We see from this expression that the forces between the spheres are exactly the same as those given by Clausius' formulæ. It would not, however, be legitimate to go and develope the laws of electrodynamics from this result in the way that Clausius does, as Clausius' conception of an electric current does not accord with that of the displacement theory. We may remark that in this case the part of the kinetic energy due to the electrification is always positive.

On theories which are based on dynamical considerations, but which neglect the action of the dielectric.

F. E. Neumann[1] was the first to develope a theory founded on the principles of the Conservation of Energy. His theory was based upon the assumption that two elements of circuit ds, ds', traversed by currents ι, ι' possess an amount of energy equal to

$$A^2\frac{\iota\,\iota'\cos\epsilon}{r}\,ds\,ds',$$

[1] 'Die mathematischen Gesetze der inducirten electrischen Ströme,' *Schriften der Berliner Academie der Wissensch.*, 1845.

where A is a constant which depends upon the unit of current, r is the distance between the elements, and ϵ the angle between their directions. F. E. Neumann showed that this assumption leads to the same law of force between two closed circuits as that given by Ampère, and also explained by means of it the induction of electric currents. v. Helmholtz[1] has investigated the most general expression for the energy possessed by two elements of current which is consistent with the condition that the force between two closed circuits should be the same as that given by Ampère's theory. We shall consider this theory in detail, as it includes all theories of this class, and we shall wish to refer to it when we come to discuss the relative merits of the various theories. v. Helmholtz begins by showing that the most general expression for the energy of two elements of circuit consistent with Ampère's laws for closed circuits is

$$\frac{1}{2}\,\frac{A^2\,\iota\,\iota'}{r}\,\{(1+k)\cos\epsilon+(1-k)\cos\theta\cos\theta'\}\,ds\,ds',$$

where θ and θ' are respectively the angles ds and ds' make with the line joining the elements, k is a constant, and the other symbols have the same meaning as before.

Let us call this quantity T; then we know that T denotes the existence of a force $d\mathrm{T}/dr$ or

$$-\frac{1}{2}\,\frac{A^2\,\iota\,\iota'}{r^2}\,\{(1+k)\cos\epsilon+(1-k)\cos\theta\cos\theta'\}\,ds\,ds'$$

along r, and a force $-d\mathrm{T}/rd\theta$ at right angles to r in the plane of ds and r, and in such a direction that it tends to diminish θ; this force equals

$$\frac{1}{2}\,\frac{A^2\,\iota\,\iota'}{r^2}(1-k)\left(\sin\theta\cos\theta'+\cos\theta\sin\theta'\,\frac{d\theta'}{d\theta}\right);$$

and since

$$\frac{d\theta'}{d\theta}=\cos\eta,$$

where η is the angle between the plane containing r and ds and that containing r and ds', the transverse force

$$=\frac{1}{2}\,\frac{A^2\,\iota\,\iota'}{r^2}(1-k)\,\{\sin\theta\cos\theta'+\cos\theta\sin\theta'\cos\eta\}.$$

We see that these forces will coincide with those assumed in Korteweg's theory if the quantities a, b, c, d, which occur in that theory, have the following values:

$$\begin{aligned}a&=-A^2\\ b&=-\tfrac{1}{2}(1+k)A^2\\ c&=\tfrac{1}{2}(1-k)A^2\\ d&=\tfrac{1}{2}(1-k)A^2.\end{aligned}$$

So that whatever be the value of k, these quantities satisfy the condition

$$2a+b+c-2d=-3A^2.$$

[1] Crelle, lxxii, p. 57; *Gesammelte Werke*, vol. i, p. 545.

According to Stefan, it is necessary if two circuits have a potential that

$$2a + b + c - 2d = 0.$$

But Stefan did not consider the couple exerted by one element of circuit on another. The couples acting on the element ds' will be as follows. There will be a couple tending to increase θ', *i.e.* a couple whose axis is at right angles to both ds' and r, equal to $dT/d\theta'$, *i.e.* to

$$\frac{1}{2}\ \frac{A^2 \iota \iota'}{r} \{(1+k) \sin \theta \cos \theta' \cos \eta - 2 \cos \theta \sin \theta'\},$$

and another couple tending to increase η, *i.e.* a couple whose axis is along the line joining the elements equal to $dT/d\eta$, *i.e.* to

$$-\frac{1}{2}\ \frac{A^2 \iota \iota'}{r}(1+k) \sin \theta \sin \theta' \sin \eta.$$

We see that these will agree with the couples in Korteweg's theory of

$$f = -\frac{A^2}{r}; \qquad g = -\frac{1}{2}A^2\frac{(1+k)}{r}; \qquad h = \frac{1}{2}A^2\frac{(1+k)}{r}.$$

Let us return to the consideration of the energy of the circuits, and suppose that, instead of currents flowing along linear circuits, we have a distribution of them throughout space. If u, v, w be the currents in the element dx, dy, dz, then the part of the energy contributed by this element will be

$$-A^2\{Uu + Vv + Ww\}\,dx\,dy\,dz,$$

where

$$U = \frac{1}{2}\iiint\left\{(1+k)\frac{u}{r} + (1-k)\frac{x-\xi}{r^2}\{u(x-\xi) + v(y-\eta) + w(z-\zeta)\}\right\} d\xi\,d\eta\,d\zeta,$$

with symmetrical expressions for V and W, where

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2.$$

We may write the expressions for U, V, W in the form

$$U = \frac{1}{2}(1-k)\frac{d\psi}{dx} + \iiint\frac{u}{r}\,d\xi\,d\eta\,d\zeta$$

$$V = \frac{1}{2}(1-k)\frac{d\psi}{dy} + \iiint\frac{v}{r}\,d\xi\,d\eta\,d\zeta$$

$$W = \frac{1}{2}(1-k)\frac{d\psi}{dz} + \iiint\frac{w}{r}\,d\xi\,d\eta\,d\zeta,$$

where $\psi = \iiint\left(u\frac{dr}{d\xi} + v\frac{dr}{d\eta} + w\frac{dr}{d\zeta}\right) d\xi\,d\eta\,d\zeta$

If u, v, w are the components of the ordinary conduction current, e the volume density of the free electricity, then

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = -\frac{de}{dt};$$

and if l, m, n be the direction cosines of the normal to a surface at which the currents become discontinuous, σ the surface density of the electricity on this surface, then

$$l\,(u-u') + m\,(v-v') + n\,(w-w') + \frac{d\sigma}{dt} = 0.$$

Remembering these equations, ψ may be transformed into

$$\iiint r\,\frac{d\rho}{dt}\,dx\,dy\,dz + \iint r\,\frac{d\sigma}{dt}\,ds\,;$$

or if ϕ denote the electrostatic potential of the free electricity, we see

$$\psi = -\frac{1}{2\pi}\iiint \frac{1}{r}\,\frac{d\phi}{dt}\,dx\,dy\,dz.$$

Substituting this value of ψ we find

$$\nabla^2 U = (1-k)\frac{d^2\phi}{dx\,dt} - 4\pi u,$$

$$\nabla^2 V = (1-k)\frac{d^2\phi}{dy\,dt} - 4\pi v,$$

$$\nabla^2 W = (1-k)\frac{d^2\phi}{dz\,dt} - 4\pi w.$$

We also see that

$$\frac{dU}{dx} + \frac{dV}{dy} + \frac{dW}{dz} = -k\frac{d\phi}{dt}.$$

In order to get the equations connecting the electromotive force with the variation of the electrodynamic potential, Neumann made use of Lenz's law, and assumed that, since by that law the electromotive force tending to increase the current in an element of circuit moving with a velocity w in the direction s would be of the same sign as

$$-Xw,$$

where X is the force along s on the element per unit of length per unit of current flowing through it, it was actually equal to this quantity multiplied by a constant c, *i.e.* to

$$-cXw;$$

but if $Ti\,ds$ be the energy of the element of current whose length is ds, and current strength i,

$$X = \frac{dT}{ds},$$

and

$$w = \frac{ds}{dt};$$

so that the electromotive force per unit of length of the element

$$= -c\,\frac{dT}{ds}\,\frac{ds}{dt}$$

$$= -c\,\frac{dT}{dt}.$$

v. Helmholtz has shown that it follows from the principle of the Conservation of Energy that if the energy in the elements dx, dy, dz, traversed by currents u, v, w, be

$$A^2 (Uu+Vv+Ww)\, dx\,dy\,dz,$$

then the components of the electromotive force parallel to the axes x, y, z respectively, due to the variation in the electrodynamic potential, will be

$$-A^2\frac{dU}{dt},\ -A^2\frac{dV}{dt},\ -A^2\frac{dW}{dt};$$

the free electricity produces an electromotive force whose components are

$$-\frac{d\phi}{dx},\ -\frac{d\phi}{dy},\ -\frac{d\phi}{dz},$$

so that the total electromotive force parallel to x, y, z

$$=-\frac{d\phi}{dx}-A^2\frac{dU}{dt}.$$

Now if σ be the specific resistance of the conductor, σu equals the electromotive force parallel to the axis of x, so that

$$\sigma u=-\frac{d\phi}{dx}-A^2\frac{dU}{dt};$$

so that by the preceding equations

$$\frac{\sigma}{4\pi}\left\{\nabla^2 U-(1-k)\frac{d^2\phi}{dx\,dt}\right\}=-\frac{d\phi}{dx}-A^2\frac{dU}{dt},$$

with similar equations for V, W. The quantities U, V, W and their first differential coefficients with respect to x, y, z are continuous, and these equations enable us to find them if we know the value of ϕ, the potential of the free electricity. Helmholtz shows that the whole energy in the field due to the currents may be written

$$\frac{A^2}{8\pi}\iiint\left\{\left(\frac{dU}{dy}-\frac{dV}{dz}\right)^2+\left(\frac{dV}{dz}-\frac{dW}{dy}\right)^2+\left(\frac{dW}{dy}-\frac{dU}{dz}\right)^2+k\left(\frac{d\phi}{dt}\right)^2\right\}dx\,dy\,dz,$$

so that if k be negative, this expression may become negative, and in that case the equilibrium would be unstable; hence we conclude that only those theories are tenable for which k is positive.

The equations written above are those which hold in a conductor, in an insulator the equations are

$$\nabla^2 U=(1-k)\frac{d^2\phi}{dx\,dt}$$

$$\nabla^2 V=(1-k)\frac{d^2\phi}{dy\,dt}$$

$$\nabla^2 W=(1-k)\frac{d^2\phi}{dz\,dt}$$

$$\nabla^2\phi=0.$$

v. Helmholtz shows that in the conductor the electrostatic potential ϕ satisfies the equation

$$\nabla^2\left\{\phi+\frac{\sigma}{4\pi}\frac{d\phi}{dt}\right\}=A^2 k\frac{d^2\phi}{dt^2},$$

so that if the conductor has an infinitely small resistance, the equation becomes

$$\nabla^2\phi = A^2 k\frac{d^2\phi}{dt^2}.$$

This represents a wave motion, the velocity of propagation of which is $1/A\sqrt{k}$. If k, as in Neumann's theory, be equal to unity, then the velocity of propagation is $1/A$, and from the value of A, found from experiments on the force between circuits conveying currents, this is nearly equal to the velocity of propagation of light. Thus, according to Neumann's theory, in a perfect conductor an electrostatic disturbance is propagated with the velocity of light. In an insulator ϕ satisfies the equation

$$\nabla^2\phi = 0\,;$$

and this represents a motion propagated with an infinite velocity, and thus, according to this theory, an electrostatic disturbance is propagated with an infinite velocity in a perfect non-conductor. In an imperfectly conducting substance the velocity of propagation of a wave motion would depend upon the length of the wave.

Let us now go on to consider, what, according to this theory, are the forces acting on an element of circuit conveying a current. Let us suppose that the element ds forms an element of a circuit through which a current ι is flowing; then the energy of the circuit will be

$$A^2\int\iota\left\{U\frac{dx}{ds}+V\frac{dy}{ds}+W\frac{dz}{ds}\right\}ds.$$

In order to find the force parallel to x, let us suppose that each element of the circuit receives an arbitrary displacement x, parallel to the axis of x; then the alteration in the energy will be

$$A^2\int\iota\left\{\frac{dU}{dx}\frac{dx}{ds}+\frac{dV}{dx}\frac{dy}{ds}+\frac{dW}{x}\frac{dz}{ds}\right\}\delta x\,ds + A^2\int\iota\,U\,\frac{d.\delta x}{ds}\,ds.$$

Integrating the second term by parts, we see that it may be written

$$[A^2\iota U\delta x] - A^2\int\iota\left\{\frac{dU}{dx}\frac{dx}{ds}+\frac{dU}{dy}\frac{dy}{ds}+\frac{dW}{dz}\frac{dz}{ds}\right\}\delta x\,ds.$$

Substituting this value for the second term, we see that the alteration in the energy,

$$= [A^2\iota U\delta x] + A^2\int\iota\left\{\frac{dy}{ds}\left(\frac{dV}{dx}-\frac{dU}{dy}\right)-\frac{dz}{ds}\left(\frac{dU}{dz}-\frac{dW}{dx}\right)\right\}\delta x\,ds\,;$$

hence we see by the Conservation of Energy that there is a force on each element of current parallel to the axis of x, equal to

$$\iota\left\{\frac{dy}{ds}\left(\frac{dV}{dx}-\frac{dU}{dy}\right)-\frac{dz}{ds}\left(\frac{dU}{dz}-\frac{dW}{dx}\right)\right\}A^2,$$

and by symmetry forces parallel to y and z equal respectively to

$$\iota\left\{\frac{dz}{ds}\left(\frac{dW}{dy}-\frac{dV}{dz}\right)-\frac{dx}{ds}\left(\frac{dV}{dx}-\frac{dU}{dy}\right)\right\}A^2\,;$$

$$\iota\left\{\frac{dx}{ds}\left(\frac{dU}{dz}-\frac{dW}{dx}\right)-\frac{dy}{ds}\left(\frac{dW}{dy}-\frac{dV}{dz}\right)\right\}A^2$$

so that the resultant of these forces is at right angles to the element. In addition to these forces there are other forces at places where the quantity ιU is discontinuous, or, since U is continuous, at places where ι is discontinuous, whose components parallel to the axes of x, y, z, are respectively

$$A^2 U\delta\iota,\ A^2 V\delta\iota,\ A^2 W\delta\iota;$$

but $\delta\iota$ equals de/dt, the rate at which the free electricity is increasing at the place, so that we have at any place where the free electricity is changing a force whose components are

$$A^2 U\frac{de}{dt},$$

$$A^2 V\frac{de}{dt},$$

$$A^2 W\frac{de}{dt}.$$

We saw before that the force acting on the circuit per unit length is at right angles at each point to the element of circuit at that point, so that, unless a circuit includes places at which the quantity of free electricity is changing, the circuit will behave as if it were acted on by forces which were everywhere normal to the elements on which they act. In the experiments which have been made to test whether the force on the element is at right angles to it, there have been no points where the free electricity is changing, so that these experiments do not contradict Neumann's theory, although, according to it, the force on an isolated element is not necessarily at right angles to that element, for in addition to the forces normal to the element we have forces equal to $A^2 U de/dt$, $A^2 V de/dt$, $A^2 W de/dt$ parallel to x, y, z respectively, acting at the ends, the resultant of these two forces is a force whose components parallel to the axes of x, y, z are respectively

$$\frac{A^2\, de\, dU\, ds}{dt\ ds},$$

$$\frac{A^2\, de\, dV\, ds}{dt\ ds},$$

$$\frac{A^2\, de\, dW\, ds}{dt\ ds},$$

and as these forces are not necessarily at right angles to the element, the resultant force is not necessarily so; the effect of these forces could not, however, be detected unless there was a discontinuity in the current.

v. Helmholtz in the memoir[1] which we have already quoted shows that, according to his extension of F. E. Neumann's theory, the forces between two elements of circuit ds and ds' may be looked upon as made up of—

(1) A repulsive force on ds due to an end of ds', equal (per unit length) to

$$-A^2\iota\frac{de'}{dt}\frac{1}{r}\frac{dr}{ds};$$

[1] *Ueber die Theorie der Elektrodynamik*, dritte Abhandlung, Crelle, lxxviii. pp. 273, 324, 1874; *Gesammelte Werke*, p. 723.

(2) A repulsive force on ds due to ds', equal per unit length to

$$-\frac{A^2\iota'}{r^2}\left\{2\cos(ds\,ds') - 3\cos(r\,ds)\cos(r\,ds')\right\};$$

(3) A repulsion between the ends of ds and ds', equal to

$$-\frac{1}{2}(1+k)A^2\frac{de}{dt}\frac{de'}{dt};$$

(4) A repulsion on ds', due to an end of ds equal per unit length to

$$-A^2\iota'\frac{de}{dt}\frac{1}{r}\frac{dr}{ds}.$$

The second of these is the only one considered in Ampère's theory. We must remember in calculating these forces that each element has two ends.

Let us now go on to find the couples acting at each point of the circuit. If the tangent to the circuit makes an angle θ with the axis of z, and the plane containing the tangent and the axis of z an angle ϕ with the plane of xz, then we may write

$$\frac{dx}{ds} = \sin\theta\cos\phi,$$

$$\frac{dy}{ds} = \sin\theta\sin\phi,$$

$$\frac{dz}{ds} = \cos\theta,$$

so that with the same notation as before the energy equals

$$A^2\int\iota\left(U\frac{dx}{ds} + V\frac{dy}{ds} + W\frac{dz}{ds}\right)ds$$

$$= A^2\int\iota\,(U\sin\theta\cos\phi + V\sin\theta\sin\phi + W\cos\theta)\,ds,$$

so that if ϕ increase by $\delta\phi$, the alteration in the energy equals

$$A^2\int\iota\,(-U\sin\theta\sin\phi + V\sin\theta\cos\phi)\,\delta\phi\,ds,$$

so that the couple tending to increase ϕ, *i.e.* the couple whose axis is parallel to the axis of z, equals

$$A^2\iota\,(V\sin\theta\cos\phi - U\sin\theta\sin\phi)$$

per unit length of current; this may be written

$$A^2\iota\left(V\frac{dx}{ds} - U\frac{dy}{ds}\right),$$

hence the couples parallel to the axes of y and x are by symmetry respectively

$$A^2\iota\left\{U\frac{dz}{ds} - W\frac{dx}{ds}\right\},$$

$$A^2\iota\left\{W\frac{dy}{ds} - V\frac{dz}{ds}\right\}.$$

The axis of the resultant couple is perpendicular to the element and to the vector whose components are U, V, W.

In another paper[1] v. Helmholtz discusses the force acting per unit of volume on a conductor traversed by electric currents; he shows that, according to the potential theory, if u, v, w are the components of current through an element $dx\,dy\,dz$, and X, Y, Z the components of the force acting on this element of volume per unit of volume, then

$$X=A^2\left[v\left(\frac{dV}{dx}-\frac{dU}{dy}\right)+w\left(\frac{dW}{dx}-\frac{dU}{dz}\right)+U\frac{de}{dt}\right]$$

$$Y=A^2\left[u\left(\frac{dU}{dy}-\frac{dV}{dx}\right)+w\left(\frac{dW}{dy}-\frac{dV}{dz}\right)+V\frac{de}{dt}\right]$$

$$Z=A^2\left[u\left(\frac{dU}{dz}-\frac{dW}{dx}\right)+v\left(\frac{dV}{dz}-\frac{dW}{dy}\right)+W\frac{de}{dt}\right]$$

He then discusses the application of the potential law to sliding contacts, that is, contacts such as those made by a wire dipping into mercury; in the derivation of the forces from the potential law it is assumed that the displacements are continuous, and it might be objected that we have no right to apply the law in this case as the motion of the wire and the mercury seems at first sight discontinuous. v. Helmholtz, however, points out that, as the wire carries the mercury with it as it moves, the motion is not really discontinuous and that Neumann's law is applicable. The question of sliding contacts comes prominently forward when we compare the various theories; we shall return to it again in this connection.

v. Helmholtz also in this paper investigates the electromotive forces acting on a conductor in motion; he shows that if the components of the velocity of the conductor at any point are α, β, γ, then P, Q, R, the components of the electromotive force, are given by the equation

$$P=\beta\left(\frac{dU}{dy}-\frac{dV}{dx}\right)+\gamma\left(\frac{dU}{dz}-\frac{dW}{dx}\right)+\frac{d}{dx}(U\alpha+V\beta+W\gamma),$$

with similar equations for Q and R.

He also investigates the difference between the results of Ampère's and Neumann's theory for the E.M.F. due to induction. The results are complicated; for practical purposes it is sufficient to notice that when there is a mechanical force tending to make the body move in a certain direction, there must be an E.M.F. when the body moves in that direction.

C. Neumann's Theory.

C. Neumann assumes that the electric potential energy is propagated with a finite velocity, and that if two electrified bodies are in motion, the mutual potential energy is not ee'/r, where r is the distance between them, but ee'/r', where r' is the distance between them at a time t before, where t is the time taken by the potential to travel from the one body to the other.

The energy considered in C. Neumann's theory is a kind of energy quite different from any that we have experience of; it is not poten-

[1] *Ueber die Theorie der Elektrodynamik*, Crelle, lxxviii. pp. 273–324, 1874; *Gesammelte Werke*, vol. ii. p. 703.

tial energy, because that at any time depends only on the position of the system at that time; it is not kinetic, because that depends only on the position and velocity of the system at the time under consideration, whilst Neumann's energy depends on the velocity and position of the system at some previous time. In spite of all this, however, Neumann applies the ordinary dynamical processes to this energy just as if it were kinetic or potential; and in this way arrives at the same expression as Weber for the force between two moving electrified bodies. The rest of the theory is the same as Weber's, except that Neumann's assumption about the nature of a current is different from Weber's. According to Weber, an electric current consists of equal quantities of positive and negative electricity, moving with equal velocities in opposite directions. According to Neumann, the positive electricity alone can move, the negative being attached to the molecules of the conductor. Riecke and Clausius have shown that with this assumption and Weber's law a steady current must exert a force upon a particle at rest and charged with electricity, and must in consequence produce an irregular distribution of electricity over any conductor in its neighbourhood.

Theories which are founded on dynamical considerations and which take into account the action of the dielectric.

In the theories we have hitherto considered, the influence of the medium which exists between the currents has been left altogether out of account. In the theories which we shall now proceed to discuss, the influence of this medium is taken into consideration. This is, perhaps, the most important step that has ever been made in the theory of electricity, though from a practical point of view it is comparatively of little importance; in fact, for practical purposes almost any one of the preceding theories will satisfy every requirement.

Faraday was the first to look upon the dielectric as an important agent in electrical phenomena; he was led to this by his desire to get rid, as far as possible, of the idea of action at a distance, which was so prevalent in his time, but to which his researches have given the death-blow. In his 'Experimental Researches,' § 1164, speaking of electrostatic induction, he says, 'I was led to suspect that common induction itself was in all cases an action of contiguous particles, and that electrical action at a distance (*i.e.* ordinary inductive action) never occurred except through the influence of surrounding matter.' And later on he gives his views as to the nature of the effect in the medium; in § 1298 of the 'Researches' he says, 'Induction appears to consist in a certain polarised state of the particles into which they are thrown by the electrified body sustaining the action, the particles assuming positive and negative points or parts, which are symmetrically arranged with respect to each other and the inducting surfaces or particles. This state must be a forced one, for it is originated and sustained only by force, and sinks to the normal or quiescent state when that force is removed. It can be continued only in insulators by the same portion of electricity, because they only can retain this state of the particles.' He gives an experimental illustration of his view in § 1350. He says, 'As an illustration of the condition of the polarised particles in a dielectric under induction I may describe an experiment. Put in a glass vessel some clear rectified oil of turpentine, and introduce two wires passing through glass tubes, when they coincide with the surface of the fluid and terminating in balls or points. Cut some very clean dry white silk into small particles, and put these also into the liquid; then electrify one of the wires by an ordinary machine and discharge by the other. The silk will immediately gather from all parts of the liquid and form a band of particles reaching from wire to wire, and if touched by a glass rod will show considerable tenacity; yet the moment the supply of electricity ceases the band will fall away and disappear by the dispersion of its parts. The conduction by the silk is in this case very small, and after the best examination I could give to the effects, the impression on my mind is that the adhesion of the whole is due to the polarity which each filament acquires, exactly as the particles of iron between the poles of a horse-shoe magnet are held together in one mass by a similar disposition of forces. The particles of silk therefore represent to me the condition of the molecules of the dielectric itself, which I assume to be polar, just as that of the silk is. In all cases of conductive discharge the contiguous polarised particles of the body are able to effect a neutralisation of their forces with greater or less facility, as the silk does also in a very slight degree. Further we are not able to carry the parallel, except in imagination; but if we could divide each particle of silk into two halves, and let each half travel until it met and united with the next half in an opposite state, it would then exert its carrying power (1307), and so far represent electrolytic discharge.'

And it is not only in statical electricity that Faraday recognised the importance of the dielectric. When he is discussing his discovery of the induction of currents, which he ascribes to the assumption of what he called the electrotonic state by the body in which induced currents are developed, he says, § 73, 'It may even exist in non-conductors,' that is, that there is an electromotive force acting on the surrounding dielectric due to the variation in the primary current. Again, in § 1661, he says, 'Now though we perceive the effects only in that portion of matter which, being in the neighbourhood, has conducting properties, yet hypothetically it is probable that the non-conducting matter has also its relations to, and is affected by, the disturbing causes, though we have not yet discovered them. Again and again the relation of conductors and non-conductors has been shown to be one, not of opposition in kind, but only in degree (1334, 1603); and therefore for this, as well as for other reasons, it is probable that what will affect a conductor will affect an insulator also, producing, perhaps, what may deserve the term of the electrotonic state (60, 242, 1114).' And though he was unable to detect these effects experimentally, the following paragraph (1728) shows that his belief in their existence was not shaken: 'But then it may be asked, What is the relation of the properties of insulating bodies, such as air, sulphur, or lac, when they intervene in the line of magnetic action? The answer to this is at present merely conjectural. I have long thought there must be a particular condition of such bodies, corresponding to the state which causes currents in metals and other conductors (26, 53, 191, 201, 213); and considering that the bodies are insulators, one could expect that state to be one of tension. I have, by rotating non-conducting bodies near magnetic poles, and poles near them, and also by causing powerful electric currents to be suddenly formed and to cease around and about insulators in various directions, endeavoured to make some such state sensible, but have not succeeded. Nevertheless as any such state must be of exceedingly low intensity, because of the feeble intensity of the currents which are used to induce it, it may well be that the state may exist, and may be discoverable by some more expert experimentalist, though I have not been able to make it sensible.'

Maxwell was the first to express Faraday's ideas in mathematical language. In his papers on 'Physical Lines of Force' in the 'Philosophical Magazine' for March, April, May, 1861, and January, February, 1862, he developes a theory of electricity according to which the energy of the electro-magnetic field resides in the dielectric as well as in the conductors; later, in the 'Philosophical Transactions' for 1865, he greatly extended Faraday's ideas as well as put them into definite mathematical language, and this without reference to any special theory of the mechanism which produces electrical phenomena. We shall devote some time to discussing Maxwell's theory, as it is freer from serious objections than any other, while at the same time it covers a much wider ground.

We shall begin by referring to Maxwell's view of the state of the dielectric in the electric field. Maxwell supposes that the dielectric is changed, and perhaps the clearest way of describing this change is that of Faraday in the extract already quoted. Maxwell's nomenclature as to this change is a little unfortunate; instead of speaking, like Faraday, of the polarisation of the dielectric, he speaks of the change as consisting of an electric displacement, which in isotropic media is in the direction of the electromotive force. Mathematically the two things are identical; we may either say of a wire that it is negatively electrified at one end A, and positively at the other end B, or else that there is a displacement of positive electricity from A to B, so that there is an excess of positive electricity at B and a deficiency at A. But though the words in a mathematical sense are identical, still the word displacement seems to connote special qualities which limit the generality of the conception in an undesirable way; the word displacement seems to imply motion in the direction of displacement, while polarisation only implies that there is a vector change of some kind in the dielectric. The condition of the dielectric is quite analogous to the state of a piece of soft iron placed in a magnetic field. The polarisation or displacement is in isotropic media in the direction of the electromotive force and proportional to it, just as the magnetic induction in isotropic media is in the direction of the magnetic force and proportional to it. It was this proportionality combined with the fact that as soon as the electromotive force is removed the dielectric springs back, as it were, to its original state, that led Maxwell to use the word displacement. He looked on the case as analogous to that of an elastic solid, which springs back to its original position when the external force is removed, and in which the displacement is proportional to the impressed force. To avoid any unnecessary definiteness we shall use the term dielectric polarisation instead of electric displacement. Thus according to this view the dielectric in the electric field is polarised. This polarisation means change of structure of some kind, and to produce this change of structure work is required. The energy in the polarised dielectric will be greater than the energy when it was unpolarised, for if the energy were less the dielectric would go into the polarised condition of itself, without the application of any external forces.

It is rather difficult to see what is meant in Maxwell's theory by the phrase 'quantity of electricity.' According to the old two-fluid theory an electrified body was supposed to contain a certain quantity of something called electricity, rules were given for measuring this quantity, and the phrase 'quantity of electricity' meant something quite definite. In Maxwell's theory, where everything is referred to the dielectric, the meaning of the phrase is not so obvious. We can, however, arrive at some idea of what is meant by the consideration of what are called 'tubes of force.' Let us suppose at first that the dielectric is air. A line of force is a line whose direction at any point coincides with the direction of the electromotive force at that point, so that we may conceive the electric field to be filled with lines of force. If we consider the lines of force passing through some small closed curve, they will form a tube, and such a tube is called a tube of force; and if the dimensions of the tube are such that the product of the cross section at any point and the electromotive force at that point is constant and equal to 4π, the tube is called a unit tube. We may thus conceive space to be filled with unit tubes of force. Since the electromotive force inside a conductor vanishes these tubes will end at the surface of a conductor. And the quantity of electricity on the conductor will be equal to the excess of the number of lines of force which leave the conductor over those which enter it. A tube is said to leave the conductor when the direction of the electromotive force is along the normal drawn outwards, and to enter it when the direction of the electromotive force is along the normal drawn inwards. As the conductor moves about it may be supposed to carry the tubes of force along with it, so that the number of tubes which end on the conductor remains constant. This way of looking at electrification is quite satisfactory as long as we keep to one dielectric air; when we have to consider different dielectrics it requires modification, because the electromotive force changes abruptly as we pass from one dielectric into another, so that a tube which was a unit tube in one dielectric is not so in another. It is easy, however, to extend the definition of unit tubes so as to meet this difficulty; for if the tubes pass from one dielectric A into another B the ratio of the product of the cross section and electromotive force is constant for all the tubes and depends only on the nature of the dielectrics; this ratio is the ratio of the specific inductive capacities in B and A. Air is taken as the standard dielectric, and the specific inductive capacity of another dielectric A is the ratio of the product of the electromotive force and cross section of a tube in air to the product of the same quantities for the same tube in the dielectric A. Thus if we amend our definition and say that a circuit tube is one such that the product of the cross section, the electromotive force, and the specific inductive capacity of the medium in which the cross section is situated is equal to 4π, then the quantity of electricity on a conductor is equal to the excess of the number of unit tubes which leave the conductor over the number of those which enter it. In this way we get an idea of what is meant by 'quantity of electricity' in Maxwell's theory. Maxwell accounts for the forces observed between electrified bodies by a system of stresses in the dielectric separating them; as, however, at present we wish to compare Maxwell's theory with other theories which do not touch upon this point, we shall discuss this part of the theory separately later on and go on to discuss those points which are involved in all the theories.

The next great point in Maxwell's theory is the development of Faraday's remark that the electrotonic state may exist even in non-conductors, *i.e.*, that the dielectric surrounding a changing current is acted on by electromotive forces which polarise it. This statement is one as to whose truth nobody seems to entertain any doubt, whilst the statement that changes in the dielectric polarisation produce effects analogous to those produced by ordinary conduction currents is by no means so universally received, and yet the one seems the necessary consequence of the other. If we regard the whole electric field as a dynamical system, and to fix our ideas consider an element a of the dielectric, and the current, which is supposed to vary, then, since a variation in the current polarises a, *i.e.*, produces a change in its structure, there must be mechanism connecting the current with the element a; but if this is so then it follows from dynamical principles that a non-uniform variation in the structure of a must produce a change in the current—in other words, that a change in the rate of change of the polarisation of a produces an electromotive force on the current, *i.e.* that the change of polarisation produces an effect analogous to that of an ordinary conduction current. We may illustrate this by a purely dynamical example. Suppose we have a dynamical system defined by two co-ordinates p and q, and let T be the kinetic energy of the system and V the potential energy; then by Lagrange's equation the force tending to increase q

$$= -\frac{dV}{dq} + \frac{dT}{dq} - \frac{d}{dt}\frac{dT}{d\dot{q}}.$$

Now if there is a force tending to alter q which depends upon the acceleration of p, there must be a term in the kinetic energy of the form

$$A\dot{p}\dot{q};$$

but if we apply Lagrange's equations to the p co-ordinates we see that this term implies the existence of a force tending to increase p equal to

$$-\frac{d}{dt}A\dot{q},$$

so that an acceleration of q will produce a force tending to alter p. To make this applicable to the case of the current and the dielectric, we have only to suppose that $\dot{p}$ represents the current, q the polarisation of the dielectric. That a change in $\dot{p}$ produces a change in q is shown by the fact that the dielectric is polarised when the current is changing, and this shows that there must be a term of the form $A\dot{p}\dot{q}$, in expression for the kinetic energy; from this it follows that a change in $\dot{q}$, *i.e.*, in the rate of change of the polarisation, will produce an E.M.F. on the circuit. As the variation of the dielectric polarisation produces the same effect as a conduction current, we must in the case, when both conduction current and alteration in the polarisation are present, look upon the true or effective current as the sum of the conduction current and the change in the polarisation.

The components f, g, h of the dielectric polarisation are defined by the equation

$$f = \frac{K}{4\pi}X \qquad g = \frac{K}{4\pi}Y \qquad h = \frac{K}{4\pi}Z,$$

where K is the specific inductive capacity of the medium, X, Y, Z the components of the electromotive force. If u, v, w are the components of the effective current, p, q, r the components of the conduction current, then Maxwell in his paper on a 'Dynamical Theory of the Electromagnetic Field,' 'Phil. Trans., 1885,' puts

$$u = p + \frac{df}{dt}, \qquad v = q + \frac{dg}{dt}, \qquad w = r + \frac{dh}{dt}.$$

Since
$$\frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = -\frac{d\rho}{dt}$$

and
$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = \rho,$$

where ρ is the volume density of the free electricity, we see that

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

If the values of the quantities in a medium A be denoting by putting the suffix 1 to the symbols representing them, and those in another dielectric B by putting the suffix 2, then if l, m, n are the direction cosines of the normal from A to B, we have at the boundary of the two media

$$l\,(p_1 - p_2) + m\,(q_1 - q_2) + n\,(r_1 - r_2) = \frac{d\sigma}{dt}$$

$$l\,(f_1 - f_2) + m\,(g_1 - g_2) + n\,(h_1 - h_2) = -\sigma,$$

where σ is the surface density of the electricity; thus

$$l\,(u_1 - u_2) + m\,(v_1 - v_2) + n\,(w_1 - w_2) = 0;$$

so that u, v, w satisfy the same equations as the components of the velocity of an incompressible fluid.

This assumption about the magnitude of the effects produced by the alteration in the dielectric polarisation makes the mathematics of the theory as simple as possible. If Maxwell had merely assumed that the alteration of the dielectric polarisation produces effects analogous to those produced by ordinary conduction currents, and that the equivalent conduction current was proportional to the rate of alteration of the dielectric polarisation, then these equations would have been

$$u = p + a\frac{dX}{dt},$$

$$v = q + a\frac{dY}{dt},$$

$$w = r + a\frac{dZ}{dt};$$

so that in a homogeneous dielectric

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = -\frac{d\rho}{dt}\left\{1 - 4\frac{a\pi}{k}\right\},$$

$$l(u_1 - u_2) + m\,(v_1 - v_2) + n\,(w_1 - w_2) = \frac{d\sigma}{dt} + a_1\frac{dN_1}{dt} - a_2\frac{dN_2}{dt}$$

where N is the component of the electromotive force normal to the surface.

Maxwell's assumption is that $a = K/4\pi$, and this makes the equations much simpler; it is, however, important to remember that Maxwell's theory of the dielectric involves the two assumptions—

1st. That alterations in the dielectric polarisation produce effects analogous to those of ordinary conduction currents;

2nd. That the magnitude of the equivalent conducting current $= d\left\{\frac{K}{4\pi}F\right\}/dt$, where F is the electromotive force at the point; this is equivalent to saying that all the currents are closed currents, and that there is no discontinuity in them.

Maxwell developes his theory by means of the principle of the Conservation of Energy.

Let us consider an electric field full of currents, whether ordinary conduction currents or polarisation ones. Then this field may be looked upon as a material system, and all the phenomena have to be explained as the effects of the motion of this system; a current must be looked upon as a change in the structure of the system, and so capable of representation by means of the differential coefficients of the co-ordinates fixing the system; we can thus represent the current at each point as the differential coefficient of some generalised co-ordinate fixing the system; the components u, v, w of the current passing through an element dx, dy, dz may be looked upon as the rates of change of some generalised co-ordinates; we may write the energy as

$$\tfrac{1}{2}\iiint\{Fu + Gv + Hw\}\,dx\,dy\,dz,$$

where F, G, H may be looked upon as momenta corresponding to u, v, w. It remains to identify F, G, H with known quantities. Maxwell does this by the aid of Faraday's result, that the electromotive force round a circuit equals the rate of diminution of the number of lines of force passing through it.

Let us consider a single linear circuit in which the current is i, or say dq/dt, then the energy

$$= \tfrac{1}{2}\int\frac{dq}{dt}\left\{F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right\}ds,$$

where ds is an element of circuit; but by Lagrange's equation the force tending to increase q, *i.e.*, the electromotive force in the circuit,

$$= -\frac{d}{dt}\int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds;$$

so that

$$\int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds$$

equals the number of lines of force passing through the circuit; but if dS be an element of surface closing up the circuit, l, m, n the direction cosines of the normal, then by Stokes' theorem

$$\int\left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right)ds$$

$$\iint\left\{l\left(\frac{dH}{dy} - \frac{dG}{dz}\right) + m\left(\frac{dF}{dz} - \frac{dH}{dx}\right) + n\left(\frac{dG}{dx} - \frac{dF}{dy}\right)\right\}dS;$$

but the number of lines of force passing through the circuit

$$=\iint(la+mb+nc)dS,$$

where a, b, c are the components of magnetic induction, so that

$$a=\frac{dH}{dy}-\frac{dG}{dz},$$

$$b=\frac{dF}{dz}-\frac{dH}{dx},$$

$$c=\frac{dG}{dx}-\frac{dF}{dy}.$$

To connect a, b, c with the current, Maxwell makes use of the principle that the line integral of the magnetic force taken round any closed curve equals the current flowing through the curve. This leads to the equations—

$$4\pi u=\frac{d\gamma}{dy}-\frac{d\beta}{dz},$$

$$4\pi v=\frac{d\alpha}{dz}-\frac{d\gamma}{dx},$$

$$4\pi w=\frac{d\beta}{dx}-\frac{d\alpha}{dy};$$

so that if μ be the coefficient of magnetic permeability,

$$4\pi\mu u=\frac{dc}{dy}-\frac{db}{dz},$$

and so on. Substituting the values of a, b, c, given above, we find

$$4\pi\mu u=\frac{d}{dx}\left\{\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}\right\}-\nabla^2F,$$

with similar equations for G and H.

Now v. Helmholtz, in his paper 'Ueber die Bewegungsgleichungen der Elektricität für ruhende leitende Körper' (Crelle, lxxii. p. 57; *Gesammelte Werke*, ii. p. 545), has investigated the most general expressions for F, G, H, consistent with the force between two closed circuits agreeing with that indicated by Ampère's theory, and he finds that if the circuits are closed circuits, as Maxwell assumes all circuits to be, then

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0,$$

and therefore

$$4\pi\mu u=-\nabla^2F,$$

with similar equations for G and H. These equations are sufficient to determine the quantities F, G, H.

Maxwell does not at once put $dF/dx+dG/dy+dH/dz=0$; he writes J for this quantity, and puts

$$\chi=\iiint\frac{J}{r}dx\,dy\,dz.$$

Then

$$F=\iiint\frac{\mu u}{r}dx\,dy\,dz+\frac{d\chi}{dx};$$

as, however, he subsequently puts $J=0$, we may at once simplify the equation by making this assumption.

Since the kinetic energy equals

$$\tfrac{1}{2}\iiint(Fu+Gv+Hw)dx\,dy\,dz,$$

we see by Lagrange's equations that the electromotive force tending to increase u

$$=-\frac{dF}{dt};$$

in addition to this there is the force arising from the electrostatic potential ϕ, so that the total electromotive force parallel to the axis of x

$$=-\frac{dF}{dt}-\frac{d\phi}{dx},$$

so that if σ be the specific resistance of the substance, K its specific inductive capacity, then

$$\sigma p=\frac{4\pi}{K}f=-\frac{dF}{dt}-\frac{d\phi}{dx};$$

but $$u=p+\frac{df}{dt}=-\frac{1}{\sigma}\left\{\frac{dF}{dt}+\frac{d\phi}{dx}\right\}-\frac{K}{4\pi}\left\{\frac{d^2F}{dt^2}+\frac{d^2\phi}{dx\,dt}\right\};$$

but we saw before that

$$4\pi\mu u=-\nabla^2F;$$

substituting for u this value, we see

$$\nabla^2F=\frac{4\pi\mu}{\sigma}\left\{\frac{dF}{dt}+\frac{d\phi}{dx}\right\}+K\mu\left\{\frac{d^2F}{dt^2}+\frac{d^2\phi}{dx\,dt}\right\},$$

thus in the dielectric the equation becomes

$$\nabla^2F=K\mu\left\{\frac{d^2F}{dt^2}+\frac{d^2\phi}{dx\,dt}\right\},$$

in the conductor

$$\nabla^2F=\frac{4\pi\mu}{\sigma}\left\{\frac{dF}{dt}+\frac{d\phi}{dx}\right\}.$$

The equation for the dielectric shows that it represents a wave-motion propagated with the velocity $1/\sqrt{K\mu}$; the numerical value of this velocity agrees very approximately with the velocity of light, and this led Maxwell to the theory that the changes in the structure of the dielectric which take place when the dielectric is polarised are of the same nature as those which constitute light. This theory, which is called the electromagnetic theory of light, might almost as justly be called the mechanical theory of dielectric polarisation. Kirchhoff, in his paper 'Ueber die Bewegung der Electricität in Drähten' (Pogg. *Ann.*, vol. c. 1857; *Gesammelte Werke*, p. 131), was the first to point out that some electrical actions are propagated with the velocity of light. In this paper he considers the motion of electricity in wires whose diameters are small compared with their length. There are three things which have to be considered in this problem—(1) the self-induction of the electric current, and if the medium be taken into account, that of the polarisation currents in the dielectric. This self-induction produces very much the same effect as if the electric current possessed momentum—(2) the electrostatic action of the free electricity which tends to bring things to a definite state, and corresponds very much to the spring in a material system. Then, lastly, there is the electrical resistance, which corresponds to friction in an ordinary system. We see from the analogy that if the resistance be small enough, the electrical system will vibrate; if, however, the resistance is large, the electrical disturbance will be propagated in the same way as heat. Kirchhoff in his paper considers the propagation of electrical disturbance along a wire under various conditions: we shall only consider here one of these cases; that of an endless wire. In his solution Kirchhoff only considers the self-induction of the current flowing along the wire; he does not consider the effects in the surrounding dielectric. He shows that if e be the quantity of electricity per unit length of the wire, and

$$e = \mathrm{X} \sin ns,$$

where s is the length of a portion of the wire measured from some fixed point, then X satisfies the differential equation

$$\frac{d^2\mathrm{X}}{dt^2} + \frac{c^2 r}{16\gamma l}\frac{d\mathrm{X}}{dt} = \frac{c^2}{2}\frac{d^2\mathrm{X}}{ds^2},$$

where c is a quantity which occurs in Weber's theory, and is the velocity with which two charged particles must move if the electrodynamic attraction between them balances the electrostatic repulsion;

r is the resistance of the wire in electrostatic measure; $\gamma = \log l/a$,

where l is the length of the wire and a the radius of its cross section. The form of the solution of this equation depends on the magnitude of

$$\frac{32\gamma}{cr\sqrt{2}}nl.$$

If this quantity be large, the solution takes the form representing the propagation of a wave along the wire with the velocity $c/\sqrt{2}$. Weber's researches show that this velocity is very nearly equal to the velocity of light. If, however, the above-mentioned quantity be small, then the solution of the equation takes the same form as the formula which expresses the conduction of heat along the wire. We must not, however, take this to mean that the electric disturbance is propagated with an infinite velocity, so that if we had an infinitely delicate electrometer at a finite distance from the source of disturbance we could detect an electrification after an indefinitely short time, for it seems obvious that the electrical resistance cannot increase the velocity of propagation any more than the resistance of the air could increase the velocity of propagation of a disturbance along a line of particles connected by an elastic string. The conditions at the end help to determine the form of the solution, and these cannot make themselves felt until the disturbance has reached it; thus the heat form of solution probably only holds after a time from the commencement of the disturbance greater than the time taken by light to travel along the wire. If we take the case of a copper wire one square centimetre in area, we shall find that the wave form of solution will hold if the wire is not more than 100 miles in length, while the heat form will correspond to wires which are much longer than this. Kirchhoff's solution only refers to the propagation of a disturbance in a conductor, while Maxwell's refers to the propagation of such a disturbance in the dielectric.

Maxwell considers the effect of the motion of the medium on the electromotive force; he shows that the electromotive force parallel to the axis of x

$$= cv - bw - \frac{dF}{dt} - \frac{d\psi}{dx},$$

where u, v, w are the components of the velocity of the medium conveying electric action. Here ψ is not the electrostatic potential merely; it is equal, as Helmholtz has shown,[1] to the electrostatic potential plus the term

$$Fu + Gv + Hw.$$

We must remark here that u, v, w are the components of the velocity of the medium conveying the electric action, *i.e.* the ether, and this need not necessarily be the same as the velocity of the dielectric.

v. Helmholtz's Dielectric Theory.

v. Helmholtz, in the paper[2] to which we have so often referred, considers the effect of the polarisation of the dielectric; he supposes that when an electromotive force X, parallel to the axis of x, acts on an element of a dielectric, it puts it into such a state that it produces the same effect as if there were electricity of surface-density $\mathfrak{x}$ on the face $dy\,dz$ of the element, and an equal quantity of electricity of the opposite sign on the parallel face, $\mathfrak{x}$ being given by the equation

$$\mathfrak{x} = \epsilon X,$$

the variations in the electromotive forces acting on the dielectric are supposed to produce the same effect as ordinary conduction currents whose components are $\dot{\mathfrak{x}}$, $\dot{\mathfrak{y}}$, $\dot{\mathfrak{z}}$, where $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ are the components of a vector quantity which in isotropic media is parallel to the electromotive force and equal to the product of ϵ and the intensity of the force. This agrees with Maxwell's assumption, provided

$$\epsilon = K/4\pi,$$

where K is the specific inductive capacity of the dielectric. If ϕ be the electrostatic potential of the free electricity, ψ the potential due to the polarisation of the dielectric, then Helmholtz shows that

$$\frac{d}{dx}\left\{(1+4\pi\epsilon)\frac{d}{dx}(\phi+\psi)\right\} + \frac{d}{dy}\left\{(1+4\pi\epsilon)\frac{d}{dy}(\phi+\psi)\right\} + \frac{d}{dz}\left\{(1+4\pi\epsilon)\frac{d}{dz}(\phi+\psi)\right\} = -4\pi E,$$

where E is the volume-density of the free electricity. The corresponding equation in Maxwell's theory is of the same form, provided

$$1 + 4\pi\epsilon = K.$$

[1] *Ueber die Theorie der Elektrodynamik; die elektrodynamische Kräfte in bewegten Leitern*, Crelle, lxxviii. p. 309; *Gesammelte Werke*, ii. p. 745.

[2] *Ueber die Theorie der Elektrodynamik*, Crelle, lxxii. p. 57; *Gesammelte Werke*, i. p. 544.

This relation seems inconsistent with the previous one; it may, however, be reconciled with it in the following way:—

The potential due to a quantity E of electricity at a point distant r from it is proportional to

$$\frac{\mathrm{E}}{(1+4\pi\epsilon)r}.$$

If ϵ_0 be the value of ϵ for air, the potential under the same circumstances in air is proportional to

$$\frac{\mathrm{E}}{(1+4\pi\epsilon_0)r};$$

if, then, we define unit potential as the potential at unit distance from unit of electricity in air, the potential due to a quantity E in another medium will be

$$\left\{\frac{1+4\pi\epsilon_0}{1+4\pi\epsilon}\right\}\frac{\mathrm{E}}{r}.$$

We see that this is equivalent to increasing the unit of potential, and therefore the unit electromotive force, $1+4\pi\epsilon_0$ times, so that if we use the new unit the equations will be

$$\mathfrak{x}=\frac{\epsilon}{1+4\pi\epsilon_0}\mathrm{X},$$

$$\frac{d}{dx}\left\{\frac{1+4\pi\epsilon}{1+4\pi\epsilon_0}\frac{d}{dx}(\phi+\psi)\right\}+\ \ldots\ldots\quad=-4\pi\mathrm{E}.$$

These will coincide with Maxwell's equation if we make ϵ and ϵ_0 each infinite and put $\mathrm{K}=\epsilon/\epsilon_0$.

Returning to Helmholtz's theory, if u, v, w are the components of the total current

$$u=p+\dot{\mathfrak{x}},$$
$$v=q+\dot{\mathfrak{y}},$$
$$w=r+\dot{\mathfrak{z}},$$

where p, q, r are the components of the conduction current.

Helmholtz puts

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=-\frac{d\rho}{dt},$$

where ρ is the volume-density of the free electricity, and if σ be the surface-density of the free electricity at any point of a surface separating two media, u_1, v_1, w_1; u_2, v_2, w_2 the components of the current in the two media, l, m, n the direction cosines of the normal to the surface drawn from the first medium to the second, then according to v. Helmholtz

$$l\,(u_1-u_2)+m\,(v_1-v_2)+n\,(w_1-w_2)=\frac{d\sigma}{dt}.$$

According to Maxwell the corresponding equations are

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0,$$

$$l\,(u_1-u_2)+m\,(v_1-v_2)+n\,(w_1-w_2)=0.$$

As it is in the difference between these equations that the difference in the theory really lies, it will be instructive to look at them from another point of view. We know of no way in which the quantity of free electricity can be altered except by electricity being conveyed by conduction currents to the place where the alteration takes place. Assuming, then, that the alteration in the density is caused by such currents

$$\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}=-\frac{d\rho}{dt},$$

$$l\,(p_1-p_2)+m\,(q_1-q_2)+n\,(r_1-r_2)=\frac{d\sigma}{dt}.$$

So that Helmholtz's equations taken in conjunction with these are equivalent to the condition

$$\frac{d\dot{\mathfrak{x}}}{dx}+\frac{d\dot{\mathfrak{y}}}{dy}+\frac{d\dot{\mathfrak{z}}}{dz}=0;$$

$$l\,(\dot{\mathfrak{x}}_1-\dot{\mathfrak{x}}_2)+m\,(\dot{\mathfrak{y}}_1-\dot{\mathfrak{y}}_2)+n\,(\dot{\mathfrak{z}}_1-\dot{\mathfrak{z}}_2)=0.$$

Thus on Helmholtz's theory the dielectric currents behave like the flow of an incompressible fluid, while on Maxwell's theory it is the total current, which is the sum of the conduction currents and the dielectric currents which behave in this way.

The equations we have arrived at for the dielectric currents seem inconsistent with Helmholtz's definition of them; for since

$$\mathfrak{x}=\epsilon X,$$

with similar equations for $\mathfrak{y}$ and $\mathfrak{z}$, and since in a medium at rest

$$X=-\frac{dU}{dt}-\frac{d\phi}{dx},$$

$$Y=-\frac{dV}{dt}-\frac{d\phi}{dy},$$

$$Z=-\frac{dW}{dt}-\frac{d\phi}{dz},$$

where U, V, W are the components of the vector potential. If we consider a surface separating two portions of the same dielectric and coated with electricity whose surface-density is σ, we have, since U, V, W are not discontinuous on crossing the surface,

$$l\,\{\dot{\mathfrak{x}}_1-\dot{\mathfrak{x}}_2\}+m\,(\dot{\mathfrak{y}}_1-\dot{\mathfrak{y}}_2)+n\,(\dot{\mathfrak{z}}_1-\dot{\mathfrak{z}}_2)=-\epsilon\,\frac{d}{dt}\left[l\,\frac{d\phi}{dx}+m\,\frac{d\phi}{dy}+n\,\frac{d\phi}{dz}\right]_1^2,$$

where $\left[l\,\frac{d\phi}{dx}+m\,\frac{d\phi}{dy}+n\,\frac{d\phi}{dz}\right]_1^2$ denotes the difference between the values of $l\,\frac{d\phi}{dx}+m\,\frac{d\phi}{dy}+n\,\frac{d\phi}{dz}$ on the two sides of the surface.

But $$\left[l\,\frac{d\phi}{dx}+m\,\frac{d\phi}{dy}+n\,\frac{d\phi}{dz}\right]_1^2=-\frac{1}{1+4\pi\epsilon}\,\sigma,$$

so that $$l\,(\dot{\mathfrak{x}}_1-\dot{\mathfrak{x}}_2)+m\,(\dot{\mathfrak{y}}_1-\dot{\mathfrak{y}}_2)+n\,(\dot{\mathfrak{z}}_1-\dot{\mathfrak{z}}_2)=\frac{\epsilon}{1+4\pi\epsilon}\,\frac{d\sigma}{dt},$$

and so cannot vanish if the surface-density of the electricity changes;

thus Helmholtz's equation seems to be inconsistent with the principle that the change in the quantity of free electricity is caused by conduction currents. In the case above considered, Maxwell's equations lead to no difficulty; it does not follow, however, that Maxwell's assumption that the total current behaves like the flow of an incompressible fluid is absolutely necessary. We shall consider later on the differences which the abandonment of this assumption will make in the theory.

We shall now go on to consider Helmholtz's equations and compare them with the corresponding ones in Maxwell's theory.

The quantities U, V, W are given by equation of the form

$$U=\frac{1}{2}(1-k)\frac{d\psi}{dx}+\iiint\frac{u}{r}d\xi\, d\eta\, d\zeta,$$

where k is the constant which we mentioned before as occurring in Helmholtz's theory, and

$$\psi=-\frac{1}{2\pi}\iiint\frac{d\phi}{dt}\frac{1}{r}\,d\xi\, d\eta\, d\zeta,$$

where ϕ is the electrostatic potential; it follows from these equations that

$$\frac{dU}{dx}+\frac{dV}{dy}+\frac{dW}{dz}=-k\frac{d\phi}{dt}.$$

The corresponding equation in Maxwell's theory is

$$\frac{dU}{dx}+\frac{dV}{dy}+\frac{dW}{dz}=0,$$

so that these equations coincide if $k=0$. We can see from the value of χ given on page 116 that, on Helmholtz's theory, this quantity would also vanish, whatever be the value of k, if the *total* current behaved like the flow of an incompressible fluid.

If α, β, γ are the components of the magnetic force, then on Helmholtz's theory

$$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=A\left\{\frac{d^2\phi}{dt\,dx}-4\pi u\right\},$$

$$\frac{d\alpha}{dz}-\frac{d\gamma}{dx}=A\left\{\frac{d^2\phi}{dt\,dy}-4\pi v\right\},$$

$$\frac{d\beta}{dx}-\frac{d\alpha}{dy}=A\left\{\frac{d^2\phi}{dt\,dz}-4\pi w\right\},$$

where A is a quantity depending on the unit of current adopted, and is such that the force between two parallel elements of currents at right angles to the line joining them is

$$\frac{1}{2}\frac{A^2}{r^2}i\,j\,ds\,ds',$$

where r is the distance between the elements, ij the current through them, and ds ds' their lengths; the corresponding equations on Maxwell's theory are

$$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=4\pi u,$$

with similar equations for v and w.

If λ, μ, ν are the intensities of magnetisation, ϑ the coefficient of induced magnetisation, the equations satisfied by the components of the dielectric and magnetic polarisation are of the type

$$\nabla^2\mathfrak{x}=\frac{4\pi\epsilon\,(1+4\pi\vartheta)}{(1+4\pi\epsilon_0)\,(1+4\pi\vartheta_0)}\mathrm{A}^2\frac{d^2\mathfrak{x}}{dt^2}+\left\{1-\frac{(1+4\pi\vartheta)\,(1+4\pi\epsilon)}{k}\right\}\frac{d}{dx}\left\{\frac{d\mathfrak{x}}{dx}+\frac{d\mathfrak{y}}{dy}+\frac{d\mathfrak{z}}{dz}\right\},$$

$$\nabla^2\lambda=\frac{4\pi\epsilon\,(1+4\pi\vartheta)}{(1+4\pi\epsilon_0)\,(1+4\pi\vartheta_0)}\mathrm{A}^2\frac{d^2\lambda}{dt^2},$$

where ϵ_0 and ϑ_0 are the values of ϵ and ϑ for air.

These equations show that the dielectric and magnetic polarisations are propagated by waves. For the dielectric polarisation longitudinal waves are propagated with the velocity

$$\frac{1}{\mathrm{A}}\left\{\frac{(1+4\pi\epsilon)\,(1+4\pi\epsilon_0)\,(1+4\pi\vartheta_0)}{4\pi\epsilon k}\right\}^{\frac{1}{2}}.$$

Transverse waves are propagated with the velocity

$$\frac{1}{\mathrm{A}}\sqrt{\frac{(1+4\pi\epsilon_0)\,(1+4\pi\vartheta_0)}{4\pi\epsilon\,(1+4\pi\vartheta)}}.$$

Longitudinal waves of magnetic disturbances are propagated with an infinite velocity, and traverse ones with the same velocity as the transverse waves of dielectric polarisation. The electrostatic potential is propagated with the velocity $1/\mathrm{A}\sqrt{k}$. In Maxwell's theory the corresponding equations are

$$\nabla^2x=\mu\mathrm{K}\frac{d^2x}{dt^2},$$

$$\nabla^2\lambda=\mu\mathrm{K}\frac{d^2\lambda}{dt^2},$$

where μ is the magnetic permeability and K the specific inductive capacity, so that for both dielectric and magnetic polarisation the velocity of the longitudinal wave is infinite, while the velocity of the transverse wave is $1/\sqrt{\mu\mathrm{K}}$. The velocity of propagation of the electrostatic potential is infinite. If in Helmholtz's theory we put $k=0$, $\vartheta_0=0$, $\epsilon/\epsilon_0=\mathrm{K}$, while both ϵ and ϵ_0 are infinite, we see that the results of his theory will in this respect agree with Maxwell's.

Though in Maxwell's theory the velocity of propagation of the electrostatic potential is infinite, and in Helmholtz's theory $1/\mathrm{A}\sqrt{k}$, the electromotive force at a point, and consequently the dielectric polarisation, does not travel with an infinite velocity in Maxwell's theory, or with the velocity $1/\mathrm{A}\sqrt{k}$ in Helmholtz's. We can see the reason of this more easily from Maxwell's theory, as the equations are simpler.

Using the notation of that theory, viz., f, g, h, for the components of the electric displacement, F, G, H for the components of the vector potential, and ϕ for the electrostatic potential, then in a dielectric the equations are

$$\frac{4\pi}{\mathrm{K}}f=-\frac{d\mathrm{F}}{dt}-\frac{d\phi}{dx}$$

$$\frac{4\pi}{K}\frac{df}{dt}=-\frac{d^2F}{dt^2}-\frac{d^2\phi}{dxdt};$$

but, since
$$4\pi\mu\frac{df}{dt}=-\nabla^2F,$$

we see that
$$\frac{1}{\mu K}\nabla^2F=\frac{d^2F}{dt^2}+\frac{d^2\phi}{dxdt}.$$

Now, since $\nabla^2\phi=0$, a particular solution of this differential equation will be

$$\frac{dF}{dt}+\frac{d\phi}{dx}=0,$$

while the general solution will be the sum of this solution and the general solution of

$$\frac{1}{\mu K}\nabla^2F=\frac{d^2F}{dt^2}.$$

The particular solution is propagated at the same rate as ϕ, while the other part of the solution represents a wave travelling with the velocity $1/\sqrt{\mu K}$. Since the part of the solution which travels at an infinite rate satisfies the equation

$$\frac{dF}{dt}+\frac{d\phi}{dx}=0$$

or
$$f=0,$$

we see that the electromotive force due to the change in the vector potential just balances the electrostatic electromotive force, so that until the part of the vector potential which travels at the rate $1/\sqrt{\mu K}$ comes up the resultant electromotive force vanishes. This explains how the electromotive force on Maxwell's theory travels at a different rate from the potential, and a similar explanation will apply to Helmholtz's theory. Helmholtz's equations for a conductor are

$$\sigma\nabla^2u=(1+4\pi\vartheta)\,4\pi A^2\frac{du}{dt}-\frac{d}{dx}\left\{\nabla^2\phi+(1+4\pi\vartheta-k)\,A^2\frac{d^2\phi}{dt^2}\right\}$$

where σ is the specific resistance of the conductor; on Maxwell's theory the equations are

$$\sigma\nabla^2u=4\pi\mu\frac{du}{dt},$$

These equations differ by terms involving the unknown constant k; but v. Helmholtz's[1] investigations on the motion of electricity along thin conducting wires show that there is not much hope of distinguishing between the theories by experiments on conductors. We have seen that we can make certain equations which occur in Helmholtz's theory coincide with the corresponding ones in Maxwell's by giving particular values to certain constants. The difference in Helmholtz's and Maxwell's views as to the continuity of the currents is too serious to let us expect that we should ever get a complete agreement between the

[1] *Ueber die Bewegungsgleichungen der Elektricität für ruhende leitende Körper. Gesammelte Werke*, vol. i. p. 603.

theories; and, in fact, make as many assumptions about the constants as we may, there are still differences between the theories.

In order to get as general a theory of these dielectric currents as possible, we shall investigate the consequences of assuming merely that these currents are proportional to the rate of change of the electromotive force, and write dielectric current$=\eta$ (rate of change of the electromotive force), where η is a constant which for the present is left indeterminate; In Maxwell's theory $\eta=K/4\pi$, where K is the specific inductive capacity of the dielectric; in Helmholtz's theory, η is also proportional to the specific inductive capacity. We shall denote the components of the dielectric currents by the symbols $\dot{f}, \dot{g}, \dot{h}$; the components of the conduction current by p, q, r, and the components of the total current by u, v, w, so that

$$u = p + \dot{f}.$$

Let us put

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = \text{P},$$

$$l\,(u_1 - u_2) + m\,(v_1 - v_2) + n\,(w_1 - w_2) = \Sigma\,;$$

on Maxwell's theory P and Σ are each zero.

If F, G, H are the components of the vector potential, then by v. Helmholtz's investigation of the most general expression possible for these quantities consistent with the condition that the forces between closed circuits should agree with those given by Ampère's laws,

$$\text{F} = \tfrac{1}{2}\,(1-k)\,\frac{d\psi}{dx} + \mu \iiint \frac{u}{r}\,d\xi\,d\eta\,d\zeta,$$

with similar expressions for G and H, where k is a constant and

$$\psi = \iiint \mu \left(u\,\frac{dr}{d\xi} + v\,\frac{dr}{d\eta} + w\,\frac{dr}{d\zeta} \right) d\xi\,d\eta\,d\zeta.$$

Transforming this expression we see, using the same notation as before, that

$$\begin{aligned}\psi &= \iint r\mu\,\{l\,(u_1-u_2) + m\,(v_1-v_2) + n\,(w_1-w_2)\}\,d\text{S} \\ &\qquad - \iiint r\mu \left(\frac{du}{d\xi} + \frac{dv}{d\eta} + \frac{dw}{d\zeta}\right) d\xi\,d\eta\,d\zeta \\ &= \iint \mu r\,\Sigma\,d\,\text{S} - \iiint \mu r\,\text{P}\,d\xi\,d\eta\,d\zeta,\end{aligned}$$

where dS is an element of a surface at which there is discontinuity in u, v, w.

Let us now consider the equations which hold in a perfectly insulating dielectric.

The rate of change of the x component of the electromotive force in a medium at rest

$$= -\frac{d^2\text{F}}{dt^2} - \frac{d^2\phi}{dt\,dx},$$

where ϕ is the electrostatic potential; it also equals $\dot{f}/\eta$, so that

$$\frac{\dot{f}}{\eta} = -\frac{d^2\text{F}}{dt^2} - \frac{d^2\phi}{dt\,dx}.$$

Since in this case there is no conduction current $u = \dot{f}$, and the preceding equation for F shows that

$$\nabla^2 F = \tfrac{1}{2}(1-k)\frac{d}{dx}\nabla^2\psi - 4\pi\mu\dot{f}.$$

substituting for $\dot{f}$

$$\nabla^2 F - \tfrac{1}{2}(1-k)\frac{d}{dx}\nabla^2\psi = 4\pi\eta\left\{\frac{d^2F}{dt^2} + \frac{d^2\phi}{dx\,dt}\right\}.$$

if $\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = \chi$, we get, by differentiating this expression, with regard to x and the corresponding equations for G and H with regard to y and z respectively, and adding

$$\nabla^2\chi - \tfrac{1}{2}(1-k)\,\nabla^4\psi = 4\pi\eta\mu\left\{\frac{d^2}{dt^2}\chi + \frac{d}{dt}\nabla^2\phi\right\}.$$

Now, as the dielectric is a perfect insulator, there are no conduction currents, so that the density of the free electricity remains constant, and therefore

$$\frac{d}{dt}\nabla^2\phi = 0.$$

From the expression for ψ we see that

$$\begin{aligned}\nabla^4\psi &= +\,8\pi P\\ &= -\,8\pi\eta\mu\left(\frac{d^2\chi}{dt^2} + \frac{d}{dt}\nabla^2\phi\right)\\ &= -\,8\pi\eta\mu\frac{d^2\chi}{dt^2}.\end{aligned}$$

Substituting this value of $\nabla^4\psi$ in the equation for χ, we get

$$\nabla^2\chi = 4\pi\eta\mu k\frac{d^2\chi}{dt^2},$$

which represents the propagation of a normal wave with the velocity

$$1/\sqrt{4\pi\eta k}.$$

The transverse wave is propagated with the velocity $1/\sqrt{4\pi\eta\mu}$, so that if the view that light consists of electric or magnetic disturbances be correct, since experiment shows that this velocity is very nearly equal to $1/\sqrt{K\mu}$, we must have $4\pi\eta = K$ or $\eta = K/4\pi$, which is Maxwell's theory. So that if we assume that light is an electric phenomenon, then in those media in which its velocity $= 1/\sqrt{\mu K}$ Maxwell's theory that the electric currents flow like an incompressible fluid must be true.

If α, β, γ are the components of the magnetic force, then, since

$$F = \tfrac{1}{2}(1-k)\frac{d\psi}{dx} + \mu\iiint\frac{u}{r}\,d\xi\,d\eta\,d\zeta,$$

we see from Ampère's formula for the magnetic force due to a circuit that

$$\mu\alpha = \frac{dH}{dy} - \frac{dG}{dz} - \mu\frac{dV}{dx},$$

where V is the magnetic potential due to the magnetism in the field both permanent and induced. From these equations we get

$$\mu\left\{\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right\}=\nabla^2\mathrm{H}-\frac{d}{dz}\left(\frac{d\mathrm{F}}{dx}+\frac{d\mathrm{G}}{dy}+\frac{d\mathrm{H}}{dz}\right)$$

$$=-4\pi\mu w+\frac{d}{dz}\left\{\tfrac{1}{2}(1-k)\nabla^2\psi-\chi\right\}$$

instead of the equation

$$\frac{d\alpha}{dy}-\frac{d\beta}{dx}=-4\pi w.$$

We have been obliged to introduce another assumption here, viz., that the magnetic force due to an element of current is given by Ampère's expression.

We could not assume Maxwell's way of connecting currents with magnetic force, viz. that the total current flowing through any closed curve is equal to the line integral of the magnetic force round the curve, for the result can only be true when the currents flow like an incompressible fluid.

Let us now go on to consider the force acting on the medium conveying the current.

If we consider a continuous distribution of currents, the kinetic energy

$$=\tfrac{1}{2}\iiint(\mathrm{F}u+\mathrm{G}v+\mathrm{H}w)\,dx\,dy\,dz.$$

If we derive the force parallel to x by the variation of the energy in the usual way we find, just as in Helmholtz's paper,[1] that the force parallel to x

$$=\left\{v\left(\frac{d\mathrm{G}}{dx}-\frac{d\mathrm{U}}{dy}\right)+w\left(\frac{d\mathrm{H}}{dx}-\frac{d\mathrm{U}}{dz}\right)-\mathrm{F}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right\},$$

or with our notation

$$=v\left\{\frac{d\mathrm{G}}{dx}-\frac{d\mathrm{U}}{dy}\right\}+w\left\{\frac{d\mathrm{H}}{dx}-\frac{d\mathrm{U}}{dz}\right\}-\mathrm{FP},$$

and that on any surface where there is a discontinuity in the values of u, v, w there is a force equal per unit of area to

$$\mathrm{F}\,\{l(u_1-u_2)+m(v_1-v_2)+n(w_1-w_2)\}$$

or $\mathrm{F}\Sigma$.

In the same paper it is shown that it follows from the principle of the Conservation of Energy that the force exerted by a distribution of currents equals the force given by Ampère's expression along with a force at the point $\xi\eta\zeta$ whose component parallel to the axis of x equals

$$\iiint\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\frac{x-\xi}{r^4}\Big(u'(x-\xi)+v'(y-\eta)+w'(z-\zeta)\Big)dx\,dy\,dz$$

$$+\iint\left\{l(u_1-u_2)+m(v_1-v_2)+n(w_1-w_2)\right\}\frac{x-\xi}{r^2}\Big(u'(x-\xi)$$

$$+v'(y-\eta)+w'(z-\zeta)\Big)d\mathrm{S}\,;$$

[1] *Die elektrodynamischen Kräfte in bewegten Leitern*, Crelle, lxxviii. p. 298. 1874, or *Gesammelte Werke*, vol. i. p. 733.

or with our notation

$$\iiint \mathrm{P}\frac{x-\xi}{r^4}\Big(u'(x-\xi)+v'(y-\eta)+w'(z-\zeta)\Big)\,dx\,dy\,dz$$

$$+\iint \Sigma\frac{x-\xi}{r^4}\Big(u'(x-\xi)+v'(y-\eta)+w'(z-\zeta)\Big)\,dS,$$

where u', v', w' are the components of the current at the point $\xi\,\eta\,\zeta$; so that in addition to Ampère's forces we have additional forces wherever P and Σ have finite values. From the above expressions we see that any element where P has a finite value exerts a repulsive force equal per unit of volume to

$$\frac{\mathrm{P}}{r}\,i\cos\theta,$$

tending from the element; where r is the distance of the element from the point at which the force is reckoned, i the intensity of the current at this point, and θ the angle between the direction of the current and r. Any element of surface where Σ has a finite value exerts a repulsive force equal per unit of surface to

$$\frac{\Sigma}{r}\,i\cos\theta,$$

where the notation is the same as before. Of course none of these forces exist in Maxwell's theory. They could be most easily detected in cases where the part of the forces given by Ampère's theory vanishes as it would for the case of an endless solenoid. In this case, though the Ampèrian forces vanish, the forces due to the discontinuity in the current do not, so that if the endless solenoid were to move under the action of external currents it would denote the existence of discontinuity in the current. An experiment of this kind has been made by Schiller; we shall discuss the results of it later.

To sum up, the differences between the most general theory which takes into account the action of the dielectric, and Maxwell's, are—

1. The existence of a normal wave in the general theory, but not in Maxwell's.
2. The difference in the velocity of propagation of the transverse wave.
3. The difference in the relation between electric currents and magnetic force.
4. The forces which arise from discontinuity in the currents.

The Experimental Evidence as to the Truth of the various Theories.

The theories we have considered may be divided into two great classes, according as they do or do not take into account the action of the dielectric surrounding the various conductors in the field. The first thing, therefore, that we have to do is to see whether experiment throws any light on this point.

When a dielectric is in an electric field it experiences a change in its structure; this is rendered evident by the alterations in its volume and elasticity observed by Quincke, by the change in its optical properties observed by Kerr, and also by the fracture of the dielectric when the field is made sufficiently intense. So that whenever an electromotive force acts on a dielectric it produces a change in its structure which we shall always speak of as polarisation. This, strictly speaking, has only been directly proved for electromotive forces produced by charges of statical electricity; but, unless we are prepared to say that the electromotive force due to statical electricity is in some way different from that due to a changing current, we must admit that when an electromotive force of the latter kind acts on a dielectric it polarises it. And we are not without experimental evidence that the electromotive force due to variations in the vector potential does produce some of the effects of the electromotive force due to a charge of statical electricity. Rowland's experiments have shown that a moving electrified body will set a magnet placed near to it in motion. It follows from this, by dynamical principles, that if we have the charged body initially at rest and move the magnet it will, if no other forces act upon it, be set in motion; so that in this case there is an electromotive force due to the motion of the magnet, *i.e.*, the variation in the vector potential produces the same effect on the electrified body as the electromotive force due to a charge of statical electricity. For this reason we shall suppose that the electromotive force due to the variation in the vector potential always produces effects on a dielectric on which it acts of the same type as those which have been observed to arise from the action of an electromotive force due to a charge of statical electricity.

Let us now consider a magnet surrounded by a dielectric. If we set the magnet in motion, we produce an electromotive force which polarises the dielectric. Let us, to fix our ideas, consider an element of the dielectric and the magnet. When the magnet moves it polarises the dielectric; it follows from dynamical principles (an extension of the principle of action and reaction),[1] that if the polarisation of the dielectric be altered, the magnet will move, so that a change in the polarisation of a dielectric produces a magnetic force.

Again, let us instead of the magnet consider a coil of wire conveying a current. A change in the rate of flow of the current produces a change in the polarisation of the dielectric; it follows that a change in the rate of change of the polarisation of the dielectric will produce a change in the current, *i.e.*, will produce an electromotive force.

It follows too, from dynamical principles, that as the change in the polarisation of an element of the dielectric due to the change in the current depends on the distance of the element from the current, there must be a force between the current and the element when the polarisation of the latter is changing. Thus we see that a change in the polarisation of the dielectric must produce all the effects of an ordinary conduction current, so that it is only absolutely necessary to consider how the experimental evidence affects those theories which take the action of the dielectric into account. As, however, the experiments which have been made are few in number, and are all concerned with interesting points, we shall consider them in their relation to all the theories, and not only to those which take the dielectric into account.

[1] See a paper by the author of this report 'On some Applications of Dynamical Principles to Physical Phenomena,' *Phil. Trans.*, 1885.

Schiller's Experiments.

The first experiment which we shall discuss is one made by Schiller, and described by him in Poggendorf's Annalen, vol. clix. pp. 456, 537; it was intended to test the potential theories of Neumann and Helmholtz. We saw that, according to these theories, in an unclosed circuit there are, in addition to the forces due to the elements of current, and which are expressed by Ampère's law, forces arising from the discontinuity of the currents at the ends of the circuit. If we have an end of a circuit where the current stops, and the electricity accumulates at the rate de/dt, it will exert on an element of current of length ds traversed by a current of intensity i a force tending to the end and equal to

$$A^2 i ds \frac{de}{dt} \frac{\cos\theta}{r}$$

where θ is the angle between the element of current and the radius drawn to it from the end. If we calculate from this expression the couple produced by an end on an endless solenoid, or on what is practically the same thing, a ring magnet, we shall find that the couple tending to turn the ring about an axis in its own place will not vanish, while the couple arising from the forces given by Ampère's law will. Thus if the ring rotates, as it should according to the potential theory, it must be from the action of the end.

In Schiller's experiment the end of the current was the end of wire connected with a Holtz machine. This was placed near to a ring magnet which was suspended by a long cocoon fibre; the magnet was protected from electrostatic influences by being enclosed in a metal box connected with the earth. Schiller determined the intensity of magnetisation of the ring magnet and the quantity of electricity passing through the point, and he calculated that if the potential theory were true, he ought to get a deflection of the magnet of about 27 scale divisions, instead of which there was no perceptible deflection.

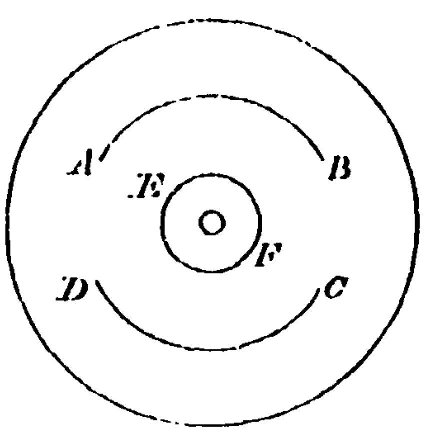
This experiment shows conclusively that the potential theory is wrong if we neglect altogether the action of the dielectric, and assume the current to stop at the end of the wire. If, however, we take the dielectric into account, the experiment tells us nothing as to whether Maxwell's theory or the more general one is true; for since the current from the Holtz machine is steady, as much electricity flows out from the end of the wires as arrives there; and thus there is really no discontinuity in the current, the only difference being that before reaching the end the current is flowing through copper and after passing it through air. The condition of things at the end of the wire remains steady, and thus the quantities which we denoted by P and Σ vanish.

The experiment might, however, be modified so as to be capable of distinguishing between the theories which take the dielectric into account. For suppose that, instead of letting the electricity escape through the point, we never let the potential at the end of the wire get so high as to allow the electricity to escape; then if the wire is initially uncharged, the condition at the end will be changing whilst the wire is charging up, and thus Σ will have a finite value; so that if the magnet were sufficiently delicate and remained undeflected, whilst the point was surrounded by dielectrics of all kinds, it would show that Maxwell's theory is correct.

I have calculated the effect which would be produced on Schiller's suspended magnet, and find that it is too small to be observed; as, however, the time of charging up the wire will be very small compared with the time of vibration of the magnet, the effect will be of the nature of an impulse, so that in this case there will be considerable advantage in having the moment of inertia of the suspended magnet small; while, as Schiller arranged the experiment, there was no such advantage, as the thing expected was a steady deflection. Thus if the ring magnet were retained it would be desirable to make the opening of it as small as possible, retaining the same cross action. I think the arrangement could be made sensitive enough to be deflected if the value of Σ were any considerable fraction of the rate of increase of the electricity at the end of the wire.

There is another way in which the continuity or discontinuity of the current might be tested, and which might perhaps be more delicate than the last. We saw on p. 141 that at any point of a current at which Σ had a finite value the mechanical force on the element is not at right angles to the element. In addition to the ordinary force at right angles to the element, there is a force in the direction of the vector potential equal in magnitude to the product of the values of the vector potential and Σ.

The existence of this force could be tested by an arrangement of the following kind:—



AB and CD are light movable segments of the same circle, having balls covered with paraffin A, B, C, D fastened to their ends. These segments are connected with a very light framework which can rotate about an axis perpendicular to the plane of the segments; the segments touch at their middle points contact-pieces which are connected with a Holtz machine. EF is the section of an electromagnet concentric with AB and CD; the whole is surrounded with a metal cylinder to screen it from external electric influences. When a current is passing through the electromagnet it produces a vector potential, whose direction is at right angles to the radius from O, the centre of the electromagnet perpendicular to its axis. Thus if Σ exists there will be a couple tending to twist the system AB, CD about its axis, but if Σ exists at all it will be when the electrical condition of the balls A, B, C, D is changing, so that unless the currents are continuous we should expect the system to rotate when the balls are being charged up. I have calculated that the system might easily be made sensitive enough to be sensibly deflected on charging or discharging, provided Σ is an appreciable fraction of the rate of change of the surface-density of the electricity on the balls.

Schiller's Second Experiment.[1]

Schiller has made another experiment, which shows that Ampère's theory fails for unclosed circuits. The first form of the experiment consisted in having a solenoid placed over a condenser one of whose plates could rotate about a vertical axis coinciding with the axis of the solenoid. One end of the solenoid was connected to one plate of the condenser and the other end to the other plate. When the solenoid is connected to a

[1] Pogg. *Ann.*, clix. p. 456; clx. p. 333.

battery the condenser will charge up and there will be radial currents of electricity in the plates; the current passing through the solenoid will produce a magnetic force which will, if Ampère's theory be true, act on the radial currents in the plate of the condenser and set it in rotation. Schiller found that this effect was too small to be observed, so he modified the experiment in the following way. Let us suppose that we have the two plates of the condenser rigidly attached to their axis and placed in a field symmetrical about its axis, in which the vertical component of the magnetic force is not uniform. Then if a current be sent through the upper plate, down through the axis, and out at the lower plate, the couple tending to twist the lower plate will not be equal and opposite to that tending to twist the upper one, as the magnetic force is not equal at the two plates, and thus the condenser will be set in rotation. Conversely, if the condenser be set in rotation in the magnetic field, and two electrodes of a galvanometer be connected with its axis, then if Ampère's theory be true there will be an electromotive force acting round the galvanometer circuit, which will produce a current, and this current could be much more easily detected than the rotation in the first form of the experiment. Schiller calculated the deflection which he ought to get if Ampère's theory were true, and found that he could easily detect it if it existed; as he was not able to see any deflection, we must conclude that Ampère's theory is not the true one.

It is easy to see that, according to the potential theory, there would be no current in the galvanometer; for, as everything is symmetrical about the axis, the potential is not altered by the rotation. The following calculation will show that, according to the dielectric theories, there should be no current through the galvanometer.

For if a, b, c are the components of magnetic induction, F, G, H those of the vector potential, X, Y, Z those of the electromotive force, then

$$\mathrm{X} = c\frac{dy}{dt} - b\frac{dz}{dt} - \frac{d}{dx}\left\{\mathrm{F}\frac{dx}{dt} + \mathrm{G}\frac{dy}{dt} + \mathrm{H}\frac{dz}{dt}\right\},$$

$$\mathrm{Y} = a\frac{dz}{dt} - c\frac{dx}{dt} - \frac{d}{dy}\left\{\mathrm{F}\frac{dx}{dt} + \mathrm{G}\frac{dy}{dt} + \mathrm{H}\frac{dz}{dt}\right\}.$$

Suppose the condenser is rotating with an angular velocity ω about the axis of Z; then the E.M.F. arising from one plate is, if R be its radius,

$$\omega\int_0^{\mathrm{R}} cr\,dr - \left(\mathrm{F}\frac{dx}{dt} + \mathrm{G}\frac{dy}{dt}\right),$$

Now

$$\mathrm{F}\frac{dx}{dt} + \mathrm{G}\frac{dy}{dt} = \omega\mathrm{R}\Theta,$$

where Θ is the component of the vector potential along the direction of motion of a point on the circumference of the plate of the condenser.

But the line integral of the vector potential round any curve equals the number of lines of magnetic force passing through it, so that, since the field is symmetrical,

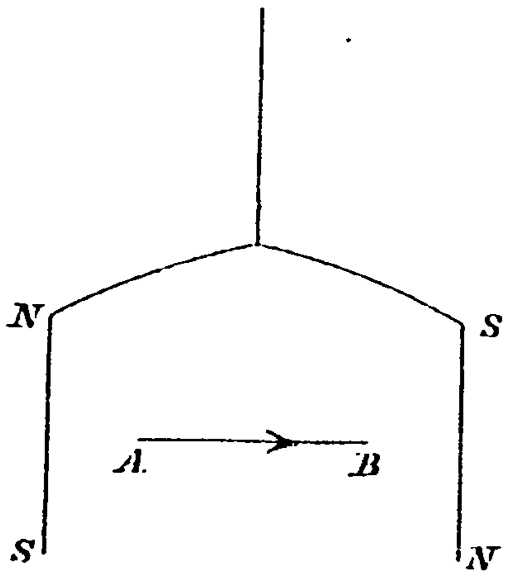
$$2\pi\int_0^{\mathrm{R}} cr\,dr = 2\pi\mathrm{R}\Theta.$$

From this equation we see that the E.M.F. due to the rotation vanishes for each plate, so that, according to this theory, there should be no current through the galvanometer.

This experiment of Schiller shows that both Grassmann's and Clausius' theories must be wrong, as well as Ampère's and Korteweg's, for we can easily see that they would make the disc rotate in the way in which Schiller first tried the experiment, and if this were so, it follows from dynamical principles that a current must be produced in the second form of the experiment.

This would seem to be the case even if we take into account the currents in the dielectric, unless we suppose that all the circuits are closed, for if all the circuits are closed then the disc will not rotate, as all the theories agree. If the circuits are not closed we may divide the currents in the disc into two parts, one part being of such magnitude as to form with the dielectric currents closed circuits; then the forces on this part and the dielectric will form a system in equilibrium; and there remains the other part of the currents, the action of the magnet on which ought to set the disc in rotation. Taking Schiller's experiments together, we may say that they show that the dielectric must be taken into account, and that some form of the potential theory is the only one of the theories we are considering which can give the expression for the forces due to a distribution of currents.

Although these two experiments of Schiller's show that of the theories we have discussed only the dielectric ones can be retained, we shall describe one or two more experiments which have been or could be made to distinguish between the various theories. Clausius' and Grassmann's theories lead to the same expression for the force between two elements of current, so that these theories stand or fall together. Grassmann in his paper[1] describes an experiment which would distinguish between his theory and Ampère's, or, in fact, any other except Clausius' which has ever been published.

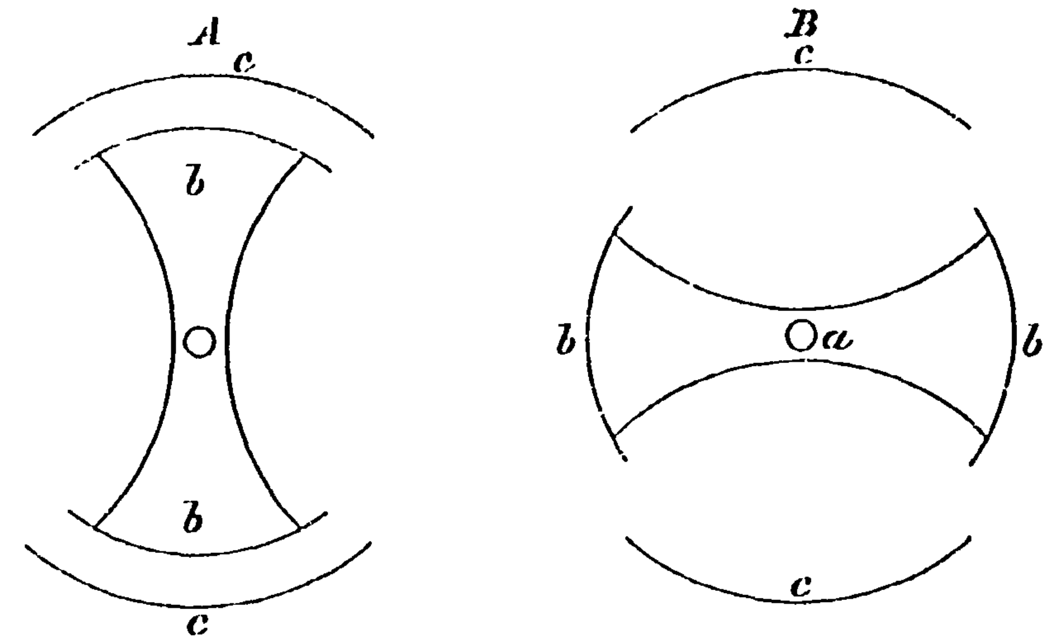


Suppose that NS and SN are two magnets whose north and south poles are denoted by N and S respectively, and that these magnets are fastened together by a rod NS, the system being suspended by a cocoon thread attached to the middle point of NS. Let AB be an unclosed circuit, say a wire joining the plates of a charged condenser; then, according to Grassmann's and Clausius' theories, the system will rotate in such a way that the sense of rotation is related to a vertical line drawn downwards like rotation and translation in a right-handed screw. According to every other theory it will rotate in the opposite direction.

Another experiment has been made by v. Helmholtz,[2] which shows that the potential theory leads to wrong results unless the action of the dielectric is taken into account. *bb* is a rotating conductor, to the ends of which large condenser plates are attached, which, when in rotation, come very near to the similar plates *c*, *c*. The plates *b* and *c* are segments

[1] Pogg., lxiv. 1, 1845. [2] *Wissenschaftliche Abhandlungen*, vol. . 783.

of coaxial cylinders. In v. Helmholtz's experiments *bb* was rotated between the poles of a powerful electromagnet. The plates *c*, *c* were connected with a commutator, which put them to earth when the rotating piece was in the position A, and to the plates of a Kohlrausch condenser when it was in the position B. Now suppose there is a difference of potential between *b* and *c*; suppose, for clearness, that *b* is at a higher potential than *c*, then when the rotating piece is in the position A the positive electricity goes to earth, and the negative is left to go to the Kohlrausch condenser, when the rotating piece gets to the position B. The change in this condenser was measured by a quadrant electrometer. v. Helmholtz found that the needle of the electrometer was deflected when the piece *bb* was rotating. Since everything is symmetrical about the axis of rotation, there would be no difference of potential between the



plates *b* and *c*, according to the potential law, if we neglect the action of the dielectric. According to Ampère's law there will be a difference of potential between *b* and *c* equal to $\Theta a\omega$, where a is the radius of the rotating piece, ω its angular velocity, and Θ the vector potential along the direction of motion of the disc. According to the dielectric theory there will also be the same difference of potential between *b* and *c* if we suppose that there is no discontinuity in the motion. We shall suppose that, instead of the velocity changing abruptly from ωa to zero as we pass from the rotating conductor to the dielectric, there is a layer of the dielectric next to the conductor in which the change of velocity is very rapid, one side of the layer moving with the velocity ωa, the other side being at rest. Then, using the same notation as before, we have—

$$X = c\frac{dy}{dt} - b\frac{dz}{dt} - \frac{d}{dx}\left\{F\frac{dx}{dt} + G\frac{dy}{dt} + H\frac{dz}{dt}\right\},$$

$$Y = a\frac{dz}{dt} - c\frac{dx}{dt} - \frac{d}{dy}\left\{F\frac{dx}{dt} + G\frac{dy}{dt} + H\frac{dz}{dt}\right\}.$$

Integrating across the thin layer of the dielectric, in which the velocity is changing rapidly, we see that the difference of potential between *b* and *c* equals

$$F\frac{dx}{dt} + G\frac{dy}{dt} + H\frac{dz}{dt},$$

where dx/dt, dy/dt, dz/dt are the velocities of a point on the boundary of the moving conductor. This equals $\Theta a\omega$, the same value as that given by Ampère's theory, so that in this case the two theories lead to identical results, which are in agreement with the result of Helmholtz's experiments.

Röntgen has recently published[1] a preliminary account of some experiments which seem to prove directly that the variations in the dielectric polarisation produce effects analogous to those due to a current.

This completes the account of the experiments which have been made to test the various theories. As the result of them we may say that they show that it is necessary to take into account the action of the dielectric, but they tell us nothing as to whether any special form of the dielectric theory, such as Maxwell's or Helmholtz's, is true or not.

I have described two experiments which would decide whether Maxwell's theory that all circuits are closed is true or not. It seems to me, however, that even if Maxwell's theory be wrong, Helmholtz's is not the only alternative. I have given a sketch of a theory in which I have tried to make as few assumptions as possible; all that I have assumed is that when a dielectric is acted on by a changing electromotive force, it behaves like a conductor conveying a current whose intensity is proportional to the rate of change of the electromotive force. We know from experiment that it produces effects of the same character, and I have assumed as the simplest assumption I could make that for the same dielectric the equivalent current is proportional to the rate of change of the electromotive force, so that equivalent current $= \eta$ (rate of change of electromotive force).

Both Maxwell and Helmholtz assume that η depends only on the specific inductive capacity of the dielectric, but I think it is preferable, until we have more experiments on this point, to look on η as the measure of a new property of a dielectric, and not to assume that it is merely a function of the specific inductive capacity, the only experimental evidence for this being the by no means perfect agreement between the refractive index and the reciprocal of the square root of the specific inductive capacity. To prove Maxwell's theory of closed circuits it would not be sufficient to prove that for one medium, say air, $\eta = K/4\pi$, for it is quite conceivable that electrical phenomena may be simpler in a dielectric like air, where the electrical behaviour of the ether seems to be but little affected by the presence of the dielectric, than in such a one as glass or other substance possessing a comparatively large specific inductive capacity, when the effect of the ether is seriously modified by the presence of the medium.

Since in the theory I have sketched the values of

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz},$$

and

$$l\,(u_1 - u_2) + m\,(v_1 - v_2) + n\,(w_1 - w_2)$$

are not zero, but arbitrary, inasmuch as they involve η, in order to find the value of the force between two circuits where there is any discontinuity in the currents we shall require to know the value of the quantity k which occurs in v. Helmholtz's theory.

The most pressing need in the theory of electrodynamics seems to

[1] *Phil. Mag.*, May, 1885.

be an experimental investigation of the question of the continuity of these dielectric currents; we have experimental proof that they exist, but we do not know whether Maxwell's assumption that they always form closed circuits with the other currents is true or not. If Maxwell's assumption should turn out to be true, we should have a complete theory of electrical action; if, on the other hand, it should turn out to be wrong, then we should have to go on to determine the quantity k. This quantity is difficult to determine, as its influence on all closed circuits disappears. It influences, as v. Helmholtz has shown, the rate of propagation of the electric potential along conducting wires, and I think we can see that it would influence the time of oscillation of an irregular distribution of electricity over a conducting shell. The easiest way, however, of determining this quantity would seem to be the straightforward one of measuring electrostatically the value of the electromotive force due to a variation in the charge of a condenser; the expression for the vector potential, as we saw on p. 140, involves k, so that if we measure the electromotive force, which is equal to the rate of variation of the vector potential, we shall determine the value of the vector potential, and consequently of k.

APPENDIX I.

Since the Report was written I have had through the kindness of the author an opportunity of seeing the advance proofs of a paper by Professor J. H. Poynting, of Mason's College, Birmingham, 'On the Connexion between Electric Current and the Electric and Magnetic Induction in the Surrounding Medium,' which is about to appear in the 'Philosophical Transactions.'

The views expressed in this paper are rather a new way of looking at Faraday and Maxwell's theory than a new theory of electrodynamic action, as however it brings the action of the dielectric into great prominence it is instructive to consider it.

The paper is largely based on a previous one by the same author on the 'Transference of Energy in the Electromagnetic Field,'[1] it is therefore necessary to give a brief account of this paper.

In it the author shows that the rate of increase of the energy inside any closed surface equals

$$\frac{1}{4\pi}\iint\{l(\mathrm{R}'\beta - \mathrm{Q}'\gamma) + m(\gamma \mathrm{P}' - a\mathrm{R}') + n(a\mathrm{Q}' - \beta \mathrm{P}')\}\,d\mathrm{S},$$

where dS is an element of surface, l, m, n the direction cosine of the normal to dS, a, β, γ the components of magnetic induction, and P′, Q′, R′ given by the following equations:—

$$\mathrm{P}' = -\frac{d\mathrm{F}}{dt} - \frac{d\psi}{dx},$$

$$\mathrm{Q}' = -\frac{d\mathrm{G}}{dt} - \frac{d\psi}{dy},$$

$$\mathrm{R}' = -\frac{d\mathrm{H}}{dt} - \frac{d\psi}{dz},$$

[1] *Phil. Trans.*, 1884, part ii.

where F, G, and H are the components of the vector potential and ψ the electrostatic potential; thus if the medium is at rest P′, Q′, R′ are the components of the electromotive force at the point.

Professor Poynting interprets this equation to mean that the components parallel to the axes of x, y, z of the flow of energy across *each* element of surface are respectively

$$\frac{1}{4\pi}(R'\beta - Q'\gamma),$$

$$\frac{1}{4\pi}(P'\gamma - R'\alpha),$$

$$\frac{1}{4\pi}(Q'\alpha - P'\beta),$$

so that according to this view the energy flows in the direction which is at right angles both to the magnetic and electromotive forces, and in the direction in which a right-handed screw would move if turned round from the positive direction of the electric intensity to the positive direction of the magnetic intensity; the quantity of energy crossing in unit time unit surface at right angles to this direction being

$$\frac{1}{4\pi} \cdot \text{Electromotive force at the point} \times \text{magnetic force} \times \text{sine of the angle between these forces.}$$

This interpretation of the expression for the variation in the energy seems open to question. In the first place it would seem impossible *à priori* to determine the way in which the energy flows from one part of the field to another by merely differentiating a general expression for the energy in any region with respect to the time, without having any knowledge of the mechanism which produces the phenomena which occur in the electromagnetic field: for although we can by means of Hamilton's or Lagrange's equations deduce from the expression for the energy the forces present in any dynamical system, and therefore the way in which the energy will move, yet for this purpose we require the energy to be expressed in terms of coordinates fixing the system, and it will not do to take any expression which happens to be equal to it. The problem of finding the way in which the energy is transmitted in a system whose mechanism is unknown seems to be an indeterminate one; thus, for example, if the energy inside a closed surface remains constant we cannot unless we know the mechanism of the system tell whether this is because there is no flow of energy either into or out of the surface, or because as much flows in as flows out. The reason for this difference between what we should expect and the result obtained in this paper is not far to seek. Though the increase in the energy inside a closed surface equals

$$\frac{1}{4\pi}\iint\{l(R'\beta - Q'\gamma) + \ldots\}dS,$$

it does not follow that the components of the flow of energy across *each* element of surface are $(R'\beta - Q'\gamma)/4\pi$, &c., for we can find quantities u, v, w which are of the dimensions of rate of change of energy per unit area, and for which

$$\iint(lu + mv + nw)\,dS = 0.$$

The following values of u, v, w satisfy this condition :—

$$u = \frac{1}{\mu}\left\{\frac{d^2}{dy\,dt}(\mathrm{FG}) - \frac{d^2}{dz\,dt}(\mathrm{HF})\right\},$$

$$v = \frac{1}{\mu}\left\{\frac{d^2}{dz\,dt}(\mathrm{GH}) - \frac{d^2}{dx\,dt}(\mathrm{FG})\right\},$$

$$w = \frac{1}{\mu}\left\{\frac{d^2}{dx\,dt}(\mathrm{HF}) - \frac{d^2}{dy\,dt}(\mathrm{GH})\right\},$$

where μ is the magnetic permeability and F, G, H are the components of the vector potential, or if ψ be the electrostatic potential

$$u = \frac{d}{dy}\left\{\frac{d\psi}{dz}\mathrm{H}\right\} - \frac{d}{dz}\left\{\frac{d\psi}{dy}\mathrm{G}\right\},$$

$$v = \frac{d}{dz}\left\{\frac{d\psi}{dx}\mathrm{F}\right\} - \frac{d}{dx}\left\{\frac{d\psi}{dz}\mathrm{H}\right\},$$

$$w = \frac{d}{dx}\left\{\frac{d\psi}{dy}\mathrm{G}\right\} - \frac{d}{dy}\left\{\frac{d\psi}{dx}\mathrm{F}\right\},$$

If the values of u, v, w which satisfy these conditions be denoted by the (u_1, v_1, w_1), (u_2, v_2, w_2) . . . then the flow across any element of surface might have for its x component—

$$\frac{1}{4\mu}(\mathrm{R}'\beta - \mathrm{Q}'\gamma) + \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \ . \ . \ .$$

where λ_1, λ_2, λ_3 are arbitrary constants, thus we see that the components of the flow of energy, instead of being uniquely determined by this process are really left quite indeterminate by it. Though this is so, it is very instructive to follow Professor Poynting's description of the way in which the energy flows in some special cases; we shall select a very simple one, the case of a current flowing along a straight wire. Here the lines of electromotive force are straight lines parallel to the wire, the lines of magnetic force are circles with their centres on the wire, and their planes at right angles to it. Then, since according to the view expressed in the paper, the energy moves at right angles both to the electric and magnetic forces, it must in this case move radially inwards to the wire where it is converted into heat. The energy, instead of being supposed to be transmitted through the wire, is regarded as transmitted by the dielectric; and though we may not regard the exact law of flow of the energy as established, still it is very important that this view should be brought into prominence. Another important point brought prominently forward in this paper is the view that magnetic force is always the sign of transference of energy, according to Professor Poynting; indeed, there must be transference of energy from one part of the field to another to give rise to magnetic force. Thus, according to his view, no magnetic force would be exerted by the discharge of a leaky condenser, because in this case he considers the energy to be confined to the space between the plates of the condenser and to be converted into heat where it stands. If the plates were connected by a metallic wire, the energy could flow out and be converted into heat in the wire and this motion of energy would give rise to magnetic forces, so that magnetic forces would be produced by the discharge of a condenser in this way, but not by leakage. In this case the theory differs from Maxwell's, as according to that theory the alteration in the electromotive force would produce magnetic forces in either case.

In Professor Poynting's second paper, which we have already mentioned, the fundamental principles of electrodynamics are described as the results of the motion of the tubes of electromotive and magnetic force. Maxwell develops electrodynamics from the principles :—

1st. That the total electromotive force round any closed curve is equal to the rate of decrease of the total magnetic induction through the curve.

2nd. The line integral of the magnetic force round any closed curve is equal to 4π times the current through the curve.

Professor Poynting restates these principles in the following way :—

1. 'Whenever electromotive force is produced by change in the magnetic field, or by motion of matter through the field, the E.M.F. per unit length is equal to the number of tubes of magnetic induction cutting or cut by the unit length per second, the E.M.F tending to produce induction in the direction in which a right-handed screw would move if turned round from the direction of motion relatively to the tubes towards the direction of the magnetic induction.'

The second principle he states in the following way :—

'Whenever magnetomotive force is produced by change in the electric field, or by motion of matter through the field, the magnetomotive force per unit length is equal to 4π times the number of tubes of electric induction cutting or cut by unit length per second, the magnetomotive force tending to produce induction in the direction in which a right-handed screw would move if turned round from the direction of the electric induction towards the direction of motion of the unit length relatively to the tubes of induction.'

By magnetomotive force is meant the line integral of the magnetomotive force round a tube of induction. This statement includes the more special one that the line integral of the magnetic force round any closed curve is equal to 4π times the number of tubes passing in or out through the curve per second.

The development of these principles leads to equations which are practically the same as those obtained by Maxwell, the chief difference being that the quantity corresponding to Maxwell's J is no longer arbitrary or rather redundant.

Professor Poynting also introduces into his equations the time integrals of the components of the magnetic force as fundamental quantities, and regards the components of the magnetic as the differential coefficients of these quantities with regard to the time. This method of representing magnetic force was also used by Professor Fitzgerald in his paper on the Electromagnetic Theory of the Reflection and Refraction of Light.[1] It has the advantage of calling attention to the dynamic character of magnetic phenomena. In Professor Poynting's paper some of the applications of his method of regarding electrical phenomena are worked out with great detail for some of the simpler cases.

[1] *Phil. Trans.*, 1880, part ii.

Appendix II.

ON THE STRESS IN THE DIELECTRIC.

In the preceding Report we have had so frequently to refer to the action of the dielectric that it may be convenient to give a very brief account of the work which has been done on the stresses which are supposed to exist in it. We shall confine ourselves to the work which has been done on the stresses in the electrostatic field; those existing in the electromagnetic field are of a similar nature, so that any remark applying to one will also apply to the other. The idea of explaining the forces in the electrostatic field by means of stresses in the dielectric seems to be due to Faraday, who describes [1] the stress in the medium by saying that the lines of force tend to contract and also to repel one another. The magnitude and distribution of this stress was investigated by Maxwell; [2] he found that in a medium whose specific inductive capacity was K, and at a point where the electromotive force is R, a tension equal to $KR^2/8\pi$ per unit area along the lines of force combined with a pressure of the same amount at right angles to these, would produce the effects observed in the electrostatic field, that is, at a point in a dielectric, the resultant of these stresses would be a force whose components, parallel to the axes of x, y, z, are eX, eY, eZ respectively, e being the charge of electricity at the point, and X, Y, Z the components of the electromotive force. It may be observed that this system of stress could not be produced by the strain in an elastic solid at rest: this points to the kinetic origin of electrostatic phenomena.

These stresses are in equilibrium at a point in a dielectric where there is no free electricity. At the junction of two media, whose specific inductive capacities are K_1 and K_2, and in which the electromotive forces are R_1 and R_2, and whose interface is perpendicular to the lines of forces, the stresses are not in equilibrium, but there is an unbalanced stress $(K_1 R_1^2 - K_2 R_2^2)/8\pi$ which will tend to make the boundary move towards the medium whose specific inductive capacity is K_1; if these dielectrics are liquids, their interface may become curved so that the forces due to surface tension balance this stress.

Quincke,[3] who has experimentally investigated the effects of electrification on various dielectrics, such, for example, as the effects on the glass of a Leyden jar, has found that the effects on different bodies are very different; he finds, for example, that though the effect of the electrification on the dielectric of the Leyden jar is generally to produce an expansion, yet in some substances, such as the fatty oils, contraction takes place.[4] This diversity in the effects of electrification on different dielectrics shows that the distribution of stress cannot be so simple as was supposed by Maxwell. It also shows that there must be forces in the electric field which are not recognised either by Maxwell's theory or the theory of action at a distance. More general theories have been given in order to meet this difficulty.

[1] *Experimental Researches*, § 1297.

[2] *Electricity and Magnetism*, 2nd edition, p. 149.

[3] *Wied. Ann.*, x. pp. 161, 374, 513; *Ibid.*, ix. p. 105; *Phil. Mag.*, vol. x. p. 30 (1880).

[4] The fatty oils are also an exception to the rule that the index of refraction equals the square root of the specific inductive capacity.