Measuring the Abundancy of Integers

RICHARD LAATSCH
Miami University
Oxford, OH 45056

The origins of the study of perfect numbers are lost in antiquity, but the concept was clearly recognized well over 2000 years ago and involves such contributors as Euclid, Fermat, Descartes, Mersenne, Legendre, and Euler. The classification of positive integers as perfect, abundant, or deficient is also an ancient idea, one which dates from before A.D. 100. It is the goal of this note to make explicit an idea used by researchers since the seventeenth century (and perhaps before) and reproduce, comment on, and extend their ideas. For the newcomer to these concepts, the basic definitions are as follows: A **perfect number** is a positive integer which is equal to the sum of its proper divisors ($1 + 2 + 3 = 6$; $1 + 2 + 4 + 7 + 14 = 28$), an **abundant number**, is one for which the sum of the proper divisors is greater than the number ($1 + 2 + 3 + 4 + 6 > 12$), and a **deficient number** has the sum of its proper divisors less than the number ($1 + 2 + 5 < 10$).

As the theory of this classification of numbers developed, investigators realized that it was computationally easier to make use of $\sigma(N)$, the sum of *all* the divisors of an integer N, including N itself. Thus a number N is abundant if $\sigma(N) > 2N$, perfect if $\sigma(N) = 2N$, and deficient if $\sigma(N) < 2N$.

Certainly the prime 41 is very deficient since the only divisors of 41 are 1 and 41, and $1 + 41$ is much less than $2 \cdot 41$, whereas 8 is, relatively speaking, just barely deficient since $1 + 2 + 4 + 8$ is close to $2 \cdot 8$. On the other hand, 360 is very abundant since the sum of its divisors, 1170, is greater than $3 \cdot 360$. By such examples we are led to a natural measure of the abundancy or deficiency of numbers.

DEFINITION. The **abundancy index** of a positive integer N is the number

$$I(N) = \frac{\sigma(N)}{N}.$$

For example, $I(41) = 42/41 = 1.024^+$, $I(8) = 15/8 = 1.875$, $I(360) = 1170/360 = 3.25$, and $I(6) = 12/6 = 2$. Thus N is perfect if $I(N) = 2$, deficient if $I(N) < 2$, and abundant if $I(N) > 2$. When $I(N)$ is an integer $r > 2$, N is said to be **multiperfect of index** r. For example, 120 is multiperfect of index 3 since $I(120) = 3$ (the smallest such example); also $I(30240) = 4$. The study of multiperfects was pursued by Descartes and others and blossomed around 1900 [**7**]; an excellent bibliography can be found in [**12**]. We shall return to the topic later, but first we establish a couple of easy theorems about the abundancy index.

THEOREM 1. *If* $k > 1$, *then* $I(kN) > I(N)$.

Proof. If $1, a_1, a_2, \dots, a_t, N$ are the divisors of N, then $1, k, ka_1, ka_2, \dots, ka_t, kN$ is a (perhaps incomplete) list of divisors of kN. Thus,

$$\begin{aligned} I(kN) &\geqslant (1 + k + ka_1 + ka_2 + \cdots + kN)/kN \\ &= 1/kN + (1 + a_1 + a_2 + \cdots + N)/N \\ &= 1/kN + I(N) > I(N). \end{aligned}$$

This theorem captures the 13th century assertions that every multiple of a perfect number is abundant and every divisor of a perfect number is deficient. It also assures us that there are infinitely many abundant numbers (all multiples of 6 are abundant) just as there are infinitely many deficient numbers (including all of the primes). What the theorem doesn't tell us is which multiples of a given N will have the larger abundancy indexes. When $N = 8$,

$$I(5 \cdot 8) > I(11 \cdot 8) > I(2 \cdot 8) > I(8).$$

We will soon make several observations about the way in which $I(N)$ grows.

The existence of arbitrarily large primes guarantees that the abundancy index has 1 as its lower bound, since

$$I(P) = \frac{P+1}{P} = 1 + \frac{1}{P}$$

for primes P. A little less obvious is the fact that the abundancy index takes on very large values.

THEOREM 2. *The function I is not bounded above.*

Proof. Let N be an integer divisible by all of the integers $1, 2, \ldots, k$. (One such choice is $N = k!$, but smaller choices are possible if $k \geqslant 4$.) Then

$$\sigma(N) \geqslant N + N/2 + N/3 + \cdots + N/k$$

and so

$$I(N) = \sigma(N)/N \geqslant 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}.$$

Since this is a partial sum of the divergent harmonic series, $I(N)$ can be made as large as desired.

This proof, although constructive, leads quickly to very, very large values of N. For example, $\sum_{i=1}^{n} 1/i$ first exceeds 4 when $n = 31$ and the smallest integer divisible by all positive integers less than or equal to 31 is about $7 \cdot 10^{13}$. However, the least integer N with $I(N) \geqslant 4$ is 27720.

We have now established that the numbers $I(N)$, all of which are rational numbers, are scattered throughout the interval $(1, \infty)$ of the real line. Before discussing how they are distributed, let us write an explicit formula for $I(N)$.

It is noted in every elementary book on number theory (see, e.g., [**8**]) that, if P is prime, then the sum of the divisors of P^n is the sum of a geometric progression

$$\sigma(P^n) = 1 + P + P^2 + \cdots + P^n = \frac{P^{n+1} - 1}{P - 1}, \tag{1}$$

and that, if $N = P^r \cdot Q^s \cdot \cdots \cdot V^t$ with $P, Q, \ldots, V$ distinct primes, then

$$\sigma(N) = \sigma(P^r) \cdot \sigma(Q^s) \cdot \cdots \cdot \sigma(V^t). \tag{2}$$

Combining (1) and (2), we obtain formula (3) below for the abundancy index $I(N)$.

THEOREM 3. *If $N = P^r \cdot Q^s \cdot \cdots \cdot V^t$ is the prime power decomposition of N, then*

$$\begin{aligned} I(N) &= \frac{1 + P + \cdots + P^r}{P^r} \cdot \frac{1 + Q + \cdots + Q^s}{Q^s} \cdot \cdots \cdot \frac{1 + V + \cdots + V^t}{V^t} \\ &= \frac{P^{r+1} - 1}{P^r(P-1)} \cdot \frac{Q^{s+1} - 1}{Q^s(Q-1)} \cdot \cdots \cdot \frac{V^{t+1} - 1}{V^t(V-1)}. \end{aligned} \tag{3}$$

We will explore here some elementary consequences of this formula; there are many additional properties of $\sigma(N)$ and $I(N)$ which involve inequalities and limits. One particularly intriguing result is that the limit of the "average" value of $\sigma(N)/N$ as N increases exists and equals $\pi^2/6$ [**13**, p. 226]. Other results of this type can be found in [**9**] and [**20**].

Prime factors and index growth

We begin our analysis of the behavior of the abundancy index by observing that, for a prime P and positive integer n,

$$I(P^n) = \frac{P^{n+1} - 1}{P^n(P-1)} = \frac{P - 1/P^n}{P - 1} \tag{4}$$

is a decreasing function of P when n is fixed, but it is an increasing function of n when P is fixed. In particular, for a fixed P, the sequence

$$\{I(P^n)\}_{n=1}^{\infty}$$

has for its first term $I(P)=(P+1)/P$ and increases to the limit $P/(P-1)$ as n becomes infinite. Thus for $P=5$, $6/5 \leqslant I(5^n) < 5/4$ for all n; and for $P=71$, $72/71 \leqslant I(71^n) < 71/70$, a very narrow range of values (about .0002 in width)—all slightly larger than 1. In other words, the contribution of a prime the size of 71 to the abundancy index of a number N having 71 as a factor is limited and does not depend strongly on the number of times the prime occurs as a factor of N. The contribution of a still larger prime is proportionally less. In fact, for primes P larger than 41, the fourth term of the sequence $I(P^4)$ agrees with the limit $P/(P-1)$ of the sequence to 8 significant digits. Note also that the ranges of these sequences (and their intervals) are disjoint for any two distinct primes P.

Given a number N, if the goal is to multiply N by a prime which is not one of its factors in order to increase the size of the abundancy index, it is clear that to get a maximum increase, the best choice is the smallest such prime. Sometimes, however, a bigger increase in the index will occur if N is multiplied by one of its prime factors. For example, given $M=360360=2^3\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13$, you could multiply M by 2, 3, 5, or 17. It turns out that 17 is the best choice since $I(17M)=3.62+$, $I(2M)=I(5M)=3.50+$, and $I(3M)=3.47+$. In contrast, given $L=90=2\cdot 3^2\cdot 5$, the choice that maximizes the index is 2 since $I(2L)=91/30$ and $I(7L)=104/35$.

In assessing the maximal contribution of the primes in the prime factorization of N to the abundancy index $I(N)$, we will make extensive use of the following theorem, which follows immediately from Theorem 3 and equation (4).

THEOREM 4. *If $N=P^r\cdot Q^s\cdot \cdots \cdot V^t$ with $P,Q,\dots,V$ distinct primes, then the least upper bound of the sequence $I(N^n)$ is*

$$\frac{P}{P-1}\cdot\frac{Q}{Q-1}\cdot\cdots\cdot\frac{V}{V-1}. \tag{5}$$

Theorem 4 yields several interesting small results. For example, *if N is an even integer but not a power of 2, then N^n is abundant for some n*. To see this, let $N=2PT$, where P is any odd prime factor of N. Then

$$\lim_{n\to\infty} I\big((2P)^n\big)=\frac{2}{2-1}\cdot\frac{P}{P-1}=\frac{2P}{P-1}>2,$$

so that some exponent r exists for which $I((2P)^r)>2$. But then, invoking Theorem 1,

$$I(N^r)=I\big((2P)^rT^r\big)\geqslant I\big((2P)^r\big)>2.$$

Similarly, *if N is not a power of 2, then 2^nN is abundant for some exponent n.*

For odd integers, we can show that *if P and Q are distinct odd primes, then $(PQ)^n$ is deficient for all n.* Since I is a decreasing function of P and 3 and 5 are the smallest odd primes, we only need to note that

$$\lim_{n\to\infty} I\big((3\cdot 5)^n\big)=(3/2)\cdot(5/4)=15/8<2.$$

From this it follows that not only is the product of any two distinct odd primes deficient, but its abundancy index is smaller than $15/8=1.875$. On the other hand,

$$\lim_{n\to\infty} I\big((3\cdot 5\cdot 7)^n\big)=35/16>2,$$

so there are abundant odd integers with three distinct prime factors. Thus we have a 1913 theorem of L. E. Dickson [**6**]: *Every abundant odd integer has at least three distinct prime factors.* The smallest odd abundant number turns out to be $945=3^3\cdot 5\cdot 7$. We note the dominant role played by the smaller primes in this kind of result: if an odd number is not divisible by either 3 or 5, then in order to be abundant, that number must have at least fifteen distinct prime factors—a fact which can be verified in a couple of minutes using Theorem 4 and a calculator.

It is an interesting historical anomaly that the first four perfect numbers (6, 28, 496, 8128) were all known by A.D. 100, and probably much earlier, but that the 13th century found scholars still unaware that odd abundant numbers such as 945 existed. The difference is partly explained by the fact that Euclid published in his *Elements* a formula for even perfect numbers.

Using formula (4), it is a simple matter to extend Dickson-type results—especially with the aid of a computer. We have prepared TABLE A, which gives, in tabular form, pairs of numbers J and K for the following statement: *Every number N with $I(N) \geqslant J$ must have at least K distinct prime factors.*

N Even		N Odd	
J	K	J	K
2	2	2	3
3	3	3	8
4	4	4	21
5	6	5	54
6	9	6	141
7	14	7	372
8	22	8	995
9	35	9	2697
10	55	10	7397
11	89	11	20502
12	142	12	57347
13	230		
14	373		
15	609		
16	996		
17	1637		
18	2698		
19	4461		
20	7398		
21	12301		
22	20503		
23	34253		
24	57348		

TABLE A. **Every number N with $I(N) \geqslant J$ must have at least K distinct prime factors.**

Given J, the number K is computed as follows:

$$\begin{aligned} &\text{for } n \text{ even:} \quad K = \min\left\{ n : \prod_{i=1}^{n} \frac{P_i}{P_i - 1} > J \right\}, \\ &\text{for } n \text{ odd:} \quad K = \min\left\{ n : \prod_{i=2}^{n} \frac{P_i}{P_i - 1} > J \right\} \end{aligned} \tag{6}$$

where P_i is the ith prime in the natural ordering of the primes ($P_1 = 2$, $P_2 = 3$, $P_3 = 5$, etc.).

The first few entries in TABLE A were known to R. D. Carmichael [4] in 1907; his paper explicitly states formula (5). Also Paul Poulet [**19**] in 1929 gave the first seven entries in the "even" table. Recently, computers have been called upon to generate this and similar tables (see, for example, [**18**]).

While compiling TABLE A for N even, it became apparent that there was a pattern in the entries in the K-column. The sequence of first differences

$$\{1,1,2,3,5,8,13,20,34,53,88,143,236,387,\ldots\}$$

bears a resemblance, which is too obvious to be ignored, to the Fibonacci sequence

$$\{1,1,2,3,5,8,13,21,34,55,89,144,233,377,\ldots\}$$

Thus, with K as defined in (6), it is natural to conjecture: If $K(J)$ is the least integer such that

$$\prod_{i=1}^{K(J)} \frac{P_i}{P_i - 1} > J$$

and f_J is the Jth Fibonacci number ($f_1 = f_2 = 1$), then $K(J+2) - K(J+1) \sim f_J$ in the sense that

$$\lim_{J\to\infty} [K(J+2) - K(J+1)]/f_J = 1.$$

This is an attractive conjecture because it relates a product of primes to the additively defined Fibonacci sequence.

Although the conjecture seems plausible for small values of J (from 2 to 16), in the range of values of J from 17 to 24 the ratio $[K(J+2) - K(J+1)]/f_J$ increases. (It is 1.304 for the last value, $J = 22$, which TABLE A allows us to compute.) The conjecture is probably false, but it may be possible to resuscitate a version of it by taking equal steps of some number $J < 1$ rather than integer steps, or by making a similar adjustment. Alternatively, perhaps $[K(J+2) - K(J+1)]/f_J$ approaches *some* limit—which, if true, would also be an interesting result.

It is reasonable to question the value of the numbers in the latter part of TABLE A. It is unlikely that anyone will ever want to exhibit a number N with abundancy index greater than 20, so the knowledge that N must contain at least 7398 distinct prime factors (all of the primes 2 through 75037, or worse!) is unlikely to be translated into a blueprint for finding N. Even for smaller values of the index there is no assurance that the *smallest* integer S with index $I(S) \geqslant J$ is obtained using the value of K given by TABLE A. In TABLE B, we give a few "smallest" integers S for given integer values of J and compare L, the number of distinct primes which actually occur in the factorization of S, with the number K in TABLE A.

J	S even	K	L
2	$6 = 2 \cdot 3$	2	2
3	$120 = 2^3 \cdot 3 \cdot 5$	3	3
4	$27720 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	4	5
5	$122522400 = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$	6	7
	S odd		
2	$945 = 3^3 \cdot 5 \cdot 7$	3	3
3	$1018976683725 = 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$	8	9

TABLE B. **The smallest integer S with $I(S) \geqslant J$.**

A much more extensive table appears in [**1**], where Alaoglu and Erdös extended some work initiated by Ramanujan and, as a part of their efforts, tabulated the first 74 **superabundant numbers**—integers whose abundancy indexes are greater than those of all smaller integers.

One final note on the growth of the abundancy index: After a certain level has been reached by the index, those index values do not occur with increasing frequency. For example, the index first reaches 4 at 27720, is equal to, but not greater than, 4 twice between there and 50000, and the number of integers with index greater than 4 in the successive intervals of length 50000 between 50000 and 500000 are, respectively, 10, 10, 11, 15, 12, 10, 12, 12, and 11. Some quantitative results on the density $A(x)$, which is the limit as $N \to \infty$ of the relative frequency of occurrence of integers N with $I(N) \geqslant x$, can be found in [**21**]. A lively and entertaining discussion of these statistics is coupled with comments on the history of perfect and abundant numbers in an article by David G. Kendall [**15**] in which philosophical attitudes and applications—ancient, medieval, and his own—are considered. One of the interesting aspects of Kendall's article is the evidence he gives to support the conjectures that 1) the density $A(2)$ of the nondeficient numbers is $1/4$, or 2) the density of the nondeficient even numbers is $1/4$. We can use TABLE B to show that these conjectures cannot both be true, for the density of odd nondeficients is at least $1/1890$ because every odd multiple of 945 is odd and abundant.

The set of abundancy indexes

Near the end of the first section of this paper, we deferred the question of the nature of the set of abundancy indexes

$$D = \{ I(N) : N \geqslant 2 \}.$$

Here we resume that discussion.

Our immediate goal is to show that this set is dense in the interval $(1, \infty)$. We first examine the behavior of the product $\prod(P_i + 1)/P_i$, where P_i is the ith prime. The behavior is quickly established, for

$$\lim_{n \to \infty} \prod_{i=1}^{n} \frac{P_i + 1}{P_i} = \infty \Leftrightarrow \lim_{n \to \infty} \prod_{i=1}^{n} 1 + \frac{1}{P_i} = \infty \Leftrightarrow \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{P_i} = \infty .$$

This second equivalence is a special case of the basic theorem on infinite products, that $\prod(1 + a_i) = \infty$ if, and only if, $\sum a_i = \infty$ (see [**14**] or [**16**]). However, $\sum 1/P_i$ is a known divergent series ([**17**, or **13**, p. 17). Thus $\prod(P_i + 1)/P_i$ is a divergent product whose partial products $\prod_{i=1}^{n}(P_i + 1)/P_i$ form a strictly increasing sequence and whose component factors $(P_i + 1)/P_i$ form a strictly decreasing sequence converging to 1. These observations yield an alternate proof of Theorem 2 and make possible the following theorem.

THEOREM 5. *The set* $D = \{I(N): N \geqslant 2\}$ *is dense in the interval* $(1, \infty)$.

Proof. The method of proving this theorem is comparable to approximating a positive number by a subseries of $\sum 1/n$. We generate an increasing sequence $\{I(N_k)\}$ converging to an arbitrary number $x > 1$, as follows: let $N_0 = P_1 P_2 \cdots P_t$, where t is the largest integer such that

$$I(N_0) = \prod_{i=1}^{t} (P_i + 1)/P_i \leqslant x.$$

(Since $I(P_1) = I(2) = 3/2$, this construction works only if $x \geqslant 3/2$, so we arbitrarily set $I(N_0) = 1$ if $x < 3/2$.) Let $d_1 = x/I(N_0)$; let Q_1 be the smallest prime such that $Q_1 > P_t$ and $(Q_1 + 1)/Q_1 \leqslant d_1$; and let $N_1 = N_0 Q_1$. Now $I(N_0) < I(N_1) \leqslant x$. Next let $d_2 = x/I(N_1)$, and let Q_2 be the smallest prime such that $Q_2 > Q_1$ and $(Q_2 + 1)/Q_2 \leqslant d_2$. Let $N_2 = N_1 Q_2$. With the proviso that we will stop if $I(N_j) = x$ at any stage (including the 0th stage), we continue this process to generate a sequence $\{N_j\}$ of integers such that $\{I(N_j)\}$ increases to x. The existence of the needed Q_j's are guaranteed by the convergence of $\{(P_i + 1)/P_i\}$ to 1, and the fact that the gap from $I(N_j)$ to x will always be bridged follows from the divergence of $\prod(P_i + 1)/P_i$ to ∞.

The numbers N_j generated in the proof are all products of distinct primes. There are infinitely many integers which would not be used to generate the sequences $\{I(N_j)\}$. Also, there are many duplications in the set of abundancy indexes; one such duplication occurs when

$$I(332640) = I(360360) = 48/11.$$

Further, some of these duplications are integers, as is evidenced by the existence of more than one perfect number. (Erdös [**9**] has done a sophisticated analysis of the frequency of occurrence of duplications.) However, the number 2 has an abundancy index which is uniquely its own. To see that $N = 2$ is the only solution of the equation $I(N) = 3/2$, note first that if N is even and $N > 2$, then $I(N) > 3/2$ (Theorem 1); second, if N is odd, then the denominator in formula (3) is odd and cannot have 2 as a factor.

Open questions

Since every $I(N)$ is a rational number and since $D = \{I(N): N \geqslant 2\}$ is dense in $(1, \infty)$, we are led to the (perhaps unanswerable) question: Is every rational number $q > 1$ the abundancy index of some integer? If we note that the construction in the proof of Theorem 5 can be carried out with only minor modifications after deleting the number 2 from the list of primes, we get another interesting question: Is every rational number $q > 1$ the abundancy index of some *odd* integer? A positive answer to the first question would be of some interest, and a positive answer to the second one would be a major advance since it would imply that there exist odd perfect numbers and odd multiperfect numbers, no examples of which are known. But the suggested modification of the proof of Theorem 5 does assert that there are odd integers whose "imperfections" are as small as we choose since we can, for any $\varepsilon > 0$, find an odd integer N with abundancy index $I(N)$ in the interval $(2 - \varepsilon, 2]$. In this context we might regard ε, or its reciprocal, as a "coefficient of frustration."

The search for multiperfect numbers, those numbers with integral abundancy indexes, was vigorously undertaken by Descartes, Fermat, and others in the 17th century. Those two were the most successful searchers of that era, and the competition between them apparently was not without rancor [**7**, pp. 33–38]. Many other multiperfects were discovered in the early 1900's by Carmichael [**4**], Mason [**5**], and Poulet [**19**] and, more recently, by Brown [**3**] and Franqui and Garcia [**10**; **11**]. These last four lists, all over 30 years old, contain seven multiperfects of index 8, each of which has from 41 to 43 distinct prime factors and is of the order of 10^{200}. Index 8 is still the greatest index achieved in the search for multiperfect numbers. Franqui and Garcia's second list also includes a multiperfect of index 7 with a 27-digit prime factor.

The ideas of the abundancy index and formula (3) have been the principal tools in the investigations of multiperfects from the beginning. Substitutions discovered and used by Descartes [**7**, p. 35], Poulet, and Carmichael and Mason have the effect of preserving the abundancy index or changing it by an integer amount to construct new multiperfect numbers from known ones. One example of such a substitution which preserves index is the replacement of $3^5 \cdot 7^3 \cdot 13$ by $3^7 \cdot 7^2 \cdot 19 \cdot 41$. A comparable substitution, $2^5 \cdot 3^3$ for $2^3 \cdot 3^2 \cdot 13$, was implicit in an earlier observation that $I(332640) = I(360360)$, or $I(2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11) = I(2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13)$. It is entirely possible that many substitutions were discovered in just this way—by comparing the prime factorizations of numbers (mostly multiperfects) known to have the same index. An excellent survey of current knowledge on multiperfect numbers and an extensive list of references can be found in [**12**].

One of the oldest and most famous (notorious) of the unanswered questions of mathematics is: Is there an odd perfect number? The theory of *even* perfect numbers was thoroughly settled by Euclid and Euler (see, for example, [**8**, Chapter 8].) Apparently, none of that theory is helpful in searching for *odd* perfect numbers.

To begin, we wish to satisfy equation (3)

$$I(N) = \frac{1 + P + P^2 + \cdots + P^r}{P^r} \cdot \frac{1 + Q + \cdots + Q^s}{Q^s} \cdot \cdots \cdot \frac{1 + V + \cdots + V^t}{V^t} = 2,$$

where the distinct prime factors $P, Q, \ldots, V$ of N are all odd. The denominator of this fraction is an odd number, so the numerator contains only a single factor of 2; but if r is an odd integer, then $1 + P + P^2 + \cdots + P^r$ is even. Thus exactly one of the exponents $r, s, \ldots, t$ is odd, and all of the other primes occur to even powers only. (The reader is invited at this point to walk in the footsteps of Euler and show that, for this prime with odd exponent, both the prime and the exponent are congruent to 1 modulo 4.) If we wish to have some 3's in the prime factorization of

N, we will need an even number of them, but the penalty for having none is that the minimum number of distinct prime factors of N increases from eight to eleven, [**25**], [**27**], [**29**].

As a variation on the approach using formula (3), we might try the approach in the proof of Theorem 5 and approximate the number $x=2$ from below by a sequence of abundancy indexes—keeping in mind the restrictions of even exponents and the like discussed above—and hope to be lucky enough to find at some stage that $2/I(N_j)$ is equal to some realizable $(P_i+1)/P_i$ or $(P_i^2+P_i+1)/P_i^2$.

In light of this discussion, it is not surprising that no odd perfect numbers have yet been found and that those adventurers who look for them can become quickly discouraged. On the other hand, the possibility that an odd perfect or multiperfect number may exist has spurred several modern investigators to develop and improve criteria and conditions such numbers must satisfy. A few of these are noted in our references; further criteria and references are available in [**2**] and [**12**].

The value of investigation

This note grew out of a "computer literacy" project for classes of humanities majors and prospective elementary teachers in which I introduced the abundancy index as a concept whose computational aspects were readily assigned to a computer. I asked them to compute a few values of $I(N)$ and search for an odd abundant number. Later findings of $\sigma(N)/N$ in older literature [**4**], [**6**] led me to an enjoyable investigation of properties of the index. The two referees of an earlier version of the paper further stimulated my interest with a number of additional references (as well as being helpful in all the ways referees can be).

The abundancy index as a hierarchical classification of numbers is an interesting concept in its own right—at least in part for its recreational value when used to investigate the general topic of abundant and deficient numbers. In addition, it has growth and density properties to intrigue both the serious and recreational students of number theory. Its analysis provides a vehicle for unifying several parts of the theory; in so doing, it suggests new unsolved problems and illuminates old ones. It is hard to ask anything more of a completely intelligible one-line definition!

References

[1] L. Alaoglu and P. Erdös, On highly composite and similar numbers, Trans. Amer. Math. Soc., 56 (1944) 448–469.

[2] A. H. Beiter, Recreations in the Theory of Numbers, 2nd ed., Dover, New York, 1966, Chapter 3.

[3] A. L. Brown, Multiperfect numbers, Scripta Math., 20 (1954) 103–106.

[4] R. D. Carmichael, A table of multiply perfect numbers, Bull. Amer. Math. Soc., 13 (1907) 383–386.

[5] R. D. Carmichael and T. E. Mason, Note on multiply perfect numbers including a table of 204 new ones and the 47 others previously published, Proc. Indiana Acad. of Sci., (1911) 257–270.

[6] L. E. Dickson, Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors, Amer. J. Math., 35 (1913) 413–422.

[7] ______, History of the Theory of Numbers, Carnegie Institute, 1919 (reprinted by Chelsea, 1966), Vol. 1, Chapter 1.

[8] U. Dudley, Elementary Number Theory, 2nd ed., Freeman, 1978.

[9] P. Erdös, Remarks on number theory. II. Some problems of the σ function, Acta Arith., 5 (1959) 171–177.

[10] B. Franqui and M. Garcia, Some new multiply perfect numbers, Amer. Math. Monthly, 60 (1953) 459–462.

[11] ______, 57 new multiply perfect numbers, Scripta Math., 20 (1954) 169–171.

[12] R. K. Guy, Unsolved Problems in Number Theory, Springer-Verlag, New York, 1981, pp. 25–30.

[13] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th ed., Oxford, 1960.

[14] E. Hille, Analytic Function Theory, Vol. 1, Ginn, 1959, p. 223.

[15] D. G. Kendall, The scale of perfection, J. Appl. Probability, 19A (1982) 125–138.

[16] K. Knopp, Theory and Application of Infinite Series, Hafner, 1947, p. 219.

[17] I. Niven and H. S. Zuckerman, An Introduction to the Theory of Numbers, 4th ed., Wiley, 1980, p. 221.

[18] K. K. Norton, Remarks on the number of factors of an odd perfect number, Acta Arith., 6 (1961) 365–374.

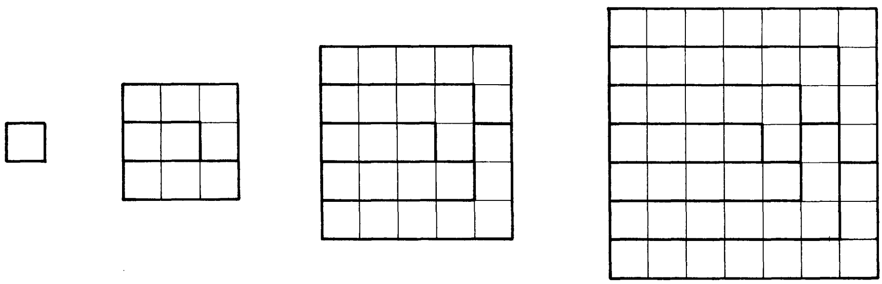
[19] P. Poulet, La Chasse aux Nombres, Fascicule I, Bruxelles, 1929, 9–27.

[20] M. Satyanarayana, Bounds of $\sigma(N)$, Math. Student, 28 (1960) 79–81.
[21] C. R. Wall, P. L. Crews, and D. B. Johnson, Density bounds for the sum of divisors function, Math. Comp., 26 (1972) 773–777.

Supplementary references on odd perfect numbers

[22] G. L. Cohen, On odd perfect numbers, Fibonacci Quart., 16 (1978) 523–527. (Improved upper bounds on $\sum_{i=1}^{t} 1/p_i$ when $N = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ is an odd perfect number.)
[23] G. L. Cohen and M. D. Hendy, On odd multiperfect numbers, Math. Chronicle, 10 (1981) 57–61. (If $\sigma(N) = KN$, K an integer $\geqslant 2$ and N odd with t distinct prime factors, then $t \geqslant (K^5 + 387)/70$.)
[24] G. G. Dandapat, J. L. Hunsucker, and C. Pomerance, Some new results on odd perfect numbers, Pacific J. Math., 57 (1975) 359–364. (A negative answer to a 1970 conjecture.)
[25] P. Hagis, Jr., Outline of a proof that every odd perfect number has at least eight prime factors, Math. Comp., 35 (1980) 1027–1032.
[26] P. Hagis, Jr., On the second largest prime divisor of an odd perfect number, Lecture Notes in Mathematics 899, Springer-Verlag, New York, 1971, pp. 254–263. (The prime of the title is > 1000.)
[27] ______, Sketch of a proof that an odd perfect number relatively prime to 3 has at least eleven prime factors, Math. Comp., 40 (1983) 399–404.
[28] P. Hagis, Jr., and W. L. McDaniel, On the largest prime divisor of an odd perfect number, Math. Comp., 29 (1975) 922–924. (The prime of the title is $\geqslant 100129$.)
[29] M. Kishore, Odd perfect numbers not divisible by 3, Math. Comp., 40 (1983) 405–411. (An alternative proof of the result of [**27**].)

Proof without words: Arithmetic progressions with sum equal to the square of the number of terms



$n = 4$

$4 + 5 + 6 + 7 + 8 + 9 + 10 = 7^2$

$$\sum_{k=n}^{3n-2} k = (2n-1)^2; \qquad n = 1, 2, 3, \ldots$$

—James O. Chilaka
Long Island University
C.W. Post Campus