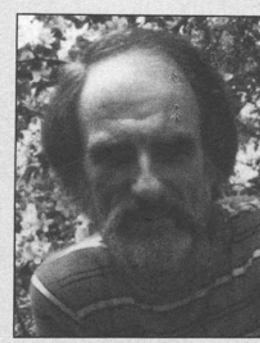
The Birth of Lie's Theory of Groups

Thomas Hawkins

In 1865 when Sophus Lie (1842–1899) completed his studies at the University of Christiania (now Oslo), Norway, he had no idea he was destined to become a mathematician. He had done well, but not brilliantly, in all subjects and was toying with the idea of becoming an observational astronomer. He even gave lectures on the subject in the student union. He had a real talent for explaining the geometry of the heavens. To support himself financially while in this state of career indecision he gave private instruction in mathematics. In this connection, Lie began to read the geometrical works by Poncelet, Chasles, and above all Plücker. Inspired by his reading, he did some original mathematical research on the real representation of imaginary quantities in projective geometry, a portion of which was accepted for publication by one of the leading mathematics journals of the time—Crelle's journal in Berlin. On the basis of this experience, he decided to devote himself to geometrical research, to become a mathematician. He was 26 years old. Five years later, during the fall of 1873, Lie made a second fateful decision: to devote himself to the enormous task of creating a theory of continuous transformation groups—a task that meant doing mathematics of a quite different sort from the geometrical work that had occupied him in his first years of mathematical research, 1869–1871—a task that ended by occupying most of his creative mathematical energies for the remainder of his career.

The purpose of this article is to explain how Lie was led from the one decision to the other. To accomplish this, I have to immerse you in the mathematical world inhabited by Lie, a world that is quite different from the one to which you are accustomed. The first two sections of this article concern the early geometrical work of Lie (1869–1871), done in close contact with Felix Klein. It was from this work that the ideas emerged which served to redirect Lie's researches. The third section briefly discusses the years 1872–1873, when the theory of first-order PDEs, particularly in the form given to it by the work of Jacobi, provided what turned out to be a fertile context for the development of the group-related ideas that had emerged during Lie's geometrical investigations. One of the reasons the history of mathematics is fascinating is that it provides insight into the dynamics by means of which ideas and concepts from diverse mathematical theories combine, in remarkable, often unexpected ways, to give birth to entirely new mathematical theories. In the course of showing you how Lie was led from his deci-



Thomas Hawkins

Thomas Hawkins has devoted his professional life to the history of mathematics—theories of integration, the origins of matrix theory, and the early years of group theory and representation theory. After getting his doctorate at the University of Wisconsin in mathematics and history of science, he taught at Swarthmore and, since 1972, at Boston University. His many visiting lectures have included the Collège de France and the 1986 International Congress of Mathematicians. His recreations include swimming, hiking, bicycling, and cross-country skiing—but above all bicycling.

sion to become a mathematician to his decision to create a new and far-reaching mathematical theory, I also hope to give you some feeling for the manner in which concepts, theorems, ideas, and viewpoints from 19th-century algebra, geometry, and analysis played roles in this creative process.

Tetrahedral Line Complexes

On the basis of the mathematical creativity exhibited by Lie in his essay on the real representation of imaginaries, he was granted a stipend to leave Norway and to travel to various centers. His first stop was Berlin, where the mathematical scene was dominated by Kummer and Kronecker and, above all, Weierstrass. Berlin was certainly one of the world's foremost centers of mathematics in 1869. However, the mathematics and the spirit in which it was done did not appeal to Lie, who found it too analytical. In Berlin he met another visitor who felt as he did. His name was Felix Klein (1849–1925). He was just 20 years old — 7 years younger than Lie — although he already had his doctorate. Klein had been Plücker's student, and after his mentor's untimely death in 1868, he had edited Plücker's lectures on line geometry for publication. Lie had, in fact, studied these lectures, and when they met in Berlin, they were both actively engaged in line-geometric research. So Klein and Lie enjoyed each other's mathematical company. They were self-styled "Synthesists" in the midst of analysts and arithmeticians.

Although Lie and Klein had interests in common, their backgrounds and personalities were quite different. Klein had a relatively solid mathematical education — first at Bonn under Plücker and then in Göttingen, where Alfred Clebsch (1833–1872) had attracted a circle of students, including, for example, Max Noether. Clebsch had been trained in Jacobian analysis and mathematical physics at the University of Königsberg. As we shall see in the third section, his contributions to Jacobi's theory of PDEs in the 1860s were to influence Lie. By the time Klein met Clebsch, however, he was working on algebraic geometry and the theory of invariants. His unexpected death from diphtheria at age 39 was a blow to Klein, who had great admiration for him.

Klein enjoyed learning about the work of others because he wanted to "understand" it from his own point of view and to place it within a more encompassing, conceptually unified picture. By contrast, Lie had a limited mathematical background (and no doctorate yet). He tended to focus rather exclusively on developing his own (highly original) ideas and became interested in the work of others only when it was clear it was relevant to his own interests. This difference in personalities dictated the way they related to each other: Lie developed his ideas, explained them to Klein, Klein reacted, Lie (sometimes) responded to Klein's reactions, and so on.

When Klein and Lie met in Berlin, Lie was studying what were known as tetrahedral line complexes. Line complexes were a basic object of study in line geometry, a geometry in which lines rather than points are taken as the basic objects. Thus, lines rather than points are coordinatized. Given a line $\ell \subset \mathbf{P}^3(\mathbf{C})$, if $(x_1, \ldots, x_4)$ and $(y_1, \ldots, y_4)$ are the homogeneous coordinates of two points on ℓ, set $p_{ij} = x_i y_j - x_j y_i$. Then $p_{12}, p_{13}, p_{14}, p_{23}, p_{42}$, and p_{34} are the six homogeneous Plücker coordinates of ℓ. They satisfy the relation

$$\Omega = p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

In terms of line coordinates, a line complex of degree d is a set of lines $\ell = (p_{12}, \ldots, p_{42})$ whose line coordinates p_{ij} satisfy a homogeneous polynomial equation of degree d (in addition to $\Omega = 0$). From an abstract viewpoint, line complexes are simply projective varieties in $\mathbf{P}^5(\mathbf{C})$, but in the mid-19th century the fact that the homogeneous coordinates corresponded to lines was always in view, and attention was focused on particular types of line complexes and their special geometrical properties.

The tetrahedral line complex studied by Lie was a special second-degree complex, which may be defined geometrically as follows. Let Δ denote a tetrahedron with faces determined by planes $\pi_1, \ldots, \pi_4$, and let k be a fixed constant. The totality $\mathcal{T}$ of all lines $\ell \subset \mathbf{CP}^3$ such that the cross ratio of $p_i = \ell \cap \pi_i$, $i = 1, \ldots, 4$, is k is called a tetrahedral complex. Other mathematicians had considered the geometry associated to a tetrahedral complex before Lie, but Lie's approach was totally original. Consider the set $\mathbf{G}$ of all projective transformations of space which fix the vertices of Δ. (Lie himself did not introduce notation such as $\mathbf{G}$; he simply spoke of the transformations in $\mathbf{G}$ as the transformations of the tetrahedron.) Then for any fixed line ℓ_0, the set $\mathcal{T}$ of all lines $T[\ell_0]$ such that $T \in \mathbf{G}$ is a tetrahedral complex. In effect, a tetrahedral complex for Lie was the orbit of some fixed line under the transformation of $\mathbf{G}$. An idea of what the transformations comprising $\mathbf{G}$ are like can be obtained by choosing for Δ the tetrahedron with vertices at the origin and the points at infinity on the coordinate axes. Then the $T \in \mathbf{G}$ are given in Cartesian coordinates by

$$x' = \lambda x, \qquad y' = \mu y, \qquad z' = \nu z. \tag{1}$$

In his study of the geometry of tetrahedral complexes, Lie took advantage of the following properties of the transformations of $\mathbf{G}$: If $T_1, T_2 \in \mathbf{G}$: (a) $T_1 \circ T_2 \in \mathbf{G}$; (b) $T_1 \circ T_2 = T_2 \circ T_1$; (c) for "general" $p, q \in \mathbf{P}^3(\mathbf{C})$, a unique $T \in \mathbf{G}$ exists such that $T(p) = q$; (d) there is a threefold infinity (∞^3) of transformations in $\mathbf{G}$. For us today, the fact that $\mathbf{G}$ is a group stands out. We see Lie using the commutativity and simple transitivity of $\mathbf{G}$. But in 1869, the theory of groups was not a part of the basic mathematics known to all active mathematicians. In 1869, "group theory" meant the theory of finite permutation groups, which was known to have a nice application to algebraic equations as shown by Galois. However, relatively few mathematicians were actively interested in this subject.

One place where permutation groups and Galois theory were the subject of lectures was the University in Christiania, where Sylow presented such lectures in 1862. In fact, Lie had actually attended these lectures! If history were more "rational," this would have been a turning point in Lie's life. But, apparently, history is not so rational — and, as a result, is more fascinating. According to Lie, he understood hardly anything that Sylow said and the lectures made no impression on him whatsoever. This is not surprising, since throughout his life Lie never displayed an interest in algebra or the theory of numbers; he was at heart a geometer. In 1862, he was still more than 5 years away from his decision to become a mathematician. Undoubtedly, Lie came away from Sylow's lectures with a vague realization that the theory of permutation groups had an important application to the resolution of algebraic equations, but there is no evidence to suggest that, before he met Klein, Lie regarded the totality of transformations **G** of the tetrahedral complex as the analogue of a group of permutations or, more importantly, that he thought this analogy was somehow significant. All the evidence suggests that it was Klein who perceived in Lie's work a continuous analogue of the theory of permutation groups, with analogous applications.

Before coming to Berlin, Klein had spent several months in Göttingen with Clebsch and his students. Shortly before Klein arrived, Clebsch had been in contact with a young French mathematician named Camille Jordan (1838–1922). Jordan was in the finishing stages of composing a systematic treatise on the theory and application of permutation groups. Jordan's book represented the first attempt to systematize Galois's cryptic ideas on polynomial equations and, in particular, to emphasize and develop their group-theoretic basis. Jordan was also interested in applying the theory of permutation groups to geometry, and this is why he made contact with Clebsch. Consider, for example, the fourth degree Kummer surface with 16 nodal points (Fig. 1). Jordan was interested in the polynomial of degree 16 whose roots yield the nodal points. From the geometrical properties of these points, he was able to determine the Galois group of the equation, and then to use its structure — the composition series — to conclude that the resolution of the equation reduced to resolving the general equation of degree 6 and some quadratic equations. Clebsch's work in enumerative algebraic geometry supplied Jordan with further geometrically relevant polynomial equations to analyze in this fashion.

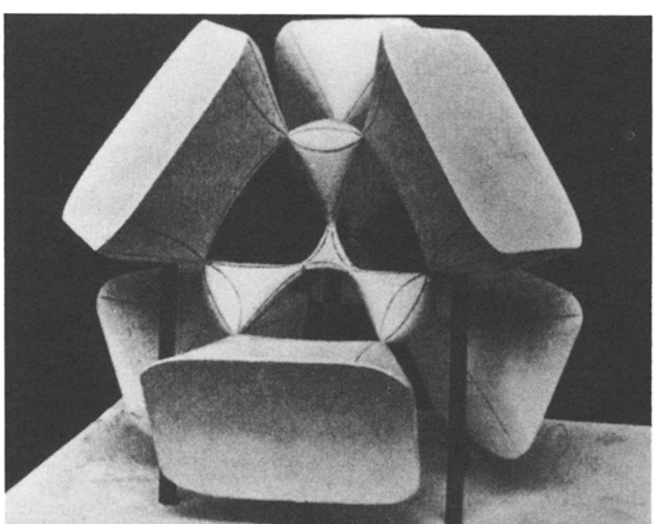


Figure 1. Kummer surface with 16 real nodal points. (Reproduced from *Kummer's Quartic Surface* by R.W.H.T. Hudson, Cambridge University Press.)

Clebsch admittedly did not fully understand the group theory Jordan was applying, but he encouraged Jordan in his efforts, apparently sensing their importance. In fact, the inaugural 1869 volume of Clebsch's new journal, *Mathematische Annalen*, contained Jordan's expository "commentary" on Galois's most famous memoir. How much group and Galois theory was discussed among Clebsch and his students, including Klein, in Göttingen is unknown, but it seems likely that there was some general talk about permutation groups, Galois's theory and Jordan's geometrical applications. In any case, Klein knew about Jordan's work on Kummer surfaces. Klein was an expert on these surfaces, which he showed could be studied completely in terms of a second-degree line complex. He cited Jordan's results about the nodal points in one of his papers, written while at Göttingen in 1869, just before he came to Berlin.

At Berlin, as Klein became acquainted with Lie and with his investigations on tetrahedral complexes, he perceived significant analogies between what Lie was doing and Galois's theory. Of primary importance in this connection was Lie's work on problems of a type considered earlier by Plücker. Plücker associated to a quadratic line complex, such as $\mathcal{T}$, a family of "cones." The cone at a point p consists of all lines $\ell \in \mathcal{T}$ which pass through p (so p is the "vertex" of the cone). Plücker had used the cones to define geometrically interesting surfaces associated to quadratic line complexes. It was in the spirit of this work that Lie posed problems such as: Determine surfaces S with the property that at each point $p \in S$, the complex cone at p "touches" S at p. Here the cone at p is said to touch the surface S at p if it meets the tangent plane to S at p in one of the lines of the cone as pictured in Figure 2.

Lie observed that a surface S defined by $z = \varphi(x,y)$ has this property if and only if φ is a solution to a certain first-order PDE, that is, an equation of the general form $f(x,y,z,p,q) = 0$, where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. From the geometry of the situation, Lie could see readily that if a surface S has the desired property, then so does any surface obtained from S by any transformation of **G**. In terms of the PDE, this says that the transformations of **G** take solutions into solutions. For future reference I shall employ the terminology later introduced by Lie to describe this property.

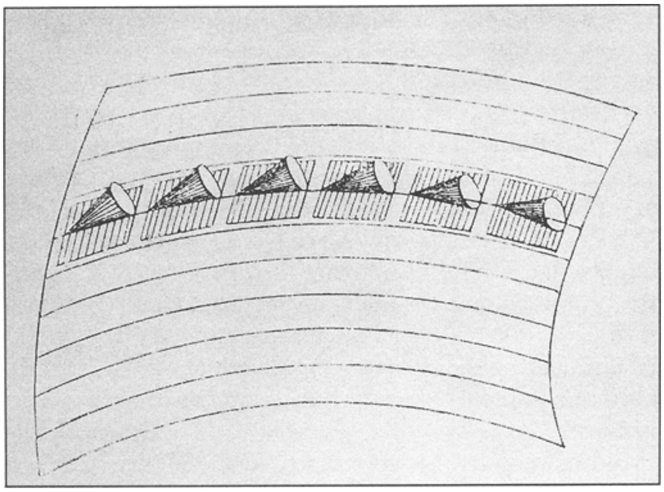


Figure 2. A surface touched by complex cones. (Reproduced with the permission of Chelsea Publishing Company.)

A PDE $f(x, y, z, p, q) = 0$ is said to admit the transformations $T \in \mathbf{G}$ if each $T \in \mathbf{G}$ takes solutions into solutions.

Using the fact that the PDE associated to his problem admits the transformations **G** of the tetrahedron, Lie was able to show how to solve it. To this end he introduced the logarithmic mapping $X = \log x$, $Y = \log y$, $Z = \log z$, which sends the $T \in \mathbf{G}$, expressed in the form (1), to the $T^* \in \mathbf{G}^*$, where T^* is defined by $X' = X + \alpha$, $Y' = Y + \beta$, $Z' = Z + \gamma$, with $\alpha = \log \lambda$, $\beta = \log \mu$, $\gamma = \log \nu$. Thus, the PDE $f(x, y, z, p, q) = 0$ transforms into a PDE, $F(X, Y, Z, P, Q) = 0$, which admits the $T^* \in \mathbf{G}^*$. Because the PDE $F(X, Y, Z, P, Q) = 0$ thereby admits all translations, Lie concluded that the function F cannot actually vary with X, Y, Z, and, hence, must be of the form $F(P, Q) = 0$, a type of PDE that had been resolved by Euler.

When Klein learned about Lie's method of solving the problem, he was struck by a consequential analogy with a result due to Abel. In a famous memoir of 1829, Abel had considered a class of polynomial equations whose roots had a commutativity property that generalized the property of the roots of cyclotomic polynomials exploited by Gauss in *Disquisitiones arithmeticae*. Abel showed that this class of polynomials could also be solved by radicals. Abel's commutativity property of the roots is equivalent to the Galois group of the polynomial being commutative. Klein's observation of this analogy made a great and lasting impression on Lie. From that moment on, Lie always had in the back of his mind the search for a systematic mathematical theory of differential equations that would resemble somewhat Galois's theory of algebraic equations. I will call it

LIE'S IDÉE FIXE. *The fact that a differential equation, or a system of such equations, admits known (possibly infinitesimal) transformations, which commute or which, more generally, form a group, should translate into information about its integration. Establish theorems showing this is the case.*

By 1871 Lie had pushed this idea to the following extent: Suppose a first-order PDE $f(x, y, z, p, q) = 0$ admits $k \leq 3$ known independent, commuting infinitesimal transformations. Then new variables X, Y, Z may be chosen so that the equation becomes (1) $F(P, Q) = 0$ if $k = 3$, (2) $F(Z, P, Q) = 0$ if $k = 2$, (3) $F(X, Y, P, Q) = 0$ if $k = 1$. In each case, the form of the transformed equation simplifies its integration. For example, regarding case (2), Lie observed that it followed from known results that the solution to such a PDE reduces to a quadrature. Lie's proof of this proposition was very vague and intuitive.

The Line-to-Sphere Mapping

Before proceeding further, an update on the travels of Lie and Klein is in order. After one semester at Berlin, they had moved to the next obvious center of mathematics — Paris, where they spent the spring and early summer of 1870. Towards the end of their stay in Paris, Lie discovered a remarkable line-to-sphere mapping, which switched the focus of his research from tetrahedral complexes to the properties and implications of his new mapping. I will have more to say about this historically important work in a moment.

The outbreak of the Franco-Prussian war caused Klein to leave Paris and return to Germany. Lie, being Norwegian, opted to hike back to Norway via Italy. He was an avid hiker with a reputation for great endurance. Lie also had some interesting hiking techniques. For example, according to Klein, in order to keep his clothing dry while hiking in the rain, he would take it off and put it in his backpack! He must have made an interesting sight. Perhaps this was one reason why he was picked up by the French authorities. He was suspected of being a German spy and thrown into prison. The letters he had written encouraged the authorities in their suspicions, for when Lie wrote (in German) about "lines" and "spheres" they thought he was writing about "infantry" and "artillery." When Lie said it was mathematics and began to explain, "Let x, y and z be rectangular coordinates. . . ," they decided he was insane! Eventually, he was released (through the intervention of Darboux), and he returned to Norway to write up the new mathematical discoveries he had made in Paris — and in prison — as his doctoral dissertation.

Lie and Klein's stay in Paris proved to be a far more congenial mathematical experience than their stay in Berlin. Camille Jordan was there and they met him. In fact, Jordan's great treatise on permutation groups and Galois's theory rolled off the press while they were in Paris. Klein at least appears to have taken a look at it and was impressed — impressed both by the monumental nature of the theory expounded by Jordan and by how little he understood of it! Klein and Lie did not spend much time talking to Jordan, and they spent no time trying to decipher his treatise. They were impressed by the existence of the theory, but they did not envision its technical details as the basis for a theory of continuous groups. By this time, the concept of an infinitesimal transformation was basic to their group-related studies connected with tetrahedral complexes and Lie's *idée fixe*, and they believed infinitesimal transformations would be fundamental to the development of a theory of continuous groups as well. Their primary research interests remained geometrical, and the person they spent most of their time talking to in Paris was a geometer, Gaston Darboux (1842–1917).

Darboux, who was Lie's age, was part of a group of French mathematicians who sought to develop an approach to geometry that combined concepts from differential geometry — for which there was a great tradition in France going back to Monge — and concepts from projective geometry in the tradition of Poncelet. Lie and Klein tended to refer to this methodological amalgam as French "metrical geometry." French metrical geometry was inspired in part by a theorem of Liouville (1846) which showed that, by contrast to the case of the plane, conformal transformations of space are less abundant: They can all be generated by composition of certain projective transformations (similarity transformations) and transformations closely related to projective transformations: what were called transformations by reciprocal radii, that is, inversions in spheres. One of the tools used by Darboux and his colleagues was a "pentaspherical coordinate" system. Darboux pointed out to them that there was a remarkable formal similarity between some of Klein's line-geometrical results (expressed in six homogeneous line coordinates) and Darboux's results (expressed in five pentaspherical coordinates). In July 1870, shortly before his departure from Paris, Lie discovered a basis for such analogies in what he called his sphere mapping.

Lie's sphere mapping was a by-product of his generalization of a basic notion in projective geometry: reciprocity, a notion that Plücker in particular had emphasized and himself generalized. One of the simplest examples is the duality between points and planes in space. It is given by

$$xX + yY + zZ + 1 = 0. \qquad (2)$$

Thus, corresponding to a fixed point whose coordinates are given by the lower case x, y, z, is the plane of points capital X, Y, Z satisfying (2). Lie's sphere mapping is based on the reciprocity determined by the two linear equations

$$(X + iY) - zZ - x = 0, \qquad z(X - iY) + Z - y = 0. \quad (3)$$

By virtue of these equations it is possible to associate various geometrical objects in complex projective space **r**, with Cartesian coordinates (x, y, z), to objects in space **R**, with Cartesian coordinates (X, Y, Z). For example, for each fixed point $P = (X, Y, Z) \in \mathbf{R}$, (3) represents a system of two linear equations in three unknowns x, y, z, and the solutions $p = (x, y, z)$ form a line $\ell \subset \mathbf{r}$. Thus, (3) establishes a correspondence $\Psi : P \to \ell$ between points of **R** and certain lines of **r**. The lines $\ell = \Psi(P)$ obtained by varying $P \in \mathbf{R}$ form a linear (=degree one) line complex $\mathcal{L} = \{\ell : \ell = \Psi(P), P \in \mathbf{R}\}$, and the mapping $\Psi : \mathbf{R} \to \mathcal{L}$ is one-to-one. Given *any* line $\ell \subset \mathbf{r}$, let $\mathcal{C}(\ell)$ denote the set of lines ℓ^* from the complex $\mathcal{L}$ which intersect the given ℓ. Then the corresponding set of points in **R**, $S(\ell) = \Psi^{-1}[\mathcal{C}(\ell)]$, is a sphere, that is, a set of points in **R** satisfying the usual Cartesian equation of a sphere, although the center and the radius may be complex numbers.

In this manner, Lie set up a correspondence between all lines in **r** and "spheres" in **R** — and, thus, between the line geometry relative to a linear line complex $\mathcal{L}$ in **r** and a "sphere geometry" in **R**. For example, lines which intersect in **r** correspond to spheres which touch in **R**. Incidentally, Lie sphere geometry (in any dimension) has recently become of interest to a number of mathematicians.

These properties of the line-to-sphere correspondence bring out another aspect of reciprocities in general to which Lie first called attention. Each of them defines what he called a contact transformation. Contact transformations of 3-space do not transform points, they transform what Lie called "surface elements." A surface element ds consists of a point $a = (x, y, z)$ and an infinitesimal surface through that point which may be identified with the tangent plane, $(z^* - z) = p(x^* - x) + q(x^* - x)$, to the surface at point $a = (x, y, z)$. In the spirit of Plücker, Lie thought of ds as coordinatized by the (x, y, z, p, q). Thus, a contact transformation is a transformation $T : (x, y, z, p, q) \to (X, Y, Z, P, Q)$ of surface elements. Lie called these transformations "contact transformations" because they preserve the contact between surfaces; that is, if two surfaces touch at a point, they are transformed by a contact transformation into surfaces which touch at the image point.

Any "point transformation" — that is, any transformation in which X, Y, Z are functions of (x, y, z) (and not p and q as well) — defines (by prolongation) a contact transformation, for point transformations evidently preserve the contact between surfaces. But many contact transformations are not generated by point transformations. An example is provided by the contact transformation $\Sigma : (x, y, z, p, q) \to (X, Y, Z, P, Q)$ associated to the reciprocity (3). Σ may be defined geometrically as follows: Given a surface element in **r** as represented by a point and a plane through it, consider all lines through that point and lying in that plane. These lines correspond in **R** to a family of spheres which touch in a common point and, hence, share a common tangent plane. This point and plane define the corresponding surface element in **R** under the contact transformation Σ. Here the point (X, Y, Z) depends not only on the point (x, y, z)

but also on the chosen plane through it and, hence, on p and q.

Lie pursued his investigations related to the sphere mapping in two directions, both of which are historically important. The first direction was encouraged by the discovery in July 1870 that, by virtue of the sphere mapping reciprocity $\mathbf{r} \leftrightarrow \mathbf{R}$, the asymptotic curves on a surface in $\mathbf{r}$ correspond to the lines of curvature on the associated surface in $\mathbf{R}$. Asymptotic curves and lines of curvature were two types of curve lying on a surface of interest to differential geometers of the time. Lie and Klein had learned in Paris that Darboux and his colleagues had recently determined the lines of curvature of a certain class of fourth-degree surfaces which arise naturally from pentaspherical coordinates and were called generalized "cyclides." Lie observed that if one thought of these cyclides as "living" in the space $\mathbf{R}$, then the corresponding surfaces in $\mathbf{r}$ were the Kummer surfaces that Klein had studied. In this way, Lie was able to transfer the results of the French on lines of curvature of generalized cyclides to a description of the asymptotic curves on Kummer surfaces—a new result at that time and one that greatly impressed Klein.

With the encouragement of Klein, Lie set out to explore the relations, established by his sphere mapping, between line geometry in the space $\mathbf{r}$ and the metrical geometry of the Darboux school in the space $\mathbf{R}$. The contact transformation Σ sets up a correspondence between transformations of $\mathbf{r}$ and transformations of $\mathbf{R}$: $t : \mathbf{r} \rightarrow \mathbf{r}$ corresponds to $T : \mathbf{R} \rightarrow \mathbf{R}$, where $T = \Sigma t [\Sigma]^{-1}$. Lie observed that various types of familiar geometrical transformations of $\mathbf{r}$ correspond, in this manner, to familiar types of geometrical transformations of $\mathbf{R}$. All of these types corresponded to groups, although Lie did not mention this fact explicitly. For example, the totality $\mathbf{L}_{10}$ of ∞^{10} projective transformations of $\mathbf{r}$ which (as line transformations) take $\mathcal{L}$ into itself corresponds to the totality $\mathbf{G}_{10}$ of ∞^{10} conformal transformations T of $\mathbf{S}$. The totality $\mathbf{M}_7$ of ∞^7 $t \in \mathbf{L}_{10}$ which fix ℓ_∞ (the line at ∞ in the xy plane) corresponds to the totality $\mathbf{H}_7$ of all similarity transformations T (generated by translations, rotations, and scaling maps).

Klein found Lie's results very stimulating. The metrical geometry of the French involved properties invariant under the group $\mathbf{G}_{10}$ of conformal transformations or, more generally, under the group $\mathbf{H}_7$ of similarity transformations. For Klein, the key to Lie's success was the mapping Ψ. He saw Lie's correlation of projective line geometry in $\mathbf{r}$ with metrical geometry in $\mathbf{R}$ as based on the existence of the map $\Psi : \mathcal{L} \rightarrow \mathbf{R}$, which brings with it $\mathbf{L}_{10} \rightarrow \mathbf{G}_{10}$ and $\mathbf{M}_7 \rightarrow \mathbf{H}_7$. Incidentally, the group $\mathbf{L}_{10}$, the projective group of a linear line complex as Lie later called it, is the projective symplectic group with structure type C_2. Symplectic groups first arose in Lie's work in this manner. As the group $\mathbf{G}_{10}$ of conformal transformations has structure type B_2, one could say (anachronistically) that Lie's correspondence between the projective line geometry of a linear complex in $\mathbf{r}$ and the metrical geometry of $\mathbf{R}$ was a reflection of the "accidental isomorphism" of C_2 and B_2.

Lie's work convinced Klein that he had become too dogmatic in his assumption, encouraged by his association with Clebsch, that projective geometry was more fundamental than other types of geometry. The equivalence established by the map Ψ put three-dimensional metrical geometry on an equal footing with the line geometry of the linear complex $\mathcal{L}$. Now the totality of all lines in 3-space coordinatized by Plücker coordinates is *four*-dimensional, $\mathcal{L}$ being of course only three-dimensional. This fact led Klein to ask whether an analogous correspondence exists between all of line geometry (which is four-dimensional) and four-dimensional metrical geometry. Klein showed the answer is yes.

Klein's answer, published in 1871, is itself less important historically than the new conception of the essence of geometry implicit in it, for Klein took the radical view that a geometry can be described by specifying (i) a manifold $\mathcal{M}$ of "elements" and (ii) a group $\mathbf{G}$ of transformations $T : \mathcal{M} \rightarrow \mathcal{M}$, which defines the invariant relations of the geometry. He regarded two geometries $(\mathcal{M}_1, \mathbf{G}_1)$ and $(\mathcal{M}_2, \mathbf{G}_2)$ as equivalent if there exists a one-to-one map Ψ from $\mathcal{M}_1$ onto $\mathcal{M}_2$ such that $\mathbf{G}_1 = \Psi^{-1}\mathbf{G}_2\Psi$. Klein regarded line geometry as determined by the pair $\mathcal{M}_1, \mathbf{G}_1$, where $\mathcal{M}_1$ consists of all $(x_1, \ldots, x_6) \in \mathbf{P}^5(\mathbf{C})$ which satisfy $\Omega = x_1^2 + \cdots + x_6^2 = 0$. Here, following Klein, Plücker coordinates p_{ij} have been replaced by those coordinates x_i in which the quadratic form Ω is expressible as $x_1^2 + \cdots + x_6^2$. Accordingly, $\mathbf{G}_1$, the group of all projective line transformations, consists of all $T \in \mathbf{PGL}(6, \mathbf{C})$ such that T takes $\mathcal{M}_1$ into itself. Four-dimensional metrical geometry was defined by analogy with the three-dimensional case as the pair $\mathcal{M}_2, \mathbf{G}_2$, where $\mathcal{M}_2 = \mathbf{P}^4(\mathbf{C})$, and $\mathbf{G}_2$ is the conformal group of 4-space, the group generated by four-dimensional similarity transformations and transformations by reciprocal radii. Having characterized line geometry and four-dimensional metrical geometry in this radically new way, Klein then showed that these two geometries are equivalent in the sense that a mapping Ψ exists.

The conception of geometry implicit in the above-described work was made explicit by Klein 2 months later in an essay "On Geometrical Methods." In it he took the view that *every* geometrical method is determined by specifying a manifold $\mathcal{M}$ and a group $\mathbf{G}$ of transformations taking it into itself. Although Klein never published this essay, the ideas were finally published in his famous Erlangen Program of 1872. As you can see, Klein owed much in the way of inspiration for these ideas to Lie's work on his sphere mapping.

Lie himself was led in another direction by the sphere mapping research. Lie's reciprocity defined by (3) was, he realized, but one example of an infinite family of reciprocities. Just as (3) defined the contact transformation Σ, so too, any type of reciprocity

(A) $F(x,y,z,X,Y,Z)=0$, (B) $\begin{matrix} F(x,y,z,X,Y,Z)=0 \\ G(x,y,z,X,Y,Z)=0 \end{matrix}$

$$\text{(C)} \quad \begin{matrix} F(x,y,z,X,Y,Z)=0 \\ G(x,y,z,X,Y,Z)=0 \\ H(x,y,z,X,Y,Z)=0 \end{matrix}$$

defines an associated contact transformation T: $(x,y,z,p,q) \rightarrow (X,Y,Z,P,Q)$. For example, take type (A). Fix X,Y,Z and regard $F=0$ as defining z as a function of x and y, to obtain, by differentiation of $F=0$, the equations $F_x + F_z p = 0$ and $F_y + F_z q = 0$, where $F_x = \partial F/\partial x, p = \partial z/\partial x$, etc. Similarly, with x,y,z held fixed, differentiation of $F=0$ yields $F_X + F_Z P = 0$ and $F_Y + F_Z Q = 0$. The 5 equations $F=0,\ldots,F_Y + F_Z Q = 0$ in the 10 unknowns x,y,z,p,q,X,Y,Z,P,Q can be solved for X,Y,Z,P,Q (assuming the requisite implicit function theorem applies) to obtain 5 equations expressing each of X,Y,Z,P,Q as a function of x,y,z,p,q. These equations define the contact transformation T associated to the reciprocity $F(x,y,z,X,Y,Z)=0$.

Plücker had called attention to type (A)—but in two dimensions and with F restricted to being a polynomial. Type (C) defines ordinary "point transformations"; for example, for a fixed point x,y,z, system (C) defines (generally) a point—the intersection of three surfaces. But it was Lie who saw all of these types as contact transformations. His sphere mapping represents a type (B) contact transformation—apparently the first of type (B) to be considered by any geometer.

By 1872 Lie had obtained a characterization of contact transformations that proved more useful in analytical reasoning and which may make the concept seem more familiar today. Contact transformations of three-dimensional space turn out to be those transformations of five variables, $T:(x,y,z,p,q) \rightarrow (X,Y,Z,P,Q)$, which leave the Pfaffian equation $dZ-(PdX+QdY)=0$ invariant, meaning that

$$dZ-(PdX+QdY)=\rho[dz-(pdx+qdy)], \quad \rho=\rho(x,y,z,p,q)\neq 0. \tag{4}$$

Although geometry had been the inspiration for the general idea of a contact transformation, Lie's interest in developing it was due mainly to its potential importance in the study of PDEs. In the mid-1860s the German mathematician Paul Du Bois-Reymond had focused attention on a very special type of contact transformation, a type which had been used occasionally in the 18th century by Euler, D'Alembert, and Legendre to transform PDEs. Typical is the transformation

$$X=-q, \quad Y=p, \quad Z=z-px-qy, \quad P=y, \quad Q=-x. \tag{5}$$

This transformation takes a first-order PDE $F(X,Y,Z,P,Q)=0$ into another such PDE, $f(x,y,z,p,q)=0$; because $Z=z-px-qy=z-Yx+Xy$, so $P=\partial Z/\partial X=y$ and $Q=\partial Z/\partial Y=-x$. In other words, (5) actually transforms a first-order PDE into another such equation because the first three equations (for X,Y,Z) imply the remaining two, so that P is actually $\partial Z/\partial X$ and Q is actually $\partial Z/\partial Y$.

Du Bois-Reymond tried to get a more general class of transformations by which a first-order PDE might be transformed into another, but he did not get very far. Certainly he had no general concept of a contact transformation. Lie realized that any contact transformation may be used to transform a first-order PDE. As Lie had learned while in Paris, Monge had showed how to develop the 18th-century theory of PDEs in terms of geometrical constructs. Lie dreamed of doing something analogous by means of the geometrical notions associated with contact transformations. One aspect of that dream was a prospective "invariant theory" of contact transformations. A general goal was to determine when two given systems of PDEs can be transformed into one another by a contact transformation. Such a transformation would take the solutions of the one system into those of the other, so that solving the one system would be tantamount to solving the other. Now, if one system has a property (P) that is preserved under all contact transformations, then it can only be transformed into another system having (P). Therefore, the problem arises to determine invariant properties of systems of PDEs in the hopes of determining necessary and sufficient conditions that one system be transformable into another by a contact transformation. The idea of this prospective invariant theory was partly inspired by the results Lie had obtained transforming various types of PDEs using the contact transformation Σ associated to the sphere mapping.

Lie was thinking about this in the closing months of 1871 when he received a copy of Klein's essay on geometrical methods. At first, Lie failed to see what Klein was proposing. But once he understood, he became very enthusiastic, because his project on invariant properties of PDEs exemplified what Klein was saying in his essay: Lie's prospective theory could be thought of as the study of the Kleinian geometrical method corresponding to the five-dimensional manifold of surface elements x,y,z,p,q and the group of all contact transformations. (Strictly speaking, these contact transformations form a Lie pseudogroup—but all group-related concepts were only vaguely articulated at this time.) Klein, in turn, was excited by Lie's observations, for he had not had anything like Lie's example in mind! When Lie and Klein were finally together again in the fall of 1872, they made their new ideas public in the following ways. Lie, with Klein's editorial assistance, announced some of his ideas on PDEs and contact transformations. Klein, with Lie's assistance, composed his Erlangen Program on geometrical methods. In the Program, Klein wrote, "Given a manifold and a group of transformations of it, develop the invariant theory related to the group"—and that is precisely what Lie intended to do for the group of contact transformations.

In the Erlangen Program, Klein called for the development of an autonomous theory of continuous transformation groups. He pointed out that in Jordan's 1870 treatise, the theory of finite groups is first developed independently, then its applications are developed. Klein and Lie were suggesting some sort of development of the theory of continuous groups, with the applications to follow. Klein probably had geometrical applications in mind, but Lie still had his *idée fixe* — an application of group theory to differential equations somewhat analogous to Galois's theory and perhaps now also involving the invariant theory of contact transformations. But who would develop the group theory? Did they actually envision carrying out such a development themselves? Certainly not in 1872. Neither of them felt prepared to do it, even though they claimed the continuous theory would be easier to develop than the "discrete" theory. After writing the Erlangen Program, Klein's interests turned to other matters, which did not involve continuous groups at all. As for Lie, he was neither prepared nor inclined to grapple with the theory of continuous groups. For example, according to his recollections, he realized at the time the Erlangen Program was written that implicit in it was the problem of classifying continuous groups. Lie, however, felt that such a classification was "absurd or impossible" — that is, so far beyond what he would be capable of resolving as to appear ridiculous. This is the same negative attitude that both Lie and Klein had shared earlier, in 1871, when a spinoff of the work on tetrahedral complexes — their theory of W-configurations — had led to a problem of classifying certain commutative subgroups of the general projective group. This classification problem they also had dismissed as too difficult to consider after attempting to resolve it in special cases. Slightly over 1 year later, however, Lie had completely changed his mind! By the end of 1873 he had decided that he could develop a theory of continuous groups, and that the effort to do so was warranted. What caused him to change his mind? In a word, the answer is: Jacobi.

Jacobi and First-Order PDEs

By the time Lie and Klein got together to write the Erlangen Program, Lie was trying to work out his geometrical theory of PDEs, based on his "invariant theory" of contact transformations. At that time, the theory of first-order PDEs was dominated by the results of Jacobi that had been inspired by his study of Hamilton's papers on dynamics. Jacobi had actually obtained these results in the late 1830s — he died in 1851 — but many of them were first published by Clebsch in the period 1862–1866. Many mathematicians were interested in developing Jacobi's theory, but Lie was unique in that he approached it with a geometrical background and with "groups" and his *idée fixe* in the back of his mind. To conclude this article, I will briefly indicate how Lie's approach was linked by Lie to Jacobi's theory, and why the link encouraged him to take on the challenge in the Erlangen Program.

Let

$$F_1(x,p) = F_1(x_1,\dots,x_n,p_1,\dots,p_n) = 0, \qquad p_i = \frac{\partial z}{\partial x_i}, \tag{6}$$

denote a first-order PDE which does not explicitly involve the dependent variable $z = \varphi(x_1,\dots,x_n)$. It was known that the solution of any first-order PDE could be reduced to this special type. Jacobi's method of solving (6) made fundamental use of the brackets that had been introduced by Poisson in 1809 in connection with Lagrangian mechanics. If G and H are any functions of $2n$ variables

$$(x,p) = (x_1,\dots,x_n,p_1,\dots,p_n),$$

then

$$(G,H) = \sum_{i=1}^{n}\left(\frac{\partial G}{\partial p_i}\frac{\partial H}{\partial x_i} - \frac{\partial G}{\partial x_i}\frac{\partial H}{\partial p_i}\right). \tag{7}$$

Jacobi observed that the PDE (6) could be completely solved if functions $F_2,\dots,F_n$ of the variables (x,p) could be determined such that $F_1,\dots,F_n$ are functionally independent and satisfy the relations $(F_i,F_j) = 0$ for all i and j. He then proceeded to devise a method for producing the functions F_i. [In other words, Jacobi's method made the Hamiltonian system defined by $H = F_1(x,p)$ completely integrable.]

Jacobi also discovered a property of the Poisson bracket (7) that Poisson and Lagrange had apparently missed:

JACOBI'S IDENTITY. *If F, G, and H are any three functions of the variables (x,p), then*

$$((F,G),H) + ((G,H),F) + ((H,F),G) = 0. \tag{8}$$

The Jacobi Identity played a fundamental role in Jacobi's method for determining the functions $F_2,\dots,F_n$. He also saw the Identity as having important implications for an earlier method he had devised for solving a first-order PDE. The earlier method solved $F_1(x,p) = 0$ by determining $2n-1$ functionally independent solutions $\Phi = F_1,\dots,F_{2n-1}$ to $(F_1,\Phi) = 0$, one of which is automatically $\Phi = F_1$. Jacobi believed that if he knew just *two* solutions $\Phi = F_2$ and $\Phi = F_3$ to $(F_1,\Phi) = 0$ such that F_1, F_2, F_3 are functionally independent, then "in general" the remaining $2n-1$ could be determined by (8). Indeed, given the solutions F_2 and F_3, it follows readily from (8) that $F_4 = (F_2,F_3)$ is another solution. The same reasoning then implies that $F_5 = (F_3,F_4)$ is another solution, and so on. Jacobi was convinced that "in general" it would be possible to produce in this manner the requisite $2n-1$ independent functions. He realized, on the other hand, that in "particular cases," such as when a certain number of solutions to $(F_1,\Phi) = 0$ are known from general mechanical principles, the above bracketing process falls short of the goal of generating $2n-1$ independent solutions. For example, it could happen that

$F_4 = (F_2, F_3)$ vanishes or is some function of F_1, F_2, and F_3. He claimed that in such cases, knowledge of these solutions could be used to simplify the problem of solving the PDE $F_1(x, p) = 0$.

Perhaps due to the poor health that plagued him after 1843 as a consequence of diabetes, Jacobi never published any detailed justification of his claims before his death in 1851. Most mathematicians doubted them. Still left open was

JACOBI'S PROBLEM. *Suppose $\Phi = F_1, \ldots, F_r$ are r functionally independent solutions to $(F_1, \Phi) = 0$ with the property that bracketing produces no more solutions, that is, for all i, j (F_i, F_j) is functionally dependent on $F_1, \ldots, F_r$:*

$$(F_i, F_j) = \Omega_{ij}(F_1, \ldots, F_r), \quad i, j = 1, \ldots, r. \qquad (9)$$

How does knowledge of $F_1, \ldots, F_r$ simplify the problem of solving $F_1(x, p) = 0$?

Although several mathematicians had dealt with this problem in the case that the equations in (9) have the special form $(F_i, F_j) = 0, \pm 1$, Lie was the first to tackle the general problem. For him, it was an expression of his *idée fixe*. This understanding of the problem was based on discoveries he had made in seeking to develop his invariant theory of contact transformations.

Before discussing these discoveries, some preparatory remarks about Lie's use of infinitesimal transformations are necessary. Infinitesimal transformations were fundamental to the conception of a continuous group that Lie had developed as an outgrowth of his work on tetrahedral complexes in 1870. For him, a group of transformations was continuous because its transformations could be generated by infinitesimal ones. In present-day formulations of Lie's theory, the infinitesimal transformations of a group are the elements of its Lie algebra. Lie himself tended to think of the infinitesimal transformations as themselves belonging to the group. He identified such a transformation

$$dT : y = (y_1, \ldots, y_m) \rightarrow (y_1 + dy_1, \ldots, y_m + dy_m),$$
$$dy_i = \eta_i(y)dt$$

with the system of ordinary differential equations

$$\frac{dy_i}{dt} = \eta_i(y), \quad i = 1, \ldots, m. \qquad (10)$$

If $y_i(t) = f_i(t, y^{(0)})$, $i = 1, \ldots, m$, is the solution to this system satisfying the initial condition $y_i(t_0) = y_i^{(0)}$, then these equations defined the one-parameter family of transformations $T_t : y^{(0)} \rightarrow y(t)$ generated by dT.

As Lie learned about the theory of first-order PDEs as it stood circa 1870, he found that solving the system (10) was equivalent to solving the linear homogeneous PDE

$$\sum_{i=1}^{m} \eta_i(y) \frac{\partial f}{\partial y_i} = 0. \qquad (11)$$

Mathematicians writing about (11) would switch to (10) without a word and vice versa: To them, (10) and (11) were two sides of the same coin. Lie also fell into this habit of identification. Thus, dT could be identified with the system (10) or with (11). Jacobi's work added another twist to this identification process. In developing his method of solving first-order PDEs, Jacobi had introduced differential-operator notation such as

$$Y(f) = \sum_{i=1}^{m} \eta_i(y) \frac{\partial f}{\partial y_i}$$

so that (11) can be expressed succinctly as $Y(f) = 0$. He also utilized the fact that if $X(f)$ is another such operator, then so is $X(Y(f)) - Y(X(f))$. Eventually, Lie identified the infinitesimal transformation dT with $Y(f)$ (which nowadays would, in turn, be identified with a vector field). If this identification is expressed by the notation $dU \leftrightarrow X(f)$, $dT \leftrightarrow Y(f)$, then Jacobi's calculus of differential operators led Lie to the further identifications:

$$\begin{aligned} dU \circ dT &\leftrightarrow X(f) + Y(f), \\ dU \circ dT \circ dU^{-1} \circ dT^{-1} &\leftrightarrow X(Y(f)) - Y(X(f)). \end{aligned} \qquad (12)$$

These identifications proved inspirational to him as he developed his invariant theory of contact transformations.

In the $(n+1)$-dimensional space of points $(x, z) = (x_1, \ldots, x_n, z)$, contact transformations are transformations $T : (x, z, p) \rightarrow (X, Z, P)$ of the $2n + 1$ variables $(x, z, p) = (x_1, \ldots, x_n, z, p_1, \ldots, p_n)$ which satisfy the higher-dimensional analog of (4). Lie discovered that in developing his theory for PDEs of the form $F(x, p) = 0$, he could restrict his attention to those with equations of the form $X_i = \xi_i(x, p)$, $Z = z + \zeta(x, p)$, $P_i = \pi_i(x, p)$. Following Lie, these contact transformations will be referred to as *transformations of* (x, p). He realized that, as transformations $(x, p) \rightarrow (X, P)$, they define canonical transformations in the sense of Hamilton–Jacobi dynamics. A link between the theory of PDEs and his *idée fixe* was provided by his discovery that $dT \leftrightarrow X(f)$ is an infinitesimal transformation of (x, p) if and only if there is a function $W(x, p)$ such that

$$X(f) = [W, f] \quad \text{for all } f = f(x, z, p), \qquad (13)$$

where the generalized Poisson brackets are defined for functions G, H of (x, z, p) by

$$[G, H] = \sum_{i=1}^{n} \frac{\partial G}{\partial p_i} \frac{DH}{\partial x_i} - \frac{DG}{\partial x_i} \frac{\partial H}{\partial p_i},$$

$$\frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z}.$$

Generalized Poisson brackets had been introduced into Jacobi's theory by his successors. They have the same basic properties as the Poisson brackets (7). In particular, the analogous Jacobi Identity holds. Lie discovered the connection between infinitesimal transformations of (x, p) and functions $W(x, p)$ by a line of reasoning typical in projective geometry: Take a simple fact (e.g., about a circle) and apply projective transformations to turn it into a general theorem (e.g., about conics). Lie's simple fact was from the theory of first-order PDEs, and contact, rather than projective, transformations were applied to obtain the general result.

Consider Jacobi's Problem. Given the solutions $\Phi = F_i$ to $(F_1, \Phi) = 0$, it follows from the chain rule that any function of $F_1, \ldots, F_r$, $F = \Theta(F_1, \ldots, F_r)$ for some function Θ of r variables, is also a solution to $(F_1, \Phi) = 0$. Thus, in Jacobi's Problem all solutions of the form $F = \Theta(F_1, \ldots, F_r)$ are known, and the problem is to use this information to simplify the integration of $F_1(x, p) = 0$. Now by virtue of (13), in Lie's mind $F = \Theta(F_1, \ldots, F_r)$ corresponded to an infinitesimal transformation $X(f) = [F, f]$. He realized that an infinitesimal transformation $X(f)$ is admitted by a PDE $F_1(x, p) = 0$ if and only if $X(F_1) = 0$ for all (x, p) satisfying $F_1(x, p) = 0$. By virtue of (13), $X(F_1) = [F, F_1] = (F, F_1) = 0$, so that $F_1 = 0$ admits the infinitesimal transformation $X(f) = [F, f]$ corresponding to $F = \Theta(F_1, \ldots, F_r)$. Lie also realized that these $X(f)$ form a Lie algebra. This property of the $X(f)$ is a consequence of the Jacobi Identity for generalized Poisson brackets. At this time, Lie did not concern himself with the question as to whether a group corresponded to this Lie algebra. He seems to have assumed it did, but, in any case, he focused his attention on the "group" of infinitesimal transformations $X(f)$, which will be denoted by $\mathcal{G}(F_1, \ldots, F_r)$. Jacobi's Problem could then be interpreted as an expression of his *idée fixe*:

> *Given that the PDE $F_1(x, p) = 0$ admits the group $\mathcal{G}(F_1, \ldots, F_r)$ of infinitesimal transformations, how does this knowledge simplify the problem of solving it?*

Lie provided an answer to this question and called the attendant theory his theory of function groups. It became the core of his invariant theory of contact transformations. Now it can be seen as a precursor of that part of symplectic geometry involving Poisson structures.

Although the concept of a function group was motivated by Jacobi's Problem, Lie's actual definition was independent of the Problem and runs as follows. Given functionally independent functions of (x, p), $F_1, \ldots, F_r$, which satisfy relations of the form (9)—but which need not be solutions to $(F_1, \Phi) = 0$—let $\mathcal{F}(F_1, \ldots, F_r)$ consist of all functions F which are functionally dependent on $F_1, \ldots, F_r$, so that $F = \Theta(F_1, \ldots, F_r)$ for some Θ. Then $\mathcal{G}(F_1, \ldots, F_r)$ consists of all infinitesimal transformations of the form $X(f) = [F, f]$ for some $F \in \mathcal{F}(F_1, \ldots, F_r)$. In his publications, written for 19th-century analysts with no understanding of group-related concepts, Lie officially designated $\mathcal{F}(F_1, \ldots, F_r)$ as an "r-term function group" and did not mention $\mathcal{G}(F_1, \ldots, F_r)$, even though he himself thought in terms of $\mathcal{G}$. Equipped with the Poisson bracket, $\mathcal{F}(F_1, \ldots, F_r)$ is a Lie algebra in its own right, and the mapping $F \to X(f) = [F, f]$ is a homomorphism with kernel consisting of the constant functions, so that the distinction between $\mathcal{F}$ and $\mathcal{G}$ is more significant conceptually than mathematically.

Using his theory of function groups, Lie was able to resolve the problem of when two systems of PDEs, $F_i(x, p) = 0$ and $G_i(x, p) = 0$, $i = 1, \ldots, r$, are transformable into each other by a contact transformation in the sense that each F_i gets transformed into the corresponding G_i. Recall from the second section that this was a major goal of his prospective invariant theory in 1871. Lie first solved the problem when the F_i and G_i form r-term function groups. In this case, the necessary and sufficient condition is that for all i and j, (F_i, F_j) is the same function of $F_1, \ldots, F_r$ as (G_i, G_j) is of $G_1, \ldots, G_r$:

$$\begin{aligned} (F_i, F_j) &= \Omega_{ij}(F_1, \ldots, F_r) \\ \text{iff } (G_i, G_j) &= \Omega_{ij}(G_1, \ldots, G_r). \end{aligned} \tag{14}$$

As we shall see, Lie discovered this theorem had an inspiring application to the group classification problem.

As a Lie algebra, $\mathcal{G}(F_1, \ldots, F_r)$ is infinite-dimensional. Jacobi's method of solving first-order PDEs was also the source of Lie's discovery that finite-dimensional groups are likewise important to the theory of first-order PDEs. In 1866 Clebsch had developed his theory of complete systems of linear homogeneous PDEs, motivated by the fact that both Jacobi's method and his own extension of it to Pfaffian equations reduced the problem to solving complete systems. (Clebsch's theory is now subsumed in the theory of completely integrable vector fields.) Lie sought to apply his *idée fixe* to complete systems just as he had to Jacobi's Problem. In the case of Jacobi's Problem he had seen that if $F_1(x, p) = 0$ admits a finite number of infinitesimal transformations, then, by applying the bracketing process, it can be assumed without loss of generality that $F_1 = 0$ admits a function group $\mathcal{G}(F_1, \ldots, F_r)$. Within the context of complete systems, somewhat analogous reasoning led him to the discovery that if a complete system admits a finite number of infinitesimal transformations, then without loss of generality it can be assumed that the system admits infinitesimal transformations $X_1(f), \ldots, X_r(f)$ which satisfy

$$X_i(X_j(f)) - X_j(X_i(f)) = \sum_{i=1}^{r} c_{ijk} X_k(f). \tag{15}$$

In modern terms this means that the $X_i(f)$ span a finite-dimensional Lie algebra. To Lie, it meant that the complete system admits a group which depends continuously on a finite number of parameters, the type of group that had arisen in his early work related to tetrahedral complexes. These groups could now be seen to be important to the general theory of PDEs. Moreover, Lie discovered a link between these groups and his theory of function groups, a link which encouraged him to believe that he was now in a position to solve, for these groups, the sort of classification problem that he had earlier dismissed as impossible.

For Lie, two transformation groups were essentially the same — "similar" was his term — if the transformations of the one could be transformed into those of the other by means of a variable change. On the infinitesimal level on which he operated, the problem of classifying all finite-dimensional groups of transformations was to determine up to similarity all finite-dimensional groups (Lie algebras) of infinitesimal transformations in m variables, $X(f) = \sum_{i=1}^{m} \eta_i(y)\,\partial f/\partial y_i$, $y = (y_1, \ldots, y_m)$. Today, Lie's problem would be formulated as that of determining all Lie algebras of vector fields up to diffeomorphisms. In the case $m = 1$, Lie was able to use (15) to conclude by elementary considerations that there are only three distinct possibilities (the projective group of the line and its subgroups). This success undoubtedly helped him appreciate the value of (15), but his confidence that he could achieve comparable success in the case of any number of variables came from another source.

As soon as the number of variables exceeds one, there are contact transformations as well as point transformations to consider. Point transformations being (as noted in the second section) special types of contact transformations, Lie's problem was to classify all Lie algebras of infinitesimal contact transformations. In this connection he had discovered that just as the study of projective transformations can be reduced to the study of linear transformations, so too the study of contact transformations can be reduced to the study of what he called "homogeneous contact transformations." On the infinitesimal level, these are transformations of (x, p) which leave z fixed. Consequently, by (13), an infinitesimal homogeneous transformation has the form $X(f) = [W, f] = (W, f)$. Lie's problem reduced to classifying all Lie algebras of infinitesimal homogeneous contact transformations $X_i(f) = (F_i, f)$. Since in this case $X(f)$ is given by a Poisson bracket, it follows from Jacobi's Identity that $X_i(X_j(f)) - X_j(X_i(f)) = ((F_i, F_j), f)$, and so the Lie algebra property (15) implies

$$(F_i, F_j) = \sum_{k=1}^{r} c_{ijk} F_k. \qquad (16)$$

Equation (16) was the primary reason why Lie changed his mind about the impossibility of group classification problems. It showed him that a finite-dimensional group has associated to it a function group. Suppose that $p \leq r$ of the functions F_i are functionally independent and that the notation is such that $F_1, \ldots, F_p$ are independent. Then for $k > p$, F_k is some function of $F_1, \ldots, F_p$, and (16) implies that (F_i, F_j) is expressible as a function of $F_1, \ldots, F_p$, which is precisely the condition (9) that $F_1, \ldots, F_p$ determine a p-term function group $\mathcal{G}(F_1, \ldots, F_p)$.

By means of this connection between the r-dimensional group and the p-term function group, Lie believed he could classify the r-dimensional groups into general similarity types by applying the results of the theory of function groups. To give some idea of how this worked — and some of the critical points Lie overlooked — it suffices to consider the case in which $F_1, \ldots, F_r$ are functionally independent, so $p = r$. Suppose $X_i(f) = (F_i, f)$ and $Y_i(f) = (H_i, f)$ define two groups of this type, and that in addition [possibly by taking suitable linear combinations of the $Y_i(f)$] the constants c_{ijk} are the same for the $Y_i(f)$. In other words, the assumption is that the two Lie algebras are isomorphic. In view of (16) this means, of course, that (F_i, F_j) is the *same* function of $F_1, \ldots, F_r$ as (H_i, H_j) is of $H_1, \ldots, H_r$. Thus, (14) is satisfied, so that by the related theorem Lie could infer the existence of a homogeneous contact transformation which takes each function H_i into the corresponding F_i. This meant that the two r-dimensional groups $X_i(f)$ and $Y_i(f)$ are similar. In other words, Lie concluded that groups of this type are similar if and only if their Lie algebras are isomorphic, so that the classification problem reduces in this case to that of classifying Lie algebras up to isomorphism. Lie envisioned a limited number of nonisomorphic possibilities and evidently felt it would not be difficult to determine them. In the fall of 1873 he, thus, glossed over an algebraic problem which is still not completely resolved today! The problem was first tackled in 1888 by Killing, who resolved it in the semisimple case.

...the task of creating a theory of continuous transformation groups, the task that became his life work.

Since Lie's classification of groups with $p < r$ also depended on the classification of Lie algebras, his overall approach was not as effective as he believed. At the time, however, it seemed that his theory of function groups, and more generally his invariant theory of contact transformations, had provided him with precisely the tools he needed to tackle the group classification problem. Not only was he now convinced of the importance of continuous groups to the general theory of PDEs but he also saw the theory of these groups as drawing on the same mathematical tools as the general theory of PDEs. These considerations gave him the courage to make the decision to commit himself completely to the task of creating a theory of continuous transformation groups, the task that became his life work.

Concluding Remarks

Lie's encounter with the work of Jacobi was evidently a decisive factor in his decision. But, as the presentation in the first two sections indicates, it was only because of his experiences during 1869–1871, when geometry was the focus of his interests, that Lie was in a position to see in Jacobi's theory something no one else saw. During 1869–1871 his work had been dominated by two successive research projects: the study of the geometry of tetrahedral complexes and the study of the sphere mapping. From the first came a fundamental concept and a fundamental idea: the concept of a continuous group of transformations and the idea of a continuous analog of Galois's theory of algebraic equations — his *idée fixe*. Likewise, a fundamental concept and a related idea originated in the sphere mapping work: the concept of a contact transformation and the idea of an invariant theory of contact transformations. It was in terms of these ideas and concepts that Lie assimilated the theory of first-order PDEs of Jacobi and his successors, and turned it into something quite different.

Further Reading

For further information about Lie's early geometrical work, see the articles by T. Hawkins ("Line Geometry, Differential Equations and the Birth of Lie's Theory of Groups") and by D. Rowe ("The Early Geometrical Works of Sophus Lie and Felix Klein") in Volume 1 of *The History of Modern Mathematics* (D. Rowe and J. McCleary, eds.), London: Academic Press (1989). For more information about Jacobi's influence on Lie, see T. Hawkins, Jacobi and the birth of Lie's theory of groups, *Arch. History Exact Sci.* (to appear). A modern presentation of Lie's group-theoretic approach to differential equations can be found in P. Olver, *Applications of Lie Groups to Differential Equations*, New York: Springer-Verlag (1986). Olver's book also contains informative historical notes, such as one on Lie's function groups and symplectic geometry (pp. 417–418). Killing's work is discussed in T. Hawkins, Wilhelm Killing and the structure of Lie algebras, *Arch. History Exact Sci.* 26 (1982), 127–192. See also: T. Hawkins, Non-Euclidean geometry and Weierstrassian mathematics: The background to Killing's work on Lie algebras, *Studies in the History of Mathematics* (E.R. Phillips, ed.), Washington DC: Mathematical Association of America (1987), pp. 21–36; A.J. Coleman, The greatest mathematical paper of all time, *Mathematical Intelligencer* 11 (3) (1989), 29–38; N.H. Ibragimov, Sophus Lie and harmony in mathematical physics, *The Mathematical Intelligencer* 16 (1) (1993), 20–28.

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Who Is Number One?[1]

Heard after dinner at the Vienna conference on operator theory:

Alexander Markus: I tell you, this young man is a very strong mathematician. You should take it seriously when I say that, but you should take it more seriously if I can state precisely some result that he has proved. Well, in this case I can state *three* of his theorems. That makes him very outstanding indeed.

Several others: This sounds like a general criterion for telling who is outstanding: How many of their results can you state?

Naum Krupnik: But if we use that criterion, I can guess whom everyone will rank number one...

[1] Compare vol. 14, no. 3, p. 23.