"Ueber die Anwendung der Jacobi-Hamilton'schen Methode auf den Fall der Anziehung nach dem electrodynamischen Gesetze von Weber," Zeit. Math. Phys. **15** (1870), 69-91.

On the application of the Jacobi-Hamilton method to the case of attraction according to Weber's electrodynamical laws.

By

GUSTAV HOLZMÜLLER

in Merseburg

Translated by D. H. Delphenich

§ 1. – Introduction.

In his lectures on dynamics, **Jacobi** confined himself to the application of **Hamilton**'s method to those problems for which the motion depended upon only the configuration of points, but not upon their velocities. However, as **Riemann** has remarked already, that method can also be applied to many problems for which the velocities come into question.

If one addresses the mutual attraction of points and one understands T to mean one-half of the *vis viva* and U to mean that function that will give the components of the force that acts at a point when it is differentiated with respect to the rectangular coordinates of that point then if U includes only the coordinates along with time, implicitly or explicitly, but not upon the velocities, then the equation:

$$\delta\int_{t_0}^{t_1}(T+U)\,dt = 0. \tag{1}$$

In this, the variation is understood to mean that the limits are not varied. This equation also preserves its validity when condition equations are present, which can include time; however, we shall ignore such things entirely.

The existence of equation (1) will be assumed to be the first requirement for the **Hamilton**ian method. It follows from the usual equations of motion, and conversely, the latter can be derived from it.

However, if U includes the velocities then it will be questionable whether the usual equations of motion would follow from equation (1). One must then examine whether that would be true in each problem. In the former case, we can start from that equation, but in the latter, we cannot. However, in the former case, if one should succeed in determining another function U, such that when one substitutes it for the previous function U in equation (1), the usual equations of motion would follow from it, then one can put that new equation at the forefront of any further investigations.

We still do not have a general method for finding that new U. **Riemann** gave some brief remarks in regard to the case in which the potential consisted of two parts, one of which included the velocities, while the other one was free of them.

If a U has been found in some way then $q_1, \ldots, q_\nu$ will be independent coordinates, and in order to extend the differential quotients, new quantities $p_1, \ldots, p_\nu$ will be introduced by the equations:

$$\frac{\partial(T+U)}{\partial q'_\mu} = p_\mu ,$$

so the differential equations of the problem will be derived from equation (1). **Hamilton**'s method replaces the system of differential equations with a first-order partial differential equation in (ν + 1) variables whose general solution V is to be sought, which then includes n arbitrary constants $\alpha_1, \ldots, \alpha_\nu$, in addition to the additive constants α.

If one then succeeds in proving that the 2ν equations:

$$\frac{\partial V}{\partial \alpha_1} = \beta_1, \ldots, \frac{\partial V}{\partial \alpha_\nu} = \beta_\nu ,$$

$$\frac{\partial V}{\partial q_1} = p_1, \ldots, \frac{\partial V}{\partial q_\nu} = p_\nu ,$$

in which the β are new arbitrary constants, are the integral equations of the above system of differential equations then the application of **Hamilton**'s method will be justified, and the problem will have been reduced to performing certain integrals. However, it we do not succeed in that proof then the results will be illusory.

One can find the general proof in **Jacobi**'s Lecture 20 for the case in which U includes only the coordinates and time, implicitly or explicitly, but not the velocities. However, a special analysis will always be necessary for those cases in which U does include the velocities.

Neumann (*) has found the expression U that should be inserted into equation (1) in a curious way for the closely-associated problem of *attraction according to Weber's electrodynamical laws*, and in so doing, gave a new meaning to those laws.

Even now, that interesting law has been the basis of many investigations, but in the opinion of many researchers, it will probably play a greater role at some later time. It is certainly quite important already, insofar as it includes the **Newton**ian case as a special case.

In what follows, we would like to examine whether the **Hamilton**ian method can be applied to this problem, and indeed, what we shall discuss first will be the *motion of a point that is attracted to a fixed center according to Weber's laws.*

(*) **Neumann**, *Prinzipien der Elektrodynamik*, Tübingen, 1868. – Cf., also **Seeger**, Inaugural dissertation: *De motu perturbationibusque planetarum etc.* Göttingen, 1864.

However, before we go on to the actual investigation, we shall first make some necessary remarks about the connection between **Weber**'s expressions and **Neumann**'s.

If r is the distance from the point x, y, z to the attracting center and m is its mass then according to **Weber**, the attraction that acts in the direction of r will be:

$$R = -\frac{m}{r^2}\left[1 - \frac{1}{c^2}\cdot\left(\frac{dr}{dt}\right)^2 + \frac{2r}{c^2}\cdot\frac{d^2 r}{dt^2}\right]. \tag{2}$$

In this, c is a very large constant, and indeed one suspects that it is roughly the speed of light.

The usual force function will then be:

$$U_1 = \frac{m}{r}\left(1 - \frac{r'^2}{c^2}\right). \tag{3}$$

It is clear that the motion of the point will take place in a plane that is determined by the direction of r and the initial push. There will then be no basis for abandoning the plane that the point determines. We therefore confine ourselves to two coordinates.

The equations of motion are:

$$\left\{\begin{aligned} m\frac{d^2 x}{dt^2} &= R\frac{x}{r} = \frac{\partial U_1}{\partial r}\cdot\frac{x}{r}, \\ m\frac{d^2 y}{dt^2} &= R\frac{y}{r} = \frac{\partial U_1}{\partial r}\cdot\frac{y}{r}, \end{aligned}\right. \tag{4}$$

and they imply that:

$$m\frac{d}{dt}\left(\frac{ds}{dt}\right)^2 = 2\frac{\partial U_1}{\partial r}\cdot\frac{dr}{dt} = 2\frac{dU_1}{dt},$$

as is easy to see, and upon integration, that:

$$T = U_1 + c_1 . \tag{5}$$

The principle of *vis viva* is therefore valid, and it cannot be applied with no further assumptions. Let us also mention, by the way, that the area principle is also true, since we are dealing with a central force.

Now, **Neumann** has found that one will come to the equations of motion (4) when one sets:

$$U = \frac{m}{r}\left(1 + \frac{r'^2}{c^2}\right) \tag{6}$$

in equation (1), instead of the usual U_1, and in fact, one will come to that U as follows: If the center attracts the point x, y, z with mass m according to **Newton**'s laws then the distance r_0 will belong to the potential m / r_0 . Now, assuming that the potential does not assert itself immediately, but comes to one with the finite velocity of the moving point (e.g., that of light or sound) then the potential at the point that belongs to the distance r_0 will first be attained when it is found at another distance r, such that m / r is the potential that acts there, whereas m / r_0, or as would emerge from **Neumann**'s analysis, the expression:

$$U = \frac{m}{r}\left(1+\frac{r'^2}{c^2}\right),$$

in which c is the speed with which the potential advances in space.

The fact that one actually goes from equation (1) to equations (4) is easy to show now. If T is one-half the *vis viva* then one will have:

$$\delta\int_{t_0}^{t_1} T\,dt = \int_{t_0}^{t_1} mx'\,\delta x'\,dt + \int_{t_0}^{t_1} my'\,\delta y'\,dt\,.$$

Integration by parts will give a new integral for every integral on the right-hand side, along with a part that is free of integrals. The latter will drop out since the end positions will not be varied, and what will remain will then be:

$$\delta\int_{t_0}^{t_1} T\,dt = -m\int_{t_0}^{t_1} (x''\,\delta x + y''\,\delta y)\,dt\,.$$

It follows further from $U = \frac{m}{r}\left(1+\frac{r'^2}{c^2}\right)$ that:

$$\delta U = \frac{\partial U}{\partial r}\delta r + \frac{\partial U}{\partial r'}\delta r' = -\frac{m}{r}\left(1+\frac{r'^2}{c^2}\right)\delta r + \frac{2mr'}{rc^2}\cdot\delta r'\,,$$

so:

$$\delta\int_{t_0}^{t_1} U\,dt = -\int_{t_0}^{t_1}\frac{m}{r}\left(1+\frac{r'^2}{c^2}\right)\delta r\,dt + \frac{2mr'}{c^2}\int_{t_0}^{t_1}\frac{r'}{r}\delta r'\,dt\,.$$

If one applies integration by parts to the last integral and combines everything then it will follow that:

$$\delta\int_{t_0}^{t_1} U\,dt = -\int_{t_0}^{t_1}\frac{m}{r}\left(1+\frac{r'^2}{c^2}+\frac{2rr'}{c^2}\right)\delta r\,dt = \int_{t_0}^{t_1} R\,\delta r\,dt\,.$$

Therefore:

$$\delta\int_{t_0}^{t_1}(T+U)\,dt = -\int_{t_0}^{t_1}\left[\left(mx''-R\frac{x}{r}\right)\delta x + \left(my''-R\frac{y}{r}\right)\delta y\right]dt = 0.$$

However, δx and δy are mutually independent, such that the following equations will be true:

$$m\,x'' = R\frac{x}{r}, \quad m\,y'' = R\frac{y}{r};$$

however, these are the equations of motion above.

We shall now take equation (1) to be the starting point of the investigation.

§ 2. – Derivation of the differential equations of the problem in independent coordinates.

r and ϑ are introduced as independent coordinates by way of the equations:

$$x = r \cos \vartheta, \qquad y = r \sin \vartheta,$$

such that $(T + U)$ will be a function of r, r', ϑ, ϑ'. If one defines:

$$\delta\int_{t_0}^{t_1} (T+U)\,dt$$

and applies integration by parts then since the parts that are free of integrals will drop out, that will yield:

$$0 = \int_{t_0}^{t_1}\left\{\left[\frac{\partial(T+U)}{\partial r} - \frac{d}{dt}\left(\frac{\partial(T+U)}{\partial r'}\right)\right]\delta r + \left[\frac{\partial(T+U)}{\partial \vartheta} - \frac{d}{dt}\left(\frac{\partial(T+U)}{\partial \vartheta'}\right)\right]\delta\vartheta\right\}dt\,,$$

in place of equation (1) in **§ 1**.

Since δr and $\delta\vartheta$ are mutually independent that will imply the equations:

$$\frac{\partial(T+U)}{\partial r} - \frac{d}{dt}\left(\frac{\partial(T+U)}{\partial r'}\right) = 0,$$

$$\frac{\partial(T+U)}{\partial \vartheta} - \frac{d}{dt}\left(\frac{\partial(T+U)}{\partial \vartheta'}\right) = 0.$$

These are the equations of motion in the new form.

The value of $(T + U)$ in the new coordinates is:

$$(T + U) = r'^2\left(\frac{m}{2} + \frac{m}{rc^2}\right) + \frac{mr^2\vartheta'}{2} + \frac{m}{r}.$$

New quantities p are introduced by the equations:

$$\frac{\partial(T+U)}{\partial r'} = p_1, \qquad \frac{\partial(T+U)}{\partial \vartheta'} = p_2 ,$$

i.e.:

$$p_1 = 2r'\, m \left(\frac{1}{2} + \frac{1}{rc^2}\right), \qquad p_2 = m\, r^2\, \vartheta' .$$

If we now denote $(T + U)$ by $| T + U |$ once the new quantities have been introduced then we will get:

$$| T + U | = \frac{p_1^2}{4m\left(\frac{1}{2} + \frac{1}{rc^2}\right)} + \frac{p_2^2}{2mr^2} + \frac{m}{r} = p_1\, r' + p_2\, \vartheta' + \frac{m}{r} .$$

The following equations are now true for this expression:

$$\frac{\partial\, |T+U|}{\partial p_1} = r', \qquad \frac{\partial\, |T+U|}{\partial p_2} = \vartheta',$$

$$\frac{\partial\, |T+U|}{\partial \vartheta} = \frac{\partial(T+U)}{\partial \vartheta} = 0,$$

$$\frac{\partial\, |T+U|}{\partial r} = -\frac{\partial(T+U)}{\partial r} - \frac{2m}{r^2} .$$

If one introduces these results into the equation of motion above then that will yield the following *differential equations for the problem:*

$$(1) \qquad \begin{cases} \alpha) & \dfrac{\partial\, |T+U|}{\partial r} = -\dfrac{dp_1}{dt} - \dfrac{2m}{r^2}, \\[2ex] \beta) & \dfrac{\partial\, |T+U|}{\partial \vartheta} = -\dfrac{dp_2}{dt} = 0, \end{cases}$$

and one will then come to:

$$(2) \qquad \begin{cases} \alpha) & \dfrac{\partial\, |T+U|}{\partial p_1} = \dfrac{dr}{dt}, \\[2ex] \beta) & \dfrac{\partial\, |T+U|}{\partial p_2} = \dfrac{d\vartheta}{dt}. \end{cases}$$

§ 3. – Presentation of the first-order partial differential equation.

If we set:

$$(1) \qquad V = \int_{t_0}^{t_1} (T+U)\, dt$$

and form δV, but in such a way that the end positions are also varied, then the expressions that are free of integrals will no longer drop out, such that we will have:

$$\delta V = \left| \frac{\partial(T+U)}{\partial r'}\delta r + \frac{\partial(T+U)}{\partial \vartheta'}\delta\vartheta \right|_{t_0}^{t_1}$$

$$+ \int_{t_0}^{t_1}\left\{\left[\frac{\partial(T+U)}{\partial r} - \frac{d}{dt}\left(\frac{\partial(T+U)}{\partial r'}\right)\right]\delta r + \left[\frac{\partial(T+U)}{\partial \vartheta} - \frac{d}{dt}\left(\frac{\partial(T+U)}{\partial \vartheta'}\right)\right]\delta\vartheta\right\}dt\,.$$

From what was said above, the last part is equal to zero. If we then let r_0 and ϑ_0 denote the initial position, while r and ϑ denote the final position then what will remain will be:

$$\delta V = \frac{\partial(T+U)}{\partial r'}\delta r - \frac{\partial(T+U)}{\partial r_0'}\delta r_0 + \frac{\partial(T+U)}{\partial \vartheta'}\delta\vartheta' - \frac{\partial(T+U)}{\partial \vartheta_0'}\delta\vartheta_0'\,.$$

However, if we consider V to be a function of the initial and final positions and the elapsed time then:

$$\delta V = \delta r_0 + \frac{\partial V}{\partial \vartheta_0}\delta\vartheta_0 + \frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial \vartheta}\delta\vartheta.$$

If one compares the last two expressions and sets the corresponding factors equal to each other, since the variations are arbitrary, then the following equations will arise:

$$\frac{\partial V}{\partial r} = \frac{\partial(T+U)}{\partial r'} = p_1\,, \qquad \frac{\partial V}{\partial \vartheta} = \frac{\partial(T+U)}{\partial \vartheta'} = p_2\,,$$

$$\frac{\partial V}{\partial r_0} = -\frac{\partial(T+U)}{\partial r_0'} = -p_{01}\,, \qquad \frac{\partial V}{\partial \vartheta_0} = -\frac{\partial(T+U)}{\partial \vartheta'} = -p_{02}\,.$$

The p can then be replaced with partial differential quotients of V in the independent coordinates.

For now, we shall consider V to be a function of the initial and final positions and the elapsed time t; the latter is included explicitly in V, as well as implicitly in the final coordinates, but not in the initial coordinates, such that:

3a) $$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial \vartheta}\cdot\frac{d\vartheta}{dt}\,.$$

Moreover, differentiating equation (1) with respect to the upper limit t will give:

3b) $$\frac{dV}{dt} = T + U.$$

By subtracting (3a) and (3b), we will get:

$$0 = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r} r' + \frac{\partial V}{\partial \vartheta} \vartheta' - (T + U),$$

or when we sets:

$$\psi = \frac{\partial V}{\partial r} r' + \frac{\partial V}{\partial \vartheta} \vartheta' - (T + U) = p_1 r' + p_2 \vartheta' - | T + U |$$

$$= 2 | T + U | - \frac{2m}{r} - | T + U | = | T + U | - \frac{2m}{r},$$

(4) $$\frac{\partial V}{\partial t} + \psi = 0.$$

We can introduce r, ϑ, p_1, p_2 into this equation in place of r, r', ϑ, and ϑ'; however, p_1 and p_2 can be replaced with $\partial V / \partial r$ and $\partial V / \partial \vartheta$, such that only r, ϑ, t, $\partial V / \partial r$, $\partial V / \partial \vartheta$ will still remain in equation (4). Under these transformations equation (4) will then go to a *first-order partial differential equation with three independent variables, and V will be defined as a function of r, t, and* ϑ *by it.*

§ 4. – Proof of the applicability of the Hamiltonian method to our problem.

We must prove the following assertion:

If V is the general solution of the first-order partial differential equation above – i.e., an integral that contains arbitrary constants α_1 and α_2 in addition to the additive constants α, and we set:

(I) $$\frac{\partial V}{\partial \alpha_1} = \beta_1, \qquad \frac{\partial V}{\partial \alpha_2} = \beta_2,$$

in which β_1 and β_2 are new arbitrary constants, and if we select the following equations from the group (2) in **§ 3** :

(II) $$\frac{\partial V}{\partial r} = p_1, \qquad \frac{\partial V}{\partial \vartheta} = p_2,$$

then we assert that these equations are the integral equations for the system of differential equations at the end of **§ 2**.

Proof: The proof will be complete when we show that differentiating equations (I) and (II) with respect to t will lead to the differential equations above.

Equations (I) are initially true identically for any arbitrary t – e.g., for $(t + dt)$, as well. If I then differentiate both sides with respect to t then I can set both sides equal to each other. Now, since t enters into $\partial V / \partial \alpha_1$ explicitly, as well as implicitly (in r and ϑ), that will yield:

(a)
$$\left\{\begin{aligned} 0 &= \frac{\partial^2 V}{\partial \alpha_1 \, \partial t} + \frac{\partial^2 V}{\partial \alpha_1 \, \partial r} \cdot \frac{dr}{dt} + \frac{\partial^2 V}{\partial \alpha_1 \, \partial \vartheta} \cdot \frac{d\vartheta}{dt}, \\ 0 &= \frac{\partial^2 V}{\partial \alpha_2 \, \partial t} + \frac{\partial^2 V}{\partial \alpha_2 \, \partial r} \cdot \frac{dr}{dt} + \frac{\partial^2 V}{\partial \alpha_2 \, \partial \vartheta} \cdot \frac{d\vartheta}{dt}. \end{aligned}\right.$$

Moreover, I can partially differentiate equation (4) in **§ 3** with respect to the α, since the α are arbitrary, so the equation will also be true identically for $(\alpha + d\alpha)$. We will then get:

b)
$$\left\{\begin{aligned} 0 &= \frac{\partial^2 V}{\partial \alpha_1 \, \partial t} + \frac{\partial \psi}{\partial \alpha_1}, \\ 0 &= \frac{\partial^2 V}{\partial \alpha_2 \, \partial t} + \frac{\partial \psi}{\partial \alpha_2}. \end{aligned}\right.$$

However:

$$\psi = \frac{\partial V}{\partial r} r' + \frac{\partial V}{\partial \vartheta} \vartheta' - (T + U),$$

in which r', ϑ', $\partial V / \partial r$, $\partial V / \partial \vartheta$ are replaced with r, ϑ, p_1, and p_2 by means of the equations:

$$r' = \frac{p_1}{2m\left(\frac{1}{2} + \frac{1}{rc^2}\right)}, \qquad \vartheta' = \frac{p_2}{mr^2},$$

$$p_1 = \frac{\partial V}{\partial r}, \qquad p_2 = \frac{\partial V}{\partial \vartheta}.$$

If one has represented ψ as a function of t, r, ϑ, p_1, and p_2 in that way then obviously α_1 and α_2 will be included in ψ only insofar as they are present in p_1 and p_2 ; equations (b) will then go to:

$$0 = \frac{\partial^2 V}{\partial \alpha_1 \, \partial t} + \frac{\partial \psi}{\partial p_1} \cdot \frac{\partial p_1}{\partial \alpha_1} + \frac{\partial \psi}{\partial p_2} \cdot \frac{\partial p_2}{\partial \alpha_1},$$

$$0 = \frac{\partial^2 V}{\partial \alpha_2 \, \partial t} + \frac{\partial \psi}{\partial p_1} \cdot \frac{\partial p_1}{\partial \alpha_2} + \frac{\partial \psi}{\partial p_2} \cdot \frac{\partial p_2}{\partial \alpha_2},$$

or to:

(e)
$$\left\{\begin{aligned} 0 &= \frac{\partial^2 V}{\partial \alpha_1 \, \partial t} + \frac{\partial^2 V}{\partial \alpha_1 \, \partial r} \cdot \frac{\partial \psi}{\partial p_1} + \frac{\partial^2 V}{\partial \alpha_1 \, \partial \vartheta} \cdot \frac{\partial \psi}{\partial p_2}, \\ 0 &= \frac{\partial^2 V}{\partial \alpha_2 \, \partial t} + \frac{\partial^2 V}{\partial \alpha_2 \, \partial r} \cdot \frac{\partial \psi}{\partial p_1} + \frac{\partial^2 V}{\partial \alpha_2 \, \partial \vartheta} \cdot \frac{\partial \psi}{\partial p_2}. \end{aligned}\right.$$

The system of equations (a) and (e) have the same coefficients. If the solution of that system gives something well-defined then that must yield the same values for the unknowns; i.e., one must have:

$$\frac{dr}{dt} = \frac{\partial \psi}{\partial p_1}, \qquad \frac{d\vartheta}{dt} = \frac{\partial \psi}{\partial p_2}, \tag{5}$$

such that we have avoided the solution of equations (a) by presenting the system (e). However, the equations will always give something well-defined (as long as their coefficients remain finite, which will always be assumed) when the determinant of the coefficients in non-zero; i.e., as long as:

$$R = \sum \pm \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial \alpha_1}\right) \cdot \frac{\partial}{\partial \vartheta}\left(\frac{\partial V}{\partial \alpha_2}\right) = \sum \pm \frac{\partial}{\partial \alpha_1}\left(\frac{\partial V}{\partial r}\right) \cdot \frac{\partial}{\partial \alpha_2}\left(\frac{\partial V}{\partial \vartheta}\right) \neq 0.$$

"However, if $R = 0$ then the quantities $\partial V / \partial \alpha_1$ and $\partial V / \partial \alpha_2$ would not be mutually independent when they are considered to be functions of r and ϑ; an equation must exist between $\partial V / \partial \alpha_1$, $\partial V / \partial \alpha_2$, α_1 , α_2 , and t that does not contain r and ϑ. One will then have an equation of the form:

$$0 = F\left(t, r, \vartheta, \frac{\partial V}{\partial r}, \frac{\partial V}{\partial \vartheta}\right),$$

i.e., a first-order partial differential equation that the assumed solution V must satisfy and that does not contain $\partial V / \partial t$. However, that will be impossible when V is actually a complete solution of equation (4)." The basis for that is word-for-word the same as the one in **Jacobi**, pp. 161, et seq.

Hence, as long as V is actually a complete solution of the equation in question, R cannot be non-zero. The conclusion that equations (5) exist will then be justified.

Now, if equations (5) are identical to equations (2) in **§ 2**, namely, to:

$$\frac{\partial |T+U|}{\partial p_1} = \frac{dr}{dt}, \qquad \frac{\partial |T+U|}{\partial p_2} = \frac{d\vartheta}{dt},$$

then equations (I) will actually be the integral equations of the latter. That will be the case if one has:

$$\psi = |\, T + U \,| - \frac{2m}{r}$$

in **§ 3**, from which, it will follow that:

$$\frac{\partial \psi}{\partial p_1} = \frac{\partial |T+U|}{\partial p_1} = \frac{dr}{dt},$$

and likewise:

$$\frac{\partial\psi}{\partial p_2} = \frac{\partial\,|T+U\,|}{\partial p_2} = \frac{d\vartheta}{dt}.$$

Hence: *Equations (I) are the integral equations for the differential equations* (2) *in* **§ 2**.

In order to prove the same thing for equations (II), we totally differentiate them with respect to *t* and get:

(d)
$$\begin{cases} \dfrac{dp_1}{dt} = \dfrac{\partial^2 V}{\partial r\,\partial t} + \dfrac{\partial^2 V}{\partial r\,\partial r}\dfrac{dr}{dt} + \dfrac{\partial^2 V}{\partial r\,\partial\vartheta}\dfrac{d\vartheta}{dt}, \\[2ex] \dfrac{dp_2}{dt} = \dfrac{\partial^2 V}{\partial\vartheta\partial t} + \dfrac{\partial^2 V}{\partial r\,\partial\vartheta}\dfrac{dr}{dt} + \dfrac{\partial^2 V}{\partial\vartheta\partial\vartheta}\dfrac{d\vartheta}{dt}. \end{cases}$$

Since:

$$\frac{\partial^2 V}{\partial r\,\partial t} = \frac{\partial p_1}{\partial r}, \qquad \frac{\partial^2 V}{\partial r\,\partial\vartheta} = \frac{\partial p_1}{\partial\vartheta}, \qquad \frac{\partial^2 V}{\partial\vartheta\partial\vartheta} = \frac{\partial p_2}{\partial\vartheta}$$

and

$$\frac{\partial^2 V}{\partial\vartheta\partial r} = \frac{\partial p_2}{\partial r},$$

and from equations (5):

$$\frac{dr}{dt} = \frac{\partial\psi}{\partial p_1}, \qquad \frac{d\vartheta}{dt} = \frac{\partial\psi}{\partial p_2},$$

moreover, the system (d) will go to:

(e)
$$\begin{cases} \dfrac{dp_1}{dt} = \dfrac{\partial^2 V}{\partial r\,\partial t} + \dfrac{\partial p_1}{\partial r}\cdot\dfrac{\partial\psi}{\partial p_1} + \dfrac{\partial p_2}{\partial r}\cdot\dfrac{\partial\psi}{\partial p_2}, \\[2ex] \dfrac{dp_2}{dt} = \dfrac{\partial^2 V}{\partial\vartheta\partial t} + \dfrac{\partial p_1}{\partial\vartheta}\cdot\dfrac{\partial\psi}{\partial p_1} + \dfrac{\partial p_2}{\partial\vartheta}\cdot\dfrac{\partial\psi}{\partial p_2}. \end{cases}$$

However, since r and ϑ first enter into $\partial V / \partial t$, but then enter into ψ explicitly, as well as implicitly in the p, the partial differentiation of equation (4) with respect to r and ϑ will give:

(f)
$$\begin{cases} 0 = \dfrac{\partial^2 V}{\partial r\,\partial t} + \dfrac{\partial\psi}{\partial p_1}\cdot\dfrac{\partial p_1}{\partial r} + \dfrac{\partial\psi}{\partial p_2}\cdot\dfrac{\partial p_2}{\partial r} + \dfrac{\partial\psi}{\partial r}, \\[2ex] 0 = \dfrac{\partial^2 V}{\partial\vartheta\partial t} + \dfrac{\partial\psi}{\partial p_1}\cdot\dfrac{\partial p_1}{\partial\vartheta} + \dfrac{\partial\psi}{\partial p_2}\cdot\dfrac{\partial p_2}{\partial\vartheta} + \dfrac{\partial\psi}{\partial\vartheta}, \end{cases}$$

such that subtracting the systems (e) and (f) will give:

(6)
$$\frac{dp_1}{dt} = -\frac{\partial\psi}{\partial r}, \qquad \frac{dp_2}{dt} = -\frac{\partial\psi}{\partial\vartheta}.$$

However, since $\psi = | T + U | - 2m / r$, it will follow from (6) that:

$$\left\{\begin{aligned} \frac{dp_1}{dt} &= -\frac{\partial | T+U |}{\partial r} - \frac{2m}{r^2}, \\ \frac{dp_2}{dt} &= -\frac{\partial | T+U |}{\partial \vartheta} = 0. \end{aligned}\right.$$

However, these are equations (1) in **§ 2.** Hence: *equations (II) are the integral equations for the corresponding differential equations.*

With that, we have produced the proof of the applicability of the **Hamiltonian** method to our problem, and indeed in a form in which it can be extended to many variables in a very simple way. We are now convinced that the integration of the first-order partial differential equation will yield the correct result.

§ 5. – Integrating the first-order partial differential equation.

If one now replaces r' and ϑ' with the p and then p_1 and p_2 with $\partial V / \partial r$ and $\partial V / \partial \vartheta$ in the equation:

$$\frac{\partial V}{\partial t} + \psi = 0$$

or

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r'} r' + \frac{\partial V}{\partial \vartheta} \vartheta' - \frac{p_1^2}{4m\left(\frac{1}{2} + \frac{1}{rc^2}\right)} - \frac{p_2^2}{2mr^2} - \frac{m}{r} = 0$$

then that equation will go to:

$$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial r}\right)^2 \cdot \frac{rc^2}{2m(rc^2+2)} + \left(\frac{\partial V}{\partial \vartheta}\right)^2 \frac{1}{2mr^2} - \frac{m}{r} = 0. \tag{4*}$$

t enters into $\partial V / \partial t$, but not explicitly, and for that reason, we heuristically set $\partial V / \partial t = \alpha_1$ or $V = \alpha_1 t + W$, in which α_1 is an arbitrary constant, but W is constant in t, such that, since $\partial W / \partial \vartheta = \partial V / \partial \vartheta$, $\partial W / \partial r = \partial V / \partial r$, the equation will go to:

$$\alpha_1 + \left(\frac{\partial W}{\partial r}\right)^2 \cdot \frac{rc^2}{2m(rc^2+2)} + \left(\frac{\partial W}{\partial \vartheta}\right)^2 \frac{1}{2mr^2} - \frac{m}{r} = 0.$$

Since ϑ enters into only $\partial W / \partial \vartheta$, but not explicitly, we set:

$$\frac{\partial W}{\partial \vartheta} = \alpha_2, \qquad W = \alpha_1 \vartheta + Z,$$

in which Z is constant in t and ϑ, so $\partial W / \partial r = \partial Z / \partial r$. The equation goes to:

$$\alpha_1 + \left(\frac{\partial Z}{\partial r}\right)^2 \cdot \frac{rc^2}{2m(rc^2+2)} + \frac{\alpha_2^2}{2mr^2} - \frac{m}{r} = 0,$$

so

$$\frac{\partial Z}{\partial r} = \sqrt{2m}\sqrt{\left(\frac{m}{r} - \alpha_1 - \frac{\alpha_2^2}{2mr^2}\right)\left(1 + \frac{2}{rc^2}\right)},$$

i.e.:

$$Z = \sqrt{2m}\int \sqrt{\left(\frac{m}{r} - \alpha_1 - \frac{\alpha_2^2}{2mr^2}\right)\left(1 + \frac{2}{rc^2}\right)}\, dr,$$

such that the general solution of equation (4*) will yield:

(A) $$V = \alpha_1\, t + \alpha_2\, \vartheta + \sqrt{2m}\int \sqrt{\left(\frac{m}{r} - \alpha_1 - \frac{\alpha_2^2}{2mr^2}\right)\left(1 + \frac{2}{rc^2}\right)}\, dr.$$

As we proved above, the integral equations for the mechanical problem are:

$$\frac{\partial V}{\partial \alpha_1} = \beta_1 = t + \frac{\partial Z}{\partial \alpha_1}, \qquad \frac{\partial V}{\partial \alpha_2} = \beta_2 = t + \frac{\partial Z}{\partial \alpha_2},$$

so:

(B) $$t = \beta_1 + m\int \frac{\sqrt{1 + \dfrac{2}{rc^2}} \cdot dr}{\sqrt{\dfrac{2m^2}{r} - \dfrac{\alpha_2^2}{r^2} - 2\alpha_1 m}}.$$

That integral gives the connection between *t* and *r*.

We further get:

(C) $$\vartheta = \beta_1 + \alpha_2 \int \frac{\sqrt{1 + \dfrac{2}{rc^2}} \cdot dr}{r^2\sqrt{\dfrac{2m^2}{r} - \dfrac{\alpha_2^2}{r^2} - 2\alpha_1 m}},$$

which will then give the connection between ϑ and *r*.

If one introduces $Z = 1 / r$ into the last integral then the equation will go to:

(C*) $$\vartheta = \beta_2 - \alpha_2 \int \frac{\sqrt{1 + \dfrac{2}{rc^2}} \cdot dZ}{\sqrt{2m^2 Z - 2\alpha_1 m - \alpha_2^2 Z^2}}.$$

§ 6. – Determination of the constants, etc.

It follows from equation (B) that:

$$dt = \frac{m\sqrt{1+\frac{2}{rc^2}}\cdot dr}{\sqrt{\frac{2m^2}{r}-\frac{\alpha_2^2}{r^2}-2\alpha_1 m}}.$$

It follows from equation (C) that:

$$d\vartheta = \frac{\alpha_1\sqrt{1+\frac{2}{rc^2}}\cdot dr}{r^2\sqrt{\frac{2m^2}{r}-\frac{\alpha_2^2}{r^2}-2\alpha_1 m}},$$

and dividing the two equations will yield:

$$\alpha_2 = m\, r^2\frac{d\vartheta}{dt}.$$

However, since $r^2\, d\vartheta$ is twice the surface area of an element that the radius vector describes, α_2 will be the same constant that gives us the law of areas. We already mentioned above that the latter will be true here.

If the initial state is given by:

$$t_0,\ r_0,\ \vartheta_0, \qquad v_0 = \left(\frac{ds}{dt}\right)_{t=t_0},$$

and the angle β_2 that the initial velocity defines with the radius r_0 is such that:

$$ds_0 \sin\beta_0 = r_0\, d\vartheta_0, \qquad ds_0 \cos\beta_0 = dr_0,$$

and therefore:

$$v_0 \cos\beta_0 = \left(\frac{dr}{dt}\right)_{t_0}, \qquad v_0 \sin\beta_0 = r_0\left(\frac{d\vartheta}{dt}\right)_{t_0},$$

then that will yield:

$$\alpha_2 = m r_0^2\left(\frac{d\vartheta}{dt}\right)_{t_0} = m\, r_0\, v_0 \sin\beta_0.$$

It further follows that α_1 will be the constant that emerges from the *vis viva* principle from the fact that:

$$v\cos\beta = \frac{dr}{dt} = \frac{\sqrt{\frac{2m^2}{r}-\frac{\alpha_2^2}{r^2}-2\alpha_1 m}}{m\sqrt{1+\frac{2}{rc^2}}},$$

$$m^2\left(1+\frac{2}{rc^2}\right)\,v^2\cos^2\beta = \frac{2m^2}{r} - \frac{\alpha_2^2}{r^2} - 2\alpha_1 m\,,$$

or when the value of α_2 is substituted:

$$\begin{aligned} 2\alpha_1 &= -\,mv^2 + \frac{2m}{r}\left(1 - v^2\cos^2\beta\,\frac{1}{rc^2}\right) \\ &= -\,mv^2 + \frac{2m}{r}\left(1 - \frac{r'^2}{c^2}\right) = -\,mv^2 + 2U_1, \end{aligned}$$

so $T = U_1 - \alpha_1$. α_1 is determined from the initial state to be:

$$2\alpha_1 = m\,v_0^2 + \frac{2m}{r_0}\left(1 - v_0^2\cos^2\beta_0\,\frac{1}{r_0\,c^2}\right).$$

At the same time, it is confirmed at this point that the *vis viva* principle is meaningful for our problem.

In order to establish the meaning of the constants β, we must give the integrals well-defined limits. If we establish a well-defined numerical value r_0 as our lower bound then that will yield:

(B) $$t = m\int \frac{\sqrt{1+\dfrac{2}{rc^2}}\cdot dr}{r^2\sqrt{\dfrac{2m^2}{r} - \dfrac{\alpha_2^2}{r^2} - 2\alpha_1 m}} + \beta_1\,,$$

and for $r = r_0$ (i.e., $t = t_0$), all that will remain on the right will be β_1, such that β_1 will be the initial value of t. It likewise follows from (C) that β_2 is the initial value of ϑ.

Since t and β must always be real, it will follow from equation (B) (in which the roots in the numerator are never imaginary, because r is always positive) that the roots in the denominator can never be imaginary, which will imply that r has a maximum and a minimum value. They are determined from the equation:

$$\frac{2m^2}{r} - \frac{\alpha_2^2}{r^2} - 2\,\alpha_1\,m = 0$$

to be

$$r_1 = \frac{m}{2\alpha_1} + \frac{1}{2\alpha_1}\sqrt{m^2 - \frac{2\alpha_1\,\alpha_2^2}{m}}$$

and

$$r_2 = \frac{m}{2\alpha_1} - \frac{1}{2\alpha_1}\sqrt{m^2 - \frac{2\alpha_1\,\alpha_2^2}{m}}$$

will then be easy to calculate from the initial states when c is given.

The possible positions of the attracted point are then restricted to a ring between two concentric circles that have radii of r_1 and r_2 around the attracting center.

As far as evaluating the elliptic integrals is concerned, we refer to **Seeger**'s dissertation, in which they are treated elegantly.

We would like to preserve the names of aphelion and perihelion for the maximum and minimum distances, resp. **Seeger** has proved that the successive perihelia and aphelia are not separated by an angle of π, as they would be with **Newton**'s law, but by something else.

This remark, with the help of the above, shall serve to provide us with a picture of the motion that takes place.

It next follows from the equation of the area principle:

$$m\, r^2 \vartheta' = \alpha_2$$

that ϑ' can never change sign, such that the motion will always proceed in the same sense. It will always enclose the small circle and be enclosed by the large one.

Furthermore, on the grounds of the same principle, the velocity will turn back from the tangent to the path in proportion to the distance from the center, such that it will be slowest at the aphelion. Finally, one gets that the same ϑ' will belong to the same r.

It follows from the *vis viva* principle by eliminating the constants α that:

$$T - T_1 = U - U_1$$

or

$$\tfrac{1}{2}\left[\left(r'^2 + r^2\vartheta'^2\right) - \left(r_1'^2 + r_1^2\vartheta_1'^2\right)\right] = \frac{1}{r^2}\left(1 - \frac{r'^2}{c^2}\right) - \frac{1}{r_1^2}\left(1 - \frac{r_1'^2}{c^2}\right).$$

For equal r that belong to the same ϑ', that will then imply that:

$$\tfrac{1}{2}\left(r'^2 - r_1'^2\right) = \frac{1}{c^2 r^2}\left(r_1'^2 - r'^2\right),$$

which is an equation that will be possible only when:

$$r' = \pm r_1'.$$

As a result, the same ϑ' and r' (up to sign) will always belong to the same r, so equal ds / dt, as well, and up to sign, the same dr / dt and dr / ($r\, d\vartheta$); in words:

If any circle around the center intersects the attracted point repeatedly then that will always happen with the same velocity and the same angle, up to sign.

That will imply the following important property of the orbit:

If P is a perihelion and A is the aphelion that follows it then the path from A to the following perihelion P_1 will be congruent to the path from P to A and will lie symmetrically with respect to the line AC when C means the center. The path from P_1 to A_1 is congruent to the one from A to P_1 and will lie symmetrically with respect to the line P_1C, etc.

The aphelia then follow each other at the same angular distances, and likewise, the perihelia follow with the same angular positions. If the latter is γ, and γ / π is irrational then the orbit will have infinitely many congruent arms, while if γ / π is rational then the number of those arms will be finite, and the orbit will turn back into itself. Initially, it has as many symmetry axes through c as perihelia and aphelia, taken together. If one additionally links a point at which two arms of the orbit intersect with the center then a symmetry axis will again come about. We also have either a finite or infinite number in the latter group.

For $c = \infty$, where the potential does not need any time then to arrive at the attracted point, the law will agree with **Newton**'s, and the orbit of the point will be an ellipse when only r_1 remains finite. The distance between the individual aphelia will then be infinitely small; i.e., $\gamma = 0$. One can then get a rough picture of the orbit for very large c (and c is very large in reality) in perhaps the following way: A point moves on an ellipse according to **Newton**'s law, while for this motion, the ellipse rotates very slowly around the focus at which the attracting force is found.

§ 7. – The same problem for three coordinates.

In order to show how fast the **Hamilton**ian method takes us to our goal, we would like to work through the same problem once more with three coordinates.

The independent coordinates are introduced by the equations:

$$x = r \cos \varphi, \qquad y = r \sin \varphi \cos \psi, \qquad z = r \sin \varphi \sin \psi,$$

such that:

$$x'^2 + y'^2 + z'^2 = r'^2 + r'^2\, \varphi'^2 + r^2 \sin^2 \varphi\, \psi'^2,$$

so:

$$T = \frac{m}{2}(r'^2 + r'^2\, \varphi'^2 + r^2 \sin^2 \varphi\, \psi'^2).$$

Furthermore:

$$U = \frac{m}{r}\left(1 + \frac{r'^2}{c^2}\right),$$

so:

$$(1) \qquad (T + U) = r'^2 m\left(\tfrac{1}{2} + \frac{1}{rc^2}\right) + \frac{mr^2}{2}\varphi'^2 + \frac{mr^2 \sin^2 \varphi}{2}\psi'^2 + \frac{m}{r}.$$

The introduction of the p comes about by way of the equations:

$$\frac{\partial(T+U)}{\partial r'} = p_1 = 2mr'\left(\tfrac{1}{2} + \frac{1}{rc^2}\right), \qquad \text{so} \qquad r' = \frac{p_1}{2m\left(\tfrac{1}{2} + \frac{1}{rc^2}\right)},$$

$$\frac{\partial(T+U)}{\partial\varphi'} = p_2 = m\,r^2\,\varphi', \qquad \text{so} \qquad \varphi' = \frac{p_2}{mr^2},$$

$$\frac{\partial(T+U)}{\partial\psi'} = p_3 = m\,r^2\sin^2\varphi\,\psi', \qquad \text{so} \qquad \psi' = \frac{p_3}{mr^2\sin^2\varphi}.$$

The transformed (T + U) will then become:

$$|\,T+U\,| = \frac{p_1^2}{4m\left(\frac{1}{2}+\frac{1}{rc^2}\right)} + \frac{p_2^2}{2mr^2} + \frac{p_3^2}{2mr^2\sin^2\varphi} + \frac{m}{r}. \tag{2}$$

This has the property that:

$$\frac{\partial\,|T+U|}{\partial p_1} = \frac{2p_1}{4m\left(\frac{1}{2}+\frac{1}{rc^2}\right)} = r', \qquad \frac{\partial\,|T+U|}{\partial p_2} = \frac{2p_2}{2mr^2} = \varphi',$$

$$\frac{\partial\,|T+U|}{\partial p_3} = \frac{2p_3}{2mr^2\sin^2\varphi} = \psi',$$

as well as the fact that:

$$\frac{\partial(T+U)}{\partial r} = -\frac{\partial\,|T+U|}{\partial r} - \frac{2m}{r^2}, \qquad \frac{\partial(T+U)}{\partial\varphi} = -\frac{\partial\,|T+U|}{\partial\varphi},$$

$$\frac{\partial(T+U)}{\partial\psi} = -\frac{\partial\,|T+U|}{\partial\psi} = 0,$$

in which the q are the independent coordinates, go to:

$$\text{(I)} \qquad \left\{\begin{aligned} &\alpha) \quad \frac{\partial\,|T+U|}{\partial r} = -\frac{dp_1}{dt} - \frac{2m}{r^2}, \\ &\beta) \quad \frac{\partial\,|T+U|}{\partial\varphi} = -\frac{dp_2}{dt}, \\ &\gamma) \quad \frac{\partial\,|T+U|}{\partial\psi} = -\frac{dp_1}{dt} = 0, \end{aligned}\right.$$

and for that reason:

$$\text{(II)} \qquad \begin{cases} \alpha) & \dfrac{\partial\,|T+U\,|}{\partial p_1} = r', \\ \beta) & \dfrac{\partial\,|T+U\,|}{\partial p_2} = \varphi', \\ \gamma) & \dfrac{\partial\,|T+U\,|}{\partial p_3} = \psi'. \end{cases}$$

If we now set:

$$\int_{t_0}^{t_1} (T+U)\,dt = V$$

then that will yield:

$$\text{(III)} \qquad \frac{\partial V}{\partial r} = \frac{\partial (T+U)}{\partial r'} = p_1, \qquad \frac{\partial V}{\partial \varphi} = \frac{\partial (T+U)}{\partial \varphi'} = p_2\,,$$

$$\frac{\partial V}{\partial \psi} = \frac{\partial (T+U)}{\partial \psi'} = p_3\,.$$

Moreover, it follows from:

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial \varphi}\cdot\frac{d\varphi}{dt} + \frac{\partial V}{\partial \psi}\cdot\frac{d\psi}{dt}$$

and

$$\frac{dV}{dt} = T + U$$

that one has the equation:

$$\frac{\partial V}{\partial t} + p_1\, r' + p_2\, \varphi' + p_3\, \psi' - (T + U) = 0,$$

or

$$\text{(IV)} \qquad \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial r}\right)^2 \frac{r\,c^2}{2m\,(2+rc^2)} + \left(\frac{\partial V}{\partial \varphi}\right)^2 \frac{1}{2m\,r^2}\cdot\frac{1}{2m\sin^2\varphi} - \frac{m}{r} = 0.$$

That is a first-order partial differential equation in four variables. In order to integrate it, we heuristically set:

$$\frac{\partial V}{\partial t} = \alpha_1\,, \qquad \text{so} \qquad V = \alpha_1\, t + W,$$

such that the equation goes to:

$$\tfrac{1}{2}\left[\left(\frac{\partial W}{\partial r}\right)^2 \frac{r\,c^2}{2+r\,c^2} + \left(\frac{\partial W}{\partial \varphi}\right)^2 \frac{1}{r^2} + \left(\frac{\partial W}{\partial \psi}\right)^2 \cdot \frac{1}{r^2\sin^2\varphi}\right] = \frac{m^2}{r} - \alpha_1\, m.$$

We add $-\alpha_2^2 / r^2$ to both sides and set:

$$\tfrac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2 \frac{r\,c^2}{2 + r\,c^2} = \frac{m^2}{r} - \alpha_1\,m - \frac{\alpha_2^2}{r^2},$$

or

$$\frac{\partial W}{\partial r} = \sqrt{2\left(\frac{m^2}{r} - \alpha_1 m - \frac{\alpha_2^2}{r^2}\right)\left(1 + \frac{2}{r\,c^2}\right)},$$

i.e.:

$$W = \int \sqrt{\left(\frac{2m^2}{r} - 2\alpha_1 m - \frac{2\alpha_2^2}{r^2}\right)\left(1 + \frac{2}{r\,c^2}\right)}\,dr + P,$$

then the equation will go to:

$$\tfrac{1}{2}\left[\left(\frac{\partial P}{\partial \varphi}\right)^2 \frac{1}{r^2} + \left(\frac{\partial P}{\partial \psi}\right)^2 \cdot \frac{1}{r^2 \sin^2 \varphi}\right] - \frac{\alpha_2^2}{r^2} = 0$$

or

$$\tfrac{1}{2}\left[\left(\frac{\partial P}{\partial \varphi}\right)^2 + \left(\frac{\partial P}{\partial \psi}\right)^2 \cdot \frac{1}{\sin^2 \varphi}\right] = \alpha_2^2.$$

If one adds – $\alpha_2^2 / \sin^2 \varphi$ to both sides and sets:

$$\tfrac{1}{2}\left(\frac{\partial P}{\partial \varphi}\right)^2 = \alpha_2^2 - \frac{\alpha_2^2}{\sin^2 \varphi},$$

so

$$P = \int \sqrt{2\alpha_2^2 - \frac{2\alpha_2^2}{\sin^2 \varphi}}\,d\varphi + Z,$$

then the equation will be converted into:

$$\tfrac{1}{2}\left(\frac{\partial Z}{\partial \psi}\right)^2 \cdot \frac{1}{\sin^2 \varphi} = \frac{\alpha_2^2}{\sin^2 \varphi},$$

or

$$Z = \int \alpha_3 \sqrt{2}\,d\psi = \psi\,\alpha_3 \sqrt{2} + (\alpha).$$

The general integral of our differential equation will then be:

(A) $$V = \alpha_1 t + \int \sqrt{\left(\frac{2m^2}{r} - 2\alpha_1 m - \frac{2\alpha_2^2}{r^2}\right)\left(1 + \frac{2}{r c^2}\right)}\, dr$$

$$+ \int \sqrt{2\alpha_2^2 - \frac{2\alpha_2^2}{\sin^2\varphi}}\, d\varphi + \psi\, \alpha_3 \sqrt{2} + (\alpha),$$

while the differential equations of the problem are:

(B) $$\beta_1 = t - m \int \frac{\sqrt{1 + \frac{2}{r c^2}}\, dr}{\sqrt{\left(\frac{2m^2}{r} - 2\alpha_1 m - \frac{2\alpha_2^2}{r^2}\right)}},$$

(C) $$\beta_2 = - m \int \frac{\sqrt{1 + \frac{2}{r c^2}}\, dr}{r^2 \sqrt{\left(\frac{2m^2}{r} - 2\alpha_1 m - \frac{2\alpha_2^2}{r^2}\right)}} + \int \frac{d\varphi}{\sqrt{2\alpha_2^2 - \frac{2\alpha_3^2}{\sin^2\varphi}}}.$$

With the substitution:

$$\cos\varphi = \sqrt{\frac{\alpha_2^2 - \alpha_3^2}{\alpha_2^2}} \cdot \cos\eta,$$

$$\sin\varphi\, d\varphi = \sqrt{\frac{\alpha_2^2 - \alpha_3^2}{\alpha_2^2}} \sin\eta\, d\eta,$$

the last integral will go to:

$$\frac{1}{\sqrt{2\alpha_2^2}} \int d\eta = \left| \frac{\eta}{\sqrt{2\alpha_2^2}} \right|_{Gr_1}^{Gr_2},$$

such that we will get:

(C*) $$\beta_2 = \int \frac{\sqrt{1 + \frac{2}{rc^2}}\, dr}{r^2 \sqrt{\frac{2m^2}{r} - 2\alpha_1 m - \frac{2\alpha_2^2}{r^2}}} + \left| \frac{\eta}{\sqrt{2\alpha_2^2}} \right|_{Gr_1}^{Gr_2}.$$

Ultimately, the final equation is:

(D) $$\psi - \beta_3 = \alpha_3 \int \frac{d\varphi}{\sin^2\varphi \sqrt{\alpha_2^2 - \frac{\alpha_3^2}{\sin^2\varphi}}}.$$

If one introduces cot φ as the variable here then that will give:

$$\psi - \beta_3 = -\alpha_3 \int \frac{d\cot\varphi}{\sqrt{\alpha_2^2 - \alpha_3^2 - \alpha_3^2 \cot^2\varphi}} = -\arccos\left(\sqrt{\frac{\alpha_3^2}{\alpha_2^2 - \alpha_2^2}} \cot\varphi\right),$$

from which, it will then follow that:

$$\text{(E)} \qquad \cos(\psi - \beta_3) = \sqrt{\frac{\alpha_3^2}{\alpha_2^2 - \alpha_2^2}} \cot\varphi,$$

such that motion will take place in a plane.

Since φ assumes only values between 0 and 180°, the integrals that are taken to be φ will imply that the minimum value of sin φ is α_3 / α_2 . Hence, if J is the inclination of the plane of the orbit with respect to the ecliptic, for which, one has φ = 90°, then the minimum of sin φ will be:

$$\frac{\alpha_3}{\alpha_2} = \sin(90^\circ - J) = \cos J.$$

Since one then has:

$$\sqrt{\frac{\alpha_3^2}{\alpha_2^2 - \alpha_3^2}} = \sqrt{\frac{\alpha_3^2 \cos^2 J}{\alpha_2^2 - \alpha_2^2 \cos^2 J}} = \pm \cot J,$$

equation (E) can also be written as:

$$\text{(E}^*\text{)} \qquad \cos(\psi - \beta_3) = \pm \cot J \cdot \cot\varphi.$$

For $\varphi = 90^\circ$, one has $\cos(\psi - \beta_3) = 0$, so $(\psi - \beta_3) = \pm 90^\circ$. However, the ψ that belongs to $\varphi = 90^\circ$ is the length of the ascending (descending, resp.) node; i.e., β_3 is the length of the ascending node $\mp 90^\circ$.

If we now give definite numerical values to the upper limit on the integral in order to understand the meaning of all the constants then it will follow from equation (D) that β_3 is the initial value of ψ. The initial position of the planet is then the lowest or highest point of the orbit. It follows from this that:

$$\varphi_0 = \left(\frac{\pi}{2} \pm J\right).$$

One can also read off the latter from equation (E*), because one will then have:

$$\psi_0 - \beta_3 = 0,$$

and it will then follow that:

$$1 = \pm \cot J \cot\varphi,$$

so since φ assumes only values between 0 and 180° :

$$\varphi_0 = \left(\frac{\pi}{2} \pm J\right).$$

From equation (C^*), β_3 would prove to be zero. In order to verify this, we (like **Jacobi**, pp. 188) can consider the following argument:

From:

$$\cos \varphi_0 = \cos\left(\frac{\pi}{2} \mp J\right) = \sqrt{\frac{\alpha_2^2 - \alpha_3^2}{\alpha_2^2}}$$

and the substitution above:

$$\cos \varphi = \sqrt{\frac{\alpha_2^2 - \alpha_3^2}{\alpha_2^2}} \cdot \cos \eta ,$$

that will imply that the initial value of η is zero, so equation (C^*) will go to:

$$\beta_3 = -\int \frac{\sqrt{1+\frac{2}{r\,c^2}} \cdot dr}{r^2\sqrt{\frac{2m^2}{r} - 2\alpha_1 m - \frac{2\alpha_2^2}{r^2}}} + \frac{\eta}{\sqrt{2\alpha_2^2}} .$$

However, φ is the hypotenuse of a rectangular spherical triangle whose cathetes are η and $(90^\circ - \eta)$ (cf., the illustration in **Jacobi**). η is then equal to 90° minus the distance from the planet to the ascending node. The integral will vanish for $r = r_0$, and will go to 90° minus the distance from the initial position of the planet to the ascending node, such that β_2 will take on the following geometric meaning:

$$\beta_2 = \frac{1}{\sqrt{2\alpha_2^2}}[90^\circ - \text{distance from the initial position of the planet to the ascending node}].$$

However, that distance amounts to 90°, so one will actually have $\beta_2 = 0$.

If one differentiates equations (B), (C), and (D) with respect to the corresponding variables and compares the results then that will imply that:

$$\frac{dt}{mr^2} = \frac{d\psi}{\alpha_3} \sin^2 \varphi,$$

or

$$\frac{\alpha_3}{m} dt = r^2 \sin^2 \varphi \, d\psi.$$

Hence, α_3 / m is the projection of twice the areal velocity onto the xy-plane, but the true value of twice the areal velocity must be:

$$\frac{\alpha_3}{m}\cdot\frac{1}{\cos J} = \frac{\alpha_3}{m}\cdot\frac{\alpha_2}{\alpha_3} = \frac{\alpha_2}{m};$$

α_2 then takes on the same meaning as in **§ 6**. α_3 is then obtained from the equation $\alpha_3 = \alpha_2 \cos J$, and finally α_1 is, in turn, the constant that arises from the *vis viva* principle. These constants are all obtained from the initial state.

As far as the matter of performing the elliptic integrals is concerned, we can refer to **Seeger**.

Appendix

Since t does not enter into our problem explicitly, and the *vis viva* principle is true, in addition, everything that **Jacobi** said in Lecture 21 can be extended to our problem, such that one can *significantly shorten the path to the partial differential equation.*

One introduces a new independent variable α by way of the equation $\partial V / \partial t = \alpha$ and sets V equal to a new function $W = V - t\,\alpha = V - t\,(\partial V / \partial t)$ then one will have $t = -\,\partial W / \partial\alpha$, and the equation:

(1) $$\frac{\partial V}{\partial t} + \psi = 0$$

will go to an equation of the form:

(2) $$\alpha + \psi\left(q_1, \ldots, q_\nu, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_\nu}\right) = 0.$$

If W is ascertained by integration then one can easily show that the integral equations will be:

(3) $$\begin{cases} \dfrac{\partial W}{\partial \alpha_1} = \beta_1, \ldots, \quad \dfrac{\partial W}{\partial \alpha_{\nu-1}} = \beta_{\nu-1}, \quad \dfrac{\partial W}{\partial \alpha} = \tau - t, \\ \dfrac{\partial W}{\partial q_1} = p_1, \ldots, \quad \dfrac{\partial W}{\partial q_{\nu-1}} = p_{\nu-1}, \quad \dfrac{\partial W}{\partial q_\nu} = p_\nu. \end{cases}$$

However, the *vis viva* principle reads:

(4) $$0 = \alpha + T - U,$$

and it can be shown that equations (2) and (4) are identical.

For all problems in which one is dealing with attraction according to **Weber**'s law, if t is not present explicitly and the *vis viva* principle remains in effect then one can find the partial differential equation in the following way:

If one applies the transformations of the ***Hamilton****ian method to the vis viva equation and sets* $p_i = \partial W / \partial q_i$ *then one will immediately obtain the first-order partial differential equation from which the integral equations of the system of differential equations will follow.*

In order to show this for our problem and employ the previous transformations for that, we introduce:

$$U = -U_1 + \frac{2m}{r}$$

into equation (4) such that it will go to:

$$T + U - \frac{2m}{r} = -\alpha_1 , \tag{5}$$

or, from equation (2) in **§ 7**:

$$\frac{p_1^2}{4m\left(\frac{1}{2} + \frac{1}{rc^2}\right)} + \frac{p_2^2}{2mr^2} + \frac{p_3^2}{2mr^2 \sin^2 \varphi} - \frac{m}{r} = -\alpha_1 .$$

If one then sets $p_i = \partial W / \partial q_i$ then that will give:

$$\tfrac{1}{2}\left[\left(\frac{\partial W}{\partial r}\right)^2 \frac{rc^2}{2 + rc^2} + \left(\frac{\partial W}{\partial \varphi}\right)^2 \cdot \frac{1}{r^2} + \left(\frac{\partial W}{\partial \psi}\right)^2 \cdot \frac{1}{r^2 \sin \varphi}\right] = \frac{m^2}{r} - m\,\alpha_1 .$$

This equation coincides with the one that was derived for W in **§ 7.**
