XXXIV. *The Theory of Magnetism and the Absurdity of Diamagnetic Polarity.* By J. PARKER, *M.A., Fellow of St. John's College, Cambridge.*

[Continued from p. 203.]

IT appears from experiment that the properties of exerting actions at a distance by a magnet are mainly situated at or near the ends of the magnet. Suppose, then, that we have two long magnets A, B, which may be considered to possess the magnetic properties only in their ends, and let these magnets be so placed that we need only take into account one end of each. Also let these two ends be so far from each other that they may be regarded as mathematical points P, Q. Then the only magnetic forces between the two magnets will be equal forces at the poles P, Q, acting along the line PQ in opposite directions.

Now let the two magnets be situated in a " vacuum " and be made to undergo a reversible cycle in which the velocities are constantly zero. To do this, they must be held by external forces equal and opposite to gravity and to the magnetic forces between P and Q. But if the equal forces between P and Q be denoted by F, a repulsion being considered positive and an attraction negative, the work done by F in a small change of the distance PQ ($\equiv r$) will be $F\,dr$. Hence the work done on the system during the cycle by the *external* forces is $-\int F\,dr$, where the two limits are identical. This must be zero, by the principles of thermodynamics, and therefore F must depend only on r, or $F \equiv f(r)$. From experiment it appears that $f(r)$ is proportional to $\frac{1}{r^2}$, so that if the force between P and Q when their distance is one centimetre be λ dynes, the force will be $\frac{\lambda}{r^2}$ dynes when the distance is r centimetres.

Now let there be any number of poles R, R′, R″, . . . , which may be treated as mathematical points, acted on simultaneously by P and Q. Then it is inferred from experiment, supported by theory, that if the two poles P, Q repel each other, the forces they exert on any one of the other poles, R, will be both repulsive or both attractive; but that if P and Q attract each other, the forces they exert on any one of the other poles will be one repulsive and the other attractive. Conversely, if P and Q both repel or both attract the pole R, they will repel each other; while if one attract and the other repel, they will attract each other; and the very same properties are true of all the poles.

Thus it appears that there are two kinds of poles, or of magnetism. Like kinds repel; unlike kinds attract. For instance, if P and Q repel each other, they are of the same kind. If both P and Q repel a third pole R, R will be of the same kind as P and Q; if both P and Q attract R, it will be of unlike kind to P and Q.

The two kinds of magnetism may be distinguished by the signs + and −. It is immaterial which kind of magnetism be considered positive; but it is generally agreed to take the kind found at that end of a soft bar of iron which, when freely suspended and in stable equilibrium, points to the north.

If the poles P, Q exert equal forces, both attractive or both repulsive, on any third pole R from which they are equally distant, the poles P, Q, or the quantities of magnetism at P and Q are said to be equal. If the forces be equal, but one attractive and the other repulsive, the poles P, Q are said to be equal and opposite, or the quantities of magnetism at P and Q are said to be numerically equal but of opposite sign. Again, if the pole P exert m times as great a force as Q, and both be attractive or both repulsive, the magnetism at P is said to be $+m$ times that at Q. If one force be attractive and the other repulsive, the magnetism at P is said to be $-m$ times that at Q. Lastly, it is inferred from experiment, supported by theory, that if two poles X, Y be at the same distance as two equal poles P, Q, and the magnetism at X be x times that at P, and that at Y y times that at Q, the force between X and Y will be xy times that between P and Q. The force between X and Y is repulsive if X and Y, or x and y, be of the same sign, that is, if the product xy be positive: the force is attractive if x and y be of opposite signs, or xy negative.

These results lead to the C.G.S. system of units. If two equal positive poles P, Q, situated at a distance of one centimetre, repel each other with a force of one dyne, the quantity of magnetism at P or Q is defined to be the unit of magnetism. It therefore follows that if two poles X, Y, at which the quantities of magnetism are m and m', be at a distance of r centimetres, the magnetic force between them will be $\frac{mm'}{r^2}$ dynes, repulsive forces being considered positive and attractive negative.

To complete the fundamental principles of magnetism, we must add the great principle of the Conservation of Magnetism, which asserts that whatever changes take place in the magnetization of a system, the quantity of magnetism remains constant.

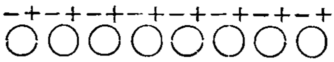
In the ordinary text-books, the fundamental definitions &c. are given in a manner which we cannot accept. Thus, let A, B be two long magnets which may be supposed to possess the magnetic properties only in their ends, and let them be so placed that we need only consider the positive pole of each, viz., P on A and Q on B. Then, if these poles are equal, and if, when they are placed "in air" at a distance of one centimetre, a force of one dyne is required to overcome the force which tends to separate them, the strength of each pole is defined to be unity, and it is asserted that at a distance of r centimetres "in air," the force which tends to separate them is $\frac{1}{r^2}$ dynes. In our method of treating the subject, we should say that the force which tends to separate the two poles is partly due to the magnetisms of the poles themselves, partly to the magnetization of the air in which the two magnets are placed, and partly to the inequalities in the pressure of the air. In some experiments, the pressure of the air is the most important factor. The so-called definitions of the text-books are therefore not definitions at all, but propositions in the Kinetic Theory of Gases, and are possibly incorrect.

Having now explained the fundamental principles of the subject, we must consider how magnetism is distributed in bodies. In the first place, it is evident that a finite quantity of magnetism cannot be concentrated into a point—that is into an indefinitely small sphere; for any two parts of the sphere would exert very great forces on each other, and the sphere would fly to pieces.

A finite quantity of magnetism can be distributed on a finite area. For if σ be the quantity of magnetism per unit area, or the surface density, on an infinite plate, this plate will exert a magnetic force $2\pi\sigma m$ on a body P with a quantity of magnetism m. If P be a second plate on which the surface-density is σ', the force exerted by the infinite plate on each unit of area of P will always have the finite value $2\pi\sigma\sigma'$.

If we break a magnet into any number of pieces, each piece is found to be a complete magnet. From this it is inferred that each atom or molecule is a complete magnet with equal quantities of positive and negative magnetism at its ends. The total quantity of magnetism on each atom or molecule is therefore zero, and the distribution on it may be supposed to be a surface distribution. To prevent any difficulty being felt with respect to surface distributions of magnetism, we have only to mention that, according to the physical theories of magnetism, all that is meant is a finite pressure or tension per unit area on the surface.

The reason why, in an ordinary bar-magnet, there is little manifestation of the magnetic properties except near the ends, is supposed to be that the positive end of one atom or molecule and the negative end of the next partially neutralize one another,

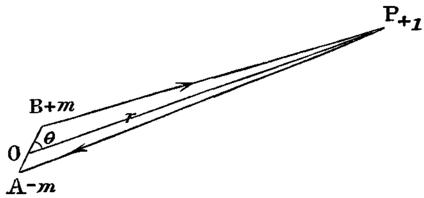


as indicated by the figure.

To calculate to what extent the magnetisms of successive molecules neutralize one another, we require some preliminary propositions.

Suppose that a very short, thin, straight magnet of length l is placed with its centre at O and let the quantities of magnetism at its two ends A, B, which may be treated as mere

Fig. 6.



points, be $-m$ and $+m$. Let a unit positive pole be situated at a point P whose distance from O is r, and let θ be the angle between OP and the line AB, or the axis of the magnet. Then the magnet AB exerts on P a repulsive force $\frac{m}{\mathrm{PB}^2}$ along BP and an attraction $\frac{m}{\mathrm{PA}^2}$ along PA. Now $\frac{m}{\mathrm{PB}^2}$ along BP is equivalent to $\frac{m}{\mathrm{PB}^2}\frac{\mathrm{OB}}{\mathrm{PB}}$ parallel to BA and $\frac{m}{\mathrm{PB}^2} \times \frac{\mathrm{PO}}{\mathrm{PB}}$ parallel to OP; and $\frac{m}{\mathrm{PA}^2}$ along PA is equivalent to $\frac{\mathrm{PA}^2}{m} \times \frac{\mathrm{AO}}{\mathrm{PA}}$ parallel to BA and $\frac{m}{\mathrm{PA}^2}\frac{\mathrm{PO}}{\mathrm{PA}}$ parallel to PO. Hence the action of the elementary magnet AB on P is equivalent to a force $\frac{ml}{2}\left(\frac{1}{\mathrm{PB}^3}+\frac{1}{\mathrm{PA}^3}\right)$ acting at P parallel to BA and a force $m\mathrm{PO}\left(\frac{1}{\mathrm{PB}^3}-\frac{1}{\mathrm{PA}^3}\right)$ at P along OP.

But, if we retain only the most important terms,

$$\frac{ml}{2}\left(\frac{1}{\mathrm{PB}^3}+\frac{1}{\mathrm{PA}^3}\right)=\frac{ml}{2}\left\{\frac{1}{\left(r-\frac{l}{2}\cos\theta\right)^3}+\frac{1}{\left(r+\frac{l}{2}\cos\theta\right)^3}\right\}$$

$$=\frac{mlr^3}{\left(r^2-\frac{l^2}{4}\cos^2\theta\right)^3},$$

$$m\,\mathrm{PO}\left(\frac{1}{\mathrm{PB}^3}-\frac{1}{\mathrm{PA}^3}\right)=mr\left\{\frac{1}{\left(r-\frac{l}{2}\cos\theta\right)^3}-\frac{1}{\left(r+\frac{l}{2}\cos\theta\right)^3}\right\}$$

$$=\frac{3mlr^3}{\left(r^2-\frac{l^2}{4}\cos^2\theta\right)^3}\cos\theta.$$

Thus if we can neglect $\frac{l^2}{r^2}$, as we certainly can when AB is comparable to the size of a molecule, the action on P reduces to $\frac{ml}{r^3}$ parallel to BA and $\frac{3ml}{r^3}\cos\theta$ along OP. These forces are exactly the same as would have been produced by another short magnet similar to AB, placed along AB with its centre at O, provided that $m'l'=ml$. Defining ml to be the magnetic moment of the elementary magnet AB, we see that two elementary magnets placed at the same point O with their axes coincident are equivalent to each other if their moments are equal.

If we take three rectangular axes Ox, Oy, Oz through O and denote the coordinates of P by (x, y, z) and the angles AB makes with the axes by (α, β, γ), the forces exerted on P by three small magnets placed at O :—

$ml\cos\alpha$ along Ox, are $-\frac{ml}{r^3}\cos\alpha$ parallel to Ox and

$$\frac{3ml\cos\alpha}{r^3}\,\frac{x}{r} \text{ along OP ;}$$

$ml\cos\beta$ along Oy, are $-\frac{ml}{r^3}\cos\beta$ parallel to Oy and

$$\frac{3ml\cos\beta}{r^3}\,\frac{y}{r} \text{ along OP ;}$$

$ml\cos\gamma$ along Oz, are $-\frac{ml}{r^3}\cos\gamma$ parallel to Oz and

$$\frac{3ml\cos\gamma}{r^3}\,\frac{z}{r}\text{ along OP.}$$

These forces combined give $\frac{ml}{r^3}$ parallel to BA and

$$\frac{3ml}{r^3}\left(\frac{x}{r}\cos\alpha+\frac{y}{r}\cos\beta+\frac{z}{r}\cos\gamma\right),$$

or $\frac{3ml}{r^3}\cos\theta$, along OP. We may therefore say that the action of a small magnet AB is equal to the sum of the actions of its components.

A molecule may be supposed made up of several elementary magnets such as AB. As each of these constituent elementary magnets is equivalent to three component magnets parallel respectively to the three rectangular axes, the whole molecule is equivalent to three elementary magnets parallel respectively to the axes, and therefore equivalent to a single resultant elementary magnet. The magnetic moment and direction of this resultant magnet may be called the magnetic moment and axis of the molecule.

As neighbouring molecules may be magnetized differently, we shall avoid the irregularities by considering a volume dv which, though very small, is still large enough to contain many molecules. Since each molecule in the volume is equivalent to three small component magnets parallel to the axes, the whole volume dv is equivalent to three small component magnets, and therefore to a single small magnet. If we denote the moment of this single magnet by $I\,dv$, I is defined to be the intensity of magnetization of the element, or at a point in the element, and the direction of I is defined to be the direction of magnetization.

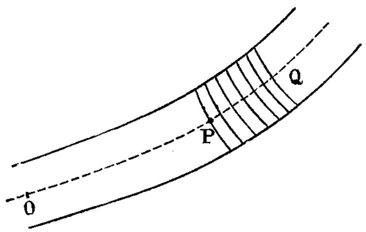
If A, B, C be the components of I parallel to the axes, it is evident that the external action of the element dv is equal to the sum of the actions of three equal volumes placed successively in the same position, whose magnetizations are respectively parallel to the axes and equal to A, B, and C.

If we draw a curve such that the tangent at any point is the direction of magnetization at that point, the curve may be called a line of magnetization. It is generally continuous so long as we keep to the same body. If at any point two consecutive tangents cut at a finite angle, we shall consider, that, for magnetic purposes, we enter a new body when we travel along the line of magnetization past the point at which the discontinuity takes place.

If the elementary volume dv be in the form of a cylinder, of small length and thickness, whose generators are lines of magnetization and whose ends are orthogonal sections, it is evident, from what has been shown, that the external magnetic action of the volume is the same as that of layers of magnetism on the ends, of surface-densities I on the positive end and $-$I on the negative end.

We shall now suppose the body divided into a vast number of elementary cylinders such as these, and we shall examine how far the magnetic layers on contiguous ends neutralize one another. Let O be a fixed point on a line of magnetization and P, Q two other points, such that the distance $OP=s$ and $OQ=s+ds$. Round the line OPQ describe a small closed curve and let a line of magnetization travel round it so as to trace out a thin tube in the body. Through P and Q draw

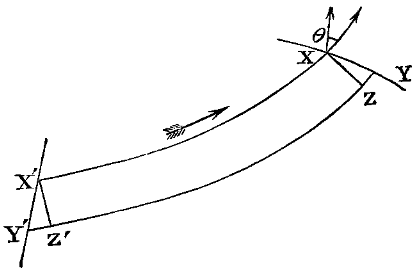
Fig. 7.



normal surfaces to the line OPQ, and let $d\alpha$ be the area of the section of the tube at P or Q. Then let the length PQ be divided into an infinite number of equal parts, each of which may be supposed considerable in comparison with the size of a molecule, and through each of the points of division draw surfaces normal to PQ, so as to divide the small cylinder PQ into an infinite number of infinitely thinner cylinders. Then, since each of these constituent cylinders of PQ is equivalent to equal surface-layers on its ends, the densities of which vary uniformly from I at P to $I+\frac{dI}{ds}ds$ at Q, it is clear that the cylinder PQ is equivalent to layers on its ends, of surface-densities $-I$ at P and $I+\frac{dI}{ds}ds$ at Q, together with a quantity of magnetism $-\frac{dI}{ds}ds\,d\alpha$ uniformly distributed throughout its volume. The density of the volume distribution is therefore $-\frac{dI}{ds}$, where the differential coefficient is found on a line of magnetization.

Now let a finite body be divided into an infinite number of thin tubes such as that surrounding the curve OPQ. Let one of these tubes meet the surface of the body in the curves XY, X′Y′, and draw two normal sections XZ, X′Z′ to the tube, entirely within the tube and just touching the curves XY, X′Y′ at the points X, X′. Then, when the section of the tube is indefinitely diminished, the external magnetic action

Fig. 8.



of XYY′X′ is ultimately the same as that of XZZ′X′, and is therefore equivalent to a volume-density whose value ρ at any point is $-\frac{dI}{ds}$, and a layer of surface-density I_X on XZ and another layer $-I_{X'}$ on X′Z′. Consider the section XZ. The layer on this section is equivalent to an equal quantity of magnetism distributed uniformly on the neighbouring small area XY. But if θ_X be the angle between the direction of magnetization and the outward drawn normal at X, the area XY = the area XZ $\times \sec\theta_X$. The surface-density on XY is therefore $I_X \cos\theta$. Similarly the density on X′Y′ is $I_{X'} \cos\theta_{X'}$. Hence we arrive at the simple result, generally obscured or made mysterious by formidable integrations, that a finite body is magnetically equivalent to a volume distribution whose density ρ at any point is $-\frac{dI}{ds}$, together with a surface-layer whose value σ at any point of the surface is $I\cos\theta$, where θ is the angle between the direction of magnetization and the outward drawn normal at the point.

The expression for ρ can be put in a more convenient form. For if three equal bodies whose magnetizations are respectively parallel to the axes and equal to A, B, C, be placed successively in the same position as the given body, the sum of their actions will be equal to that of the given body. Hence, since the body whose magnetization is parallel to Ox is equivalent to a volume distribution $-\frac{dA}{dx}$ and a surface-layer whose density at a point P where the outward drawn normal makes angles (λ, μ, ν) with the axes, is $A\cos\lambda$, we obtain

$$\rho = -\left(\frac{dA}{dx} + \frac{dB}{dy} + \frac{dC}{dz}\right),$$

and

$$\sigma = A\cos\lambda + B\cos\mu + C\cos\nu,$$

or, if the direction of magnetization make angles (α, β, γ) with the axes,

$$\begin{aligned}\sigma &= I(\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu) \\ &= I\cos\theta,\end{aligned}$$

as before.

We may now find the energy U and the entropy ϕ of any magnetized system at rest, with its magnetization in equilibrium, stable or unstable. For this purpose we shall first obtain the energy U' and the entropy ϕ' of a magnetized system identical with the given system except that it is broken up into an infinite number of small pieces.

Without altering the internal conditions or the magnetic distribution of any part of the system (U', ϕ'), let all its elementary portions be removed to infinite distances from one another, and left without velocity. Suppose that in thus preventing the forces acting between the various elements from producing velocities, the work obtained from the system is $Y+W$, where the part Y is due to the magnetization of the system and W to gravitation. Then, since the operation is clearly reversible and unattended by any thermal phenomenon, the energy will now be $U'-Y-W$, and the entropy ϕ'. Also, since the values of the energy before and after the operation $(U', U'-Y-W)$ depend only on the two states, it is clear that $Y+W$, and therefore Y alone, is independent of the manner in which the change of state is effected.

Let us now consider one of the elements after it has been removed to an infinite distance from all the other elements. Its energy will be proportional to its volume dv, if that volume is small enough; and, if the substance be homogeneous (that is non-crystalline), will be independent of the angle the direction of magnetization makes with any line fixed in the element. If, therefore, the element be homogeneous, and we suppose, for simplicity, that its state depends only on the intensity of magnetization I and the absolute temperature θ, the energy of the element may be written $F(I, \theta)dv$. Calling C the value of $F(I, \theta)$ when $I=0$, which is clearly finite, we may put $F(I, \theta)dv$ in the form $Cdv+\{F(I, \theta)-C\}dv$, or $Cdv+f(I, \theta)dv$, where $f(I, \theta)=0$ when $I=0$. We have then

$$U'-Y-W=\int Cdv+\int f(I, \theta)dv,$$

or

$$U'=Y+W+\int Cdv+\int f(I, \theta)dv.$$

Now if U_0' be the value of U' when the system (in its original state) is deprived of its magnetization, but otherwise unchanged, we shall have, since both Y and $\int f(I, \theta)dv$ vanish when $I=0$ and W does not alter,

$$U_0'=W+\int Cdv.$$

Hence

$$U'=U_0'+Y+\int f(I, \theta)dv.$$

If, therefore, we assume that $U'-U_0'$ is the same as if the system was not broken up, or equal to $U-U_0$, we obtain

$$U=U_0+Y+\int f(I, \theta)dv. \quad . \quad . \quad . \quad . \quad (1)$$

Similarly we may obtain

$$\phi'=\int Ddv+\int h(I, \theta)dv,$$

$$\phi_0'=\int Ddv,$$

and therefore

$$\phi'=\phi_0'+\int h(I, \theta)dv,$$

from which we may infer

$$\phi=\phi_0+\int h(I, \theta)dv. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The very simple expressions (1) and (2) are due, I believe, to Duhem, by whom they were given in 1888. Before making use of them, I will show how the energy of a magnetized system is discussed in the ordinary text-books.

The principle of the conservation of magnetism being taken for granted, it is first assumed that magnetization may be separated from material bodies; in other words, that the property of matter of exerting actions at a distance may exist apart from matter. It is next assumed that there is an infinite store of positive and negative magnetism at infinity at which we can be supplied gratis with as much as we require. Lastly, it is assumed that the attraction or repulsion between any two small quantities of magnetism m, m', dissevered from matter, is exactly the same as the attraction or repulsion between two small portions of matter at the same distance, magnetized with the same quantities of magnetism m, m'. Then, in order to find the energy U of any magnetized system, we suppose its magnetism made up of an infinite number of elements which were originally at an infinite distance from the given material system and from one another, and unassociated with matter. If, therefore, we denote by U_0 the energy that the given system would have if it were deprived of its magnetization, but otherwise unchanged, and imagine some agent capable of bringing the magnetic elements from infinity up to the given system, and there placing them in the positions they are to occupy, without exerting more force than is just necessary to overcome the attractions and repulsions between them, we are supposed to get

$$U = U_0 + Y.$$

The ordinary text-books make no attempt to find the entropy of a magnetized system. In fact, until the appearance of Duhem's book in 1888, the rigid methods of thermodynamics do not seem to have been thought necessary.

In order to find the condition of magnetic stability on a homogeneous body of uniform temperature θ, we suppose the body incapable of receiving or losing heat except at the constant temperature θ. Then we imagine the magnetization of a single volume element dv to change slightly in direction, and to increase from I to $I + \delta I$; and we suppose that when the temperature has again become equal to θ, no other change has been made in the system.

If δQ be the heat absorbed in the process, the principles of thermodynamics require that

$$\delta Q < \theta \delta \phi,$$

or, since no work has been done on the system during the operation,

$$\delta U < \theta \delta \phi.$$

Hence

$$\delta Y + \left(\frac{dI}{df} - \theta \frac{dh}{dI} \right) dv \, \delta I < 0,$$

or

$$\delta Y + \psi(I, \theta) dv \, \delta I < 0, \text{ (say).} \quad . \quad . \quad . \quad (3)$$

Let us now imagine a system identical with the given system before the change in dv, and let this particular element be removed to infinity without causing any other change in the system. Then if w be the work so obtained, we have clearly

$$\delta\mathrm{Y} = \delta w.$$

To find δw, we may take the volume dv of any form we please. Suppose it is a cylinder with its ends perpendicular to the axis, and the axis parallel to I. Then, by the principles of the potential, if $d\omega$ be the section of the cylinder and ds its length, we have

$$w = \mathrm{I}d\omega ds \frac{d\mathrm{V}}{ds},$$

or

$$w = \mathrm{I}\frac{d\mathrm{V}}{ds}dv,$$

where, in finding $\frac{d\mathrm{V}}{ds}$, we travel on the line of magnetization. Now since the potential V at any point (x, y, z) is a function only of the three coordinates of that point, we obtain, if (α, β, γ) be the angles the direction of magnetization at the point (x, y, z) makes with the axes,

$$\begin{aligned}\frac{d\mathrm{V}}{ds} &= \frac{d\mathrm{V}}{dx}\frac{dx}{ds} + \frac{d\mathrm{V}}{dy}\frac{dy}{ds} + \frac{d\mathrm{V}}{dz}\frac{dz}{ds} \\ &= \cos\alpha\frac{d\mathrm{V}}{dx} + \cos\beta\frac{d\mathrm{V}}{dy} + \cos\gamma\frac{d\mathrm{V}}{dz},\end{aligned}$$

and therefore

$$\mathrm{I}\frac{d\mathrm{V}}{ds} = \mathrm{A}\frac{d\mathrm{V}}{dx} + \mathrm{B}\frac{d\mathrm{V}}{dy} + \mathrm{C}\frac{d\mathrm{V}}{dz}.$$

Thus, since the potential at any point of the element dv, and therefore the values of $\frac{d\mathrm{V}}{dx}, \frac{d\mathrm{V}}{dy}, \frac{d\mathrm{V}}{dz}$ are independent of the magnetization of that particular element when it is small enough, we obtain

$$\delta w = \left(\delta\mathrm{A}\frac{d\mathrm{V}}{dx} + \delta\mathrm{B}\frac{d\mathrm{V}}{dy} + \delta\mathrm{C}\frac{d\mathrm{V}}{dz}\right)dv.$$

If the element dv be to any extent magnetically "rigid," its magnetization will not be fully able to obey the directing causes, and there will be relations between δA, δB, and δC; but if the element be "perfectly soft," we may consider δA, δB, δC independent. In the latter case, if we put δB and δC both zero, equation (3) gives

$$\delta A \frac{dV}{dx} + \psi(I, \theta)\delta I < 0.$$

But since $I^2 = A^2 + B^2 + C^2$, we have, when B and C are constant,

$$I\,\delta I = A\,\delta A.$$

Hence, for all values of δA, we have

$$\left\{\frac{dV}{dx} + \frac{A}{I}\psi(I, \theta)\right\}\delta A < 0.$$

If the quantity $\frac{dV}{dx} + \frac{A}{I}\psi$, within $\{\ldots\}$, be positive, A can only decrease; if it be negative, A can only increase; if it be zero, A can neither increase nor decrease. We have, therefore, in stable equilibrium, at every point of a "perfectly soft" substance,

$$\frac{1}{A}\frac{dV}{dx} = \frac{1}{B}\frac{dV}{dy} = \frac{1}{C}\frac{dV}{dz} = -\frac{1}{I}\psi(I, \theta). \quad . \quad . \quad . \quad (4)$$

We must now explain the meaning of the differential coefficients of V. We know that if at any external point $P(x, y, z)$, a unit positive pole be placed without disturbing the magnetization of any part of the given material system, $\left(-\frac{dV}{dx}, -\frac{dV}{dy}, -\frac{dV}{dz}\right)$ will be the magnetic forces (X, Y, Z), parallel to the axes, exerted on the unit pole at P by the given system. When the point P is within the given system, we cannot place a unit pole there without disturbing the system. We therefore imagine a small right circular cylinder, whose axis coincides with the direction of magnetization and whose ends are perpendicular to the axis, removed from about the point P; and suppose that no change is made in the system beyond the removal of the contents of the cylinder. If the point P is in the midst of a liquid or gas, a thin substance, the magnetization of which may be neglected, must be used as a lining for the cylinder, so that the interior of the cylinder is vacuous. Then if V' be the potential at P of the new system obtained by removing the contents of the cylinder from the original system, $\left(-\frac{dV'}{dx}, -\frac{dV'}{dy}, -\frac{dV'}{dz}\right)$ will be the magnetic forces parallel to the axes, exerted by the new system on a unit positive pole placed at P without disturbing the system. But if V'' was the potential at P of the contents of the cylinder before removal, and V the potential of the whole of the given system, we should have

$$V = V' + V'',$$

and therefore

$$-\frac{dV}{dx} = -\frac{dV'}{dx} - \frac{dV''}{dx}, \text{ \&c., \&c.}$$

Now the contents of the small cylinder, before being cut away, were magnetically equivalent to layers on the ends, of densities $+I$ on the positive end and $-I$ on the negative end. Thus $-\frac{dV''}{dx}$ is simply the force, parallel to the axis of x, arising from these two layers. But if we take a circular layer of uniform density I, the force it exerts on a unit pole in the axis of the layer at a point where the radius of the layer subtends an angle α, is $2\pi I(1-\cos\alpha)$, and may therefore be neglected when α is small. Hence, if the radius of the right circular cylinder be infinitely small in comparison with the length, the differential coefficients of V'' will be zero. Consequently, $\left(-\frac{dV}{dx}, -\frac{dV}{dy}, -\frac{dV}{dz}\right)$ are the magnetic forces parallel to the axes, exerted by the new system V' on a unit pole placed at P without disturbing that system. These forces are written (X, Y, Z), and are called *the forces* of the *given* system at P.

If F be the resultant of (X, Y, Z), or the resultant force of the given system at P, equations (4) become

$$\frac{X}{A} = \frac{Y}{B} = \frac{Z}{C} = \pm\frac{F}{I} = \frac{1}{I}\psi(I, \theta).$$

Now it has been shown by Duhem (*Des Corps Diamagnétiques*) that $\psi(I, \theta)$ must always be positive. We must therefore always take the positive sign before $\frac{F}{I}$, and may write

$$\frac{A}{X} = \frac{B}{Y} = \frac{C}{Z} = \frac{I}{F} = \chi(I, \theta), \quad . \quad . \quad . \quad . \quad (5)$$

where $\chi(I, \theta)$ is always positive.

The meaning of equations (5) is that, at any point of a "perfectly soft" homogeneous substance, the magnetization, when in stable equilibrium, coincides in direction with the force at that point. If there is any magnetic "rigidity" about the substance, the magnetization at a point may, of course, make a finite angle with the force at that point.

We can now explain Weber's hypothesis of magnetism. He considers that a body which appears to be neutral is as much magnetized as when it exhibits active magnetic properties, only that, in the former case, the magnetized molecules have their axes pointing in all directions so as exactly to neutralize one another. He then supposes that the act of magnetization merely consists in giving the axes of the magnetized molecules a definite direction. In fact, if we suppose an elementary magnet suspended freely by the centre of mass, it is clear that it will set its axis in the direction of the external magnetic force which acts upon it.

In the common theory of magnetism it is admitted that in a "perfectly soft" homogeneous substance, the magnetization at any point is in the same straight line as the force; but it is supposed that in the so-called diamagnetic homogeneous "soft substances," ψ, or χ is negative, or that the magnetization is in the opposite direction to the force. This gives rise to a difficulty in Weber's theory; for it appears to follow that when an elementary magnet is freely suspended by the centre of mass, it may permanently set its axis in the opposite direction to the external magnetic force. To escape from this difficulty it might be assumed that in every diamagnetic body a number of molecules form a kind of lock-work, similar to that of a gun, and that the first act of the external magnetizing force is to set the lock. In this way, it might be thought, we should have a means of setting the magnetized molecules in the opposite direction to the force and keeping them there; but it would follow that a diamagnetic body could not be magnetized until the external magnetizing force exceeded a certain value, and would not lose its magnetization when the force was withdrawn. As this appears to be contrary to experiment, we conclude that our explanation of the difficulty due to the common theory of diamagnetism must be insufficient.

In the case of a quasi-homogeneous substance, like air or any gas, the mass-density will vary from point to point. Denoting the mass-density by ρ, and putting $I=\frac{\rho}{\rho_0}I'$, where ρ_0 is a standard fixed value of ρ, the state of the air or gas at any point may be defined by the three variables (I', ρ, θ). If we put $A=\frac{\rho}{\rho_0}A'$, &c. &c., it is easy to see that equations (5), which hold for the stable distribution of magnetization, become

$$\frac{A'}{X}=\frac{B'}{Y}=\frac{C'}{Z}=\frac{I'}{F}=\chi'(I', \rho, \theta), \quad . \quad . \quad . \quad (6)$$

where χ' is always positive.

Now let the air be mapped out by equipotential surfaces and lines of force, just as in electrostatics, and imagine a small right circular cylinder described in the air with its ends at right angles to the axis and the radius of its normal section very small in comparison with the length of the axis. Then, by taking the axis of the cylinder tangential to an equipotential surface, it is easily seen that the pressure and density of the air have constant values all over the same equipotential surface. If the axis coincide with a line of force, and we suppose ourselves to travel in the direction of the force, we obtain, since the force exerted at any point by the neighbouring molecules is zero,

$$dp=I\frac{dF}{ds}ds=I'\frac{\rho}{\rho_0}\frac{dF}{ds}ds,$$

or, if we assume the simple law of gases, $p=R\rho\theta$, where R is a constant for the same gas,

$$dp=I'\frac{p}{p_0}\frac{dF}{ds}ds,$$

or

$$\frac{dp}{p}=\frac{I'}{p_0}\frac{dF}{ds}ds.$$

Integrating this equation, we get

$$\log\frac{p_2}{p_1}=\frac{1}{p_0}\int_1^2 I'\frac{dF}{ds}ds.$$

Now it is usually assumed that for the feebly magnetic substances, the positive quantity χ is practically constant, and its value is written k. We have then, in a homogeneous soft body (liquid or solid), $A=kX=-k\frac{dV}{dx}$, &c. &c., so that if ρ be the volume-density of the magnetism in the interior of the body,

$$\rho=-\left(\frac{dA}{dx}+\frac{dB}{dy}+\frac{dC}{dz}\right)=k\nabla^2V.$$

Hence, since $\nabla^2V=-4\pi\rho$, we have

$$\rho(1+4\pi k)=0,$$

and therefore $\rho=0$, or the body is magnetically equivalent to a layer of magnetism on the surface.

A similar result would follow for a gas, provided the mass-density be nearly uniform.

In air or a gas,

$$I' = kF,$$

or

$$I = k\frac{\rho}{\rho_0}F,$$

and, therefore, along a line of force in air,

$$\log\frac{p_2}{p_1} = \frac{k}{2p_0}(F_2^2 - F_1^2),$$

or

$$\frac{p_2}{p_1} = e^{\frac{k}{2p_0}(F_2^2 - F_1^2)},$$

or approximately, since k is very small,

$$\frac{p_2 - p_1}{p_1} = \frac{k}{2p_0}(F_2^2 - F_1^2),$$

that is, nearly,

$$p_2 - p_1 = \frac{k}{2}(F_2^2 - F_1^2). \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In a liquid, $I = kF$, and therefore, along a line of force,

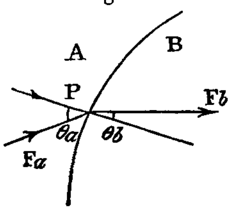
$$dp = I\frac{dF}{ds}ds = kF\frac{dF}{ds}ds,$$

and therefore

$$p_2 - p_1 = \frac{k}{2}(F_2^2 - F_1^2). \quad . \quad . \quad . \quad . \quad . \quad (7')$$

The important simplification effected by putting χ constant makes it easy to determine the abrupt change which takes place in the force when we pass from one soft body to another. For let F_a be the force just inside a soft body A and F_b the force just inside another soft body B, near any point P of their common surface; and let θ_a, θ_b be the angles F_a and F_b make with the common normal at P, drawn from A to B. Then the density at P of the surface-layer of A will be $I_a \cos\theta_a$, and of the layer of B, $-I_b\cos\theta_b$. Hence, in passing from A to B, we have

Fig. 9.



$$F_b\cos\theta_b - F_a\cos\theta_a = 4\pi\{-I_b\cos\theta_b + I_a\cos\theta_a\}.$$

If both A and B be liquid or solid, we obtain

$$(1+4\pi k_b)F_b \cos\theta_b = (1+4\pi k_a)F_a \cos\theta_a.$$

If A be air or any gas, and ρ be the mass density at P,

$$(1+4\pi k_b)F_b \cos\theta_b = \left(1+4\pi k_a \frac{\rho}{\rho_0}\right) F_a \cos\theta_a.$$

If A be a perfect "vacuum," we put $k_a=0$.

This result, which is even more important in electricity than in magnetism, can be written in a very brief form. For if F_{an}, F_{bn} be the normal components of F_a and F_b, and if μ_a stand for $1+4\pi k_a$ or $1+4\pi k_a \frac{\rho}{\rho_0}$ and μ_b for $1+4\pi k_b$, we have

$$\mu_b F_{bn} = \mu_a F_{an}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The foregoing is the usual method of stating the result; but if we keep to the convention of supposing every *normal* to be drawn *outwards*, we shall have

$$\mu_a F_{an} + \mu_b F_{bn} = 0.$$

Since for many substances k is very small (being less than $\frac{1}{400,000}$), it follows from the preceding investigation that the abrupt change of the force in crossing the boundary of two soft bodies may generally be neglected. Hence, if a number of soft, feebly magnetic bodies be magnetized by permanent steel magnets, we may suppose, without sensible error, that the force at any point is entirely due to the permanent magnets; in other words, we may neglect the force due to the magnetization induced in the soft bodies. This may also be shown as follows:—Let B be any soft, feebly magnetic body. Then the force at any point is the resultant of two forces—F_b due to B and F_a due to the rest of the system. To make a rough comparison between F_b and F_a, we take the point close to the surface of B, in which case it is evident that F_b is comparable with $2\pi I_b$, that is, with $2\pi k_b F$, or with the resultant of $2\pi k_b F_b$ along F_b and $2\pi k_b F_a$ along F_a. Thus F_b is comparable with $2\pi k_b F_a$, or if $k_b = \frac{1}{400,000}$, with the 60,000th of F_a; and we draw the same conclusion as before.

Let us now suppose that a soft, feebly magnetic body B, which is either a solid or a liquid contained in a bag, is magnetized inductively by a permanent steel magnet situated to the left; and let us imagine, for the sake of simplicity, that the magnetic force within the body B is everywhere parallel to the axis of x, so that as we travel parallel to Ox in the positive direction, the force diminishes *numerically*, whether it be the positive or the negative pole of the permanent magnet which acts on B. Then if we consider a parallelopiped on the base $dy\,dz$, the force acting on it parallel to Ox will be

$$dy\,dz\int \mathrm{I}\frac{d\mathrm{F}}{dx}dx,$$

or

$$k_b dy\,dz\int \mathrm{F}\frac{d\mathrm{F}}{dx}dx,$$

or

$$\tfrac{1}{2}k_b dy\,dz\int\frac{d\mathrm{F}^2}{dx}dx,$$

which is always negative, since F^2 diminishes as x increases, and k_b is positive. Hence the soft body B is always attracted by the permanent magnet. The same result would have been obtained if B had been air or a gas contained in a bag.

If the body B be immersed in air or in a gas, or in a soft liquid, the pressures on the two ends of the small parallelopiped will, by equations (7) and (7′), give a force in the opposite direction to Ox of

$$\tfrac{1}{2}k_a dy\,dz\int\frac{d\mathrm{F}^2}{dx}dx,$$

where F has the same meaning as before.

Hence if k_b be greater than k_a, or the body B more magnetic than the gaseous or liquid medium by which it is surrounded, the attraction of the permanent magnet will overpower the pressure on the surface and the body B will be drawn towards the pole of the magnet; but if k_b be less than k_a, or B less magnetic than the surrounding medium, the attraction of the permanent magnet will be overpowered by the pressure on the surface, and the body B will appear to be repelled by the permanent magnet.

We may now sum up the analogies we have found between magnetism and gravitation in the case of homogeneous bodies. First of all, every soft substance is attracted when placed near one pole of a magnet; and every body is attracted to the earth. Secondly, if a number of soft bodies be magnetized by a steel magnet, we may neglect the action of the magnetized soft bodies on one another; and if a number of small bodies be placed near the earth, we may neglect their gravitational attraction on one another in comparison with that of the whole earth. Lastly, if a soft body be immersed in a gas or liquid, and then placed near the pole of a magnet, it will appear to be attracted or repelled according as it is more or less magnetic than the gas or liquid by which it is surrounded; and if any body be immersed in a gas or liquid, it will appear to be attracted or repelled by the earth according as it is heavier or lighter than the gas or liquid in which it is placed.

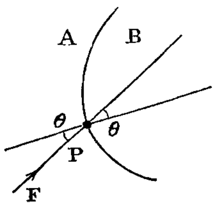
The theory we have given is beautifully illustrated and confirmed by the following experiments of Faraday's, described in Tyndall's 'Diamagnetism.' Theory and experiment fit together so exquisitely that we cannot but wonder the true theory should not have been seen from the first.

"If a weak solution of protosulphate of iron, *m*, be put into a selected thin glass tube about an inch long, and one third or one fourth of an inch in diameter, and sealed up hermetically, and be then suspended horizontally between the magnetic poles in the air, it will point axially, and behave in other respects like iron; if instead of air between the poles, a solution of the same kind as *m*, but a little stronger, *n*, be substituted, the solution in the tube will point equatorially, or as bismuth. A like solution somewhat weaker than *m*, to be called *l*, enclosed in a similar tube, will behave like bismuth in air but like iron in water."

It now remains to describe how it is generally attempted to gloss over the imaginary difficulties of diamagnetism.

It is generally admitted that the apparent magnetic or diamagnetic properties of a soft body B immersed in air or any other gas or a liquid, are merely differential—that is, depend on the algebraic excess of the coefficient k_b of the body B over the coefficient k_a of the substance A in which it is immersed. This result is thought to be "proved" in the following way:—Since every soft body is magnetically equivalent to a layer of magnetism on its surface, it follows that there are two layers on the common surface of A and B, one belonging to A, the other to B. It is then supposed that the layer on this surface which properly belongs to A, does not really belong to A at all, but to B. The body A being magnetically equivalent to a layer on its surface, it is assumed that, as the layer on the common surface of A and B is supposed transferred to B, we may treat A as unmagnetized. With this assumption the pressure of A would be uniform, and the behaviour of B would be entirely determined by the supposed compound layer on its surface. Now if θ be the angle at any point P of the common surface of A and B between the normal at P, supposed drawn from A to B, and the force, which may be considered continuous in crossing the bounding surface, the superficial density of the layer at P which belongs to B will be $-I_b \cos\theta$, and of that which belongs to A, $I_a \cos\theta$. Hence the density at P of the compound layer is $(I_a - I_b)\cos\theta$. The ratio of this to the density at P of the layer which properly belongs to B, is $\frac{I_b - I_a}{I_b}$, which is equal to a constant $\frac{k_b - k_a}{k_b}$. Thus, since the attraction of the permanent magnet on B, due to the surface-layer which properly belongs to B, may be written k_bG, where G would have the same value for any soft body of the same shape and size as B, when placed in the same position, the attraction of the permanent magnet on B, when immersed in A, will be $(k_b - k_a)$ G, or equal to the force due to the layer which properly belongs to B, diminished by what this force would be for the gas or liquid, A, displaced by B.

Fig. 10.



According to the remarkable caricature of reasoning just noticed, it follows that we do not need to know the absolute value of the coefficient k belonging to any soft substance, but merely the algebraic excess of the coefficient over that of some standard substance. This standard "substance" is often chosen to be a "vacuum," and its coefficient is put zero. Then, since many bodies are apparently repelled by a magnet pole in a comparatively slight "vacuum" of 2 to 3 millimetres of mercury, it is concluded that the coefficients of these bodies, or, rather, the excesses of their coefficients over that of a vacuum, are negative.

Granting, for the present, the first part of this so-called reasoning, we must point out that a vacuum can only be obtained by removing the air completely from the interior of a closed vessel, and not by merely reducing the pressure to 2 or 3 millimetres of mercury. If we were allowed to consider such a comparatively slight reduction of density as constituting a vacuum, we could prove the existence of diagravitation; for if we could find a gas 100 times as light as hydrogen, a balloon could be made which would float in this so-called vacuum.

We must now consider the two layers on the common surface of two soft bodies A, B, A being a gas or liquid. If the surface-molecules of A were provided with sharp points and were caused by the smallest amount of magnetization to stick to B, it might be thought that both surface-layers would then belong to B; but a little consideration tells us that the molecules which stick to B would take with them two layers of opposite signs, and it is clear that the remainder of A would still have a surface-layer of its own, adjoining the modified surface of B. In order, therefore, to cause the two surface-layers both to belong to B, we must make the following assumptions:—It must be supposed that every molecule of A is provided with a sharp point, and that the act of magnetization causes each molecule to be broken into two halves, on one of which is the positive magnetism, on the other the negative. Then it must be supposed that those half-molecules on which are the surface-layers of A stick to the bodies B . . . , with which they happen to be in contact, and lastly, that the other half-molecules stick together in pairs in such a way that their magnetisms neutralize one another. In this way, we should have both surface-layers belonging to B, and we might treat the free part of the gas or liquid A as unmagnetized. A difficulty would, however, arise when the magnetizing force was withdrawn, unless we had some means of reminding the half-molecules to take partners. We might avoid the difficulty by imagining the two halves of each molecule tied together by a piece of thread, but then we should introduce the absurdity that magnetization changed the gas or liquid A into a solid. Lastly, we should be obliged to conclude that when a system is once magnetized, it is impossible to increase the magnetization—a conclusion which, of course, is necessarily false.

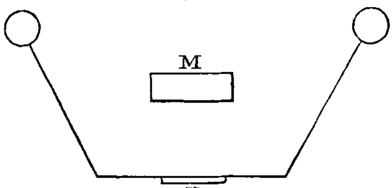
There is one other way of treating diamagnetism which requires to be noticed. This is the method of induced electric currents used by Weber, by which it is thought to be proved that k is negative for bismuth and some other substances. To this I reply that I have already sufficiently disproved the common theory of diamagnetism; and secondly, it will be proved in a future paper that the common theory of induced currents generally involves an absurdity, and can seldom be correct.

The rest of the paper will be occupied with a brief discussion of a few important problems in the light of the new theory.

I. If a body P, placed near a number of fixed bodies X, Y, Z, ... , be subject to no actions at a distance, but those of magnetism and gravitation arising from the fixed bodies X, Y, Z . . . . , it can easily be shown that no state of equilibrium of the body P can be stable unless it be in material contact with one or more of the fixed bodies. This proposition is proved in Duhem's *L'aimantation per influence.* When P is in material contact with one or more of the fixed bodies, its equilibrium may, of course, be stable. The most useful case to consider is the following :—

Imagine a small bar of bismuth B, suspended from two small balloons by threads, as in the figure, and suppose that the mass of the system is slightly less than the mass of the air which it displaces when unmagnetized and near the ground. Then, if the system be set free, it will ascend in the air, and, of course, a state of stable equilibrium will be attained when it has risen high enough, if the weather be calm. We may, however, obtain a state of stable equilibrium in a more convenient way. For if a permanent strong steel magnet M be fixed some distance from the ground, just over the bar B, the system of the bismuth and balloons, when let go, will ascend until the motion is checked by the increased density and pressure of the air about M, and will ultimately take up a state of stable equilibrium suspended in the air at a moderate distance from the ground.

Fig. 11.



The problem just given was, we believe, first considered by Sir W. Thomson, to whom it was suggested by the story of the coffin of Mahomet.

II. We will next consider the thermal phenomena due to the motion of a soft body B in presence of a permanent magnet M. For simplicity, let M be held at rest and suppose its magnetism "rigid." Also let B be homogeneous, and let our magnetized system consist merely of the two bodies B and M, situated in a vacuum. Then if the temperature, in every state of equilibrium, is uniform and equal to θ, and if no heat can be absorbed or given out except at the temperature θ, we shall have, in any small reversible operation,

$$\delta Q = \theta \delta \phi.$$

Now the formula for the entropy is

$$\phi = \phi_0 + \int h(I, \theta) dv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We have, therefore, if no appreciable change takes place in the form or size of B,

$$\delta Q = \theta \int \frac{dh}{dI} \delta I \, dv,$$

where the integral refers only to the soft body B, since no change can take place in M.

If the operation consist in moving B nearer to M, I will increase or δI be positive. Hence, if $\frac{dh}{dI}$ be positive for all values of I, the operation will cause an absorption of heat, or would cool the body B, if heat was not supplied from without: if $\frac{dh}{dI}$ be always negative, there will be an evolution of heat, or the operation would heat the body B.

III. We will, last of all, examine, with Duhem, the method proposed by Jamin for the determination of the distribution of magnetism on a permanent magnet.

A small piece of soft iron, B, being placed in contact with the permanent magnet at any point P, the smallest force required to detach it is measured, and it is supposed by Jamin that this force is proportional to $\left(\frac{dV}{dn}\right)^2$, where V is the potential of the permanent magnet at P and dn an element of the outward drawn normal.

We observe, in the first place, that the small piece of soft iron B is magnetically equivalent to a layer on its surface. Consequently, the magnetic force at any point is the resultant of that due to the surface-layer of B and of that due to the permanent magnet. Within the body B, the force due to the surface-layer of B is the greater of the two. This will complicate the problem, and so, for the sake of argument, we will agree to ignore the surface-layer of B. With this assumption, it follows that the equipotential surfaces and lines of force will be due entirely to the permanent magnet, and that the *total* magnetic force exerted on B by the permanent magnet acts along the line of force at P, and is equal to $I \frac{dF}{ds} dv$, where dv is the volume of B, and ds an element of the line of force *in the positive direction of* F. If, for simplicity, we put $I = kF$, this result becomes $\frac{1}{2}k \frac{dF^2}{ds} dv$. Now if the line of force at P be directed outwards, $\frac{dF^2}{ds}$ will be negative, and if it be directed inwards, $\frac{dF^2}{ds}$ will be positive. Thus in both cases, the total magnetic force exerted by the permanent magnet on B is an attraction; and its component along the outward-drawn normal is $\frac{1}{2}k\frac{dF^2}{dn}\,dv$.

Thus even with all our assumptions, the force which Jamin requires to be measured is proportional, not to $\left(\frac{dV}{dn}\right)^2$, that is to F^2, but to $\frac{dF^2}{dn}$.

The preceding three examples, and many others, are discussed in Duhem's *L'aimantation par influence*—a book which seems to contain the first systematic application of the principles of thermodynamics to magnetism.

XXXV. *The Expansion of Chlorine by Light as applied to the Measurement of the Intensity of Rays of High Refrangibility.* By Dr. A. Richardson, *Lecturer on Chemistry, University College, Bristol**.

[Plates III. & IV.]

IT has been shown by Budde (Phil. Mag. iv. 1871; Pogg. *Ann.* Ergbd. vi. 1873) that when chlorine is exposed to the influence of sunlight, an expansion of the gas occurs which is independent of the direct heating-effects due to the light; the volume to which the gas first expands is maintained during exposure provided that the intensity of the light remains constant, contraction to the original volume taking place when the gas is shaded. He further found that the rays of high refrangibility were influential in promoting this change, no expansion being occasioned by the rays at the red end of the spectrum. The application of this property of chlorine to the measurement of the "actinic"† intensity of light was suggested by Budde many years ago, but no further steps appear to have been taken in this direction.

Some experiments on which I am at present engaged have rendered it necessary that the actinic intensity of light should be measured during periods of many months together, and it seemed possible that the expansion of chlorine by light might be applied to this purpose. As, however, the researches of Bunsen and Roscoe (Trans. Roy. Soc. 1887, p. 381) led them to the conclusion that no change in volume occurred in chlorine, when exposed to light, other than that due to direct heating-effects, it became necessary to repeat some of Budde's experiments so as if possible to decide this point. In order to do this

* Communicated by the Physical Society: read June 26, 1891.

† The term "actinic" is used for brevity to denote rays at the violet end of the spectrum.