Applied Mathematical Sciences
Volume 154

Springer
New York
Berlin
Heidelberg
Barcelona
Hong Kong
London
Milan
Paris
Singapore
Tokyo

Applied Mathematical Sciences

1. *John:* Partial Differential Equations, 4th ed.
2. *Sirovich:* Techniques of Asymptotic Analysis.
3. *Hale:* Theory of Functional Differential Equations, 2nd ed.
4. *Percus:* Combinatorial Methods.
5. *von Mises/Friedrichs:* Fluid Dynamics.
6. *Freiberger/Grenander:* A Short Course in Computational Probability and Statistics.
7. *Pipkin:* Lectures on Viscoelasticity Theory.
8. *Giacaglia:* Perturbation Methods in Non-linear Systems.
9. *Friedrichs:* Spectral Theory of Operators in Hilbert Space.
10. *Stroud:* Numerical Quadrature and Solution of Ordinary Differential Equations.
11. *Wolovich:* Linear Multivariable Systems.
12. *Berkovitz:* Optimal Control Theory.
13. *Bluman/Cole:* Similarity Methods for Differential Equations.
14. *Yoshizawa:* Stability Theory and the Existence of Periodic Solution and Almost Periodic Solutions.
15. *Braun:* Differential Equations and Their Applications, 3rd ed.
16. *Lefschetz:* Applications of Algebraic Topology.
17. *Collatz/Wetterling:* Optimization Problems.
18. *Grenander:* Pattern Synthesis: Lectures in Pattern Theory, Vol. I.
19. *Marsden/McCracken:* Hopf Bifurcation and Its Applications.
20. *Driver:* Ordinary and Delay Differential Equations.
21. *Courant/Friedrichs:* Supersonic Flow and Shock Waves.
22. *Rouche/Habets/Laloy:* Stability Theory by Liapunov's Direct Method.
23. *Lamperti:* Stochastic Processes: A Survey of the Mathematical Theory.
24. *Grenander:* Pattern Analysis: Lectures in Pattern Theory, Vol. II.
25. *Davies:* Integral Transforms and Their Applications, 2nd ed.
26. *Kushner/Clark:* Stochastic Approximation Methods for Constrained and Unconstrained Systems.
27. *de Boor:* A Practical Guide to Splines: Revised Edition.
28. *Keilson:* Markov Chain Models—Rarity and Exponentiality.
29. *de Veubeke:* A Course in Elasticity.
30. *Sniatycki:* Geometric Quantization and Quantum Mechanics.
31. *Reid:* Sturmian Theory for Ordinary Differential Equations.
32. *Meis/Markowitz:* Numerical Solution of Partial Differential Equations.
33. *Grenander:* Regular Structures: Lectures in Pattern Theory, Vol. III.
34. *Kevorkian/Cole:* Perturbation Methods in Applied Mathematics.
35. *Carr:* Applications of Centre Manifold Theory.
36. *Bengtsson/Ghil/Källén:* Dynamic Meteorology: Data Assimilation Methods.
37. *Saperstone:* Semidynamical Systems in Infinite Dimensional Spaces.
38. *Lichtenberg/Lieberman:* Regular and Chaotic Dynamics, 2nd ed.
39. *Piccini/Stampacchia/Vidossich:* Ordinary Differential Equations in $\mathbf{R}^n$.
40. *Naylor/Sell:* Linear Operator Theory in Engineering and Science.
41. *Sparrow:* The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors.
42. *Guckenheimer/Holmes:* Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.
43. *Ockendon/Taylor:* Inviscid Fluid Flows.
44. *Pazy:* Semigroups of Linear Operators and Applications to Partial Differential Equations.
45. *Glashoff/Gustafson:* Linear Operations and Approximation: An Introduction to the Theoretical Analysis and Numerical Treatment of Semi-Infinite Programs.
46. *Wilcox:* Scattering Theory for Diffraction Gratings.
47. *Hale/Magalhães/Oliva:* Dynamics in Infinite Dimensions, 2nd ed.
48. *Murray:* Asymptotic Analysis.
49. *Ladyzhenskaya:* The Boundary-Value Problems of Mathematical Physics.
50. *Wilcox:* Sound Propagation in Stratified Fluids.
51. *Golubitsky/Schaeffer:* Bifurcation and Groups in Bifurcation Theory, Vol. I.
52. *Chipot:* Variational Inequalities and Flow in Porous Media.
53. *Majda:* Compressible Fluid Flow and System of Conservation Laws in Several Space Variables.
54. *Wasow:* Linear Turning Point Theory.
55. *Yosida:* Operational Calculus: A Theory of Hyperfunctions.
56. *Chang/Howes:* Nonlinear Singular Perturbation Phenomena: Theory and Applications.
57. *Reinhardt:* Analysis of Approximation Methods for Differential and Integral Equations.
58. *Dwoyer/Hussaini/Voigt (eds):* Theoretical Approaches to Turbulence.
59. *Sanders/Verhulst:* Averaging Methods in Nonlinear Dynamical Systems.

(continued following index)

George W. Bluman   Stephen C. Anco

Symmetry and Integration Methods for Differential Equations

With 18 Illustrations

Springer

George W. Bluman
Department of Mathematics
The University of British Columbia
Vancouver, British Columbia V6T 1Z2
Canada
bluman@math.ubc.ca

Stephen C. Anco
Department of Mathematics
Brock University
St. Catharines, Ontario L2S 3A1
Canada
sanco@brocku.ca

Mathematics Subject Classification (2000): 22E25, 22E65, 22E70, 34-01, 34A05, 35K05, 35L05, 35Q20, 58F35, 58F37, 58G35, 58G37, 70H35

Library of Congress Cataloging-in-Publication Data
Bluman, George W., 1943–
Symmetry and integration methods for differential equations/George W. Bluman, Stephen C. Anco.
p. cm. — (Applied mathematical sciences; v. 154)
Includes bibliographical references and index.
ISBN 0-387-98654-5 (alk. paper)
1. Differential equations—Numerical solutions. 2. Differential equations, Partial—Numerical solutions. 3. Lie groups. I. Bluman, George W., 1943– Symmetries and differential equations. II. Title. III. Applied mathematical sciences (Springer-Verlag New York, Inc.); v.154.
QA1 .A647 no. 154 2002
[QA372]
510 s—dc21
[515'.35] 2001054908

ISBN 0-387-98654-5 Printed on acid-free paper.

Printed in the United States of America.

9 8 7 6 5 4 3 2 1 SPIN 10696756

Typesetting: Pages created by the authors using Microsoft Word.

www.springer-ny.com

Springer-Verlag New York Berlin Heidelberg
A member of BertelsmannSpringer Science+Business Media GmbH

Contents

Preface

This book is a significant update of the first four chapters of *Symmetries and Differential Equations* (1989; reprinted with corrections, 1996), by George W. Bluman and Sukeyuki Kumei. Since 1989 there have been considerable developments in symmetry methods (group methods) for differential equations as evidenced by the number of research papers, books, and new symbolic manipulation software devoted to the subject. This is, no doubt, due to the inherent applicability of the methods to nonlinear differential equations. Symmetry methods for differential equations, originally developed by Sophus Lie in the latter half of the nineteenth century, are highly algorithmic and hence amenable to symbolic computation. These methods systematically unify and extend well-known ad hoc techniques to construct explicit solutions for differential equations, especially for nonlinear differential equations. Often ingenious tricks for solving particular differential equations arise transparently from the symmetry point of view, and thus it remains somewhat surprising that symmetry methods are not more widely known. Nowadays it is essential to learn the methods presented in this book to understand existing symbolic manipulation software for obtaining analytical results for differential equations. For ordinary differential equations (ODEs), these include reduction of order through group invariance or integrating factors. For partial differential equations (PDEs), these include the construction of special solutions such as similarity solutions or nonclassical solutions, finding conservation laws, equivalence mappings, and linearizations.

A large portion of this book discusses work that has appeared since the above-mentioned book, especially connected with finding first integrals for higher-order ODEs and using higher-order symmetries to reduce the order of an ODE. Also novel is a comparison of various complementary symmetry and integration methods for an ODE.

The present book includes a comprehensive treatment of dimensional analysis. There is a full discussion of aspects of Lie groups of point transformations (point symmetries), contact symmetries, and higher-order symmetries that are essential for finding solutions of differential equations. No knowledge of group theory is assumed. Emphasis is placed on explicit algorithms to discover symmetries and integrating factors admitted by a given differential equation and to construct solutions and first integrals resulting from such symmetries and integrating factors.

This book should be particularly suitable for applied mathematicians, engineers, and scientists interested in how to find systematically explicit solutions of differential equations. Almost all examples are taken from physical and engineering problems including those concerned with heat conduction, wave propagation, and fluid flow.

Chapter 1 includes a thorough treatment of dimensional analysis. The well-known Buckingham Pi-theorem is presented in a manner that introduces the reader concretely to the notion of invariance. This is shown to naturally lead to generalizations through invariance of boundary value problems under scalings of variables. This prepares the reader to consider the still more general invariance of differential equations under Lie groups of transformations in the third and fourth chapters. Basically, the first chapter gives the reader an intuitive grasp of some of the subject matter of the book in an elementary setting.

Chapter 2 develops the basic concepts of Lie groups of transformations and Lie algebras that are necessary in the following two chapters. By considering a Lie group of point transformations through its infinitesimal generator from the point of view of mapping functions into functions with their independent variables held fixed, we show how one is able to consider naturally other local transformations such as contact transformations and higher-order transformations. Moreover, this allows us to prepare the foundation for consideration of integrating factors for differential equations.

Chapter 3 is concerned with ODEs. A reduction algorithm is presented that reduces an nth-order ODE, admitting a solvable r-parameter Lie group of point transformations (point symmetries), to an $(n-r)$th-order differential equation and r quadratures. We show how to find admitted point, contact, and higher-order symmetries. We also show how to extend the reduction algorithm to incorporate such symmetries. It is shown how to find admitted first integrals through corresponding integrating factors and to obtain reductions of order using first integrals. We show how this simplifies and significantly extends the classical Noether's Theorem for finding conservation laws (first integrals) to any ODE (not just one admitting a variational principle). In particular, we show how to calculate integrating factors by various algorithmic procedures analogous to those for calculating symmetries in characteristic form where only the dependent variable undergoes a transformation. We also compare the distinct methods of reducing order through admitted local symmetries and through admitted integrating factors. We show how to use invariance under point symmetries to solve boundary value problems. We derive an algorithm to construct special solutions (invariant solutions) resulting from admitted symmetries. By studying their topological nature, we show that invariant solutions include separatrices and singular envelope solutions.

Chapter 4 is concerned with PDEs. It is shown how to find admitted point symmetries and how to construct related invariant solutions. There is a full discussion of the applicability to boundary value problems with numerous examples involving scalar PDEs and systems of PDEs.

Chapters 2 to 4 can be read independently of the first chapter. Moreover, a reader interested in PDEs can skip the third chapter.

Every topic is illustrated by examples. All sections have many exercises. It is essential to do the exercises to obtain a working knowledge of the material. The Discussion section at the end of each chapter puts its contents into perspective by summarizing major results, by referring to related works, and by introducing related material.

Within each section and subsection of a given chapter, we number separately, and consecutively, definitions, theorems, lemmas, and corollaries. For example, Definition 2.3.3-1 refers to the first definition and Theorem 2.3.3-1 to the first theorem in Section 2.3.3. Exercises appear at the conclusion of each section; Exercise 2.4-2 refers to the second problem of Exercises 2.4.

We thank Benny Bluman for the illustrations and Cecile Gauthier for typing several drafts of Sections 3.5 to 3.8.

Vancouver, British Columbia, Canada — George W. Bluman
St. Catharines, Ontario, Canada — Stephen C. Anco