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George W. Bluman Stephen C. Anco

Symmetry and Integration Methods for Differential Equations

With 18 Illustrations



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Preface

This book is a significant update of the first four chapters of *Symmetries and Differential Equations* (1989; reprinted with corrections, 1996), by George W. Bluman and Sukeyuki Kumei. Since 1989 there have been considerable developments in symmetry methods (group methods) for differential equations as evidenced by the number of research papers, books, and new symbolic manipulation software devoted to the subject. This is, no doubt, due to the inherent applicability of the methods to nonlinear differential equations. Symmetry methods for differential equations, originally developed by Sophus Lie in the latter half of the nineteenth century, are highly algorithmic and hence amenable to symbolic computation. These methods systematically unify and extend well-known ad hoc techniques to construct explicit solutions for differential equations, especially for nonlinear differential equations. Often ingenious tricks for solving particular differential equations arise transparently from the symmetry point of view, and thus it remains somewhat surprising that symmetry methods are not more widely known. Nowadays it is essential to learn the methods presented in this book to understand existing symbolic manipulation software for obtaining analytical results for differential equations. For ordinary differential equations (ODEs), these include reduction of order through group invariance or integrating factors. For partial differential equations (PDEs), these include the construction of special solutions such as similarity solutions or nonclassical solutions, finding conservation laws, equivalence mappings, and linearizations.

A large portion of this book discusses work that has appeared since the above-mentioned book, especially connected with finding first integrals for higher-order ODEs and using higher-order symmetries to reduce the order of an ODE. Also novel is a comparison of various complementary symmetry and integration methods for an ODE.

The present book includes a comprehensive treatment of dimensional analysis. There is a full discussion of aspects of Lie groups of point transformations (point symmetries), contact symmetries, and higher-order symmetries that are essential for finding solutions of differential equations. No knowledge of group theory is assumed. Emphasis is placed on explicit algorithms to discover symmetries and integrating factors admitted by a given differential equation and to construct solutions and first integrals resulting from such symmetries and integrating factors.

This book should be particularly suitable for applied mathematicians, engineers, and scientists interested in how to find systematically explicit solutions of differential equations. Almost all examples are taken from physical and engineering problems including those concerned with heat conduction, wave propagation, and fluid flow.

Chapter 1 includes a thorough treatment of dimensional analysis. The well-known Buckingham Pi-theorem is presented in a manner that introduces the reader concretely to the notion of invariance. This is shown to naturally lead to generalizations through invariance of boundary value problems under scalings of variables. This prepares the reader to consider the still more general invariance of differential equations under Lie groups of transformations in the third and fourth chapters. Basically, the first

chapter gives the reader an intuitive grasp of some of the subject matter of the book in an elementary setting.

Chapter 2 develops the basic concepts of Lie groups of transformations and Lie algebras that are necessary in the following two chapters. By considering a Lie group of point transformations through its infinitesimal generator from the point of view of mapping functions into functions with their independent variables held fixed, we show how one is able to consider naturally other local transformations such as contact transformations and higher-order transformations. Moreover, this allows us to prepare the foundation for consideration of integrating factors for differential equations.

Chapter 3 is concerned with ODEs. A reduction algorithm is presented that reduces an n th-order ODE, admitting a solvable r -parameter Lie group of point transformations (point symmetries), to an $(n - r)$ th-order differential equation and r quadratures. We show how to find admitted point, contact, and higher-order symmetries. We also show how to extend the reduction algorithm to incorporate such symmetries. It is shown how to find admitted first integrals through corresponding integrating factors and to obtain reductions of order using first integrals. We show how this simplifies and significantly extends the classical Noether's Theorem for finding conservation laws (first integrals) to any ODE (not just one admitting a variational principle). In particular, we show how to calculate integrating factors by various algorithmic procedures analogous to those for calculating symmetries in characteristic form where only the dependent variable undergoes a transformation. We also compare the distinct methods of reducing order through admitted local symmetries and through admitted integrating factors. We show how to use invariance under point symmetries to solve boundary value problems. We derive an algorithm to construct special solutions (invariant solutions) resulting from admitted symmetries. By studying their topological nature, we show that invariant solutions include separatrices and singular envelope solutions.

Chapter 4 is concerned with PDEs. It is shown how to find admitted point symmetries and how to construct related invariant solutions. There is a full discussion of the applicability to boundary value problems with numerous examples involving scalar PDEs and systems of PDEs.

Chapters 2 to 4 can be read independently of the first chapter. Moreover, a reader interested in PDEs can skip the third chapter.

Every topic is illustrated by examples. All sections have many exercises. It is essential to do the exercises to obtain a working knowledge of the material. The Discussion section at the end of each chapter puts its contents into perspective by summarizing major results, by referring to related works, and by introducing related material.

Within each section and subsection of a given chapter, we number separately, and consecutively, definitions, theorems, lemmas, and corollaries. For example, Definition 2.3.3-1 refers to the first definition and Theorem 2.3.3-1 to the first theorem in Section 2.3.3. Exercises appear at the conclusion of each section; Exercise 2.4-2 refers to the second problem of Exercises 2.4.

We thank Benny Bluman for the illustrations and Cecile Gauthier for typing several drafts of Sections 3.5 to 3.8.

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