Angular Momentum in General Relativity: A New Definition

Anthony Rizzi
Physics Department, Princeton University, Princeton, New Jersey 08540
(Received 25 March 1998)

Although considerable progress has been made in generalizing the concept of angular momentum to general relativity, until now no satisfactory definition that allows for the exchange of angular momentum has been given. I here give the first such definition. It is a definition at null infinity, the place and time where gravity waves reach in the limit far from all masses. The definition applies to any isolated system of masses *including* those that change their angular momentum L by emitting gravity waves. $\dot{L}$ is given solely in terms of parameters in principle measurable *directly* by Michelson interferometer gravitational wave detectors such as LIGO or LISA. [S0031-9007(98)06673-3]

PACS numbers: 04.20.Cv, 04.20.Ha

Introduction.—Soon after Einstein conceived relativity in 1916, he addressed the problem of energy-momentum conservation and the related problem of angular momentum. It became clear that a local definition (that includes gravity) of these quantities was not possible in general relativity. Nonlocal spatial definitions of all three conserved quantities were found. Spatial definitions apply in a time slice far from an isolated source. However, spatial definitions do not allow for exchange of the conserved quantity. The spatial definition of angular momentum, for instance, does *not* allow one to say that when a body with angular momentum L emits a pulse of gravity waves with angular momentum ΔL, the body will then have $L - \Delta L$ (cf. Fig. 1). In the 1960s, the first definitions of energy and linear momentum at null infinity, i.e., definitions that allow an isolated system to change its E or $\vec{P}$ by emitting gravity waves, were given by Bondi, Van der Burg, and Metzger [1], Sachs [2], and others. Penrose and Newman [3,4] and others (listed in [5]) made advances toward a definition of angular momentum at null infinity, but no satisfactory definition was found. More recently, the work of Christodoulou and Klainerman [6] shed a powerful light on null infinity. We will utilize the power of the Christodoulou formalism especially as in [6–8].

Angular momentum at null infinity.—The two problems responsible for the difficulty in defining conserved quantities in general relativity are the inability to localize the gravitational field and the general lack of Poincaré symmetries in general relativity. One overcomes these problems by, respectively, integrating over a sphere around the source and working far from all mass in a strongly asymptotically flat [6] space-time. The latter is the mathematical formulation of the intuitive concept of an "isolated system." Such a space-time is developed from an initial Cauchy hypersurface surface Σ, in which all the mass is in a compact region and in which the space-time metric on Σ, g_{ij}, and extrinsic curvature of Σ, k_{ij}, fall off far from the source in a prescribed way (cf. p. 11 [6]), giving a well behaved Arnowitt-Desser-Misner (ADM) energy, momentum, and angular momentum.

We want the definition to be the limit of an integral over a sphere as that sphere approaches null infinity. To facilitate this limit, we foliate the space-time near null infinity with two-spheres. We use a foliation, called an affine foliation, made of constant affine parameter slices.

The foliation is created in the following way which is illustrated in Fig. 2. Start with an initial maximal (that is, with $\mathrm{tr}(k_{ij}) = 0$, making it as close to "flat" as possible) hypersurface Σ. Solve Laplace's equation for an electric charge on Σ; as one proceeds away, far from all mass, the surfaces of constant potential become more and more nearly spherical till "at infinity" they are precisely so. Take one of these surfaces far from all mass, call it S_{-k}, and send out light rays into the space-time with initial tangent vector given by $\underline{l} = T - N$, where T is the time direction unit normal to Σ and N is the unit normal in Σ perpendicular to S_{-k}. These light rays form a cone (topology $S^2 \times R$) called C_s^-; the tangents to the null rays define $\underline{l}$ on the cone. Define an affine parameter u such that $\underline{l}u = 1$ and $u = -k$ on S_{-k} with k large. All the S^2 surfaces that foliate the

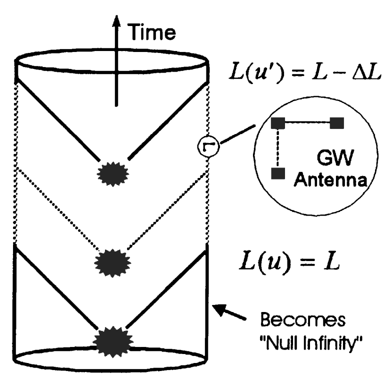

FIG. 1. The definition allows one, for the first time, to specify unambiguously the angular momentum L of a star before, during, and after gravity waves carry away ΔL to null infinity leaving the star with $L - \Delta L$.

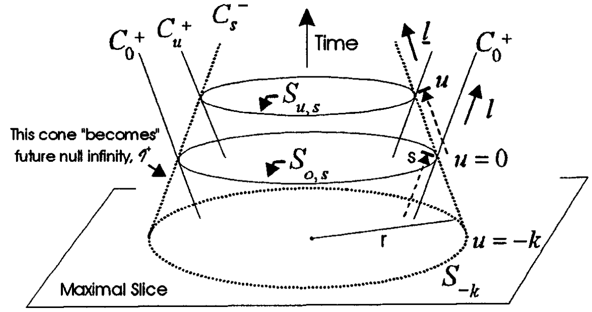

FIG. 2. The construction of the affine foliation. Let $k, s \to \infty$. C_∞^- is null infinity. $S_{-\infty}$ is spatial infinity. Actual S^2 surfaces $S_{u,\infty}$ foliate null infinity.

cone are labeled by the same affine parameter s such that $\underline{l}s = 1$ and $s = r \equiv \sqrt{(\text{surface area of } S_{0,s})/4\pi}$, that is, using the areal radius of the $u = 0$ surface, $S_{0,s}$. Next, define the outgoing null vector l at each point on the cone by requiring $\underline{l} \cdot l = -2$. For each constant u surface on $C_s^-, S_{u,s}$, use l to send light rays into the space-time to generate a new cone C_u^+. The topologically S^2 surfaces given by s and u constant constitute leafs of a foliation of space-time near C_s^-. C_s^- becomes null infinity as the radius coordinate defined by Laplace's equation goes to infinity [9]. Hence, the foliation is a "near null infinity" foliation. Finally, on each point of each leaf, we put a set of appropriately normalized null tetrads; two spatial vectors, e_A, where $A \in \{1,2\}$ and we already have two null vectors: $e_3 = \underline{l}, e_4 = l$ (tangent vector of the appropriate geodesic of C_u^+). The connection coefficients $\Gamma_{abc} = e_a \cdot \nabla_{e_c} e_b$, where ∇ is the covariant derivative with respect to the space-time metric $g_{\mu\nu}$ and $a, b, c \in \{1,2,3,4\}$, are called the Christoffel symbols when a coordinate basis is used. With a null tetrad system, they are called the null Ricci rotation coefficients γ_{abc}. In an affine foliation they are [5]

$$\begin{aligned} H_{AB} &= \langle D_A e_4, e_B\rangle = \chi_{AB}\,, \quad \underline{H}_{AB} = \langle D_A e_3, e_B\rangle = \underline{\chi}_{AB}\,, \\ Z_A &= \tfrac{1}{2}\langle D_3 e_4, e_A\rangle = \zeta_A\,, \quad \underline{Z}_A = \tfrac{1}{2}\langle D_4 e_3, e_A\rangle = -\zeta_A\,, \\ Y_A &= \tfrac{1}{2}\langle D_4 e_4, e_A\rangle = 0\,, \quad \underline{Y}_A = \tfrac{1}{2}\langle D_3 e_3, e_A\rangle = 0\,, \\ \Omega &= \tfrac{1}{4}\langle D_4 e_4, e_3\rangle = 0\,, \quad \underline{\Omega} = \tfrac{1}{4}\langle D_3 e_3, e_4\rangle = 0\,, \\ V_A &= \tfrac{1}{2}\langle D_A e_4, e_3\rangle = \zeta_A\,. \end{aligned} \tag{1}$$

By inspection, it is clear that these quantities exist on an S^2 leaf.

One can now incorporate the Einstein equations by using a Newman-Penrose-like formalism [6,10]. These equations, the structure equations (Ref. [6], p. 168), and Bianchi identities (Ref. [6], p. 161) consist of many coupled differential equations relating the various γ_{abc} and Riemann curvature terms. To simplify, we use the diffeomorphism $\phi_{u,s}: S^2 \mapsto S_{u,s}$ to pull back all quantities from the space-time's topologically S^2 leaf to an actual two-sphere. Hereafter, we work only on actual two-spheres by using these "pulled back quantities." These then are the tools; we now look for the definition.

We begin by defining properties that are essential to the concept of angular momentum. The properties are given in pedagogical order not in order of importance. (1) It should tell the rate of "twist." (2) At null infinity, $\dot{L}$ should be expressible solely in terms of measurable parameters of gravity waves [11] (where the dot over the L refers to the derivative with respect to the affine parameter u, which can also be considered as the retarded time). (3) It should give the intuitive answer expected in Minkowski and Kerr space-times and in the radiative quadrupole approximation. (4) It should be unique up to an initial choice of origin and rest frame. (5) It should, in some sense, be conserved.

The abstract definition and the pulled back quantities.—In an affine foliation, the rate of twisting is given by the null Ricci rotation coefficient called the torsion, $\zeta_A = \frac{1}{2}\langle D_A e_4, e_3\rangle$. To get the appropriate components of L, one must project out one of the three independent components of the angular momentum by dotting with one of the three independent rotation vector fields $\Omega_{(i)}$ (cf. Fig. 3) on the given actual two-sphere and integrate that over the sphere. Mathematically, the definition is

$$L(\Omega_{(i)}) = \frac{1}{8\pi} \lim_{s\to\infty} \int_{S^2} \zeta_A\, \Omega_{(i)}^A\, dS_\gamma\,. \tag{2}$$

Here, s is the affine parameter taking one to null infinity shown in Fig. 2. dS_γ is the area element associated with the S^2 metric γ. The $A = \{1,2\}$ on ζ_A and Ω^A means the e_Ath component on the S^2 surface. To work on actual spheres of unit radius at null infinity, in terms of the pulled back quantities, we use the appropriate diffeomorphism, divide by the appropriate power of the luminosity radius defined by $r \equiv \sqrt{(\text{surface area})/4\pi}$ and take $s \to \infty$. For example, the metric γ becomes the standard one on the unit sphere γ^0; this can be done for each two-sphere foliating null infinity [7] showing that it is foliated by actual two-spheres. The important variables and their pulled back quantities are shown in Table I.

L and $\dot{L}$ in terms of measurable parameters.—One uses the structure equations, cf. Sect. 3.5 [5], that relate the pulled back quantities at null infinity to write L and $\dot{L}$

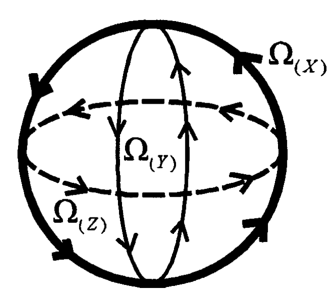

FIG. 3. The three rotation vector fields on a two-sphere.

TABLE I. Pulled back quantities.

Name	Description	Pulled back limit $\lim_{s\to\infty} \phi^*_{u,s}(\,)$
γ	Metric on two-surface	$(r^{-2}\gamma) \to \gamma^0$
$\hat{\chi}$	Traceless part of extrinsic curvature (l)	$(\hat{\chi}) \to \Sigma$ l shear
h	r tr $\chi - 2$	$rh \to H$
$\underline{\hat{\chi}}$	Traceless part of extrinsic curvature ($\underline{l}$)	$(r^{-1}\underline{\hat{\chi}}) \to \Xi$ $\underline{l}$ shear
ζ	Torsion	$(r\zeta) \to Z$
β	$\beta_A \equiv \frac{1}{2} R(e_A, l, \underline{l}, l)$	$r^3\beta \to I$ (Kerr)

in terms of measurable parameters. For L one obtains

$$L(\Omega_{(i)}) = \frac{1}{8\pi} \int (\Sigma_{AB} \nabla\!\!\!/_C \cdot \Sigma^{CB} + I_A)\, \Omega^A_{(i)}\, dS_{\gamma^0}\,. \quad (3)$$

We can use the Bianchi identities, cf. Sect. 3.5 [5], to obtain $\dot{L}$ in terms of measurable parameters.

$$\frac{\partial L(\Omega)}{\partial u} = \frac{1}{8\pi} \int_S dS_{\gamma^0} \{ -\Xi_{AB} \nabla\!\!\!/_C \Sigma^{CB} + \frac{1}{2} (\Sigma_{AB} \nabla\!\!\!/_C \Xi^{CB} - \Sigma^C_B \nabla\!\!\!/^B \Xi_{CA}) \} \Omega^A$$

$\nabla\!\!\!/$ is the derivative intrinsic to the sphere. (4)

The new symbols above are defined in Table I. In a free-fall Fermi-normal frame, changes in Σ correspond to changes in separation of the Michelson interferometer test masses of a gravity wave detector shown in Fig. 4, and Ξ is proportional to the speed of that separation. Mathematically, one has

$$\Sigma_{AB} = \frac{r}{d_0} \Delta x^A_{(B)} \quad \text{and} \quad \Xi_{AB} = \frac{2r}{d_0} \Delta \dot{x}^A_{(B)} \quad (5)$$

which follows from $\Delta\Sigma_{AB} = rh_{AB}, \dot{\Sigma}_{AB} = -\frac{1}{2}\Xi_{AB}$, and $\Sigma_{AB}(-\infty) = \Xi_{AB}(-\infty) = 0$. h_{AB} is the perturbation to the Minkowski metric. Using Eqs. (3) and (4), one can show that the correct results are obtained for the three test cases.

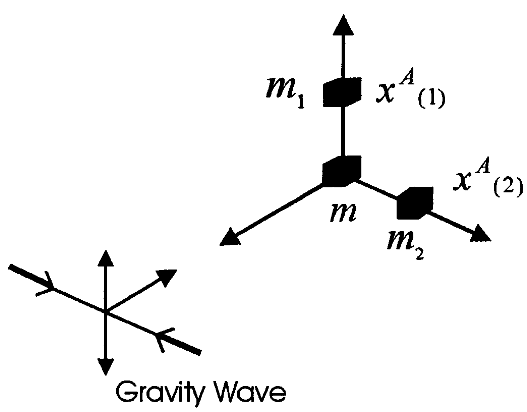

FIG. 4. A Michelson interferometer gravity wave detector, similar to LIGO or LISA, is shown. The masses begin a distance d_0 from the center mass.

An ambiguity in the definition and its resolution.—We have not picked the surface over which to do this integral. At each point in time, one must know what surface on null infinity to do the integral over. We will now find the two conditions that will pick the correct surface. The first condition restricts Σ. That is, one can show that the Σ can be decomposed as

$$\nabla\!\!\!/^B \cdot \Sigma_{BA} = \nabla\!\!\!/_A \phi_{\rm odd} + \varepsilon_{AB} \nabla\!\!\!/^B \phi_{\rm even}\,, \quad (6)$$

where the $\phi_{\rm odd}, \phi_{\rm even}$ are functions on S^2.

Substituting into the appropriate structure equations for pulled back quantities at null infinity (Sect. 4.1.2 [5]) gives

$$-\Delta\!\!\!/ \phi_{\rm even} = Q - \tfrac{1}{2} \Sigma \wedge \Xi\,, \quad (7)$$

$$\Delta\!\!\!/ (\phi_{\rm odd} - \tfrac{1}{2} H) = \underline{N} + P - \tfrac{1}{2} \Sigma \cdot \Xi\,. \quad (8)$$

Here P and Q are Riemann curvature terms, and $\underline{N}$ is a mass term. In Minkowski space-time, these equations imply that $\phi_{\rm even}$ is constant and, hence, $\Sigma_{\rm even}$ is forced to zero by the space-time structure, but because of the H freedom $\Sigma_{\rm odd}$ is free. Hence, to create a shear free surface one sets the only free variable: $\Sigma_{\rm odd} = 0$. We can generalize this condition to a curved space-time since $\Sigma_{\rm odd}$ remains free, leaving $\Sigma_{\rm even}$ again to be fixed by the space-time structure. In Minkowski space-time, the surface obtained by setting $\Sigma_{\rm odd} = 0$ can be thought of as emanating from a point in the space-time. In curved space-time, the point of emanation becomes an abstract point. In short, the rotation vector fields on each $\Sigma_{\rm odd} = 0$, S^2 surface induce a unique origin. For the integral above to represent angular momentum, it must be the sum of angular momenta all taken with respect to the same point. Hence, we pick out a single origin by requiring $\Sigma_{\rm odd} = 0$ [12]. So if, for example, we start at spatial infinity with this condition, we must now find a prescription for going to the next slice such that $\Sigma_{\rm odd}$ is kept constant. The lapse function $\psi(\theta, \phi)(u) = du/du'$, which tells us, for a given angular point on the sphere, how much du' must elapse for a given du to get to the next S^2 slice (cf. Fig. 5), is the variable to be determined. Using the structure equation "l" in Appendix I of [5], with $l \to \psi^{-1} l$ and $\underline{l} \to \psi \underline{l}$, gives pulled back coordinates:

$$2 \frac{\partial \Sigma_{AB}}{\partial u'} = 2 \nabla\!\!\!/_A \nabla\!\!\!/_B \psi - \gamma^0_{AB} \Delta\!\!\!/ \psi - \psi \Xi_{AB}\,, \quad (9)$$

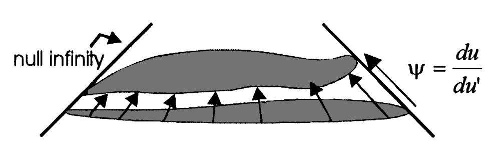

FIG. 5. The lapse function ψ defines how to get from one leaf of a foliation of null infinity to the next.

where Σ, Ξ are with respect to affine frame.

One finds ψ must satisfy the following equation if $\Sigma_{\text{odd}} = 0$ along each $u' =$ constant surface:

$$\Delta\!\!\!/^2\psi + 2\Delta\!\!\!/\psi = \nabla\!\!\!/^A\nabla\!\!\!/^B(\psi \cdot \Xi_{AB}). \tag{10}$$

Ξ is the news function. It can be proved [13] that this equation has a solution with four constants of integration, $c_t(u)$, $c_x(u)$, $c_y(u)$, $c_z(u)$, *at least* when $|\Xi|^2 < 16$, i.e., [14],

$$\frac{\partial M_{\text{isolated-system}}}{\partial u} < 10^6 M_\odot/\mathrm{s}. \tag{11}$$

By analogy from Minkowski space, one can show that each c_i is related to a boost in the $\hat{i}$ direction. Hence, we make an initial choice of origin and of rest frame; then we choose not to boost off this choice by keeping $c_{x,y,z} = 0$ and $c_t = 1$. This second condition makes the surfaces mathematically unique by noting that angular momentum requires one to pick an origin and stay with that choice.

Conclusion.—Hence, the definition given in Eq. (3), coupled with the gauge conditions on the cuts of null infinity: (1) The odd parity part of the shear is zero and (2) $c_{x,y,z} = 0$ and $c_t = 1$, satisfies all five properties one wants for angular momentum. With the existence of the definition established comes interesting questions about the definition's relation to the spatial definition, to the Bondi momentum, and to an invariance of the action; it also incites an interest in more general than quadrupole tests of the definition. Many new pieces come together to make the new definition, but two are key. First, the identification of the torsion dotted with a chosen rotation vector field as the local rate of twisting (per unit surface area) in the chosen direction gives the definition sound physical footing. Second, the gauge condition is physically required and not extraneous or arbitrary since it is necessary to ensure that essential aspects of angular momentum are respected. The new definition fully incorporates the Einstein equations and involves no approximations. Also, the major impediment to previous definition, the supertranslation ambiguity, is resolved.

I would like to acknowledge Demetrios Christodoulou, my thesis [5] advisor, for his invaluable assistance through the use of his personal notes and through insightful conversations.

[1] H. Bondi, M. G. J. Van der Burg, and A. W. K. Metzner, Proc. R. Soc. London A **269**, 21–52 (1962).

[2] R. K. Sachs, Proc. R. Soc. London A **270**, 103–126 (1962).

[3] E. T. Newman and R. Penrose, J. Math. Phys. (N.Y.) **7**, 863–870 (1966).

[4] B. Aronson, R. Lind, J. Messmer, and E. Newman, J. Math. Phys. (N.Y.) **12**, 2462–2467 (1971).

[5] A. Rizzi, Ph.D. thesis, Princeton University, 1997.

[6] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space* (Princeton University Press, Princeton, NJ, 1993).

[7] D. Christodoulou, Phys. Rev. Lett. **67**, 1486–1489 (1991).

[8] D. Christodoulou, in *Proceedings of the 6th Marcel Grossmann Meeting* (World Scientific, Singapore, 1992).

[9] Taking the limit makes all the previous construction precise. One also lets $S_{-k} \to S^*_{-\infty}$ so that $-\infty < u < \infty$ on null infinity. Further, note the foliation construction picks out an origin on the initial spatial slice.

[10] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, 1983).

[11] As soon as one has an expression for $\dot{L}$, a conservation law is established. Hence, properties 1 and 5 are interrelated.

[12] This induction from Minkowski to curved space is strongly motivated and leaves little, if any, room for maneuvering, but nonetheless remains an induction not a deduction.

[13] See Appendix III of [5]; the fundamentals of this proof are due to D. Christodoulou.

[14] It can be shown by order of magnitude calculation that this condition is roughly equivalent to saying that there is not sufficient concentration of gravity wave energy anywhere near null infinity to result in the formation of a black hole.