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The complete Kepler group can be derived by Lie group analysis

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It is shown that the complete symmetry group for the Kepler problem, as introduced by Krause, can be derived by Lie group analysis. The same result is true for any autonomous system. © *1996 American Institute of Physics.*
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I. INTRODUCTION

In Ref. 1, a new concept of symmetry group for ordinary differential equations, which have $x_k = x_k(t)$, $(k=1,2,\dots,N)$ as unknown functions, was introduced. Krause defined a complete symmetry group by adding two properties to the definition of Lie symmetry group:

(i) the manifold of solutions is a homogeneous space of the group;
(ii) the group is specific to the system (no other system admits it).

This definition of the complete symmetry group needed a new type of symmetry to be introduced. For example, neither Lie point symmetries nor contact symmetries give rise to a complete symmetry group for the Kepler problem. The generator of the new symmetry was defined to be

$$Y = \left[\int \xi(t,x_1,\dots,x_N)\,dt \right] \partial_t + \sum_{k=1}^{N} \eta_k(t,x_1,\dots,x_N)\,\partial_{x_k}, \tag{1}$$

which is different from the generator of a Lie point symmetry group[2–11] because of the appearance of the integral of ξ.

Here, we show that if the system under study is autonomous then the complete symmetry group can be recovered by Lie group analysis. In particular, the extra symmetries that Krause claimed not to be found by Lie group analysis for the Kepler problem can be so determined. This is explained by the following observation. If the system is autonomous, then one of the unknown functions can be taken to be the new independent variable and the system consequently can be rewritten. Thus, Lie group analysis applied to the transformed system leads to different results, and in particular the extra symmetries which were found by applying Krause's method can be retrieved.

II. OUTLINE OF THE METHOD

Let us consider the following autonomous system of N second-order ordinary differential equations

$$\ddot{x}_k = F_k(x_1,x_2,\dots,x_N,\dot{x}_1,\dots,\dot{x}_N) \quad (k=1,2,\dots,N). \tag{2}$$

A generator of a Lie point symmetry group for this system has the form

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$$X=\tau(t,x_1,\ldots,x_N)\partial_t+\sum_{k=1}^{N}\eta_k(t,x_1,\ldots,x_N)\partial_{x_k}. \tag{3}$$

System (2) can be converted into the following autonomous system of $2N$ first-order ordinary differential equations[10]

$$\dot{u}_k=u_{N+k}, \tag{4}$$

$$\dot{u}_{N+k}=F_k(u_1,u_2,\ldots,u_N,u_{N+1},\ldots,u_{2N}). \tag{5}$$

Now, we can choose one of the dependent variables to be the new independent variable y. For example, let us take $u_N=y$. Then, system (4)–(5) becomes the following nonautonomous system of $2N-1$ first-order ordinary differential equations with independent variable y

$$\frac{du_j}{dy}=\frac{u_{N+j}}{u_{2N}}, \tag{6}$$

$$\frac{du_{N+j}}{dy}=\frac{F_j(u_1,u_2,\ldots,u_{N-1},y,u_{N+1},\ldots,u_{2N})}{u_{2N}}, \tag{7}$$

$$\frac{du_{2N}}{dy}=\frac{F_N(u_1,u_2,\ldots,u_{N-1},y,u_{N+1},\ldots,u_{2N})}{u_{2N}}, \tag{8}$$

where $(j=1,2,\ldots,N-1)$. From Eqs. (6) we can deduce that

$$u_{N+j}=u_{2N}\frac{du_j}{d_y},$$

which when substituted into Eqs. (7) and (8) yields the following system of $N-1$ second-order ordinary differential equations and one first-order equation for the unknowns $u_j=u_j(y)$, and $u_{2N}=u_{2N}(y)$

$$\begin{aligned}u_j''=\frac{1}{u_{2N}^2}[&F_j(u_1,u_2,\ldots,u_{N-1},y,u_1',\ldots,u_{N-1}',u_{2N})\\&-F_N(u_1,u_2,\ldots,u_{N-1},y,u_1',\ldots,u_{N-1}',u_{2N})u_j'],\end{aligned} \tag{9}$$

$$u_{2N}'=\frac{F_N(u_1,u_2,\ldots,u_{N-1},y,u_1',\ldots,u_{N-1}',u_{2N})}{2_{2N}}, \tag{10}$$

where $'$ denotes differentiation by y. A generator of a Lie point symmetry group for this system is of the form:

$$Z=V(y,u_1,\ldots,u_{N-1},u_{2N})\partial_y+\sum_{j=1}^{N-1}G_j(y,u_1,\ldots,u_{N-1},u_{2N})\partial_{u_j}+G_{2N}(y,u_1,\ldots,u_{N-1},u_{2N})\partial_{u_{2N}}, \tag{11}$$

which can be transformed into the operator Y in (1) by substituting u_j,y,u_{2N} with $x_j,x_N,\dot{x}_N$, respectively, and solving the following system for ξ and η_k

$$Y(x_j)\equiv\eta_j=G_j, \tag{12}$$

$$Y(x_N) \equiv \eta_N = V, \tag{13}$$

$$Y^{(1)}(\dot{x}_N) \equiv \frac{d\,\eta_N}{dt} - \xi \dot{x}_N = G_{2N}, \tag{14}$$

where $Y^{(1)}$ denotes the first prolongation of Y. Therefore, Krause's symmetries can be recovered by means of Lie group analysis applied to system (9), or other nonautonomous systems which can be deduced from (4)–(5) by choosing $u_j = y$. Vice versa, Y in (1) can be transformed into Z in (11) by means of (12)–(14).

However, it should be noticed that Z in (11) could lead to a more general operator than Y in (1) because of the appearance of $\dot{x}_N$. In this case, a sort of contact symmetry generalization of Y would be needed.

Finally, Z can be transformed into a generator of a Lie point symmetry group X in (3), with $\tau = \int \xi\, dt$, if (12)–(14) yield that η_k do not depend on $\dot{x}_N$, and either ξ is constant or $\xi = f(x_N)\dot{x}_N$, with f arbitrary function of x_N.

III. KEPLER PROBLEM

The Kepler problem provides a good example of the method outlined in the previous paragraph. The original Kepler problem is given by the following system:

$$\ddot{x}_1 = \frac{-Kx_1}{(x_1^2+x_2^2+x_3^2)^{3/2}}, \quad \ddot{x}_2 = \frac{-Kx_2}{(x_1^2+x_2^2+x_3^2)^{3/2}}, \quad \ddot{x}_3 = \frac{-Kx_3}{(x_1^2+x_2^2+x_3^2)^{3/2}}. \tag{15}$$

It is well known[7] that Lie group analysis applied to (15) yields a five-dimensional Lie symmetry algebra generated by

$$\begin{gathered} X_1 = \partial_t, \quad X_2 = 3t\partial_t + 2x_1\partial_{x_1} + 2x_2\partial_{x_2} + 2x_3\partial_{x_3}, \\ X_3 = x_2\partial_{x_1} - x_1\partial_{x_2}, \quad X_4 = x_3\partial_{x_1} - x_1\partial_{x_3}, \quad X_5 = x_3\partial_{x_2} - x_2\partial_{x_3}. \end{gathered} \tag{16}$$

Instead, the eight-dimensional complete symmetry group of (15) is generated by X_1, X_2, X_3, X_4, X_5, and

$$\begin{aligned} Y_1 &= 2\left(\int x_1\ dt\right)\partial_t + x_1^2\partial_{x_1} + x_1x_2\partial_{x_2} + x_1x_3\partial_{x_3}, \\ Y_2 &= 2\left(\int x_2\ dt\right)\partial_t + x_1x_2\partial_{x_1} + x_2^2\partial_{x_2} + x_2x_3\partial_{x_3}, \\ Y_3 &= 2\left(\int x_3\ dt\right)\partial_t + x_1x_3\partial_{x_1} + x_2x_3\partial_{x_2} + x_3^2\partial_{x_3}, \end{aligned} \tag{17}$$

which can be obtained by means of the new generator (1) with $N=3$.[1] Krause stated that Y_1, Y_2, and Y_3 cannot be recovered by Lie group analysis. It is true that they are not included in (16), but they can be retrieved by applying Lie group analysis to the equivalent nonautonomous systems.

Let us choose x_3 to be the new independent variable y. Then, the new system (9) becomes

$$u_1'' = \frac{-K(u_1 - yu_1')}{(u_1^2+u_2^2+y^2)^{3/2}u_6^2}, \quad u_2'' = \frac{-K(u_2 - yu_2')}{(u_1^2+u_2^2+y^2)^{3/2}u_6^2}, \quad u_6' = \frac{-Ky}{(u_1^2+u_2^2+y^2)^{3/2}u_6}, \tag{18}$$

where $'$ denotes differentiation by y. Note that, because of the symmetric form of (15), a system similar to (18) is derived, whatever dependent variable we choose as the new independent variable. Lie group analysis applied to (18) yields a three-dimensional Lie symmetry algebra generated by

$$Z_1 = y^2\partial_y + yu_1\partial_{u_1} + yu_2\partial_{u_2},$$
$$Z_2 = 2y\partial_y + 2u_1\partial_{u_1} + 2u_2\partial_{u_2} - u_6\partial_{u_6}, \quad Z_3 = u_2\partial_{u_1} - u_1\partial_{u_2}. \tag{19}$$

If we transform these operators into the form given by Y in (1), then Z_1 becomes Y_3 in (17), and Z_2, Z_3 become X_2, X_3 in (16), respectively. In fact, transforming Z_1 into an operator of the form given by Y in (1) corresponds to solve (12)–(14), i.e.,

$$\eta_1 = x_1 x_3 \equiv G_1, \tag{20}$$

$$\eta_2 = x_2 x_3 \equiv G_2, \tag{21}$$

$$\eta_3 = x_3^2 \equiv V, \tag{22}$$

$$\frac{d\eta_3}{dt} - \xi\dot{x}_3 = 0 \equiv G_{2N}. \tag{23}$$

Substituting (22) into Eq. (23) yields $2x_3\dot{x}_3 - \xi\dot{x}_3 = 0$, which implies $\xi = 2x_3$.

It is easy to show that Y_1 $[Y_2]$ can be obtained by applying Lie group analysis to the equivalent nonautonomous system, which has x_1 $[x_2]$ as the new independent variable.

We have used our own interactive Reduce programs[12] to perform Lie group analysis, and apply Krause's method with the generator (1).

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