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Aiming to bypass the equation of the Lorentz force, this study analyzes Maxwell's equations from the perspective of a receiver at rest. This approach is necessary because experimental results suggest that the general validity of the Lorentz force is questionable in non-stationary cases. Calculations in the receiver's rest frame are complicated, and thus, rarely performed. However, after a Lorentz boost, the resulting force should be identical to the force obtained when the problem is considered in the rest frame of the transmitter with the Lorentz force applied, as is commonly done. Yet, this is not the case. Instead, Maxwell's equations lead to Weber electrodynamics. The present article demonstrates this result by deriving and solving the inhomogeneous wave equation from Maxwell's equations. Subsequently, it is shown that the resulting force is a relativistic generalization of the Weber force. Furthermore, the Hertzian dipole, i.e., a simple antenna, is mathematically investigated and discussed from the viewpoint of Weber electrodynamics for the first time.

I. Introduction

The four Maxwell equations have been the basis of electrodynamics for more than 150 years. In addition to Maxwell's equations, however, there is a fifth equation that often receives far less attention, but is of great importance: the Lorentz force.

In contrast to Maxwell's equations, the Lorentz force is not a differential equation. It does not depend on time or location, which is most likely why it is overshadowed by the Maxwell equations. The Lorentz force combines the electric and magnetic field into a total electromagnetic force on a point-like test charge, which serves as the receiver of the force. However, because only forces or voltages can be measured, the Lorentz force is the mediator between the calculated fields obtained from Maxwell's equations and the measured experimental result, highlighting the decisive importance of the Lorentz force.

Thus, it is surprising that the literature has focused almost exclusively on electric and magnetic fields and almost never addresses the field of the force. It is also remarkable that, in contrast to Maxwell's equations, the time at which the law of the Lorentz force was developed is not known. Allegedly, the formula was used for the first time in 1895 by H. A. Lorentz, indicating that it must have been developed much later than Maxwell's equations. This is surprising because Maxwell's equations provide only field strengths and do not describe how these fields affect electric charges.

It is known, however, that at the time of the origin of Maxwell's equations, a number of different formulas for the force between current elements existed, which are all equivalent if the current elements are connected to a closed conductor loop. J. C. Maxwell wrote on this subject in 1873 [1, p. 161], stating that there are considerable degrees of freedom for these formulas and four parameters can be chosen independently. His conclusion was that the original formula of A. M. Ampère from 1822 is the most reasonable formula, but he also mentioned other formulas, such as that of H. Graßmann, which is used in contemporary electrodynamics, because Ampère's formula is not compatible with the Lorentz force[1].

It is clearly an established fact that the Lorentz force is correct for electro- and magnetostatic problems under non-relativistic conditions. Electro- and magnetostatic problems are problems in which the displacement current in Maxwell's equations can be neglected. This case corresponds to closed circuits with direct current (DC) because open conductors change their net charge if current flows, which leads to a time-varying electric field. However, a time-varying electric field implies the presence of a displacement current, which in turn means that the full set of Maxwell's equations is needed.

When the displacement current is not neglected, the complete set of Maxwell's equations is very powerful because it describes how electromagnetic fields propagate in space and time. As one can assert, the addition of the displacement current by J. C. Maxwell ushered in the age of technological modernity. However, thought experiments and actual experiments seem to show that the Lorentz force might be invalid in the presence of a displacement current [3]. In other words, by adding the displacement current, degrees of freedom are lost in the force law, and it is not self-evident that the Graßmann formula, which is perfectly appropriate for electro- and magnetostatics, retains its validity.

Fortunately, Maxwell's equations can also be used without the Lorentz force because, due to the principle of relativity, a uniformly moving test charge, i.e., a measuring probe or the receiver of a force, can be considered to be at rest, and instead, the field-generating charge, i.e., the transmitter of the force, can be interpreted as moving. This is the method applied in this article. It follows the approach of H. Dodig, who recently demonstrated that Maxwell's equations are a direct

[1]It should be noted that also in Maxwell's work formulas can be found which formally correspond to the Lorentz force, but do not refer to point charges and and can be interpreted as a variant of Graßmann's force (e.g. [2, p. 485] or [1, p. 226]).

consequence of Coulomb's law and the fact that the force in the rest frame of the receiver always moves at speed c [4].

The objective of this article is to determine whether solving Maxwell's equations in the receiver's rest frame and then transforming the force to the transmitter's rest frame gives the same result obtained by solving Maxwell's equations in the transmitter's rest frame and then applying the Lorentz force. For this purpose, the inhomogeneous wave equation is initially derived and solved in the receiver's rest frame. On the basis of this solution, it becomes clear that the Maxwell equations contain a hidden Galilean transformation. The use of the Lorentz transformation seems inappropriate in regard to this finding and leads then actually to contradictory results.

However, using the Galilean transformation, we obtain the force formula of W. Weber, derived in 1846 from the force formula of A. M. Ampère, which was preferred by J. C. Maxwell. This result shows that Weber electrodynamics, which has received increasing attention in the past decades, due to the work of A. K. T. Assis and others [5]–[14], are closely related to Maxwell's equations and are based on the inhomogeneous wave equation given by Maxwell's equations. The discovery of this relation allows the Weber force to be generalized relativistically and extended to a fully-fledged field theory for studying the propagation of electromagnetic waves. This is demonstrated in this article by means of the Hertzian dipole.

II. Nomenclature notes

Please note that in this article, electric charges are called *transmitters* (of the force) when their property of generating an electromagnetic field is the focus of attention. Conversely, electric charges are referred to as *receivers* (of the force) when their reaction to an existing electromagnetic field is the focus. In particular, these terms are used when electric charges move uniformly with respect to each other and no waves are present. Furthermore, in this article, standard electrodynamics is referred to as Lorentz-Einstein electrodynamics in order to be distinguished from Weber electrodynamics, which, as will become evident, is also a Maxwellian electrodynamics.

III. The inhomogeneous wave equation

The Maxwell equations in vacuum

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3}$$

$$\nabla \times \boldsymbol{B} = \mu_0\, \boldsymbol{j} + \frac{1}{c^2}\,\frac{\partial \boldsymbol{E}}{\partial t}. \tag{4}$$

are the starting point of this article. Together with the Lorentz force

$$\boldsymbol{F} = q_d\, \boldsymbol{E} + q_d\, \boldsymbol{v}_d \times \boldsymbol{B}, \tag{5}$$

Maxwell's equations form the basic set of equations and mathematical framework of what is currently considered valid electrodynamics, denoted as Lorentz-Einstein electrodynamics in this article.

This article will demonstrate that Maxwell's equations also hold in Weber electrodynamics, provided that one solves the equations rigorously in the rest frame of the receiver. However, instead of the Lorentz force (5), the following equation must be used:

$$\boldsymbol{F} = q_{d0}\, \gamma(\boldsymbol{v})\, \boldsymbol{E} = q_d\, \boldsymbol{E}, \tag{6}$$

where $\boldsymbol{v}$ is the differential velocity between the transmitter and receiver, not, as $\boldsymbol{v}_d$ is in the Lorentz force (5), the measured velocity of the receiver in the rest frame of the observer (laboratory system). q_{d0} is the rest charge, i.e., the charge obtained by measuring the force applied to a known resting charge using Coulomb's law.

It should be noted that the rest charge is introduced only for convenience, ensuring that the subsequent calculations for Lorentz-Einstein electrodynamics and Weber electrodynamics are identical. The charge does not change its value when in motion; rather, the force depends on the differential velocity. As usual, γ is the Lorentz factor, which is defined by

$$\gamma(\boldsymbol{v}) := \frac{1}{\sqrt{1 - \|\boldsymbol{v}\|^2/c^2}}. \tag{7}$$

The Maxwell equations and the force equations (5) and (6) can be combined to a single partial differential equation – the inhomogeneous wave equation – which reveals the essence of Maxwell' equations in a particularly clear form. To obtain this equation, we exploit the fact that the laboratory system is allowed to move at the same velocity $\boldsymbol{v}_d$ as the receiver in Lorentz-Einstein electrodynamics, due to the principle of relativity. In this case, the Lorentz force simplifies to

$$\boldsymbol{F} = q_d\, \boldsymbol{E}, \tag{8}$$

and the term with the cross product is omitted. Thus, in Lorentz-Einstein electrodynamics, one does not actually need formula (5). Instead, one uses this equation only because it is convenient.

However, this approach does not apply for equation (6) because if the laboratory system moves synchronously with the receiver, the differential velocity between the transmitter and receiver remains unaffected. Thus, the force laws (6) and (8) are now formally equivalent. As will be shown below, this equivalency makes the subsequent calculations valid for Weber electrodynamics as well.

The next step is to calculate the derivative of the fourth Maxwell equation (4) with respect to time t. One obtains

$$\nabla \times \frac{\partial \boldsymbol{B}}{\partial t} = \mu_0\, \frac{\partial \boldsymbol{j}}{\partial t} + \frac{1}{c^2}\,\frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \tag{9}$$

By inserting the third Maxwell equation (3) and equation (6) or (8), one obtains

$$\frac{1}{c^2}\,\frac{\partial^2 \boldsymbol{F}}{\partial t^2} + \nabla \times (\nabla \times \boldsymbol{F}) = -q_d\, \mu_0\, \frac{\partial \boldsymbol{j}}{\partial t}, \tag{10}$$

which, because of $\nabla\times(\nabla\times\boldsymbol{F}) = \nabla(\nabla\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}$, corresponds to

$$\Box\,\boldsymbol{F} + \nabla(\nabla\cdot\boldsymbol{F}) = -q_d\,\mu_0\,\frac{\partial\boldsymbol{j}}{\partial t}. \tag{11}$$

The operator $\Box$ denotes the d'Alembert operator:

$$\Box := \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2. \tag{12}$$

Please note that the sign of the d'Alembert operator is not used uniformly in the literature.

Substituting equation (8) into the first Maxwell equation (1) gives

$$\nabla\cdot\boldsymbol{F} = \frac{q_d\,\rho}{\varepsilon_0}. \tag{13}$$

Substituting this into equation (11), one finally arrives at the inhomogeneous wave equation [15, p. 246, eq. (6.49)]:

$$\Box\,\boldsymbol{F} = -\frac{q_d}{\varepsilon_0}\left(\frac{1}{c^2}\frac{\partial\boldsymbol{j}}{\partial t} + \nabla\rho\right). \tag{14}$$

This wave equation is valid in Lorentz-Einstein electrodynamics only for receivers at rest because the additional cross product term must be considered due to the Lorentz force for moving receivers, which takes the magnetic field into account. However, as previously explained, this is not a restriction of generality: because of the principle of relativity, a uniformly moving receiver may be viewed as being at rest if instead the transmitter is considered to be moving.

Equation (14) is valid for arbitrary charge and current distributions, as long as they satisfy the continuity equation. Particularly important, however, are those cases in which the field-generating charge distribution is point-like. Here, the charge density is given by

$$\rho = q_s\,\delta(\boldsymbol{r}_r) \tag{15}$$

with

$$\boldsymbol{r}_r := \boldsymbol{r} - \boldsymbol{r}_s, \tag{16}$$

where $\boldsymbol{r}_s$ represents the trajectory of the charge q_s and $\boldsymbol{r}$ is the location of the receiver at rest q_d. The corresponding current density is

$$\boldsymbol{j} = \boldsymbol{v}_s\,\rho, \tag{17}$$

where

$$\boldsymbol{v}_s := \dot{\boldsymbol{r}}_s \tag{18}$$

is the velocity of the charge q_s from the perspective of the charge q_d at rest.

Thus, the wave equation

$$\boxed{\Box\,\boldsymbol{F} = -\frac{q_d\,q_s}{\varepsilon_0}\left(\frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{v}_s\,\delta(\boldsymbol{r}_r) + \nabla\,\delta(\boldsymbol{r}_r)\right)} \tag{19}$$

follows from equation (14). This equation represents the force exerted by the transmitter q_s with trajectory $\boldsymbol{r}_s$ on a receiver q_d at rest at location $\boldsymbol{r}$ in Lorentz-Einstein electrodynamics. As mentioned above, calculating the force in the receiver's frame of rest does not lead to a restriction of generality, as it should always be possible to perform a Lorentz boost into the rest frame of the transmitter.

To calculate the acceleration $\ddot{\boldsymbol{r}}$ for a receiver of force $\boldsymbol{F}$ in classical mechanics, one needs only Newton's second law:

$$\boldsymbol{F} = m\,\ddot{\boldsymbol{r}}. \tag{20}$$

In Lorentz-Einstein electrodynamics, however, equation (20) does not apply. Instead, one needs the following equation:

$$\boldsymbol{F} = \gamma(\dot{\boldsymbol{r}})\,m\,\ddot{\boldsymbol{r}} + \frac{m}{c^2}\,\gamma(\dot{\boldsymbol{r}})^3\;(\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}})\;\dot{\boldsymbol{r}}, \tag{21}$$

which gives

$$\ddot{\boldsymbol{r}} = \frac{1}{\gamma(\dot{\boldsymbol{r}})\,m}\left(\boldsymbol{F} - \frac{1}{c^2}\,\dot{\boldsymbol{r}}\;(\boldsymbol{F}\cdot\dot{\boldsymbol{r}})\right) \tag{22}$$

when solved for the acceleration.

For Weber electrodynamics, this approach is not necessary. As will be shown below, Newton's second law continues to be valid in the original form (20). It is remarkable that this results in an electrodynamics that can describe the propagation of electromagnetic waves and satisfy Einstein's postulates without requiring a Lorentz transformation. At the same time, magnetism as a dual force of electromagnetism disappears and becomes a residual effect, due to the inability to shield the electric force in all reference frames simultaneously for a multi-particle system with varying differential velocities.

IV. Solution of the wave equation

A. General solution in the rest frame of the receiver

Because of its importance, this section demonstrates how the inhomogeneous wave equation (19) can be solved. For the inhomogeneous wave equation

$$\Box\,\boldsymbol{F}(\boldsymbol{r},t) = \boldsymbol{f}(\boldsymbol{r},t), \tag{23}$$

the solution is [15, p. 243-245]

$$\boldsymbol{F}(\boldsymbol{r},t) = \iiint_V \frac{\boldsymbol{f}\left(\boldsymbol{r}', t - \frac{1}{c}\|\boldsymbol{r}-\boldsymbol{r}'\|\right)}{4\,\pi\,\|\boldsymbol{r}-\boldsymbol{r}'\|}\,\mathrm{d}\boldsymbol{r}'. \tag{24}$$

It is noted that solutions of the homogeneous wave equation $\Box\,\boldsymbol{F}(\boldsymbol{r},t) = 0$ can be added to equation (24). However, these solutions are not related to the cause $\boldsymbol{f}(\boldsymbol{r},t)$ of the field $\boldsymbol{F}(\boldsymbol{r},t)$ and therefore are of no relevance.

In equation (19), we have

$$\boldsymbol{f}(\boldsymbol{r},t) = -\frac{q_d\,q_s}{\varepsilon_0}\left(\frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{v}_s\,\delta(\boldsymbol{r}_r) + \nabla\,\delta(\boldsymbol{r}_r)\right). \tag{25}$$

Therefore, using the definitions (16) and

$$t' := t - \frac{1}{c}\|\boldsymbol{r} - \boldsymbol{r}_s(t')\|, \tag{26}$$

we obtain

$$\begin{aligned}&\boldsymbol{F}(\boldsymbol{r},t) = \\ &-\frac{\partial}{\partial t}\iiint_V \frac{q_d\,q_s\,\boldsymbol{v}_s(t')\,\delta(\boldsymbol{r}'-\boldsymbol{r}_s(t'))}{4\,\pi\,\varepsilon_0\,c^2\,\|\boldsymbol{r}-\boldsymbol{r}'\|}\,\mathrm{d}\boldsymbol{r}' - \\ &\nabla\iiint_V \frac{q_d\,q_s\,\delta(\boldsymbol{r}'-\boldsymbol{r}_s(t'))}{4\,\pi\,\varepsilon_0\,\|\boldsymbol{r}-\boldsymbol{r}'\|}\,\mathrm{d}\boldsymbol{r}'.\end{aligned} \tag{27}$$

Despite the Dirac function, the integrals cannot be solved directly, because t' is a function of $\boldsymbol{r}'$ according to formula

(26). However, by using the Jacobian determinant D, this equation can be converted into a formally solvable form [16, p. 616-617]:

$$\boldsymbol{F}(\boldsymbol{r},t) = -\frac{\partial}{\partial t}\iiint_V \frac{q_d\, q_s\, \boldsymbol{v}_s(t')\,\delta(\boldsymbol{s})}{4\,\pi\,\varepsilon_0\, c^2\, \|\boldsymbol{r}-\boldsymbol{r}'\|}\,\frac{1}{D}\,\mathrm{d}\boldsymbol{s} - \nabla\iiint_V \frac{q_d\, q_s\, \delta(\boldsymbol{s})}{4\,\pi\,\varepsilon_0\, \|\boldsymbol{r}-\boldsymbol{r}'\|}\,\frac{1}{D}\,\mathrm{d}\boldsymbol{s}, \tag{28}$$

by using the definitions

$$\boldsymbol{s} := \boldsymbol{r}' - \boldsymbol{r}_s\left(t - \frac{1}{c}\|\boldsymbol{r}-\boldsymbol{r}'\|\right) \tag{29}$$

and

$$D := \det\begin{vmatrix} \frac{\partial s_x}{\partial r'_x} & \frac{\partial s_x}{\partial r'_y} & \frac{\partial s_x}{\partial r'_z} \\ \frac{\partial s_y}{\partial r'_x} & \frac{\partial s_y}{\partial r'_y} & \frac{\partial s_y}{\partial r'_z} \\ \frac{\partial s_z}{\partial r'_x} & \frac{\partial s_z}{\partial r'_y} & \frac{\partial s_z}{\partial r'_z} \end{vmatrix}. \tag{30}$$

The integrations in equation (28) can now be performed, and one obtains

$$\boldsymbol{F}(\boldsymbol{r},t) = -\frac{\partial}{\partial t}\frac{q_d\, q_s\, \boldsymbol{v}_s(t')}{4\,\pi\,\varepsilon_0\, c^2\, \|\boldsymbol{r}-\boldsymbol{r}'\|}\,\frac{1}{D} - \nabla\frac{q_d\, q_s}{4\,\pi\,\varepsilon_0\, \|\boldsymbol{r}-\boldsymbol{r}'\|}\,\frac{1}{D} \tag{31}$$

under the constraint $\boldsymbol{s} = \boldsymbol{0}$. This result can be further simplified, as the calculation of the Jacobian determinant yields

$$D = 1 - \frac{\boldsymbol{v}_s\,(t')\,(\boldsymbol{r}-\boldsymbol{r}')}{c\,\|\boldsymbol{r}-\boldsymbol{r}'\|}. \tag{32}$$

Substituting this into equation (31) gives

$$\boldsymbol{F}(\boldsymbol{r},t) = -q_d\left(\frac{\partial}{\partial t}\boldsymbol{A} + \nabla\,\Phi\right) \tag{33}$$

with scalar potential

$$\Phi := \frac{q_s}{4\,\pi\,\varepsilon_0\,\|\boldsymbol{r}-\boldsymbol{r}'\|\left(1 - \frac{\boldsymbol{v}_s(t')(\boldsymbol{r}-\boldsymbol{r}')}{c\,\|\boldsymbol{r}-\boldsymbol{r}'\|}\right)} \tag{34}$$

and vector potential

$$\boldsymbol{A} := \frac{1}{c^2}\,\boldsymbol{v}_s(t')\,\Phi. \tag{35}$$

Equation (29) must be zero because of the constraint; hence, it follows that $\boldsymbol{r}' = \boldsymbol{r}_s(t')$. Moreover, because of equation (26),

$$\|\boldsymbol{r}-\boldsymbol{r}'\| = c\,(t-t') \tag{36}$$

holds. If we insert both relations into the scalar potential and rearrange slightly, we finally obtain

$$\Phi = \frac{q_s\, c}{4\,\pi\,\varepsilon_0\,\left(c^2\,(t-t') - \boldsymbol{v}_s(t')\cdot\boldsymbol{r}_r(t')\right)}. \tag{37}$$

Clearly, this potential depends only on the retarded distance vector

$$\boldsymbol{r}_r(t') = \boldsymbol{r} - \boldsymbol{r}_s(t') \tag{38}$$

and the retarded velocity

$$\boldsymbol{v}_s(t') = \dot{\boldsymbol{r}}_s(t'). \tag{39}$$

Here,

$$t' := t - \frac{\|\boldsymbol{r}-\boldsymbol{r}_s(t')\|}{c} \tag{40}$$

is the time at which the force has left the transmitter q_s at location $\boldsymbol{r}_s(t')$ to meet the receiver q_d at location $\boldsymbol{r}$ at time t.

As previously mentioned, the scalar potential (37) and the vector potential (35) are valid in both electrodynamics, i.e., Lorentz-Einstein electrodynamics and Weber electrodynamics. However, Weber electrodynamics is unique in that a Lorentz transformation is not required to give relativistically correct results. Instead, a Galilean transformation can be used. Nevertheless, the force in the rest frame of each receiver propagates at the speed of light c, even if the receivers themselves move with respect to each other.

This sounds like a logical fallacy, because it seems impossible for a physical entity such as a force field to move at the same speed c for two receivers moving at different speeds. Expressed another way: how is the transmitter supposed to know the speed at which it must emit the wave so that it has speed c in the receiver's rest frame? What if there are two or more receivers moving at different speeds? Would not the transmitter then have to send out a suitable and different wave for each receiver?

However, there is no contradiction because this phenomenon is most likely caused by a very simple physical mechanism that does not require any concepts beyond Newtonian mechanics, such as the spacetime continuum or luminiferous ether, for its explanation. Further details can be found in [17]. At this point, we simply describe the basic idea: a transmitter emits a field of electromagnetic force carriers with random emission velocities. The paradox is resolved because each receiver can perceive only those force carriers whose propagation velocities in its own rest frame are not faster than c. One can then show that, although the waves propagate at all wave velocities, only the portion with a speed of exactly c remains in each rest frame. All other wave components either interfere destructively or are not perceptible, because they are too fast.

B. Uniformly moving point charges

With the potential (37), it is now possible to calculate retarded electromagnetic forces between arbitrarily moving point charges. In this section, we discuss the simplest case and assume that there is no acceleration or that the acceleration can be neglected. For this purpose, we consider a uniformly moving point charge q_s with the trajectory $\boldsymbol{r}_s(t) = \boldsymbol{v}\,t$, which exerts a force on a stationary point charge at location $\boldsymbol{r}$.

To calculate the force according to equation (33), the potential (37) must be known. Thus, one must solve equation (40), which has two solutions in this particular case, with only one solution satisfying causality $t \geq t'$:

$$t' = \frac{c^2\,t - \boldsymbol{r}\cdot\boldsymbol{v} - \sqrt{c^2\,\|\boldsymbol{r}-\boldsymbol{v}\,t\|^2 - \|\boldsymbol{r}\times\boldsymbol{v}\|^2}}{c^2 - v^2}. \tag{41}$$

This solution can be substituted into equation (37) to obtain

$$\Phi = \frac{q_s\, c}{4\,\pi\,\varepsilon_0\,\sqrt{\|\boldsymbol{r}-\boldsymbol{v}\,t\|^2\,(c^2 - v^2) + ((\boldsymbol{r}-\boldsymbol{v}\,t)\cdot\boldsymbol{v})^2}}, \tag{42}$$

using the relations $\boldsymbol{r}_r(t') = \boldsymbol{r} - \boldsymbol{v}\,t'$ and $\boldsymbol{v}_s(t') = \boldsymbol{v}$. The potential can then be inserted into equation (35) and thereafter into equation (33). After calculating the derivatives and summing up all terms, we obtain

$$\boldsymbol{F}(\boldsymbol{r},t) = \frac{c^3\, q_s\, q_d\, (\boldsymbol{r}-\boldsymbol{v}\,t)\,\gamma(\boldsymbol{v})^{-2}}{4\,\pi\,\varepsilon_0\,\left(\|\boldsymbol{r}-\boldsymbol{v}\,t\|^2\,(c^2-v^2)+((\boldsymbol{r}-\boldsymbol{v}\,t)\cdot\boldsymbol{v})^2\right)^{3/2}}. \quad (43)$$

Formula (43) gives us the force from the perspective of an observer at rest together with the receiver at location $\boldsymbol{r}$. In equation (43) it is remarkable that the term $\boldsymbol{r}-\boldsymbol{v}\,t$ occurs several times. This term corresponds to the location of the receiver at time t from the perspective of the transmitter. This means consequently, (i) the force is moving in conjunction with the transmitter, and (ii) the wave equation (19), which follows from Maxwell's equations, seems to contain a hidden *Galilean transformation*.

Therefore, it is reasonable to transform the force into the rest frame of the transmitter by means of a Galilean transformation, rather than a Lorentz boost. For this purpose, we apply the substitution $\boldsymbol{r} \to \boldsymbol{r}+\boldsymbol{v}\,t$ in equation (43), resulting in

$$\boxed{\boldsymbol{F} = \frac{c^3\, q_s\, q_d\, \boldsymbol{r}\,\gamma(\boldsymbol{v})^{-2}}{4\,\pi\,\varepsilon_0\,\left(r^2\,(c^2-v^2)+(\boldsymbol{r}\cdot\boldsymbol{v})^2\right)^{3/2}},} \quad (44)$$

which has no time dependence and describes a force exerted by a stationary transmitter q_s on a moving receiver q_d at location $\boldsymbol{r}$.

We note that this force does not correspond to the force we would obtain in Lorentz-Einstein electrodynamics in the rest frame of the transmitter, because a point charge at rest does not have a magnetic field, but only an electric field, namely the Coulomb field. Moreover, one must calculate the acceleration effect on a receiver with the dynamics of special relativity by using equation (22), in which we substitute the Coulomb force for $\boldsymbol{F}$. Thus, the acceleration depends not only on the force, but also on the velocity of the receiver.

However, it is remarkable that the Lorentz-Einstein electrodynamics do not generally yield the Coulomb force if one performs the following substitutions[2]:

$$\begin{aligned} \boldsymbol{r} &\to \boldsymbol{r} + (\gamma(v)-1)\,\boldsymbol{r}\cdot\boldsymbol{v}\frac{\boldsymbol{v}}{v^2} + \gamma(v)\,\boldsymbol{v}\,t \\ t &\to \gamma(v)\left(t + \frac{1}{c^2}\boldsymbol{r}\cdot\boldsymbol{v}\right) \end{aligned} \quad (45)$$

in function (43), i.e., if one applies the (inverse) Lorentz transformation [16, p. 635]. Instead, one obtains the Coulomb force only if the condition $\boldsymbol{r} \parallel \boldsymbol{v}$ is satisfied. Otherwise, the expression after the Lorentz boost still depends on the velocity. For the special case $\boldsymbol{r}\cdot\boldsymbol{v} = 0$, one obtains the Coulomb force multiplied by a Lorentz factor, which is incorrect. Unfortunately, the cause of this discrepancy is unclear because equation (43) describes the force generated by a uniformly moving transmitter on a receiver at rest in Lorentz-Einstein electrodynamics, and a different but correct calculation approach should not lead to a different result.

[2]For the Lorentz transformation, (45) maintains $\|\boldsymbol{r}\|^2 = c^2\,t^2$ before and after the substitution.

However, because we want to assume that Maxwell's equations (1)-(4) and the wave equation (19) are correct, we return to equation (44) and apply the equation $q_d = q_{d0}\,\gamma(\boldsymbol{v})$. Then, we expand the equation into a Taylor series and obtain a second-order approximation:

$$\boldsymbol{F} \approx \frac{q_s\, q_{d0}}{4\,\pi\,\varepsilon_0}\left(1+\frac{v^2}{c^2}-\frac{3}{2}\left(\frac{\boldsymbol{r}}{r}\cdot\frac{\boldsymbol{v}}{c}\right)^2\right)\frac{\boldsymbol{r}}{r^3}. \quad (46)$$

This formula was first documented in 1846 by W. Weber. Obviously, the Weber formula is an approximation of the solution of Maxwell's equations and the inhomogeneous wave equation (19). Thus, the Weber force is closely connected to Maxwell's equations and represents more than just a lucky fit to empirical data [18].

It should be mentioned that formula (44) can be brought into a form that is more geometrically readable [16, p. 619] by applying the angle α between $\boldsymbol{r}$ and $\boldsymbol{v}$. For this angle,

$$(\boldsymbol{r}\cdot\boldsymbol{v})^2 = r^2\,v^2\,\cos(\alpha)^2 \quad (47)$$

holds. Substituting this expression into formula (44), we obtain

$$\boldsymbol{F} = \frac{q_s\, q_d\, \boldsymbol{r}}{4\,\pi\,\varepsilon_0\, r^3}\,\frac{1-\frac{v^2}{c^2}}{\left(1-\frac{v^2}{c^2}\,\sin(\alpha)^2\right)^{3/2}} \quad (48)$$

after some rearrangement of the terms. Clearly, the force (44) and thus the Weber force (46) are central forces because the field lines are always straight lines starting at the transmitter q_s. The strength of the force, however, depends on the angle.

If the charge q_s is moving directly toward or away from the charge q_d, $\sin(\alpha) = 0$ and the force is weaker than the Coulomb force by a factor of $1/\gamma(v)^2$. However, if the charges are moving exactly sideways past each other, then $\sin(\alpha) = 1$ and the force is stronger than the Coulomb force by a factor of $\gamma(v)$. This angular dependence becomes more pronounced for high velocities. For very high speeds $v \to c$, the force is eventually nonzero only when α has a value of 90°.

Clearly, the force (44) is symmetrical, similar to the Weber force (46), because the transmitter exerts a force on the receiver that is equal in magnitude and opposite in direction to the force produced by the receiver on the transmitter. Therefore, Newton's third law is satisfied for charges with uniform motion. Consequently, all conservation laws for *uniformly* moving charges and charge distributions are also fulfilled. However, this is not the case in Lorentz-Einstein electrodynamics. Obviously, the reason for this difference does not lie in the Maxwell equations, but in the Lorentz force.

Another point should also be addressed: as shown by equation (44), the force between the two point charges appears to act directly and without a loss of time, indicating that the force propagates instantaneously. This apparent paradox has repeatedly raised questions in Lorentz-Einstein electrodynamics [18], [19]. However, there is no paradox, because equation (44) results from the wave equation (14), which has a wave velocity of c. The best way to understand this result is by considering the force-carrier mechanism, which seems to be the cause of the relativistic effects [17].

C. General solution for arbitrary inertial frames

As can be seen from equation (44), the force of a resting point charge on other uniformly moving charges is independent of time. Therefore, one might assume that it would have been mathematically simpler to calculate the force directly in the rest frame of the transmitter right from the start. However, equations (33) and (37) apply explicitly only to the force in the rest frame of the receiver. Yet, one can generalize these equations so that they hold for a uniformly moving observer with velocity $\boldsymbol{u}$ relative to the receiver at rest. For uniformly moving transmitters, one can then choose $\boldsymbol{u} = \boldsymbol{v}$ to set the velocity of the observer such that the transmitter is resting and the receiver is moving.

To apply this approach, we must first consider that the potential (37) is ultimately a function of t and $\boldsymbol{r}$. In section IV-B, the derivatives of the potential $\Phi(\boldsymbol{r}, t)$ were first determined according to equation (33) to obtain the force (43) in the rest frame of the receiver. Subsequently, a Galilean transformation $\boldsymbol{r} \rightarrow \boldsymbol{r} + \boldsymbol{u}\, t$ was performed, i.e., all occurrences of $\boldsymbol{r}$ were replaced with $\boldsymbol{r} + \boldsymbol{u}\, t$, and for the special case $\boldsymbol{v} = \boldsymbol{u}$, the force (44) was obtained.

However, it would be mathematically convenient if the Galilean transformation of the potentials could be calculated first and the derivatives calculated afterwards. In equation (33), two differential operators occur. For the gradient,

$$\Big(\nabla\Phi(\boldsymbol{r}, t)\Big)\Big|_{\boldsymbol{r}\rightarrow\boldsymbol{r}+\boldsymbol{u}\,t} = \nabla\Phi(\boldsymbol{r} + \boldsymbol{u}\, t, t) \tag{49}$$

is valid, i.e., in this case, the order in which one performs the Galilean transformation and calculates the derivative is irrelevant. For the time derivative, however, this does not apply. Here, we have

$$\Big(\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r}, t)\Big)\Big|_{\boldsymbol{r}\rightarrow\boldsymbol{r}+\boldsymbol{u}\,t} = \left(\frac{\partial}{\partial t} - (\boldsymbol{u}\cdot\nabla)\right)\boldsymbol{A}(\boldsymbol{r} + \boldsymbol{u}\, t, t), \tag{50}$$

where the differential operator $\boldsymbol{u}\cdot\nabla$ is defined by

$$\boldsymbol{u}\cdot\nabla := u_x\,\frac{\partial}{\partial x} + u_y\,\frac{\partial}{\partial y} + u_z\,\frac{\partial}{\partial z}. \tag{51}$$

By applying this result to equation (33), we obtain

$$\begin{aligned}\boldsymbol{F}\,\Big|_{\boldsymbol{r}\rightarrow\boldsymbol{r}+\boldsymbol{u}\,t} = & -q_d\left(\frac{\partial}{\partial t} - (\boldsymbol{u}\cdot\nabla)\right)\boldsymbol{A}(\boldsymbol{r} + \boldsymbol{u}\, t, t) - \\ & q_d\nabla\Phi(\boldsymbol{r} + \boldsymbol{u}\, t, t),\end{aligned} \tag{52}$$

where $\Phi(\boldsymbol{r}, t)$ and $\boldsymbol{A}(\boldsymbol{r}, t)$ are the potentials in the rest frame of the receiver. Thus, when calculating the force, it is now possible to choose the inertial frame that is most convenient for the calculation.

To demonstrate the advantage of this formula by an example, the calculation performed in section IV-B is repeated in the rest frame of the transmitter, i.e., for $\boldsymbol{u} = \boldsymbol{v}$. First, the potential (42) is translated into the rest frame of the transmitter using the Galilean transformation $\boldsymbol{r} \rightarrow \boldsymbol{r} + \boldsymbol{v}\, t$, which gives

$$\Phi = \frac{q_s\, c}{4\,\pi\,\varepsilon_0\,\sqrt{r^2\,(c^2 - v^2) + (\boldsymbol{r}\cdot\boldsymbol{v})^2}}, \tag{53}$$

i.e., the true scalar potential of a point charge at rest, which corresponds to the Coulomb potential only for $\boldsymbol{v} = 0$. The vector potential is calculated from the scalar potential (53) using equation (35) as

$$\boldsymbol{A} = \frac{\boldsymbol{v}}{c^2}\,\Phi. \tag{54}$$

We note that the differential velocity $\boldsymbol{v}$ between the transmitter and receiver does not depend on the reference frame in a Galilean transformation and therefore is not transformed. Both potentials (53) and (54) can now be substituted into equation (52). The time derivative is now zero because of the time independence of the potentials, and thus, it follows that

$$\begin{aligned}\boldsymbol{F} &= q_d\,(\boldsymbol{v}\cdot\nabla)\,\frac{\boldsymbol{v}}{c^2}\,\Phi - q_d\,\nabla\Phi \\ &= q_d\,\frac{\boldsymbol{v}}{c^2}\,(\boldsymbol{v}\cdot\nabla)\,\Phi - q_d\,\nabla\Phi \\ &= q_d\,\frac{\boldsymbol{v}}{c^2}\,(\boldsymbol{v}\cdot\nabla\,\Phi) - q_d\,\nabla\Phi\end{aligned} \tag{55}$$

because $\boldsymbol{u} = \boldsymbol{v}$. Now, only the gradient of the scalar potential must be calculated and inserted. After combining all terms, one obtains the force (44), which represents the relativistic generalized Weber force.

D. The Hertzian dipole

Another important special case is a bound particle that oscillates within itself. Such a bound particle may be electrically neutral, but inside, it consists of two charge quantities, $+q_s$ and $-q_s$. These charge quantities can oscillate with respect to each other so that the point charge $+q_s$ moves upward while the other point charge $-q_s$ moves downward. Of course, the strong attractive electric force between the two charge quantities prevents them from separating too far and causes the direction of motion to periodically reverse. Finally, an oscillation of the form $\boldsymbol{p}_0\,\sin(\omega\, t)$ arises. Here, $\boldsymbol{p}_0$ is the polarization vector, which specifies the spatial direction of the oscillation and the maximum distance between the two charge quantities that we want to model as point charges. ω is the angular frequency of the oscillation.

In electrodynamics, such an object is called a Hertzian dipole. This dipole is a standard study object for understanding electromagnetic waves and has the same importance in classical electrodynamics as the hydrogen atom in atomic physics, as it represents the simplest possible antenna. Hence, corresponding calculations can be found in numerous textbooks of classical electrodynamics.

The solutions found in textbooks always assume that the center of the bound particle does not move and is located at the coordinate origin. The solution then consists of two fields $\boldsymbol{E}$ and $\boldsymbol{B}$, which clearly indicate that the waves move at speed c for a resting receiver.

To calculate the force on a moving receiver, one applies the same fields $\boldsymbol{E}$ and $\boldsymbol{B}$ with the same time dependencies and wave velocities by substituting them into the Lorentz force (5). However, the Lorentz force is a very simple formula, and the time dependencies are not affected in $\boldsymbol{E}$ and $\boldsymbol{B}$. For example, if the receiver is moving away from the transmitter on a straight line with velocity v, it would observe a wave velocity of $c - v$ in its own rest frame. Because this contradicts experimental findings, it seemed necessary to H. A. Lorentz to introduce the Lorentz transformation.

In Weber electrodynamics, however, this problem does not arise because *every* moving receiver always perceives the same field as if it were moving *in its own* rest frame with c. Although there are clear physical and logical reasons for this interpretation, these reasons were never recognized or correctly interpreted by physicists. The reason for this is that the Graßmann force was implicitly but unjustifiably generalized during the transition from electrostatics and magnetostatics to electrodynamics. Later one thought that the resulting problems were solved with Lorentz force, Lorentz transformation and special theory of relativity.

To obtain the field of the Hertzian dipole, we consider a point charge $q_s = +q$ moving along the following trajectory:

$$\boldsymbol{r}_s(t) = \boldsymbol{v}\, t + \boldsymbol{s}(t), \tag{56}$$

where $\boldsymbol{s}(t)$ is any arbitrary but small spatial oscillation. The solution of the Hertzian dipole is obtained by considering a second point charge $-q$ following the trajectory $\boldsymbol{v}\, t - \boldsymbol{s}(t)$. For $\boldsymbol{s}(t) = \boldsymbol{p}_0 \sin(\omega\, t)$, the sum of the forces of the two charges gives the field of the Hertzian dipole, with its center moving along trajectory $\boldsymbol{v}\, t$. Hence, to calculate the force of the Hertzian dipole, it is sufficient to initially calculate only the force exerted by a transmitter $+q$ with trajectory (56) on a receiver resting at location $\boldsymbol{r}$.

We obtain this force by first calculating the potential (37). For this step, we need the retarded distance vector (38) and the retarded velocity (39). In this case, we obtain

$$\boldsymbol{r}_r(t') = \boldsymbol{r} - (\boldsymbol{v}\, t' + \boldsymbol{s}(t')) \approx \boldsymbol{r} - \boldsymbol{v}\, t' \tag{57}$$

and

$$\boldsymbol{v}_s(t') = \boldsymbol{v} + \dot{\boldsymbol{s}}(t'), \tag{58}$$

where the approximation in equation (57) is valid only if the spatial displacement of the oscillation $\boldsymbol{s}(t)$ is small enough for all times t to be negligible. Equations (57) and (58) can now be substituted into the potential (37), and we arrive at the scalar potential

$$\Phi_+ = \frac{q\, c}{4\, \pi\, \varepsilon_0\, \left(c^2\, (t - t') - (\boldsymbol{v} + \dot{\boldsymbol{s}}(t')) \cdot (\boldsymbol{r} - \boldsymbol{v}\, t')\right)}, \tag{59}$$

which is caused by the positive charge. The vector potential is obtained by substituting the scalar potential (59) into equation (35). In this case, we obtain

$$\boldsymbol{A}_+ = \frac{q\, (\boldsymbol{v} + \dot{\boldsymbol{s}}(t'))}{4\, \pi\, \varepsilon_0\, c\, \left(c^2\, (t - t') - (\boldsymbol{v} + \dot{\boldsymbol{s}}(t')) \cdot (\boldsymbol{r} - \boldsymbol{v}\, t')\right)}. \tag{60}$$

To further simplify the calculation, we can again exploit the fact that the displacement caused by the oscillation $\boldsymbol{s}(t)$ is small and that therefore the scalar potential and vector potential can be approximated with respect to amplitude via a Taylor series. For the scalar potential, the first-order approximation is

$$\Phi_+ \approx \frac{q\, c}{4\, \pi\, \varepsilon_0\, \left(c^2\, (t - t') - \boldsymbol{r} \cdot \boldsymbol{v} + t'\, v^2\right)} + \frac{q\, c\, \left((\boldsymbol{r} - \boldsymbol{v}\, t') \cdot \dot{\boldsymbol{s}}(t')\right)}{4\, \pi\, \varepsilon_0\, \left(c^2\, (t - t') - \boldsymbol{r} \cdot \boldsymbol{v} + t'\, v^2\right)^2}. \tag{61}$$

For the vector potential, we obtain

$$\boldsymbol{A}_+ \approx \frac{\boldsymbol{v}}{c^2}\, \Phi_+ + \frac{q\, \dot{\boldsymbol{s}}(t')}{4\, \pi\, \varepsilon_0\, c\, \left(c^2\, (t - t') - \boldsymbol{r} \cdot \boldsymbol{v} + t'\, v^2\right)}. \tag{62}$$

Now, the potentials generated by the positive charge are known. However, the Hertzian dipole consists of two charges, where the oscillation of the negative charge is exactly inverse to that of the positive charge. The potentials Φ_- and $\boldsymbol{A}_-$ are obtained by substituting $q \to -q$, $\boldsymbol{s}(t') \to -\boldsymbol{s}(t')$ and $\dot{\boldsymbol{s}}(t') \to -\dot{\boldsymbol{s}}(t')$ in equations (61) and (62). Therefore, the total potentials are the sums of the partial potentials, i.e., $\Phi = \Phi_+ + \Phi_-$ and $\boldsymbol{A} = \boldsymbol{A}_+ + \boldsymbol{A}_-$. Consequently, the scalar potential of the Hertzian dipole is

$$\Phi = \frac{q\, c\, \left((\boldsymbol{r} - \boldsymbol{v}\, t') \cdot \dot{\boldsymbol{s}}(t')\right)}{2\, \pi\, \varepsilon_0\, \left(c^2\, (t - t') - \boldsymbol{r} \cdot \boldsymbol{v} + t'\, v^2\right)^2}. \tag{63}$$

For the vector potential, we find

$$\boldsymbol{A} = \frac{\boldsymbol{v}}{c^2}\, \Phi + \frac{q\, \dot{\boldsymbol{s}}(t')}{2\, \pi\, \varepsilon_0\, c\, \left(c^2\, (t - t') - \boldsymbol{r} \cdot \boldsymbol{v} + t'\, v^2\right)}. \tag{64}$$

To use the potentials (63) and (64), we still need the retarded time t' as a function of $\boldsymbol{r}$ and t. This term can be obtained by solving equation (40) and by assuming that the amplitude of the oscillation $\boldsymbol{s}(t')$ is very small compared with the distance $\boldsymbol{r}$ and thus can be neglected. Therefore, the solution t' for the Hertzian dipole is given by equation (41).

Now, to determine the force in the receiver's rest frame, we could substitute the retarded time (41) into the potentials (63) and (64) and then apply equation (33). However, it is easier to make use of equation (52) with $\boldsymbol{u} = \boldsymbol{v}$. For this step, we need to transform the potentials (63) and (64) into the rest frame of the transmitter by replacing all occurrences of $\boldsymbol{r}$ in equation (41), (63), and (64) with $\boldsymbol{r} + \boldsymbol{v}\, t$. Equation (41) then becomes

$$t' = t - \tau \tag{65}$$

with

$$\tau := \frac{\boldsymbol{r} \cdot \boldsymbol{v} + \sqrt{c^2\, r^2 - \|\boldsymbol{r} \times \boldsymbol{v}\|^2}}{c^2 - v^2}. \tag{66}$$

For the potentials (63) and (64), we obtain

$$\Phi = \frac{q\, c\, (\boldsymbol{r} + \boldsymbol{v}\, \tau) \cdot \dot{\boldsymbol{s}}(t - \tau)}{2\, \pi\, \varepsilon_0\, \left(c^2\, r^2 - \|\boldsymbol{r} \times \boldsymbol{v}\|^2\right)} \tag{67}$$

and

$$\boldsymbol{A} = \frac{\boldsymbol{v}}{c^2}\, \Phi + \frac{q\, \dot{\boldsymbol{s}}(t - \tau)}{2\, \pi\, \varepsilon_0\, c\, \sqrt{c^2\, r^2 - \|\boldsymbol{r} \times \boldsymbol{v}\|^2}} \tag{68}$$

by applying the Galilean transformation $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{v}\, t$ and using equations (65) and (66).

The potentials (67) and (68) can now be substituted into equation (52) to obtain the force on a receiver q_d moving in the rest frame of the transmitter with velocity $-\boldsymbol{v}$. First, from equation (52), we obtain

$$\boldsymbol{F} = -q_d \frac{\partial}{\partial t}\boldsymbol{A} + q_d\ (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{A} - q_d\,\nabla\Phi \tag{69}$$

because $\boldsymbol{u} = \boldsymbol{v}$.

The calculations of the spatial derivatives can be greatly simplified by considering that the oscillation $\boldsymbol{s}(t)$ is a generic function without a given direction of oscillation. For this reason, $\boldsymbol{r} = r_x\,\boldsymbol{e}_x$ with $r_x > 0$ can be assumed without a loss of generality. This assumption simplifies equation (69) to

$$\boldsymbol{F} = -q_d \frac{\partial}{\partial t}\boldsymbol{A} + q_d\, v_x \frac{\partial}{\partial r_x}\boldsymbol{A} - q_d\,\boldsymbol{e}_x \frac{\partial}{\partial r_x}\Phi. \tag{70}$$

For equation (66), we have

$$\tau = \frac{r_x}{v_\tau} \tag{71}$$

with

$$v_\tau := -v_x + \sqrt{c^2 - (v^2 - v_x^2)} \tag{72}$$

as the velocity, which is independent of r_x. The potentials become significantly simpler as well, and we obtain

$$\Phi = \frac{q\,c\,\left(\boldsymbol{e}_x + \frac{\boldsymbol{v}}{v_\tau}\right)\cdot\dot{\boldsymbol{s}}\left(t - \frac{r_x}{v_\tau}\right)}{2\,\pi\,\varepsilon_0\ (c^2 - (v^2 - v_x^2))\ r_x} \tag{73}$$

for the scalar potential (67) by using equation (71). For the vector potential (68), we find

$$\boldsymbol{A} = \frac{\boldsymbol{v}}{c^2}\,\Phi + \frac{q\,\dot{\boldsymbol{s}}\left(t - \frac{r_x}{v_\tau}\right)}{2\,\pi\,\varepsilon_0\,c\,r_x\ \sqrt{c^2 - (v^2 - v_x^2)}}. \tag{74}$$

As can be seen, the spatial derivatives of the potentials are now as simple to calculate as the temporal derivatives. In particular, we obtain the following equations:

$$\frac{\partial}{\partial r_x}\boldsymbol{A} = -\left(\frac{\boldsymbol{A}}{r_x} + \frac{1}{v_\tau}\frac{\partial}{\partial t}\boldsymbol{A}\right) \tag{75}$$

and

$$\frac{\partial}{\partial r_x}\Phi = -\left(\frac{\Phi}{r_x} + \frac{1}{v_\tau}\frac{\partial}{\partial t}\Phi\right). \tag{76}$$

Equations (75) and (76) can be further simplified if one is interested only in the far-field. The terms $\boldsymbol{A}/r_x$ and Φ/r_x decrease with $1/r_x^2$ and therefore do not affect the far-field. Thus, the following approximations apply:

$$\frac{\partial}{\partial r_x}\boldsymbol{A} \approx -\frac{1}{v_\tau}\frac{\partial}{\partial t}\boldsymbol{A} \tag{77}$$

and

$$\frac{\partial}{\partial r_x}\Phi \approx -\frac{1}{v_\tau}\frac{\partial}{\partial t}\Phi. \tag{78}$$

The far-field approximations (77) and (78) can now be substituted into equation (70), which gives

$$\boldsymbol{F} = -q_d\left(\left(1 + \frac{v_x}{v_\tau}\right)\frac{\partial\boldsymbol{A}}{\partial t} - \frac{\boldsymbol{e}_x}{v_\tau}\frac{\partial\Phi}{\partial t}\right). \tag{79}$$

This equation contains only time derivatives, which are easy to calculate.

Remarkably, with the result (79), it becomes apparent that the restriction to $\boldsymbol{r} = r_x\,\boldsymbol{e}_x$ is actually not necessary. This conclusion results from the fact that equation (79) depends only on the direction-independent quantity v, the direction vector $\boldsymbol{e}_x = \boldsymbol{r}/r_x$, and the projection $v_x = \boldsymbol{v}\cdot\boldsymbol{r}/r_x$. At the same time, however, the oscillation $\boldsymbol{s}(t)$ is generic and consequently independent of direction. For this reason, r_x can be replaced by the distance r, and we obtain

$$\boldsymbol{F} = -q_d\left(\left(1 + \tau\,\frac{\boldsymbol{v}\cdot\boldsymbol{r}}{r^2}\right)\frac{\partial\boldsymbol{A}}{\partial t} - \tau\,\frac{\boldsymbol{r}}{r^2}\frac{\partial\Phi}{\partial t}\right) \tag{80}$$

because $v_\tau = r/\tau$. For the potentials, we can apply the general functions (67) and (68), and for the time τ, we can apply equation (66). We again note that formula (80) is the force of the dipole in the rest frame of the transmitter and that, because of the approximations used, the wave of the electromagnetic force is only correctly represented in the far-field.

Formula (80) now needs to be analyzed. In the simplest case, $\boldsymbol{v} = \boldsymbol{0}$. Because $\tau = r/c$, Equation (80) becomes

$$\boldsymbol{F} = -q_d\left(\frac{\partial\boldsymbol{A}}{\partial t} - \frac{\boldsymbol{r}}{r\,c}\frac{\partial\Phi}{\partial t}\right). \tag{81}$$

For $v = 0$, equations (67) and (68) simplify to

$$\Phi = \frac{q\,\boldsymbol{r}\cdot\dot{\boldsymbol{s}}\left(t - \frac{r}{c}\right)}{2\,\pi\,\varepsilon_0\,c\,r^2} \tag{82}$$

and

$$\boldsymbol{A} = \frac{q\,\dot{\boldsymbol{s}}\left(t - \frac{r}{c}\right)}{2\,\pi\,\varepsilon_0\,c^2\,r}. \tag{83}$$

Substituting the potentials (82) and (83) into equation (81) gives

$$\begin{aligned}\boldsymbol{F} &= -\frac{q_d\,q}{2\,\pi\,\varepsilon_0\,c^2\,r}\left(\ddot{\boldsymbol{s}}\,(t-\tau) - \frac{\boldsymbol{r}}{r}\cdot\ddot{\boldsymbol{s}}\,(t-\tau)\ \frac{\boldsymbol{r}}{r}\right)\\ &= \frac{q_d\,q}{2\,\pi\,\varepsilon_0\,c^2\,r}\left(\frac{\boldsymbol{r}}{r}\times\left(\frac{\boldsymbol{r}}{r}\times\ddot{\boldsymbol{s}}\,(t-\tau)\right)\right).\end{aligned} \tag{84}$$

A reader familiar with electrodynamics will certainly recognize that this is the type of field one would expect for a Hertzian dipole. For $\boldsymbol{s}(t) = \boldsymbol{e}_z\,p_0/(2\,q)\ \sin(\omega\,t)$, this expression corresponds exactly to the field one can usually find in textbooks (e.g., [16, p. 470]).

Equation (84) gives a good insight into the essential properties of a point-like transmitter. In particular, the force vanishes for $\boldsymbol{r} \parallel \boldsymbol{s}(t)$, indicating that there is no radiation perpendicular to the direction of oscillation. Furthermore, it becomes obvious that the field in the far-field is always aligned parallel to the direction of the dipole oscillation and propagates in the shape of a ring perpendicular to the axis of oscillation. This ring-wave propagation explains why the amplitude of the wave decreases with $1/r$ and not with $1/r^2$, as would be the case for a spherical wave. Interestingly, the term $\ddot{\boldsymbol{s}}\,(t-\tau)$ shows that the information contained in the oscillation $\boldsymbol{s}(t)$ propagates at the speed of light c.

However, this is only true if the receiver is at rest relative to the center of gravity of the transmitter. For $\boldsymbol{v} \neq \boldsymbol{0}$ and $v \ll c$, one obtains the first-order approximations as

$$\tau \approx \frac{r}{c} + \frac{\boldsymbol{r}\cdot\boldsymbol{v}}{c^2}, \tag{85}$$

$$\Phi \approx \frac{q}{2\pi\varepsilon_0 c\, r}\left(\frac{\boldsymbol{r}}{r}+\frac{\boldsymbol{v}}{c}\right)\cdot \dot{\boldsymbol{s}}\,(t-\tau)\,, \tag{86}$$

and

$$\boldsymbol{A} \approx \frac{q}{2\pi\varepsilon_0 c^2 r}\left(\dot{\boldsymbol{s}}\,(t-\tau)+\frac{\boldsymbol{r}}{r}\cdot \dot{\boldsymbol{s}}\,(t-\tau)\,\frac{\boldsymbol{v}}{c}\right) \tag{87}$$

by performing a Taylor series expansion of equations (66), (67), and (68). Substituting equation (85) into equation (80) yields the relation

$$\boldsymbol{F} = -q_d\left(\frac{\partial \boldsymbol{A}}{\partial t}-\frac{\boldsymbol{r}}{r\,c}\frac{\partial \Phi}{\partial t}\right)\left(1+\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right)-q_d\left(\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right)^2\frac{\partial \boldsymbol{A}}{\partial t}. \tag{88}$$

The quadratic term $((\boldsymbol{r}\cdot\boldsymbol{v})/(r\,c))^2$ can be neglected for very small relative speeds v, and we arrive at

$$\boldsymbol{F} \approx -q_d\left(\frac{\partial \boldsymbol{A}}{\partial t}-\frac{\boldsymbol{r}}{r\,c}\frac{\partial \Phi}{\partial t}\right)\left(1+\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right). \tag{89}$$

Now, the two potentials (86) and (87) can be substituted, and we obtain

$$\begin{aligned}\boldsymbol{F} = &-\frac{q_d\, q\left(1+\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right)}{2\pi\varepsilon_0 c^2 r}\left(\ddot{\boldsymbol{s}}\,(t-\tau)-\frac{\boldsymbol{r}}{r}\cdot\ddot{\boldsymbol{s}}\,(t-\tau)\,\frac{\boldsymbol{r}}{r}\right)-\\ &\frac{q_d\, q\left(1+\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right)}{2\pi\varepsilon_0 c^2 r}\left(\frac{\boldsymbol{r}}{r}\cdot\ddot{\boldsymbol{s}}\,(t-\tau)\,\frac{\boldsymbol{v}}{c}-\frac{\boldsymbol{v}}{c}\cdot\ddot{\boldsymbol{s}}\,(t-\tau)\,\frac{\boldsymbol{r}}{r}\right).\end{aligned} \tag{90}$$

This expression can be simplified by using Graßmann's identity, and we finally obtain

$$\boxed{\begin{aligned}\boldsymbol{F} = &\frac{q_d\, q\left(1+\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right)}{2\pi\varepsilon_0 c^2 r}\left(\frac{\boldsymbol{r}}{r}\times\left(\frac{\boldsymbol{r}}{r}\times\ddot{\boldsymbol{s}}\,(t-\tau)\right)\right)+\\ &\frac{q_d\, q\left(1+\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{r\,c}\right)}{2\pi\varepsilon_0 c^2 r}\left(\left(\frac{\boldsymbol{v}}{c}\times\frac{\boldsymbol{r}}{r}\right)\times\ddot{\boldsymbol{s}}\,(t-\tau)\right).\end{aligned}} \tag{91}$$

This equation describes the force perceived by the receiver in the rest frame of the transmitter. It does *not* describe the force perceived by the transmitter itself! The transmitter does not move in relation to itself. Therefore, the second term is zero for the transmitter. The same conclusion applies to every observer moving synchronously with the transmitter, because an observer in the rest frame of the transmitter is nothing more than a receiver resting in relation to the transmitter. This interpretation clarifies that the wave velocity needs to be c only in the reference frame of a receiver, because only receivers are able to measure the speed of light.

We now return to Lorentz-Einstein electrodynamics. Here, the field of the force generated by the oscillating dipole is calculated differently. In Lorentz-Einstein electrodynamics, one uses the potentials (82) and (83), which are only valid for a receiver at rest with respect to the center of gravity of the transmitter. One then applies these potentials in the formula of the Lorentz force (5), which – using potentials instead of fields – reads

$$\boldsymbol{F} = -q_d\,\frac{\partial}{\partial t}\boldsymbol{A}+q_d\,\boldsymbol{v}\times(\nabla\times\boldsymbol{A})-q_d\,\nabla\Phi \tag{92}$$

and differs from the force law (69) for $v \neq 0$.

Clearly, this approach does not use the wave equation in a correct manner, because the potentials (67) and (68) are Liénard-Wiechert potentials, i.e., solutions of the inhomogeneous wave equation that consider the transit time of the force from the transmitter to the receiver due to the trajectory of the transmitter. However, in the simplified form (82) and (83), these potentials apply only when the differential velocity between the center of mass of the transmitter and the receiver is zero. To be precise, the error consists in neglecting the differential velocity between the transmitter and receiver in the current density term in the wave equation (19).

For the case in which the differential velocity is not zero, the wave equation (19), which follows directly from the full set of Maxwell' s equations, provides many additional effects, but these are ignored because rest frame potentials are used in the Lorentz force (92). For example, this approach eliminates the Doppler effect, the trajectory, and the correct time characteristic. Additionally, attenuation and amplification effects of the electromagnetic force due to the differential velocity are not properly reproduced. Even the force directions are not completely correct. This can be seen from equation (91), which after performing substitution $\boldsymbol{r} \to \boldsymbol{r}-\boldsymbol{v}\,t$ is the *correct solution in Lorentz-Einstein electrodynamics* in the rest frame of the receiver for the far-field. In particular, there are force components that are parallel and proportional to $\boldsymbol{v}$, which contradicts the Lorentz force [3].

Because Lorentz-Einstein electrodynamics is currently the standard version of electrodynamics and has unfortunately served as the origin for the development of modern physics, the force of the oscillating dipole is now calculated again using the Lorentz force. This calculation is not very difficult because, as previously mentioned, we need only the simple rest frame potentials (82) and (83). We insert these potentials into the Lorentz force (92), and for the far-field, i.e., by neglecting all terms of order $1/r^2$, we obtain the following approximation:

$$\boldsymbol{F} \approx \frac{q_d\, q}{2\pi\varepsilon_0 c^2 r}\left(\left(\frac{\boldsymbol{r}}{r}-\frac{\boldsymbol{v}}{c}\right)\times\left(\frac{\boldsymbol{r}}{r}\times\ddot{\boldsymbol{s}}\left(t-\frac{r}{c}\right)\right)\right), \tag{93}$$

which can also be found in textbooks[3].

The result (93) can now be compared with the force (91). As can be seen, the two forces are identical only for $v = 0$. Notably, the meaning of $\boldsymbol{v}$ in the two formulas differs in its sign. In the Lorentz force, $\boldsymbol{v}$ is the velocity of the receiver from the perspective of the transmitter; in contrast, in formula (91), $\boldsymbol{v}$ is the velocity of the transmitter from the perspective of the receiver. However, even when the sign is adjusted, the two equations differ significantly for $v \neq 0$.

The most essential difference between equations (91) and (93) is not the structure of the terms with the cross products, but the argument of the function $\ddot{\boldsymbol{s}}(.)$. In Lorentz-Einstein electrodynamics, the argument is $t-\boldsymbol{r}/c$. In equation (91), however, the argument is $t-\tau$, where τ has a value such that the wave has speed c in the reference frame of the receiver.

Thus, when we solve Maxwell's equations in the rest frame of the transmitter and use the Lorentz force afterwards, the

[3]This expression is usually split into magnetic and electric fields and is often given in spherical coordinates.

problem arises that the wave moves at c for a resting receiver, but not for a moving receiver. Indeed, if one were to perform a Galilean transformation into the receiver's rest frame, the wave would generally have a propagation velocity different from c. For this reason, the Lorentz transformation is necessary in Lorentz-Einstein electrodynamics. In contrast, if one applies Maxwell's equations rigorously in the rest frame of the receiver without the Lorentz force, as described in this article, the velocity of the wave is always exactly c, independent of its velocity relative to the transmitter. This result has also been recently shown by H. Dodig [4].

It is important to realize that these findings are perfectly sufficient to satisfy Einstein's two postulates:

1) There is no absolute velocity, only relative velocities. A velocity, like a voltage, always needs a reference point.
2) The propagation velocity of an electromagnetic wave is equal to c for every observer in vacuum.

The principle of relativity is clearly satisfied in Weber electrodynamics because, in this case, $\boldsymbol{v}$ is the differential velocity between the transmitter and receiver and does not depend on the observer. The second point is satisfied without the Lorentz transformation because an observer can *principally* measure the wave velocity only in his own rest frame. Thus, the observer is a receiver himself. If the observer moves with respect to the transmitter, this motion is explicitly considered in the wave equation (19), and the calculated force propagates for the receiver with speed c. If the receiver does not move with respect to the transmitter, then this lack of motion is also taken into account by the wave equation (19), and the propagation speed of the wave is again exactly c. Thus, it does not matter how fast the receiver is moving with respect to the transmitter because, after the wave equation is solved, the force travels for him and for any other receiver or observer at speed c.

The above explanation is the mathematical point of view. Logically, however, this explanation seems deeply contradictory. Yet, this contradiction is only present at first sight. In fact, the phenomenon can be explained physically by assuming that the electric force is mediated by force carriers emitted by the transmitter with stochastic velocities. A sparkler can serve as an illustrative analogy: the force carriers correspond to the radiated sparks and have no uniform emission velocity, but are radially symmetric with respect to the transmitter. If one moves the sparkler, the center of radiation moves as well. Obviously, the wave equation (19), which follows from Maxwell's equations, describes exactly such a phenomenon.

However, there is a special feature: for *each* receiver, only force carriers that do not move faster than c relative to the receiver have an effect. If an observer is moving away from the transmitter, he can perceive force carriers that are faster than c relative to the transmitter. Yet, for the receiver, the carriers move at exactly c. If the observer moves toward the source, the opposite effect occurs.

The force-carrier model explains special relativity but offers also new opportunities for interpreting quantum effects due to its field quantization. Unfortunately, little investigation has been conducted on the consequences of the mechanism postulated here for phenomena outside of classical electrodynamics. However, it is known that one can derive from this model not only magnetism, but most likely gravitation and inertial effects as well.

Here, we include a brief discussion on magnetism, as this topic cannot be fully treated due to length constraints. It is worth noting that the magnetic field was originally introduced to describe force effects on permanent magnets. This original meaning has no relation to the Lorentz force. In 1820, A. M. Ampère recognized that permanent magnets and DC conductor loops are equivalent. From this realization followed the Lorentz force, which is completely correct for electro- and magnetostatic scenarios.

For electrodynamic situations, however, permanent magnets and DC conductor loops are not equivalent in Lorentz-Einstein electrodynamics. This conclusion can be verified by (i) calculating the force on a permanent magnet using the $\boldsymbol{B}$ field and (ii) calculating the force on a current in a small conductor loop using the Lorentz force. However, because Weber electrodynamics does not use the Lorentz force, the magnetic field can be defined again in its original form, namely as the field of the force that would act on imaginary magnetic monopoles caused by the DC current in a very small conductor loop. If one were to follow this approach, one would most likely find that permanent magnets and DC conducting loops act in the same way, especially in the electrodynamic context. Further explorations of this aspect will be conducted as future work.

V. Summary and conclusions

This article showed that the inhomogeneous wave equation (19), which follows from Maxwell's equations for the rest frame of a receiver, is valid for both Lorentz-Einstein electrodynamics and Weber electrodynamics. This finding is evidenced by the fact that the solution of the wave equation for transmitters and receivers moving uniformly with respect to each other corresponds to the Weber force (46) for differential velocities that are not too high, if one uses equation (6) instead of the Lorentz force (5).

Until now, this result was unknown, and the Weber force seemed to be detached from modern electrodynamics. However, this article shows that Weber electrodynamics has a close connection to Maxwell's equations and that the difference between Lorentz-Einstein and Weber electrodynamics primarily lies in how one interprets and applies the field equations and the resulting wave equation.

In Lorentz-Einstein electrodynamics, one usually solves the wave equation in the reference frame of the transmitter, as this seems to correspond to the natural viewpoint and allows one to apply the established methods electro- and magnetostatics. However, this approach implies that the wave generated by the transmitter moves at velocity c only with respect to the transmitter. However, numerous experiments have shown that the wave velocity is c in the reference frame of any receiver. This contradiction finally led to the development of the Lorentz transformation.

However, if one solves the wave equation rigorously in the rest frame of the receiver, this contradiction does not exist and one obtains solutions showing that the wave indeed propagates at velocity c for any receiver. In particular, for the simple case of a uniformly moving transmitter and receiver, one obtains the force formula (44), which agrees with the Weber force for the limit of small relative velocities. Furthermore, the force formula in the receiver's frame of reference (43) indicates that this solution and thus the wave equation itself is based on a Galilean transformation, even at relativistic velocities. If one nevertheless applies the Lorentz transformation, one obtains inconsistencies in Lorentz-Einstein electrodynamics, i.e. one finds that the results are different depending on the calculation method.

Finally, the far-field approximation (91) of a Hertzian dipole was calculated from the viewpoint of Weber electrodynamics. This approximation was analyzed and compared with the known solution (93) for Lorentz-Einstein electrodynamics. This comparison clarified that there is no need for a Lorentz transformation and that the solution for Weber electrodynamics is physically more reasonable.

The main conclusion of this article is that Weber electrodynamics is also a Maxwellian electrodynamics, as it can be derived from Maxwell's equations and the corresponding wave equation. The difference for Lorentz-Einstein electrodynamics lies in the fact that the solution of Maxwell's equations must always be obtained in the rest frame of the receiver in Weber electrodynamics. This approach avoids the usage of the Lorentz force, whose validity seems questionable outside of electro- and magnetostatics.

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