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# On the Incompleteness of Ibragimov's Conservation Law Theorem and Its Equivalence to a Standard Formula Using Symmetries and Adjoint-Symmetries

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**Abstract:** A conservation law theorem stated by N. Ibragimov along with its subsequent extensions are shown to be a special case of a standard formula that uses a pair consisting of a symmetry and an adjoint-symmetry to produce a conservation law through a well-known Fréchet derivative identity. Furthermore, the connection of this formula (and of Ibragimov's theorem) to the standard action of symmetries on conservation laws is explained, which accounts for a number of major drawbacks that have appeared in recent work using the formula to generate conservation laws. In particular, the formula can generate trivial conservation laws and does not always yield all non-trivial conservation laws unless the symmetry action on the set of these conservation laws is transitive. It is emphasized that all local conservation laws for any given system of differential equations can be found instead by a general method using adjoint-symmetries. This general method is a kind of adjoint version of the standard Lie method to find all local symmetries and is completely algorithmic. The relationship between this method, Noether's theorem and the symmetry/adjoint-symmetry formula is discussed.

**Keywords:** conservation law; symmetry; adjoint-symmetry; Fréchet derivative identity; Ibragimov's theorem

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## 1. Introduction

The most well-known method for finding conservation laws of differential equations (DEs) is Noether's theorem [1], which is applicable to any system of one or more DEs admitting a variational formulation in terms of a Lagrangian. Noether's theorem shows that every local symmetry preserving the variational principle of a given variational system yields a non-trivial local conservation law. Moreover, for variational systems that do not possess any differential identities, every non-trivial local conservation law arises from some local symmetry that preserves the variational principle.

However, there are many physically and mathematically interesting DEs that are not variational systems, and this situation has motivated much work in the past few decades to look for some generalization of Noether's theorem which could be applied to non-variational DEs. One direction of work has been to replace the need for a variational principle by introducing some other structure, but still making use of the local symmetries of a given DE system to produce local conservation laws. In fact, a general formula is available that yields local conservation laws from local symmetries combined with solutions of the adjoint of the symmetry determining equations. This formula first appears (to the knowledge of the author) in a 1986 paper by Caviglia [2] and was later derived independently in a 1990 Russian paper by Lunev [3], as well as in a 1997 paper by the author and Bluman [4]. In the latter paper, solutions of the adjoint of the symmetry determining equations were called adjoint-symmetries; these solutions are also known as cosymmetries in the literature on integrable systems [5]. Essentially the same formula appears in a more abstract form in the cohomological framework for finding conservation laws, summarized in References [6–8].

In recent years, a similar conservation law formula has been popularized by Ibragimov [9–13] and subsequently extended by others [14–17], where a “nonlinear self-adjointness” condition is required to hold for the given DE system. However, in several papers [17–19], this formula sometimes is seen to produce only trivial conservation laws, and sometimes, the formula does not produce all admitted conservation laws. Furthermore, in a number of papers [14–17,20–22], the use of translation symmetries is mysteriously avoided, and other more complicated symmetries are used instead, without explanation.

The purpose of the present paper is to make several relevant remarks:

- (1) Ibragimov’s conservation law formula is a simple re-writing of a special case of the earlier formula using symmetries and adjoint-symmetries;
- (2) Ibragimov’s “nonlinear self-adjointness” condition in its most general form is equivalent to the existence of an adjoint-symmetry for a general DE system and reduces to the existence of a symmetry in the case of a variational DE system;
- (3) this formula does not always yield all admitted local conservation laws, and it produces trivial conservation laws whenever the symmetry is a translation and the adjoint-symmetry is translation-invariant;
- (4) the computation to find adjoint-symmetries (and, hence, to apply the formula) is just as algorithmic as the computation of local symmetries;
- (5) most importantly, if all adjoint-symmetries are known for a given DE system (whether or not it has a variational formulation), then they can be used directly to obtain all local conservation laws, providing a kind of generalization of Noether’s theorem to general DE systems.

All of these remarks have been pointed out briefly in Reference [23], and Remark (2) has been discussed in References [16,17], but it seems worthwhile to give a comprehensive discussion for all of the remarks (1)–(5), with examples, as the formula continues to be used in recent papers when a complete, general method for finding all local conservation laws is available instead. In particular, for any given DE system, a full generalization of the content of Noether’s theorem is provided by a direct method using adjoint-symmetries, based on the framework shown in References [24,25] and presented in an algorithmic fashion in References [4,26–28]. In the case when a DE system is variational, adjoint-symmetries reduce to symmetries, and the direct method reproduces the relationship between symmetries and conservation laws in Noether’s theorem, but without the need for a Lagrangian. A detailed review and further development of this general method appears in Reference [29]. Consequently, there is no need for any kind of special methods to find local conservation laws, just as there is no need to use special methods to find local symmetries, because the relevant determining equations can be solved in a direct algorithmic manner.

The remainder of the present paper is organized as follows. Remarks (1) and (2) will be demonstrated in Section 2. Remark (3), along with some further consequences and properties related to the action of symmetries, will be explained in Section 3. Remarks (4) and (5) will be briefly discussed in Section 4. Throughout, the class of DEs  $u_{tt} - u_{xx} + a(u)(u_t^2 - u_x^2) + b(u)u_t + c(u)u_x + m(u) = 0$  will be used as a running example to illustrate the main points, and the notation in Ibragimov’s work will be used to allow the simplest possible comparison of the results. Some concluding remarks are made in Section 5.

Many examples of conservation laws of wave equations and other evolution equations can be found in References [24,30,31] and the references therein.

## 2. Symmetries, Adjoint-Symmetries and “Nonlinear Self-Adjointness”

As preliminaries, a few basic tools from variational calculus will be reviewed. Let  $x = (x^1, \dots, x^n)$  be  $n \geq 1$  independent variables and  $u = (u^1, \dots, u^m)$  be  $m \geq 1$  dependent variables, and let  $\partial^k u$  denote all  $k$ -th order partial derivatives of  $u$  with respect to  $x$ . Introduce an index notation for the components of  $x$  and  $u$ :  $x^i$ ,  $i = 1, \dots, n$ ;

and  $u^\alpha$ ,  $\alpha = 1, \dots, m$ . In this notation, the components of  $\partial^k u$  are given by  $u_{i_1 \dots i_k}^\alpha$ ,  $\alpha = 1, \dots, m$ ,  $i_q = 1, \dots, n$ , with  $q = 1, \dots, k$ . Summation is assumed over each pair of repeated indices in any expression. The coordinate space  $J = (x, u, \partial u, \partial^2 u, \dots)$  is called the jet space associated with the variables  $x, u$ . A *differential function* is a locally smooth function of finitely many variables in  $J$ . Total derivatives with respect to  $x$  applied to differential functions are denoted  $D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots$ .

The necessary tools that will now be introduced are the Fréchet derivative and its adjoint derivative, the Euler operator and its product rule, and the Helmholtz conditions.

Given a set of  $M \geq 1$  differential functions  $f_a(x, u, \partial u, \dots, \partial^N u)$ ,  $a = 1, \dots, M$ , with differential order  $N \geq 1$ , the *Fréchet derivative* is the linearization of the functions as defined by

$$\begin{aligned} (\delta_w f)_a &= \left( \frac{\partial}{\partial \epsilon} f_a(x, u + \epsilon w, \partial(u + \epsilon w), \dots, \partial^N(u + \epsilon w)) \right) \Big|_{\epsilon=0} \\ &= w^\alpha \frac{\partial f_a}{\partial u^\alpha} + w_i^\alpha \frac{\partial f_a}{\partial u_i^\alpha} + \dots + w_{i_1 \dots i_N}^\alpha \frac{\partial f_a}{\partial u_{i_1 \dots i_N}^\alpha}. \end{aligned} \quad (1)$$

This linearization can be viewed as a local directional derivative in jet space, corresponding to the action of a generator  $\hat{X} = w^\alpha \partial_{u^\alpha}$  in characteristic form,  $\hat{X}(f) = \delta_w f$ , where  $w = (w^1(x, u, \partial u, \dots, \partial^k u), \dots, w^m(x, u, \partial u, \dots, \partial^k u))$  is a set of  $m$  arbitrary differential functions.

It is useful also to view the Fréchet derivative as a linear differential operator acting on  $w$ . Then, integration by parts defines the *Fréchet adjoint derivative*

$$(\delta_v^* f)_\alpha = v^a \frac{\partial f_a}{\partial u^\alpha} - D_i \left( v^a \frac{\partial f_a}{\partial u_i^\alpha} \right) + \dots + (-1)^N D_{i_1} \dots D_{i_N} \left( v^a \frac{\partial f_a}{\partial u_{i_1 \dots i_N}^\alpha} \right) \quad (2)$$

which is a linear differential operator acting on a set of  $M \geq 1$  arbitrary differential functions  $v = (v^1(x, u, \partial u, \dots, \partial^k u), \dots, v^M(x, u, \partial u, \dots, \partial^k u))$ .

These two derivatives (1) and (2) are related by

$$v^a (\delta_w f)_a - w^\alpha (\delta_v^* f)_\alpha = D_i \Psi^i(w, v; f) \quad (3)$$

where the associated vector  $\Psi^i(w, v; f)$  is given by the explicit formula

$$\begin{aligned} \Psi^i(w, v; f) &= w^\alpha v^a \frac{\partial f_a}{\partial u_i^\alpha} + (D_j w^\alpha) v^a \frac{\partial f_a}{\partial u_{ji}^\alpha} - w^\alpha D_j \left( v^a \frac{\partial f_a}{\partial u_{ji}^\alpha} \right) + \dots \\ &+ \sum_{q=1}^N (-1)^{q-1} (D_{j_1} \dots D_{j_{N-q}} w^\alpha) D_{i_1} \dots D_{i_{q-1}} \left( v^a \frac{\partial f_a}{\partial u_{j_1 \dots j_{N-q} i_1 \dots i_{q-1}}^\alpha} \right). \end{aligned} \quad (4)$$

The *Euler operator*  $E_{u^\alpha}$ , or variational derivative, is defined in terms of the Fréchet derivative through the variational relation

$$\delta_w f = w^\alpha E_{u^\alpha}(f) + D_i \Phi^i(w; f) \quad (5)$$

which is obtained from integration by parts, yielding

$$E_{u^\alpha}(f) = \frac{\partial f}{\partial u^\alpha} - D_i \left( \frac{\partial f}{\partial u_i^\alpha} \right) + \dots + (-1)^N D_{i_1} \dots D_{i_N} \left( \frac{\partial f}{\partial u_{i_1 \dots i_N}^\alpha} \right) = \frac{\delta f}{\delta u^\alpha} \quad (6)$$

where

$$\Phi^i(w; f)v = \Psi^i(w, v; f). \quad (7)$$

Here, for simplicity,  $f(x, u, \partial u, \dots, \partial^N u)$  is a single differential function. In particular, an explicit formula for  $\Phi^i(w; f)$  is given by

$$\begin{aligned} \Phi^i(w; f) = & w^\alpha \frac{\partial f}{\partial u_i^\alpha} + (D_j w^\alpha) \frac{\partial f}{\partial u_{ji}^\alpha} - w^\alpha D_j \frac{\partial f}{\partial u_{ji}^\alpha} + \dots \\ & + \sum_{q=1}^N (-1)^{q-1} (D_{j_1} \dots D_{j_{N-q}} w^\alpha) D_{i_1} \dots D_{i_{q-1}} \frac{\partial f}{\partial u_{j_1 \dots j_{N-q} i_1 \dots i_{q-1} i}^\alpha} \end{aligned} \tag{8}$$

from expression (4).

The Euler operator (6) has the following three important properties: First, it obeys the product rule

$$E_{u^\alpha}(fg) = (\delta_\sigma^* f)_\alpha + (\delta_f^* g)_\alpha. \tag{9}$$

Second, its kernel

$$E_{u^\alpha}(f) = 0 \tag{10}$$

is given by total divergences

$$f = D_i F^i \tag{11}$$

holding for some differential vector function  $F^i$ . Third, its image consists of differential functions

$$E_{u^\alpha}(f) = g_\alpha \tag{12}$$

characterized by the Helmholtz conditions

$$(\delta_w g)_\alpha = (\delta_w^* g)_\alpha \tag{13}$$

where  $w^\alpha$  is a set of arbitrary differential functions.

There are several common alternative notations for the Fréchet derivative and its adjoint:  $\delta_w f = f'(w)$  and  $\delta_v^* f = f'^*(v)$  appear in the literature on integrable systems and in Reference [29];  $\delta_w f = D_w f$  and  $\delta_v^* f = D_v^* f$  are used in Olver's book [24];  $\delta_w f = L[u]w$  is used in the early work of Anco and Bluman [4,26,27] and in the book [31]. In contrast, Ibragimov [9,12] uses  $\delta_v^* f = f^*[u, v]$ .

### 2.1. Conservation Laws and Symmetries

Consider an  $N$ -th-order system of  $M \geq 1$  DEs

$$F = (F_1(x, u, \partial u, \dots, \partial^N u), \dots, F_M(x, u, \partial u, \dots, \partial^N u)) = 0. \tag{14}$$

The space of solutions  $u(x)$  of the system will be denoted  $\mathcal{E}$ . When the number of independent variables  $x$  is  $n = 1$ , each DE is an ordinary differential equation (ODE), whereas when the number of independent variables  $x$  is  $n \geq 2$ , each DE is a partial differential equation (PDE). The number,  $m$ , of dependent variables  $u$  need not be the same as the number,  $M$ , of DEs in the system.

A local infinitesimal symmetry [24,31,32] of a given DE system (14) is a generator

$$\mathbf{X} = \xi^i(x, u, \partial u, \dots, \partial^r u) \partial / \partial x^i + \eta^\alpha(x, u, \partial u, \dots, \partial^r u) \partial / \partial u^\alpha \tag{15}$$

whose prolongation leaves invariant the DE system

$$\text{pr}\mathbf{X}(F)|_{\mathcal{E}} = 0 \tag{16}$$

which holds on the whole solution space  $\mathcal{E}$  of the system. (In this determining equation, the notation  $\mathcal{E}$  means that the given DE system, as well as its differential consequences, are to be used). The differential functions  $\xi^i$  and  $\eta^\alpha$  in the symmetry generator are called the *symmetry characteristic functions*.

When acting on the solution space  $\mathcal{E}$ , an infinitesimal symmetry generator can be formally exponentiated to produce a one-parameter group of transformations  $\exp(\epsilon \text{pr}\mathbf{X})$ , with parameter  $\epsilon$ , where the infinitesimal transformation is given by

$$u^\alpha(x) \rightarrow u^\alpha(x) + \epsilon(\eta^\alpha(x, u(x), \partial u(x), \dots, \partial^r u(x)) - u_i^\alpha(x) \zeta^i(x, u(x), \partial u(x), \dots, \partial^r u(x))) + O(\epsilon^2) \quad (17)$$

for all solutions  $u(x)$  of the DE system.

Two infinitesimal symmetries are equivalent if they have the same action (17) on the solution space  $\mathcal{E}$  of a given DE system. An infinitesimal symmetry is thereby called *trivial* if it leaves all solutions  $u(x)$  unchanged. This occurs iff its characteristic functions satisfy the relation

$$\eta^\alpha|_{\mathcal{E}} = (u_i^\alpha \zeta^i)|_{\mathcal{E}}. \quad (18)$$

The corresponding generator (15) of a trivial symmetry is thus given by

$$\mathbf{X}_{\text{triv.}}|_{\mathcal{E}} = \zeta^i \partial / \partial x^i + \zeta^i u_i^\alpha \partial / \partial u^\alpha \quad (19)$$

which has the prolongation  $\text{pr}\mathbf{X}_{\text{triv.}}|_{\mathcal{E}} = \zeta^i D_i$ . Conversely, any generator of this form on the solution space  $\mathcal{E}$  represents a trivial symmetry. Thus, any two generators that differ by a trivial symmetry are equivalent. The differential order of an infinitesimal symmetry is defined to be the smallest differential order among all equivalent generators.

Any symmetry generator is equivalent to a generator given by

$$\hat{\mathbf{X}} = \mathbf{X} - \mathbf{X}_{\text{triv.}} = P^\alpha \partial / \partial u^\alpha, \quad P^\alpha = \eta^\alpha - \zeta^i u_i^\alpha, \quad (20)$$

under which  $u$  is infinitesimally transformed while  $x$  is invariant, due to the relation

$$\text{pr}\mathbf{X} - \text{pr}\hat{\mathbf{X}} = \zeta^i D_i. \quad (21)$$

This generator (20) defines the *characteristic form* for the infinitesimal symmetry. The symmetry invariance (16) of the DE system can then be expressed by

$$\text{pr}\hat{\mathbf{X}}(F)|_{\mathcal{E}} = 0 \quad (22)$$

holding on the whole solution space  $\mathcal{E}$  of the given system. Note that the action of  $\text{pr}\hat{\mathbf{X}}$  is the same as a Fréchet derivative (1), and hence, an equivalent, modern formulation [24,29,31] of this invariance (22) is given by the *symmetry determining equation*

$$(\delta_P F)_a|_{\mathcal{E}} = 0. \quad (23)$$

(Recall, the notation  $\mathcal{E}$  means that the given DE system, as well as its differential consequences, are to be used in these determining equations.)

In jet space  $J$ , a group of transformations  $\exp(\epsilon \text{pr}\mathbf{X})$  with a non-trivial generator  $\mathbf{X}$  in general will not act in a closed form on  $x, u$  and derivatives  $\partial^k u$  up to a finite order, except [24,31] for point transformations acting on  $(x, u)$  and contact transformations acting on  $(x, u, \partial u)$ . Moreover, a contact transformation is a prolonged point transformation when the number of dependent variables is  $m > 1$  [24,31]. A *point symmetry* is defined as a symmetry transformation group on  $(x, u)$ , whose generator is given by characteristic functions of the form

$$\mathbf{X} = \zeta(x, u)^i \partial / \partial x^i + \eta^\alpha(x, u) \partial / \partial u^\alpha \quad (24)$$

corresponding to the infinitesimal point transformation

$$x^i \rightarrow x^i + \epsilon \zeta^i(x, u) + O(\epsilon^2), \quad u^\alpha \rightarrow u^\alpha + \epsilon \eta^\alpha(x, u) + O(\epsilon^2). \quad (25)$$

Likewise, a *contact symmetry* is defined as a symmetry transformation group on  $(x, u, \partial u)$  whose generator corresponds to an infinitesimal transformation that preserves the contact relations  $u_i^\alpha = D_i u^\alpha$ . The set of all admitted point symmetries and contact symmetries for a given DE system comprises its group of *Lie symmetries*. The corresponding generators of this group comprise a Lie algebra [24,31,32].

A *local conservation law* of a given DE system (14) is a divergence equation

$$D_i C^i|_{\mathcal{E}} = 0 \quad (26)$$

which holds on the whole solution space  $\mathcal{E}$  of the system, where

$$C = (C^1(x, u, \partial u, \dots, \partial^r u), \dots, C^n(x, u, \partial u, \dots, \partial^r u)) \quad (27)$$

is the *conserved current vector*. In the case when one of the independent variables represents a time coordinate and the remaining  $n - 1$  independent variables represent space coordinates, namely  $x = (t, x^1, \dots, x^{n-1})$ , then  $C^1 = T$  is a *conserved density* and  $(C^2, \dots, C^n) = \vec{X}$  is a *spatial flux vector*, while the conservation law has the form of a local continuity equation  $(D_t T + \text{Div} \vec{X})|_{\mathcal{E}} = 0$ . (Similarly to the symmetry determining equation, the notation  $\mathcal{E}$  here means that the given DE system, as well as its differential consequences, are to be used).

A conservation law (26) is *locally trivial* if

$$C^i|_{\mathcal{E}} = D_j \Theta^{ij} \quad (28)$$

holds for some differential antisymmetric tensor function  $\Theta^{ij}(x, u, \partial u, \dots, \partial^{r-1} u)$  on  $\mathcal{E}$ , since any total curl is identically divergence free,  $D_i(D_j \Theta^{ij}) = D_i D_j \Theta^{ij} = 0$  due to the commutativity of total derivatives. Two conservation laws are said to be *locally equivalent* if, on the solution space  $\mathcal{E}$ , their conserved currents differ by a locally trivial current (28). The *differential order of a conservation law* is defined to be the smallest differential order among all locally equivalent conserved currents. (Sometimes a local conservation law is itself defined as the equivalence class of locally equivalent conserved currents).

For a given DE system (14), the set of all non-trivial local conservation laws (up to local equivalence) forms a vector space on which the local symmetries of the system have a natural action [24,31,33]. In particular, the infinitesimal action of a symmetry (15) on a conserved current (27) is given by [24]

$$C_{\mathbf{X}}^i = \text{pr}\mathbf{X}(C^i) + C^i D_i \zeta^i - C^j D_j \zeta^i. \quad (29)$$

When the symmetry is expressed in characteristic form (20), its action has the simple form

$$C_{\hat{\mathbf{X}}}^i = \text{pr}\hat{\mathbf{X}}(C^i) = \delta_P C^i. \quad (30)$$

The conserved currents  $C_{\mathbf{X}}^i$  and  $C_{\hat{\mathbf{X}}}^i$  are locally equivalent,

$$(C_{\hat{\mathbf{X}}}^i - C_{\mathbf{X}}^i)|_{\mathcal{E}} = D_j \Theta^{ij} \quad (31)$$

with

$$\Theta^{ij} = \zeta^i C^j - \zeta^j C^i \quad (32)$$

which follows from the relation (21).

A DE system is *variational* if it arises as the Euler–Lagrange equations of a local Lagrangian. This requires that the number of equations in the system is the same as the number of dependent

variables,  $M = m$ , and that the differential order  $N$  of the system is even, in which case the system is given by

$$F_\alpha = E_{u^\alpha}(L), \quad \alpha = 1, \dots, M = m \quad (33)$$

where the Lagrangian is a differential function

$$L(x, u, \partial u, \dots, \partial^{N/2} u). \quad (34)$$

The necessary and sufficient conditions [24,29,31,32] for a given DE system (14) to be variational consist of the Helmholtz conditions (13), which are given by

$$(\delta_w F)_\alpha = (\delta_w^* F)_\alpha \quad (35)$$

where  $w^\alpha$  is a set of arbitrary differential functions. Note that these conditions (35) are required to hold identically in jet space  $J$  (and not just on the solution space  $\mathcal{E}$  of the DE system).

## 2.2. Ibragimov's Conservation Law Formula

The starting point is the well-known observation [24] that any  $N$ -th-order system of  $M \geq 1$  DEs (14) can be embedded into a larger system by appending an "adjoint variable" for each DE in the system, where this set of  $M \geq 1$  variables  $v = (v^1, \dots, v^M)$  is taken to satisfy the adjoint of the linearization of the original DE system. Specifically, the enlarged DE system is given by

$$F_a(x, u, \partial u, \dots, \partial^N u) = 0, \quad a = 1, \dots, M \quad (36)$$

$$(\delta_v^* F)_\alpha = F_\alpha^*(x, u, v, \partial u, \partial v, \dots, \partial^N u, \partial^N v) = 0, \quad \alpha = 1, \dots, m \quad (37)$$

for  $u^\alpha(x)$  and  $v^\alpha(x)$ , in Ibragimov's notation. This system (36)–(37) comprises the Euler–Lagrange equations of the Lagrangian function

$$L = v^a F_a(x, u, \partial u, \dots, \partial^N u) \quad (38)$$

since, clearly,

$$E_{v^a}(L) = F_a, \quad E_{u^\alpha}(L) = (\delta_v^* F)_\alpha \quad (39)$$

through the product rule (9).

All solutions  $u(x)$  of the original DE system (36) give rise to solutions of the Euler–Lagrange system (39) by letting  $v(x)$  be any solution (for instance  $v = 0$ ) of the DEs (37). Conversely, all solutions  $(u(x), v(x))$  of the Euler–Lagrange system (39) yield solutions of the original DE system (36) by projecting out  $v(x)$ .

This embedding relationship can be used to show that every symmetry of the original DE system (36) can be extended to a variational symmetry of the Euler–Lagrange system (39). The proof is simplest when the symmetries are formulated in characteristic form (20).

Let

$$\hat{X} = P^\alpha(x, u, \partial u, \dots, \partial^r u) \partial / \partial u^\alpha \quad (40)$$

be any local symmetry generator (in characteristic form) admitted by the DE system (36). Under some mild regularity conditions [29] on the form of these DEs, the symmetry determining Equation (23) implies that the characteristic functions  $P^\alpha$  satisfy

$$(\delta_P F)_a = R_P(F)_a \quad (41)$$

where

$$R_P = R_P^b + R_P^{bi} D_i + R_P^{bij} D_i D_j + \dots + R_P^{bi_1 \dots i_r} D_{i_1} \dots D_{i_r} \quad (42)$$



is some linear differential operator whose coefficients  $R_{P_a}^b, R_{P_a}^{bi}, \dots, R_{P_a}^{bi_1 \dots i_r}$  are differential functions that are non-singular on solution space  $\mathcal{E}$  of the DE system (14). Now, consider the action of this symmetry generator (40) on the Lagrangian (38). From the operator relation (41) followed by integration by parts, the symmetry action is given by

$$\text{pr}\hat{\mathbf{X}}(L) = v^a R_P(F)_a = F_a R_P^*(v)^a + D_i \hat{\Theta}^i \tag{43}$$

where

$$R_P^* = R_{P_a}^{*b} - R_{P_a}^{*bi} D_i + R_{P_a}^{*bij} D_i D_j + \dots + (-1)^r R_{P_a}^{*bi_1 \dots i_r} D_{i_1} \dots D_{i_r} \tag{44}$$

is the adjoint of the operator (42), with the non-singular coefficients

$$\begin{aligned} R_{P_a}^{*b} &= R_{P_a}^b - (D_j R_{P_a}^{bj}) + \dots + (-1)^r (D_{j_1} \dots D_{j_r} R_{P_a}^{bj_1 \dots j_r}), \\ R_{P_a}^{*bi} &= R_{P_a}^{bi} - \binom{2}{1} (D_j R_{P_a}^{bji}) + \dots + (-1)^{r-1} \binom{r}{r-1} (D_{j_1} \dots D_{j_{r-1}} R_{P_a}^{bj_1 \dots j_{r-1}i}), \\ R_{P_a}^{*bij} &= R_{P_a}^{bij} - \binom{3}{1} (D_k R_{P_a}^{bkij}) + \dots + (-1)^{r-2} \binom{r}{r-2} (D_{j_1} \dots D_{j_{r-2}} R_{P_a}^{bj_1 \dots j_{r-2}ij}), \\ &\vdots \\ R_{P_a}^{*bi_1 \dots i_r} &= R_{P_a}^{bi_1 \dots i_r} D_{i_1} \dots D_{i_r}. \end{aligned} \tag{45}$$

Although the Lagrangian is not preserved, the expression (43) for the symmetry action shows that if the symmetry is extended to act on  $v$  via

$$\hat{\mathbf{X}}^{\text{ext.}} = P^\alpha \partial / \partial u^\alpha - R_P^*(v)^a \partial / \partial v^a, \tag{46}$$

then, under this extended symmetry, the Lagrangian will be invariant up to a total divergence,

$$\text{pr}\hat{\mathbf{X}}^{\text{ext.}}(L) = v^a R_P(F)_a - F_a R_P^*(v)^a = D_i \hat{\Theta}^i. \tag{47}$$

This completes the proof. A useful remark is that the vector  $\hat{\Theta}^i$  in the total divergence (47) is a linear expression in terms of  $F_a$  (and total derivatives of  $F_a$ ), and hence, this vector vanishes whenever  $u(x)$  is a solution of the DE system (36). Consequently,  $\hat{\Theta}^i$  is a trivial current for the Euler–Lagrange system (39).

Some minor remarks are that the proof given by Ibragimov [9] does not take advantage of the simplicity of working with symmetries in characteristic form and also glosses over the need for some regularity conditions on the DE system so that the symmetry operator relation (41) will hold. Moreover, that proof is stated only for DE systems in which the number of equations is the same as the number of dependent variables,  $M = m$ .

Now, since the extended symmetry (46) is variational, Noether’s theorem can be applied to obtain a corresponding conservation law for the Euler–Lagrange system (39), without the need for any additional conditions. The formula in Noether’s theorem comes from applying the variational identity (5) to the Lagrangian (38), which yields

$$\text{pr}\hat{\mathbf{X}}(L) = \phi^a F_a + v^a (\delta_P F)_a = \hat{\mathbf{X}}(v^a) E_{v^a}(L) + \hat{\mathbf{X}}(u^\alpha) E_{u^\alpha}(L) + D_i \Phi^i(P; L) \tag{48}$$

for any generator

$$\hat{\mathbf{X}} = P^\alpha \partial / \partial u^\alpha + \phi^a \partial / \partial v^a. \tag{49}$$



The total divergence term  $D_i\Phi^i(P;L)$  is given by the formula (8) derived using the Euler operator (6). This yields

$$\begin{aligned} \Phi^i(P;L) &= P^\alpha v^a \frac{\partial F_a}{\partial u_i^\alpha} + (D_j P^\alpha) v^a \frac{\partial F_a}{\partial u_{ji}^\alpha} - P^\alpha D_j \left( v^a \frac{\partial F_a}{\partial u_{ji}^\alpha} \right) + \dots \\ &+ \sum_{q=1}^N (-1)^{q-1} (D_{j_1} \dots D_{j_{N-q}} P^\alpha) D_{i_1} \dots D_{i_{q-1}} \left( v^a \frac{\partial F_a}{\partial u_{j_1 \dots j_{N-q} i_1 \dots i_{q-1} i}^\alpha} \right). \end{aligned} \tag{50}$$

When this variational identity (48) is combined with the action (47) of the variational symmetry (46) on the Lagrangian, the following Noether relation is obtained:

$$D_i(\hat{\Psi}^i - \Phi^i(P;L)) = \phi^a F_a + P^\alpha F_\alpha^*, \quad \phi^a = -R_p^*(v)^a \tag{51}$$

where  $F_\alpha^*$  is expression (37). Since  $F_a, F_\alpha^*$  and  $\hat{\Psi}^i$  vanish when  $(u(x), v(x))$  is any solution of the Euler–Lagrange system (39), the Noether relation (51) yields a local conservation law

$$D_i \hat{C}^i|_{\mathcal{E}(u,v)} = 0, \quad \hat{C}^i = \Phi^i(P;L) \tag{52}$$

where  $\mathcal{E}(u, v)$  denotes the solution space of the system (39) (including its differential consequences). This conservation law is locally equivalent to the conservation law formula underlying Ibragimov’s work [9,12], which is given by

$$D_i C^i|_{\mathcal{E}(u,v)} = 0, \quad C^i = \hat{C}^i - \zeta^i L \tag{53}$$

where  $C^i|_{\mathcal{E}(u)} = \hat{C}^i|_{\mathcal{E}(u)}$  since  $L|_{\mathcal{E}(u)} = 0$ . Strangely, nowhere does Ibragimov (or subsequent authors) point out that the term  $\zeta^i L$  in the conserved current trivially vanishes on all solutions  $(u(x), v(x))$  of the Euler–Lagrange system!

Hence, the following result has been established.

**Proposition 1.** Any DE system (36) can be embedded into a larger Euler–Lagrange system (39) such that every symmetry (40) of the original system can be extended to a variational symmetry (46) of the Euler–Lagrange system. Noether’s theorem then yields a conservation law (52) for all solutions  $(u(x), v(x))$  of the Euler–Lagrange system (39).

A side remark is that the locally equivalent conservation law (53) also can be derived from Noether’s theorem if the extended symmetry (46) is expressed in canonical form

$$\mathbf{X}^{\text{ext.}} = \zeta^i \partial / \partial x^i + \eta^\alpha \partial / \partial u^\alpha + (\zeta^i v_i^a - R_p^*(v)^a) \partial / \partial v^a \tag{54}$$

as obtained from relations (20)–(21). In particular, the corresponding form of the variational identity (48) becomes

$$\text{pr}\mathbf{X}^{\text{ext.}}(L) + (D_i \zeta^i) L = \hat{\mathbf{X}}(v^a) E_{v^a}(L) + \hat{\mathbf{X}}(u^\alpha) E_{u^\alpha}(L) + D_i(\zeta^i L + \Phi^i(P;L)) \tag{55}$$

where  $P^\alpha = \eta^\alpha - \zeta^i u_i^\alpha$  and  $Q^a = -R_p^*(v)^a$ , while the action of the symmetry (54) on the Lagrangian is given by

$$\begin{aligned} \text{pr}\mathbf{X}^{\text{ext.}}(L) + (D_i \zeta^i) L &= (D_i(\zeta^i v^a) - R_p^*(v)^a) F_a + v^a \text{pr}\mathbf{X}^{\text{ext.}}(F^a) \\ &= (D_i(\zeta^i v^a) - R_p^*(v)^a) F_a + v^a (R_p(F)^a + \zeta^i D_i F^a) = D_i \Theta^i \end{aligned} \tag{56}$$

where  $\Theta^i$  vanishes whenever  $u(x)$  is a solution of the DE system (36). Hence, the Noether relation obtained from combining Equations (55) and (56) yields the conserved current  $C^i = \Phi^i(P, v; F) - \zeta^i L$  modulo the locally trivial current  $\Theta^i$ . If the original symmetry (40) being used is a point symmetry,

then this trivial current  $\Theta^i$  can be shown to vanish identically, which is the situation considered in Ibragimov’s papers [9,12] and in nearly all subsequent applications in the literature.

### 2.3. “Nonlinear Self-Adjointness”

The conservation law (52) holds for all solutions  $(u(x), v(x))$  of the Euler–Lagrange system (39). It seems natural to restrict this to solutions of the original DE system (36) for  $u(x)$  by putting  $v = 0$ . However, the resulting conserved current is trivial,  $\Phi^i(P; L)|_{v=0} = \Phi^i(P; 0) = 0$ , because  $L$  is a linear expression in terms of  $v$ . Consequently, some other way must be sought to project the solution space  $\mathcal{E}(u, v)$  of the Euler–Lagrange system onto the solution space  $\mathcal{E}$  of the original DE system (36).

Ibragimov’s first paper [9] proposes to put  $v = u$ , which is clearly a significant restriction on the form of the original DE system (36). In particular, this requires that  $F_\alpha^*|_{v=u} = F_a$  hold identically, where the DE system is assumed to have the same number of equations as the number of dependent variables,  $M = m$ , which allows the indices  $a = \alpha$  to be identified. He calls such a DE system  $F_a = 0$  “strictly self-adjoint”. This definition is motivated by the case of a linear DE system, since linearity implies that  $(\delta_u F)_a = F_a$  and  $(\delta_u^* F)_\alpha = F_\alpha^*|_{v=u}$  are identities, whereby a linear DE system with  $M = m$  is “strictly self-adjoint” iff it satisfies  $(\delta F)_\alpha = (\delta^* F)_\alpha$ , which is the condition for the self-adjointness of a linear system. However, for nonlinear DE systems, the definition of “strictly self-adjoint” conflicts with the standard of definition [6,24] in variational calculus that a general DE system  $F_a = 0$  is self-adjoint iff its associated Fréchet derivative operator is self-adjoint,  $(\delta F)_a = (\delta^* F)_\alpha$ , which requires  $M = m$ .

Ibragimov subsequently [10] proposed to have  $v = \phi(u)$ , which he called “quasi-self-adjointness”. A more general proposal  $v = \phi(x, u)$  was then introduced first in Reference [14] and shortly later appears in Ibragimov’s next paper [12], with the condition that  $F_\alpha^*|_{v=\phi(x,u)} = \lambda_\alpha^\beta F_\beta$  must hold for some coefficients  $\lambda_\alpha^\beta$ , again with  $M = m$ . This condition is called “weak self-adjointness” in Reference [14] and “nonlinear self-adjointness” in Reference [12]. Ibragimov also mentions an extension of this definition to  $v = \phi(x, u, \partial u, \dots, \partial^s u)$ , but does not pursue it. Later, he applies this definition in Reference [13] to a specific PDE, where  $\lambda_\alpha^\beta$  is extended to be a linear differential operator. However, unlike in the previous papers, no conservation laws are found from using this extension. A subsequent paper [15] then uses this extension, which is called “nonlinear self-adjointness through a differential substitution”, to obtain conservation laws for several similar PDEs. Finally, the same definition is stated more generally in Reference [17] for DE systems with  $M = m$ :

$$F_\alpha^*|_{v=\phi(x,u,\partial u,\dots,\partial^s u)} = \lambda_\alpha^\beta F_\beta + \lambda_\alpha^{\beta i} D_i F_\beta + \dots + \lambda_\alpha^{\beta i_1 \dots i_p} D_{i_1} \dots D_{i_p} F_\beta \tag{57}$$

where the coefficients  $\lambda_\alpha^\beta, \lambda_\alpha^{\beta i}, \dots, \lambda_\alpha^{\beta i_1 \dots i_p}$  are differential functions.

These developments lead to the following conservation law theorem, which is a generalization of Ibragimov’s main theorem [9,12] to arbitrary DE systems (not restricted by  $M = m$ ), combined with the use of a differential substitution [12,15,17].

**Theorem 1.** *Suppose a system of DEs (14) satisfies*

$$F_\alpha^*|_{v=\phi} = \lambda_\alpha^a F_a + \lambda_\alpha^{ai} D_i F_a + \dots + \lambda_\alpha^{ai_1 \dots i_p} D_{i_1} \dots D_{i_p} F_a \tag{58}$$

for some differential functions  $\phi^a(x, u, \partial u, \dots, \partial^s u)$  and  $\lambda_\alpha^a(x, u, \partial u, \dots, \partial^s u)$ ,  $\lambda_\alpha^{ai}(x, u, \partial u, \dots, \partial^s u), \dots, \lambda_\alpha^{ai_1 \dots i_p}(x, u, \partial u, \dots, \partial^s u)$  that are non-singular on the solution space  $\mathcal{E}$  of the DE system, where  $F_\alpha^*$  is the adjoint linearization (2) of the system. Then, any local symmetry

$$\mathbf{X} = \zeta^i(x, u, \partial u, \dots, \partial^r u) \partial / \partial x^i + \eta^\alpha(x, u, \partial u, \dots, \partial^r u) \partial / \partial u^\alpha \tag{59}$$

admitted by the DE system yields a local conservation law (26) given in an explicit form by the conserved current (50) with  $v^a = \phi^a$  and  $P^\alpha = \eta^\alpha - \zeta^i u_i^\alpha$ .

An important remark is that all of the functions  $\phi^a, \lambda_\alpha^a, \lambda_\alpha^{ai}, \dots, \lambda_\alpha^{ai_1 \dots i_p}$  must be non-singular on  $\mathcal{E}$ , as otherwise, the condition (58) can be satisfied in a trivial way. This point is not mentioned in any of the previous work [9,12,14,15,17].

The “nonlinear self-adjointness” condition (58) turns out to have a simple connection to the determining equations for symmetries. This connection is somewhat obscured by the unfortunate use of non-standard definitions and non-standard notation in References [9,12]. Nevertheless, it is straightforward to show that Equation (58) is precisely the adjoint of the determining Equation (23) for symmetries formulated as an operator Equation (41).

#### 2.4. Adjoint-Symmetries and a Formula for Generating Conservation Laws

For any given DE system (14), the adjoint of the symmetry determining Equation (23) is given by

$$(\delta_Q^* F)_\alpha|_{\mathcal{E}} = 0 \quad (60)$$

for a set of differential functions  $Q^\alpha(x, u, \partial u, \dots, \partial^r u)$ . (Similarly to the symmetry determining equation, the notation  $\mathcal{E}$  here means that the given DE system, as well as its differential consequences, are to be used). These differential functions are called an *adjoint-symmetry* [4], in analogy to the characteristic functions of a symmetry (40), and so, Equation (60) is called the *adjoint-symmetry determining equation*. As shown in Reference [29], this analogy has a concrete geometrical meaning in the case when a DE system is an evolutionary system  $F_\alpha = u_t^\alpha - f_\alpha(x, u, \partial_x u, \dots, \partial_x^N u) = 0$  with  $M = m$  and  $x = (t, x^1, \dots, x^{n-1})$ , where  $t$  is a time coordinate and  $x^i, i = 1, \dots, n-1$ , are space coordinates. In this case,  $Q^\alpha$  can be viewed as the coefficients of a one-form or a covector  $Q^\alpha \mathbf{d}u^\alpha$ , in analogy to  $P^\alpha$  being the coefficients of a vector  $P^\alpha \partial / \partial u^\alpha$ . The condition for  $P^\alpha \partial / \partial u^\alpha$  to be a symmetry can be formulated as  $(\mathcal{L}_f P^\alpha \partial / \partial u^\alpha)|_{\mathcal{E}} = 0$  where  $\mathcal{L}_f$  denotes the Lie derivative [24,29] with respect to the time evolution vector  $\hat{X} = f_\alpha \partial / \partial u^\alpha$ . Then, the condition for  $Q^\alpha \mathbf{d}u^\alpha$  to be an adjoint-symmetry is equivalent to  $(\mathcal{L}_f Q^\alpha \mathbf{d}u^\alpha)|_{\mathcal{E}} = 0$ . (Note the awkwardness in the index positions here comes from Ibragimov’s choice of index placement  $F_\alpha$  for a DE system with  $M = m$ . A better notation would be  $F^\alpha$  and  $F^a$  when  $M \neq m$ , which is used in References [4,26,27,31].)

In the case when a DE system is variational (33), the symmetry determining equation is self-adjoint, since  $(\delta_Q^* F)_\alpha = (\delta_Q F)_\alpha$ . Then, the adjoint-symmetry determining Equation (60) reduces to the symmetry determining Equation (23), with  $Q^\alpha(x, u, \partial u, \dots, \partial^r u) = P^\alpha(x, u, \partial u, \dots, \partial^r u)$ , where the indices  $a = \alpha$  can be identified, due to  $M = m$ . Consequently, adjoint-symmetries of any variational DE system are the same as symmetries.

Other aspects of adjoint-symmetries and their connection to symmetries are discussed in Reference [34].

Now, under some mild regularity conditions [29] on the form of a general DE system (14), the adjoint-symmetry determining Equation (60) implies that the functions  $Q^\alpha$  satisfy

$$(\delta_Q^* F)_\alpha = R_Q(F)_\alpha \quad (61)$$

where

$$R_Q = R_{Q_\alpha}^b + R_{Q_\alpha}^{bi} D_i + R_{Q_\alpha}^{bij} D_i D_j + \dots + R_{Q_\alpha}^{bi_1 \dots i_r} D_{i_1} \dots D_{i_r} \quad (62)$$

is some linear differential operator whose coefficients  $R_{Q_\alpha}^b, R_{Q_\alpha}^{bi}, \dots, R_{Q_\alpha}^{bi_1 \dots i_r}$  are differential functions that are non-singular on the solution space  $\mathcal{E}$  of the DE system (14). In Ibragimov’s notation  $F_\alpha^*(x, u, v, \partial u, \partial v, \dots, \partial^N u, \partial^N v) = (\delta_v^* F)_\alpha$ , the adjoint-symmetry Equation (61) coincides with the “nonlinear self-adjointness” condition (58) in Theorem 1, where the operator on the right-hand side of Equation (58) is precisely the adjoint-symmetry operator (62).

Therefore, the following equivalence has been established.

**Proposition 2.** For a general DE system (14), the condition (58) of “nonlinear self-adjointness” coincides with the condition of existence of an adjoint-symmetry (60). When a DE system is variational (33), these conditions reduce to the condition of the existence of a symmetry.

One remark is that the formulation of “nonlinear self-adjointness” given here is more general than what appears in References [12,15,17] since those formulations assume that the DE system has the same number of equations as the number of dependent variables,  $M = m$ . Another remark is that the meaning of “nonlinear self-adjointness” shown here in the case of variational DE systems has not previously appeared in the literature.

Example: Consider the class of semilinear wave equations  $u_{tt} - u_{xx} + a(u)(u_t^2 - u_x^2) + b(u)u_t + c(u)u_x + m(u) = 0$  for  $u(t, x)$ , with a nonlinearity coefficient  $a(u)$ , damping coefficients  $b(u), c(u)$  and a mass-type coefficient  $m(u)$ . In Reference [20], the conditions under which a slightly more general family of wave equations is “nonlinearly self-adjoint” (58) are stated for  $v = \phi(u)$ . These results will be generalized here by considering  $v = \phi(t, x, u)$ . A first observation is that this class of wave equations admits an equivalence transformation  $u \rightarrow \tilde{u} = f(u)$ , with  $f' \neq 0$ , which can be used to put  $a = 0$  by  $f(u) = \int \exp(A(u))du$  where  $A' = a$ . (Equivalence transformations were not considered in Reference [20], and so, their results are considerably more complicated than is necessary). This transformation gives

$$u_{tt} - u_{xx} + b(u)u_t + c(u)u_x + m(u) = 0. \quad (63)$$

In Ibragimov’s notation, the condition of “nonlinear self-adjointness” with  $v = \phi(t, x, u)$  is given by  $0 = E_u(vF)|_{v=\phi}$  where

$$F = u_{tt} - u_{xx} + b(u)u_t + c(u)u_x + m(u). \quad (64)$$

This yields

$$(D_t^2\phi - D_x^2\phi - bD_t\phi - cD_x\phi + m'\phi)|_{F=0} = 0. \quad (65)$$

For comparison, the determining Equation (23) for local symmetries  $\hat{X} = P(t, x, u, u_t, u_x, \dots)\partial/\partial u$  (in characteristic form) is given by

$$(D_t^2P - D_x^2P + bD_tP + cD_xP + (u_t b' + u_x c' + m')P = 0)|_{F=0} = 0. \quad (66)$$

Its adjoint is obtained by multiplying by  $Q(t, x, u, u_t, u_x, \dots)$  and integrating by parts, which yields  $(D_t^2Q - D_x^2Q - D_t(bQ) - D_x(cQ) + (u_t b' + u_x c' + m')Q = 0)|_{F=0} = 0$ . After the  $D_x$  terms are expanded out, this gives the determining Equation (60) for local adjoint-symmetries

$$(D_t^2Q - D_x^2Q - bD_tQ - cD_xQ + (b'u_t + c'u_x + m')Q)|_{F=0} = 0 \quad (67)$$

which coincides with the “nonlinear self-adjointness” condition (65) extended to differential substitutions [12,14,16] given by  $v = Q(t, x, u, u_t, u_x, \dots)$ . All adjoint-symmetries of lowest-order form  $Q(t, x, u)$  can be found in a straightforward way. After  $Q(t, x, u)$  is substituted into the determining Equation (67) and  $u_{tt}$  is eliminated through the wave Equation (63), the determining equation splits with respect to the variables  $u_t$  and  $u_x$ , yielding a linear overdetermined system of four equations (after some simplifications):

$$Q_{tt} - Q_{xx} - bQ_t - cQ_x - mQ_u + m'Q = 0, \quad (68)$$

$$Q_{tu} - bQ_u = 0, \quad Q_{xu} + cQ_u = 0, \quad (69)$$

$$Q_{uu} = 0. \quad (70)$$

It is straightforward to derive and solve this determining system by Maple. Hereafter, the conditions

$$b' \neq 0, \quad c' \neq 0, \quad m'' \neq 0, \quad m(0) = 0 \quad (71)$$

will be imposed, which corresponds to studying wave equation (63) whose lower-order terms are nonlinear and homogeneous. The general solution of the determining system (68)–(70) then comprises three distinct cases (as obtained using the Maple package ‘rifsimp’), after merging. This leads to the following complete classification of solution cases shown in Table 1. The table is organized by listing each solution  $Q$  and the conditions on  $b, c, m$  for which it exists. (From these conditions, a classification of maximal linear spaces of multipliers can be easily derived). Note that if the transformation  $u \rightarrow \tilde{u} = \int \exp(A(u))du$  is inverted, then  $Q$  transforms to  $\tilde{Q} = \exp(A(u))Q$ . (Also note that, under the restriction  $Q = \phi(u)$  considered in Reference [20], the classification reduces to just the first case with  $m = \text{const.}$  and  $Q = 1$ ).

**Table 1.** Adjoint-symmetries (“nonlinear self-adjointness”).

$Q(t, x, u)$	$b(u)$	$c(u)$	$m(u)$	Conditions
$e^{m_2 t + m_3 x}$	arb.	arb.	$m_1 u + \int (m_2 b + m_3 c) du$	$m_1 = m_3^2 - m_2^2$
$e^{\alpha x + \beta t}$	$b_0 + b_1 m'$	$c_0 + c_1 m'$	arb.	$b_1 \beta + c_1 \alpha = 1$ $\beta(\beta - b_0) = \alpha(\alpha + c_0)$
$e^{\gamma x} q(x \mp t)$	$b_0 + b_1 m'$	$c_0 + c_1 m'$	arb.	$\gamma = \pm b_0 = -c_0,$ $b_1 = 1/b_0, c_1 = -1/c_0,$ $q(\xi) = \text{arb.}$

The Fréchet derivative operator in the symmetry determining Equation (23) and the adjoint of this operator in the adjoint-symmetry determining Equation (60) are related by the integration-by-parts formula (3). For a general DE system (14), this formula is given by

$$Q^a (\delta_P F)_a - P^\alpha (\delta_Q^* F)_\alpha = D_i \Psi^i(P, Q; F) \quad (72)$$

where the vector  $\Psi^i(P, Q; F)$  is given by the explicit expression (4) with  $v = Q$ ,  $w = P$ , and  $f = F$ . As shown in References [2–4], this vector  $\Psi^i(P, Q; F)$  will be a conserved current

$$D_i \Psi^i(P, Q; F)|_{\mathcal{E}} = 0 \quad (73)$$

whenever the differential functions  $P^\alpha$  and  $Q^a$  respectively satisfy the symmetry and adjoint-symmetry determining equations. Moreover, it is straightforward to see

$$\Psi^i(P, Q; F) = \Phi^i(P; L)|_{v=Q}, \quad (74)$$

which follows from relation (7), where  $\Phi^i(P; L)$  is the Noether conserved current (50) and  $L$  is the Lagrangian (38). Alternatively, the equality (74) can be derived indirectly by applying formula (72) to the variational identity (48) with  $v = Q$ , giving

$$\begin{aligned} \text{pr}\hat{\mathbf{X}}(L) &= \phi^a F_a + v^a (\delta_P F)_a = \phi^a F_a + P^\alpha (\delta_Q^* F)_\alpha + D_i \Psi^i(P, v; F) \\ &= \hat{\mathbf{X}}(v^a) E_{v^a}(L) + \hat{\mathbf{X}}(u^\alpha) E_{u^\alpha}(L) + D_i \Psi^i(P, v; F) \end{aligned} \quad (75)$$

which implies  $\Psi^i(P, v; F) = \Phi^i(P; L)$  holds (up to the possible addition of a total curl).

When the relation (74) is combined with Propositions 1 and 2, the following main result is obtained.

**Theorem 2.** For any DE system (14) admitting an adjoint-symmetry (60) (namely, a “nonlinearly self-adjoint system” in the general sense), the conserved current (50) derived from applying Noether’s theorem to the extended Euler–Lagrange system (39) using any given symmetry (46) is equivalent to the conserved current obtained using the adjoint-symmetry/symmetry formula (72).

This theorem shows that the “nonlinear self-adjointness” method based on Ibragimov’s theorem as developed in papers [9,12,14,15,17] for DE systems with  $M = m$  is just a special case of the adjoint-symmetry/symmetry formula (72) introduced for general DE systems in prior papers References [2–4], which were never cited. Moreover, the adjoint-symmetry/symmetry formula (72) has the advantage that there is no need to extend the given DE system by artificially adjoining variables to get a Euler–Lagrange system.

Another major advantage of the adjoint-symmetry/symmetry formula is that it can be used to show how the resulting local conservation laws are, in general, not necessarily non-trivial and comprise only a subset of all of the non-trivial local conservation laws admitted by a given DE system. In particular, in many applications of Theorem 1, it is found that some non-trivial symmetries, particularly translation symmetries, only yield trivial conservation laws [17–19], and that some local conservation laws are not produced even when all admitted symmetries are used. These observations turn out to have a simple explanation through the equivalence of Theorem 1 and the adjoint-symmetry/symmetry formula (72), as explained in the next section.

Example: For the semilinear wave Equation (63), the extended Euler–Lagrange system in Ibragimov’s notation consists of

$$F = u_{tt} - u_{xx} + b(u)u_t + c(u)u_x + m(u) = \frac{\delta L}{\delta v} = 0, \quad (76)$$

$$F^* = v_{tt} - v_{xx} - bv_t - cv_x + (b'u_t + c'u_x + m')v = \frac{\delta L}{\delta u} = 0, \quad (77)$$

where  $F^*$  is defined by the adjoint-symmetry Equation (67) with  $Q = v$ , and where the Lagrangian (38) is simply  $L = vF = v(u_{tt} - u_{xx} + b(u)u_t + c(u)u_x + m(u))$  in terms of the variables  $u$  and  $v$ . Consider any point symmetry of the wave Equation (76) for  $u$ , given by a generator

$$\mathbf{X} = \tau(t, x, u)\partial/\partial t + \zeta(t, x, u)\partial/\partial x + \eta(t, x, u)\partial/\partial u. \quad (78)$$

Its equivalent characteristic form is  $\hat{\mathbf{X}} = P\partial/\partial u$ , with  $P = \eta - \tau u_t - \zeta u_x$  satisfying the symmetry determining Equation (66) on the space of solutions  $u(x)$  of the wave Equation (76). Every point symmetry can be extended to a variational symmetry (54) admitted by the Euler–Lagrange system, which is given by the generator  $\mathbf{X}^{\text{ext}} = \mathbf{X} + (\tau v_t + \zeta v_x - R_p^*(v))\partial/\partial v$  where  $R_p^*$  is the adjoint of the operator  $R_p$  defined by relation (41) for the point symmetry holding off of the solution space of the wave Equation (76). In particular,  $R_p$  can be obtained by a straightforward computation of  $\delta_p F = R_p(F)$ , where the terms in  $\delta_p F$  are simplified by using the equations  $\tau_u = \zeta_u = 0$ ,  $\tau_t = \zeta_x$  and  $\tau_x = \zeta_t$  that arise from splitting the determining Equation (66). This yields

$$R_p = -\tau D_t - \zeta D_x + \eta_u - (\tau_t + \zeta_x), \quad (79)$$

and thus

$$R_p^* = \tau D_t + \zeta D_x + \eta_u. \quad (80)$$

Hence, the variational symmetry is simply

$$\mathbf{X}^{\text{ext}} = \tau\partial/\partial t + \zeta\partial/\partial x + \eta\partial/\partial u - \eta_u v\partial/\partial v \quad (81)$$

which is a point symmetry.

The action of this variational symmetry on the Lagrangian  $L = vF$  is given by

$$\text{pr}\mathbf{X}^{\text{ext}}(L) = -\eta_u vF + v\text{pr}\mathbf{X}(F) = -(\tau_t + \zeta_x)vF = -(D_t\tau + D_x\zeta)L \quad (82)$$



since  $\text{pr}\mathbf{X}(F) = \tau D_t F + \xi D_x F + R_P(F) = (\eta_u - (\tau_t + \xi_x))F$ . This symmetry action then can be combined with the variational identity (55) to get the Noether relation

$$D_t(\tau L + \Phi^t(P; L)) + D_x(\tau L + \Phi^x(P; L)) = -\hat{\mathbf{X}}^{\text{ext.}}(v)F - \hat{\mathbf{X}}^{\text{ext.}}(u)F^* \quad (83)$$

using  $F = E_v(L)$  and  $F^* = E_u(L)$ , where

$$\Phi^t(P; L) = v D_t P (b(u)v - v_t)P, \quad \Phi^x(P; L) = -v D_x P + (c(u)v + v_x)P \quad (84)$$

are obtained from formula (50). This yields a conservation law

$$(D_t C^t + D_x C^x)|_{\mathcal{E}(u,v)} = 0, \quad C^t = \Phi^t(P; L) - \tau L, \quad C^x = \Phi^x(P; L) - \xi L \quad (85)$$

on the solution space  $\mathcal{E}(u, v)$  of the Euler-Lagrange system  $F = 0, F^* = 0$ . Since  $L|_{\mathcal{E}(u,v)} = 0$ , this conservation law is locally equivalent to the conservation law (52) which is given by

$$(D_t \hat{C}^t + D_x \hat{C}^x)|_{\mathcal{E}(u,v)} = 0, \quad \hat{C}^t = \Phi^t(P; L), \quad \hat{C}^x = \Phi^x(P; L). \quad (86)$$

Moreover, from the identity (72) relating the symmetry Equation (66) and the adjoint-symmetry Equation (67), the conserved current  $(\hat{C}^t, \hat{C}^x)$  in the conservation law (86) is the same as the conserved current  $(\Psi^t, \Psi^x)$  in the adjoint-symmetry/symmetry formula

$$\Psi^t(P, Q; F)|_{Q=v} = \Phi^t(P; L), \quad \Psi^x(P, Q; F)|_{Q=v} = \Phi^x(P; L) \quad (87)$$

where

$$\begin{aligned} \Psi^t(P, Q; F) &= Q D_t P + (b(u)Q - D_t Q)P, \\ \Psi^x(P, Q; F) &= -Q D_x P + (c(u)Q + D_x Q)P. \end{aligned} \quad (88)$$

In Reference [20], the conservation law formula (85) is used to obtain a single local conservation law for a special case of the wave Equation (63) given by  $b = -c = -\ln(u)$  and  $d = 0$ , corresponding to  $\tilde{u}_{tt} - \tilde{u}_{xx} - (\tilde{u}_t^2 - \tilde{u}_x^2) + \tilde{u}(\tilde{u}_t - \tilde{u}_x) = 0$  after an equivalence transformation  $u \rightarrow \tilde{u} = e^{-u}$  is made. The formula is applied to the adjoint-symmetry  $\tilde{Q} = e^{-\tilde{u}}$  and the point symmetry  $\tilde{\mathbf{X}} = e^{(t+x)/2} \partial / \partial \tilde{u}$  with characteristic  $\tilde{P} = e^{(t+x)/2}$ , which respectively correspond to  $Q = 1$  and  $\mathbf{X} = e^{(t+x)/2} u \partial / \partial u$  with  $P = e^{(t+x)/2} u$ . The likely reason why the obvious translation symmetries  $\tilde{\mathbf{X}} = \partial / \partial t$  and  $\tilde{\mathbf{X}} = \partial / \partial x$  were not considered in Reference [20] is that these symmetries lead to locally trivial conservation laws when  $\tilde{Q} = e^{-\tilde{u}}$  is used.

To illustrate the situation, consider the translation symmetries

$$\mathbf{X}_1 = \partial / \partial t, \quad \mathbf{X}_2 = \partial / \partial x \quad (89)$$

admitted by the wave Equation (63) for arbitrary  $b(u), c(u), m(u)$ . The characteristic functions of these two symmetries are, respectively,  $P = -u_t$  and  $P = -u_x$ . Local conservation laws can be obtained by applying formula (85), or its simpler equivalent version (86), with  $v = Q(t, x, u)$  being the adjoint-symmetries classified in Table 1. The resulting conserved currents  $(\Psi^t, \Psi^x)$ , modulo locally trivial currents, are shown in Table 2.



**Table 2.** Conserved currents from the adjoint-symmetry/symmetry formula.

Conditions	Q	X = ∂/∂t	X = ∂/∂x
		Ψ <sup>t</sup> , Ψ <sup>x</sup>	Ψ <sup>t</sup> , Ψ <sup>x</sup>
$m = m_1u + m_2 \int b \, du$ $+ m_3 \int c \, du$ $m_1 = m_3^2 - m_2^2$	$e^{m_3x+m_2t}$	$m_2Q(u_t - m_2u + \int b \, du),$ $m_2Q(m_3u - u_x + \int c \, du)$	$m_3Q(u_t - m_2u + \int b \, du),$ $m_3Q(m_3u - u_x + \int c \, du)$
$b = b_0 + b_1m'$ $c = c_0 + c_1m'$ $b_1\beta + c_1\alpha = 1$ $\beta(\beta - b_0) = \alpha(\alpha + c_0)$	$e^{\alpha x + \beta t}$	$\beta Q(u_t - \beta u + \int b \, du),$ $\beta Q(\alpha u - u_x + \int c \, du)$	$\alpha Q(u_t - \beta u + \int b \, du),$ $\alpha Q(\alpha u - u_x + \int c \, du)$
$b = \pm(\gamma + \frac{1}{\gamma}m')$ $c = -\gamma + \frac{1}{\gamma}m'$	$e^{\gamma x}q(x \mp t)$	$-e^{\gamma x}(q''u \pm q'(u_t + \int b \, du)),$ $\pm e^{\gamma x}((\gamma q' - q'')u$ $+ q(u_x \mp \int b \, du))$	$e^{\gamma x}(\pm(q'' + \gamma q')u$ $+ (q' + \gamma q)(u_t + \int b \, du)),$ $e^{\gamma x}((q'' - \gamma^2 q)u$ $-(q' + \gamma q)(u_x \mp \int b \, du))$

Notice that for  $Q = \text{const.}$  the conserved currents  $(\Psi^t, \Psi^x)$  obtained from the two translation symmetries vanish. This implies that Ibragimov’s theorem (85) yields just trivial conserved currents  $(\Psi^t, \Psi^x)$  for some cases of the wave Equation (63) when a non-trivial conserved current exists. A full explanation of why this occurs will be given in the next section.

### 3. Properties of Conservation Laws Generated by the Adjoint-Symmetry/Symmetry Formula and Ibragimov’s Theorem

To determine when a conserved current is locally trivial or when two conserved currents are locally equivalent, it is useful to have a characteristic (canonical) form for local conservation laws, in analogy to the characteristic form for local symmetries.

Any local conservation law (26) can be expressed as a divergence identity [24]

$$D_i C^i = R_C^a F_a + R_C^{ai} D_i F_a + \dots + R_C^{ai_1 \dots i_r} D_{i_1} \dots D_{i_r} F_a \tag{90}$$

by moving off of the solution space  $\mathcal{E}$  of the system, where  $R_C^a, R_C^{ai}, \dots, R_C^{ai_1 \dots i_r}$  are some differential functions that are non-singular on  $\mathcal{E}$ , under some mild regularity conditions [29] on the form of the DEs (14). Integration by parts on the terms on the right-hand side in this identity (90) then yields

$$D_i \tilde{C}^i = Q_C^a F_a \tag{91}$$

with

$$Q_C^a = R_C^a - D_i R_C^{ai} + \dots + (-1)^r D_{i_1} \dots D_{i_r} R_C^{ai_1 \dots i_r}, \tag{92}$$

where

$$\tilde{C}^i|_{\mathcal{E}} = C^i|_{\mathcal{E}} \tag{93}$$

reduces to the conserved vector in the given conservation law (26). Hence,

$$(D_i \tilde{C}^i)|_{\mathcal{E}} = 0 \tag{94}$$

is a locally equivalent conservation law. The identity (91) is called the *characteristic equation* [24] for the conservation law (26), and the set of differential functions (92) is called the *conservation law multiplier* [24]. In general, a set of functions  $f^a(t, x, u, \partial u, \partial^2 u, \dots, \partial^s u)$  will be a multiplier if it is non-singular on  $\mathcal{E}$  and its summed product with the DEs  $F_a$  in the system has the form of a total divergence.

For a given local conservation law, the multiplier arising from the integration by parts formula (92) will be unique iff the coefficient functions in the characteristic Equation (90) are uniquely determined by the conserved vector  $C^i$ . This uniqueness holds straightforwardly for any DE system consisting of a single equation that can be expressed in a solved form for a leading derivative [29]. For DE systems containing more than one equation, some additional technical requirements are necessary [24]. In particular, it is necessary that a DE system have no differential identities [24], and it is sufficient that a DE system have a generalized Cauchy–Kovalevskaya form [24–27]. A concrete necessary and sufficient condition, which leads to the following uniqueness result, is stated in Reference [29].

**Proposition 3.** *For any closed DE system (14) having a solved form in terms of leading derivatives and having no differential identities, a conserved current is locally trivial (28) iff its corresponding multiplier (92) vanishes when evaluated on the solution space of the system.*

This class of DE systems includes nearly all systems of physical interest, apart from systems such as the Maxwell equations and the incompressible fluid equations, which possess differential identities. Often the distinction between systems with and without differential identities is overlooked in the literature on conservation law multipliers.

The importance of Proposition 3 is that, in a wide class of DE systems, it establishes that a unique characteristic form for locally equivalent conservation laws is provided by multipliers. From this result, it is now straightforward to derive a simple condition to detect when a local conservation law given by the adjoint-symmetry/symmetry formula (72) is locally trivial (28).

Let  $P^\alpha(x, u, \partial u, \dots, \partial^r u)$  be the characteristic functions defining a symmetry (23), and let  $Q^a(x, u, \partial u, \dots, \partial^s u)$  be a set of differential functions defining an adjoint-symmetry (60). Then, the adjoint-symmetry/symmetry formula (72) yields a local conservation law (73). The characteristic equation of this conservation law is given by substituting the symmetry identity (41) and the adjoint-symmetry identity (61) into the formula (72) to get

$$D_i \Psi^i(P, Q; F) = Q^a R_P(F)_a - P^\alpha R_Q(F)_\alpha. \quad (95)$$

Integration by parts gives

$$D_i \tilde{\Psi}^i(P, Q; F) = (R_P^*(Q)^a - R_Q^*(P)^a) F_a \quad (96)$$

where

$$\tilde{\Psi}^i(P, Q; F)|_{\mathcal{E}} = \Psi^i(P, Q; F)|_{\mathcal{E}}. \quad (97)$$

Hence, the conservation law multiplier is given by [35]

$$Q_\Psi^a = R_P^*(Q)^a - R_Q^*(P)^a. \quad (98)$$

This yields the following result.

**Proposition 4.** *The adjoint-symmetry/symmetry formula (72) for a given DE system (14) produces a locally trivial conservation law if the condition*

$$(R_P^*(Q)^a - R_Q^*(P)^a)|_{\mathcal{E}} = 0 \quad (99)$$

holds for the given symmetry and adjoint-symmetry pair, where  $P^\alpha(x, u, \partial u, \dots, \partial^r u)$  is the set of characteristic functions of the symmetry (23) and  $Q^a(x, u, \partial u, \dots, \partial^s u)$  is the set of functions defining the adjoint-symmetry (60). This condition (99) is also sufficient whenever the DE system (14) belongs to the class stated in Proposition 3.

Through the equivalence stated in Theorem 2, which relates the adjoint-symmetry/symmetry formula (72) and the generalized version of Ibragimov's conservation law formula in Theorem 1, it follows that "nonlinear self-adjointness" through a differential substitution with  $v = Q$  produces a conservation law (50) that is locally trivial when  $Q^a(x, u, \partial u, \dots, \partial^s u)$  and  $P^a(x, u, \partial u, \dots, \partial^r u)$  satisfy condition (99).

A useful remark is that the triviality condition (99) can be checked directly, without the need to derive the local conservation law itself.

Example: For the semilinear wave Equation (63), consider the conserved currents obtained in Table 2, which are generated from the three adjoint-symmetries

$$Q_1 = e^{m_2 t + m_3 x}, \quad Q_2 = e^{\alpha x + \beta t}, \quad Q_3 = e^{\gamma x} q(x \mp t), \quad (100)$$

and the two translation symmetries (89). The operators  $\delta_P F = R_P(F)$  associated with the characteristics

$$P_1 = -u_t, \quad P_2 = -u_x \quad (101)$$

of these two symmetries (89) are given by the formula (79), which yields

$$R_{P_1} = -D_t, \quad R_{P_2} = -D_x. \quad (102)$$

For adjoint-symmetries of the form  $Q(t, x, u)$ , the operator  $\delta_Q^* F = R_Q(F)$  is easily found to be

$$R_Q = Q_u. \quad (103)$$

Hence, the operators associated with the three adjoint-symmetries (100) are simply

$$R_{Q_1} = R_{Q_2} = R_{Q_3} = 0. \quad (104)$$

The triviality condition (99) is then given by

$$R_{P_1}^*(Q_l) - R_{Q_l}^*(P_1) = D_t Q_l = 0, \quad R_{P_2}^*(Q_l) - R_{Q_l}^*(P_2) = D_x Q_l = 0, \quad l = 1, 2, 3. \quad (105)$$

This shows that the two conserved currents obtained from  $Q_1$  will be trivial when  $m_2 = 0$  and  $m_3 = 0$  hold, respectively, and that likewise, the two conserved currents obtained from  $Q_2$  will be trivial when  $\beta = 0$  and  $\alpha = 0$  hold, respectively. Similarly, for  $Q_3$ , the first conserved current will be trivial when  $q' = 0$  holds, while the second conserved current will be trivial when  $q' + \gamma q = 0$  holds, corresponding to  $q = e^{-\gamma(x \pm t)}$ . These trivial cases can be seen to occur directly from the explicit expressions for the conserved currents  $(\Psi^t, \Psi^x)$  in Table 2.

In general, while the adjoint-symmetry/symmetry formula (72) (and, hence, Ibragimov's theorem) looks very appealing, it has major drawbacks that in many examples [14–17,20–22] the selection of a symmetry must be fitted to the form of the adjoint-symmetry to produce a non-trivial conservation law, and that in other examples [17–19] no non-trivial conservation laws are produced when only translation symmetries are available. More importantly, it is *not* (as is sometimes claimed) a generalization of Noether's theorem to non-variational DE systems.

As a reinforcement of these statements, consider the situation of variational DE systems, where adjoint-symmetries coincide with symmetries. Then, the adjoint-symmetry/symmetry formula (72) produces a conserved current directly from any pair of symmetries admitted by a given variational DE system. However, from Noether's theorem, this conserved current must also arise directly from some variational symmetry of the system. Moreover, if the pair of symmetries being used are variational symmetries that happen to commute with each other, then the resulting conserved current turns out to be trivial, as shown in Reference [36].

To understand these aspects and other properties of the formula, the determining equations for multipliers are needed.

### 3.1. Multiplier Determining equations

All conservation law multipliers for any given DE system can be determined from the property (10) and (11) that a differential function is a total divergence iff it is annihilated by the Euler operator (6). Specifically, when this property is applied directly to the characteristic Equation (91) for local conservation laws, it yields the determining equations

$$E_{u^\alpha}(Q_C^a F_a) = 0, \quad \alpha = 1, \dots, m \tag{106}$$

which are necessary and sufficient [24] for a set of differential functions  $Q_C^a(x, u, \partial u, \dots, \partial^s u)$  to be a multiplier for a local conservation law (26). Note the Equation (106) must hold identically in jet space (and not just on the solution space  $\mathcal{E}$  of the DE system).

The multiplier determining the Equation (106) have a close connection to the determining Equation (60) for adjoint-symmetries. This can be immediately seen from the product rule (9) obeyed by the Euler operator, which gives

$$0 = E_{u^\alpha}(Q_C^a F_a) = (\delta_{Q_C}^* F)_\alpha + (\delta_F^* Q_C)_\alpha \tag{107}$$

holding identically in jet space  $J(x, u, \partial u, \partial^2 u, \dots)$ . Notice that if this Equation (107) is restricted to the solution space  $\mathcal{E} \subset J$  of the given DE system (14), then it coincides with the adjoint-symmetry determining Equation (60). Hence, every conservation law multiplier is an adjoint-symmetry. This is a well-known result [4,24,26,27,29,31]. What is not so well known are the other conditions [4,29] that an adjoint-symmetry must satisfy to be a conservation law multiplier. These conditions arise from splitting the determining Equation (107) with respect to  $F_a$  and its total derivatives. As shown in Reference [29], the splitting can be derived by using the adjoint-symmetry identity (61) combined with the expression

$$(\delta_F^* Q)_\alpha = F_a \frac{\partial Q^a}{\partial u^\alpha} - D_i \left( F_a \frac{\partial Q^a}{\partial u_i^\alpha} \right) + \dots + (-1)^s D_{i_1} \dots D_{i_s} \left( F_a \frac{\partial Q^a}{\partial u_{i_1 \dots i_s}^\alpha} \right). \tag{108}$$

Then, in the determining Equation (107), the coefficients of  $F_a$ ,  $D_i F_a$ , and so on yield the system of equations [29]:

$$(\delta_{Q_C}^* F)_\alpha|_{\mathcal{E}} = 0 \tag{109}$$

and

$$R_{Q_\alpha}^a + E_{u^\alpha}(Q_C^a) = 0 \tag{110}$$

$$R_{Q_\alpha}^{a i_1 \dots i_q} + (-1)^q E_{u^\alpha}^{(i_1 \dots i_q)}(Q_C^a) = 0, \quad q = 1, \dots, s \tag{111}$$

where  $R_{Q_\alpha}^a$  and  $R_{Q_\alpha}^{a i_1 \dots i_q}$  are the coefficient functions of the linear differential operator (62) determined by Equation (109), and where  $E_{u^\alpha}$  is the Euler operator (6) and  $E_{u^\alpha}^{(i_1 \dots i_q)}$  is a higher-order Euler operator defined by [24,29]

$$E_{u^\alpha}^{(i_1 \dots i_q)}(f) = \frac{\partial f}{\partial u_{i_1 \dots i_q}^\alpha} - \binom{q+1}{1} D_j \left( \frac{\partial f}{\partial u_{i_1 \dots i_q j}^\alpha} \right) + \dots + (-1)^r \binom{q+r}{r} D_{j_1} \dots D_{j_r} \left( \frac{\partial f}{\partial u_{i_1 \dots i_q j_1 \dots j_r}^\alpha} \right), \quad q = 1, 2, \dots \tag{112}$$

for an arbitrary differential function  $f(x, u, \partial u, \dots, \partial^s u)$ . This system (109)–(111) constitutes a determining system for conservation law multipliers. Its derivation requires the same technical conditions on the form of the DE system (14) as stated in Proposition 3.

**Theorem 3.** *The determining Equation (107) for conservation law multipliers of a general DE system (14) is equivalent to the linear system of equations (109)–(111). In particular, multipliers are adjoint-symmetries (109) satisfying Helmholtz-type conditions (110)–(111) which are necessary and sufficient for an adjoint-symmetry to have the variational form (92) derived from a conserved current.*

This well known result [4,26,27,29] gives a precise relationship between adjoint-symmetries and multipliers, or equivalently between “nonlinear self-adjointness” and multipliers. In particular, it provides necessary and sufficient conditions for an adjoint-symmetry to be a multiplier. The simplest situation is when adjoint-symmetries of the lowest-order form  $Q^\alpha(x, u)$  are considered, which corresponds to “nonlinear self-adjointness” without differential substitutions. In this case, the only condition is Equation (110), which reduces to  $R_{Q^\alpha} + \frac{\partial Q^\alpha}{\partial u^\alpha} = 0$ . This condition is, in general, non-trivial. (Unfortunately, some recent work [16] incorrectly asserts that, for any DE system, every adjoint-symmetry of the form  $Q^\alpha(x, u)$  is a multiplier).

When Theorem 3 is applied to variational DE systems, it yields the following well-known connection [4,26,27,29] with Noether’s theorem.

**Corollary 1.** *For a variational DE system (33), the multiplier determining system (109)–(111) reduces to a determining system for variational symmetries. In particular, the determining equation for adjoint-symmetries (109) coincides with the determining equation for symmetries (23), and the Helmholtz-type conditions (110)–(111) coincide with the necessary and sufficient conditions for a symmetry to be variational (namely, that  $\text{pr}\hat{X}(L) = D_i\Gamma^i$  holds for some differential vector function  $\Gamma^i$ , where  $L$  is the Lagrangian (34)).*

Note that, in this modern formulation of Noether’s theorem, the use of a Lagrangian is completely by-passed through the Helmholtz-type conditions (110) and (111).

Example: For the semilinear wave Equation (63), the determining equation for multipliers of lowest-order form  $Q_C(t, x, u)$  is given by

$$0 = E_u(Q_C F) = \delta_{Q_C}^* F + \delta_F^* Q_C. \quad (113)$$

Since  $Q_C(t, x, u)$  does not depend on derivatives of  $u$ , this determining equation splits with respect to the variables  $u_t, u_x, u_{tt}, u_{xx}$ , giving an overdetermined linear system which can be derived and solved directly by Maple. This provides the simplest computational route to finding all multipliers of lowest-order form. The connection between multipliers and adjoint-symmetries arises when the determining Equation (113) is instead split into the two terms  $\delta_{Q_C}^* F$  and  $\delta_F^* Q_C$ , which are given by

$$\delta_{Q_C}^* F = D_t^2 Q_C - D_x^2 Q_C - b D_t Q_C - c D_x Q_C + (b' u_t + c' u_x + m') Q_C = R_{Q_C}(F) \quad (114)$$

$$\delta_F^* Q_C = \frac{\partial Q_C}{\partial u} F = E_u(Q_C) F \quad (115)$$

where the operator  $R_{Q_C}$  is obtained from expression (103). Hence, on the solution space  $\mathcal{E}$  of the wave Equation (63), the multiplier determining equation reduces to the adjoint-symmetry Equation (67). Off of the solution space  $\mathcal{E}$ , the multiplier determining equation then becomes

$$0 = R_{Q_C}(F) + E_u(Q_C) F = 2 \frac{\partial Q_C}{\partial u} F \quad (116)$$

which splits with respect to  $F$ , yielding

$$\frac{\partial Q_C}{\partial u} = 0. \quad (117)$$

This Helmholtz-type Equation (117) together with the adjoint-symmetry Equation (67) constitutes the determining system (109)–(111) for finding all lowest-order multipliers  $Q_C(t, x, u)$  admitted by the wave Equation (63).

The Helmholtz-type Equation (117) directly shows that all adjoint-symmetries of the form  $Q(t, x)$  are conservation laws multipliers  $Q_C(t, x)$ , and so, the three adjoint-symmetries (100) each determine a non-trivial conserved current through the characteristic equation

$$Q_C F = D_t \hat{C}^t + D_x \hat{C}^x. \tag{118}$$

These conserved currents  $(\hat{C}^t, \hat{C}^x)$  can be derived in terms of the multipliers  $Q_C(t, x, u)$  in several different ways. One simple way is by applying integration by parts to the terms in  $Q_C F$  to get a total time derivative  $D_t \hat{C}^t$  plus a total space derivative  $D_x \hat{C}^x$ , which yields  $(\hat{C}^t, \hat{C}^x)$ . Another way is by taking  $\hat{C}^t(t, x, u, u_t, u_x)$  and  $\hat{C}^x(t, x, u, u_t, u_x)$  as unknowns and splitting the characteristic equation with respect to  $u_{tt}, u_{tx}, u_{xx}$  to get a linear system of determining equations that can be integrated. The resulting conserved currents are shown in Table 3. The specific relationship between these conserved currents and the conserved currents derived in Table 2 will be explained in the next subsection.

Table 3. Conserved currents.

Conditions	$Q_C$	$\hat{C}^t$	$\hat{C}^x$
$m = m_1 u + m_2 \int b \, du$ $+ m_3 \int c \, du$ $m_1 = m_3^2 - m_2^2$	$e^{m_3 x + m_2 t}$	$e^{m_3 x + m_2 t} (u_t - m_2 u + \int b \, du)$	$e^{m_3 x + m_2 t} (m_3 u - u_x + \int c \, du)$
$b = b_0 + b_1 m'$ $c = c_0 + c_1 m'$ $b_1 \beta + c_1 \alpha = 1$ $\beta(\beta - b_0) = \alpha(\alpha + c_0)$	$e^{\alpha x + \beta t}$	$e^{\alpha x + \beta t} (u_t - \beta u + \int b \, du)$	$e^{\alpha x + \beta t} (\alpha u - u_x + \int c \, du)$
$b = \pm(\gamma + \frac{1}{\gamma} m')$ $c = -\gamma + \frac{1}{\gamma} m'$	$e^{\gamma x} q(x \mp t)$	$e^{\gamma x} (q(u_t + \int b \, du) \pm q' u)$	$e^{\gamma x} ((q' - \gamma q) u - (u_x \mp \int b \, du))$

### 3.2. Conservation Laws Produced by a Multiplier/Symmetry Pair

From Theorem 3 and Proposition 3, every multiplier admitted by a given DE system determines, up to local equivalence, a conserved current for the system. Since multipliers are adjoint-symmetries, the adjoint-symmetry/symmetry formula (72) can be applied by using any multiplier (92) together with any symmetry (40). The resulting conserved current produced this way is given by

$$Q_C^a (\delta_P F)_a - P^\alpha (\delta_{Q_C}^* F)_\alpha = D_i \Psi^i(P, Q_C; F) \tag{119}$$

where  $P^\alpha(x, u, \partial u, \dots, \partial^r u)$  is a given symmetry characteristic and  $Q_C^a(x, u, \partial u, \dots, \partial^s u)$  is a given multiplier. The following result characterizing these conserved currents will now be established for DE systems in the class stated in Proposition 3. The case of DE systems consisting of a single DE has appeared previously in Reference [23].

**Theorem 4.** Let  $\Psi^i(P, Q_C; F)$  be the conserved current produced from the adjoint-symmetry/symmetry formula (72) by using any multiplier  $Q_C^a(x, u, \partial u, \dots, \partial^s u)$  together with any symmetry characteristic  $P^\alpha(x, u, \partial u, \dots, \partial^r u)$ . This conserved current  $\Psi^i(P, Q_C; F)$  is locally equivalent to a conserved current (30) that is given by the infinitesimal action of the symmetry  $\hat{X}_P = P^\alpha \partial / \partial u^\alpha$  applied to the conserved current  $C^i$  determined by the multiplier  $Q_C^a$ . In particular,  $\Psi^i(P, Q_C; F)$  and  $C^i$  are related by

$$(\Psi^i(P, Q_C; F) - \text{pr} \hat{X}_P(C^i))|_{\mathcal{E}} = D_j \Theta^{ij} \tag{120}$$

for some differential antisymmetric tensor function  $\Theta^{ij}(x, u, \partial u, \dots, \partial^k u)$ .

The proof consists of showing that both conserved currents  $\Psi^i(P, Q_C; F)$  and  $\text{pr}\hat{X}_P(C^i)$  have the same multiplier. Consider the local conservation law determined by the multiplier  $Q_C^a$ . The symmetry  $\hat{X}_P$  applied to the characteristic Equation (91) of this conservation law yields

$$\text{pr}\hat{X}_P(D_i\tilde{C}^i) = \text{pr}\hat{X}(Q_C^a F_a) = \delta_P(Q_C^a F_a) = (\delta_P Q_C)^a F_a + Q_C^a (\delta_P F_a). \tag{121}$$

The second term in this equation can be expressed as

$$Q_C^a (\delta_P F_a) = Q_C^a R_P(F)_a = F_a R_P^*(Q_C)^a + D_i \Gamma^i(Q_C, F; P) \tag{122}$$

using the symmetry identity (41) combined with integration by parts, where  $\Gamma^i(Q_C, F; P)|_{\mathcal{E}} = 0$ . Next, the first term in Equation (121) can be expressed as

$$\begin{aligned} (\delta_P Q_C)^a F_a &= P^\alpha (\delta_F^* Q_C)_\alpha + D_i \Psi^i(P, Q_C; F) = -P^\alpha (\delta_{Q_C}^* F)_\alpha + D_i \Psi^i(P, Q_C; F) \\ &= -P^\alpha R_{Q_C}(F)_\alpha + D_i \Psi^i(P, Q_C; F) \end{aligned} \tag{123}$$

through the Fréchet derivative identity (3) combined with the multiplier determining Equation (107) and the adjoint-symmetry identity (61). Integration by parts then yields

$$(\delta_P Q_C)^a F_a = -F_a R_{Q_C}^*(P)^a + D_i (\Psi^i - \Gamma^i(P, F; Q_C)) \tag{124}$$

where  $\Gamma^i(P, F; Q_C)|_{\mathcal{E}} = 0$ . Substitution of expressions (124) and (122) into Equation (121) gives

$$\text{pr}\hat{X}_P(D_i\tilde{C}^i) = (R_P^*(Q_C)^a - R_{Q_C}^*(P)^a) F_a + D_i (\Psi^i(P, Q_C; F) + \Gamma^i(Q_C, F; P) - \Gamma^i(P, F; Q_C)). \tag{125}$$

Finally, since  $\text{pr}\hat{X}_P$  commutes with total derivatives [24,29], this yields

$$D_i (\text{pr}\hat{X}_P(C^i) + \tilde{\Gamma}^i) = Q_\Psi^a F_a \tag{126}$$

where  $\tilde{\Gamma}^i|_{\mathcal{E}} = 0$  is a locally trivial conserved current, and where

$$Q_\Psi^a = R_P^*(Q_C)^a - R_Q^*(P)^a \tag{127}$$

is the multiplier (98) of the local conservation law (96) from the adjoint-symmetry/symmetry formula (72) with  $Q^a = Q_C^a$ . This completes the proof.

Theorem 4 is a generalization of a similar result [2,3,36] for variational DE systems, where the adjoint-symmetry/symmetry formula (72) reduces to a formula using any pair of symmetries.

**Corollary 2.** For a variational DE system, let  $\Psi^i(P, Q_C; F)$  be the conserved current produced from the adjoint-symmetry/symmetry formula (72) by using any symmetry characteristic  $P^\alpha(x, u, \partial u, \dots, \partial^r u)$  together with any multiplier  $Q_C^\alpha(x, u, \partial u, \dots, \partial^s u)$  given by a variational symmetry characteristic. The conserved current  $\Psi^i(P, Q_C; F)$  is locally equivalent to a conserved current (30) that is given by the infinitesimal action of the symmetry  $\hat{X}_P = P^\alpha \partial / \partial u^\alpha$  applied to the conserved current  $C^i$  determined by the multiplier  $Q_C^\alpha$ . Moreover, through Noether's theorem, the multiplier of this conserved current  $\Psi^i(P, Q_C; F)$  is the characteristic of a variational symmetry given by the commutator of the symmetries  $\hat{X}_P = P^\alpha \partial / \partial u^\alpha$  and  $\hat{X}_{Q_C} = Q_C^\alpha \partial / \partial u^\alpha$ .

Several basic properties of the adjoint-symmetry/symmetry formula (72) can be deduced from Theorem 4, as first shown in Reference [23] for DE systems consisting of a single DE.

**Theorem 5.** (i) For a given DE system (14), let  $Q_C^a$  be the multiplier for a local conservation law in which the components of the conserved current  $C^i$  have no explicit dependence on  $x$ . Then, using any translation symmetry  $\mathbf{X} = a^i \partial / \partial x^i$ , with characteristic  $P^\alpha = -a^i u_i^\alpha$  where  $a^i$  is a constant vector, the conserved current  $\Psi^i(P, Q_C; F)$  is locally trivial. (ii) For a given DE system (14) that possesses a scaling symmetry  $\mathbf{X} =$



$a_{(i)}x^i\partial/\partial x^i + b_{(\alpha)}u^\alpha\partial/\partial u^\alpha$ , where  $a_{(i)}, b_{(\alpha)}$  are constants, let  $Q_C^a$  be the multiplier for a local conservation law in which the components of the conserved current  $C^i$  are scaling homogeneous. Then, using the characteristic  $P^\alpha = b_{(\alpha)}u^\alpha - a_{(i)}x^i u_i^\alpha$  of the scaling symmetry, the conserved current  $\Psi^i(P, Q_C; F)$  is locally equivalent to a multiple  $w$  of the conserved current  $C^i$  determined by  $Q_C^a$ . This multiple,  $w = \text{const.}$ , is the scaling weight of the conserved integral given by  $\int_{\partial\Omega} C^i dS_i$  where  $\Omega$  is any closed domain in  $\mathbb{R}^n$  and  $\partial\Omega$  is its boundary surface.

The proof is a straightforward extension of the proof in Reference [23] and will be omitted.

Part (i) of this theorem explains the observations made in many recent papers in which Ibragimov's theorem gave only trivial local conservation laws. This will happen whenever the only local symmetries admitted by a DE system are translations and the only admitted adjoint-symmetries have no dependence on  $x$ .

Part (ii) of the theorem first appeared in Reference [28]. It shows that the local conservation laws admitted by any DE system with a scaling symmetry can be obtained from an algebraic formula using the conservation law multipliers. This explains why in many recent papers, the use of scaling symmetries in Ibragimov's theorem has produced non-trivial local conservation laws.

A more important point comes from putting together Theorems 3 and 4. Together, these two theorems show that the adjoint-symmetry/symmetry formula (72) cannot produce any "new" local conservation laws, since any local conservation law admitted by a given DE system must already arise directly from a multiplier. Moreover, for this formula to generate all of the local conservation laws for a given DE system, it seems plausible that the set of admitted symmetries needs to act transitively on a set of admitted local conservation laws, so then every multiplier arises from some symmetry applied to some multiplier. The need for a transitive action is especially clear from Corollary 2, since if a pair of commuting variational symmetries is used in the formula, then the resulting local conservation law will have a trivial multiplier, and hence, will be a locally trivial conservation law.

These significant deficiencies should discourage the unnecessary use of the adjoint-symmetry/symmetry formula (72), and consequently the unnecessary use of Ibragimov's theorem, when local conservation laws are being sought for a given DE system. It is much simpler and more direct to find all multipliers and then to derive the conserved currents determined by these multipliers, as will be explained further in the next section.

Example: For the semilinear wave Equation (63), Table 2 shows the conserved currents obtained from the adjoint-symmetry/symmetry formula (72). Each of these conserved currents can be checked to satisfy the characteristic Equation (118) with  $\hat{C}^t = \Psi^t$  and  $\hat{C}^x = \Psi^x$ , where the resulting multipliers  $Q_\Psi$  are shown in Table 4. There is a simple relationship (127) between each multiplier  $Q_\Psi$  and the adjoint-symmetry/symmetry pair  $Q, P$  used to generate the conserved current  $(\Psi^t, \Psi^x)$ . In particular, from expressions (100)–(104) for the symmetry characteristics, adjoint-symmetries and their associated operators  $R_P$  and  $R_Q$ , the relationship (127) yields

$$\begin{aligned} Q_{\Psi(P_1, Q_l; F)} &= R_{P_1}^*(Q_l) - R_{Q_l}^*(P_1) = D_t Q_l, \quad l = 1, 2, 3 \\ Q_{\Psi(P_2, Q_l; F)} &= R_{P_2}^*(Q_l) - R_{Q_l}^*(P_2) = D_x Q_l, \quad l = 1, 2, 3 \end{aligned} \quad (128)$$

in accordance with Table 4.

In particular, consider the case when  $m(u)$  is zero and both  $b(u), c(u)$  are arbitrary, so then the only admitted multiplier of lowest-order form  $Q(t, x, u)$  is  $Q = 1$  (up to a multiplicative constant), as shown by Table 1. In this case, it is straightforward to show that (by solving the relevant determining equations) there are no first-order multipliers and that the only admitted point symmetries are generated by the translations (89). Consequently, when the set of multipliers  $Q(t, x, u, u_t, u_x)$  is considered, a single non-trivial conservation law  $C^t = u_t + \int b(u) du, C^x = -u_x + \int c(u) du$  is admitted by the wave equation  $u_{tt} - u_{xx} + b(u)u_t + c(u)u_x = 0$  with  $b(u)$  and  $c(u)$  arbitrary, whereas all of the conserved currents  $(\Psi^t, \Psi^x)$  obtained from Ibragimov's theorem (85) or from the simpler equivalent adjoint-symmetry/symmetry formula (86), are trivial! Note that, correspondingly, the symmetry

action on the set of non-trivial conservation laws given by the set of multipliers  $Q(t, x, u, u_t, u_x)$  is not transitive. This example succinctly illustrates the incompleteness of these formulas for generating conservation laws.

**Table 4.** Multipliers from the adjoint-symmetry/symmetry formula.

$P$	$Q$	$\Psi^t, \Psi^x$	$Q_\Psi$
$-u_t$	$e^{m_3x+m_2t}$	$m_2e^{m_3x+m_2t}(u_t - m_2u + \int b \, du),$ $m_2e^{m_3x+m_2t}(m_3u - u_x + \int c \, du)$	$m_2e^{m_3x+m_2t}$ $= D_t(e^{m_3x+m_2t})$
$-u_x$	$e^{m_3x+m_2t}$	$m_3e^{m_3x+m_2t}(u_t - m_2u + \int b \, du),$ $m_3e^{m_3x+m_2t}(m_3u - u_x + \int c \, du)$	$m_3e^{m_3x+m_2t}$ $= D_x(e^{m_3x+m_2t})$
$-u_t$	$e^{\alpha x+\beta t}$	$\beta e^{\alpha x+\beta t}(u_t - \beta u + \int b \, du),$ $\beta e^{\alpha x+\beta t}(\alpha u - u_x + \int c \, du)$	$\beta e^{\alpha x+\beta t}$ $= D_t(e^{\alpha x+\beta t})$
$-u_x$	$e^{\alpha x+\beta t}$	$\alpha e^{\alpha x+\beta t}(u_t - \beta u + \int b \, du),$ $\alpha e^{\alpha x+\beta t}(\alpha u - u_x + \int c \, du)$	$\alpha e^{\alpha x+\beta t}$ $= D_x(e^{\alpha x+\beta t})$
$-u_t$	$e^{\gamma x}q$	$-e^{\gamma x}(q''u \pm q'(u_t + \int b \, du)),$ $\pm e^{\gamma x}((\gamma q' - q'')u + q(u_x \mp \int b \, du))$	$\mp e^{\gamma x}q$ $= D_t(e^{\gamma x}q)$
$-u_x$	$e^{\gamma x}q$	$e^{\gamma x}(\pm(q'' + \gamma q')u + (q' + \gamma q)(u_t + \int b \, du)),$ $e^{\gamma x}((q'' - \gamma^2 q)u - (q' + \gamma q)(u_x \mp \int b \, du))$	$e^{\gamma x}(q' + \gamma q)$ $= D_x(e^{\gamma x}q)$

#### 4. A Direct Construction Method to Find All Local Conservation Laws

The results stated in Proposition 3 and Theorems 3, 4 and 5 have been developed in References [4,23,26–28] and extended in References [29,35]. This collective work provides a simple, algorithmic method to find *all* local conservation laws for any given system of DEs. The method is based on the general result that all local conservation laws arise from multipliers as given by the solutions of a linear system of determining equations, where the multipliers are simply adjoint-symmetries subject to certain Helmholtz-type conditions.

Consequently, all multipliers can be found by either of the two following methods [4]: (1) directly solve the full determining system for multipliers; or (2) first, solve the determining equation for adjoint-symmetries, and next, check which of the adjoint-symmetries satisfy the Helmholtz-type conditions. The adjoint-symmetry determining equation is simply the adjoint of the symmetry determining equation, and hence, it can be solved by the standard algorithmic procedure used for solving the symmetry determining equation [24,31,32]. Likewise, the same procedure works equally well for solving the multiplier determining system.

A natural question is, in practice, at which differential orders  $s \geq 0$  will multipliers or adjoint-symmetries  $Q^a(x, u, \partial u, \dots, \partial^s u)$  be found?

One answer is that the same situation arises for symmetries. Normally, point symmetries are sought first, since many DE systems admit point symmetries, and since relatively fewer DE systems admit contact symmetries or higher-order symmetries. Indeed, the existence of a sufficiently high-order symmetry is one main definition of an integrable system [37], as this can indicate the existence of an infinite hierarchy of successively higher-order symmetries. For multipliers, the most physically important conserved currents always have a low differential order. Based on numerous examples, a concrete definition of a *low-order multiplier* that seems to characterize these physically important conserved currents, and distinguishes them from higher-order conserved currents arising for integrable systems, has been introduced in recent work [29,35].

Another answer is that it is straightforward just to find all multipliers or adjoint-symmetries with a specified differential order  $s = 0, 1, 2, \dots$ , going up to any desired maximum finite order. Moreover, in some situations, a standard descent/induction argument [38–40] can be used to find the multipliers or adjoint-symmetries to all orders  $s \geq 0$ .

Once a set of multipliers has been found for a given DE system, the corresponding conserved currents are straightforward to find in an explicit form. Several different methods are available.

One algorithmic method is the direct integration of the characteristic equation [31,41] defining the conserved current. Another algorithmic method is the use of a homotopy integral formula. This method has several versions [4,24,26,27,29], all of which involve trade-offs between the simplicity of the integration versus the flexibility of avoiding singularities (if any) in the integrand.

However, purely algebraic methods for the construction of conserved currents from multipliers are known. One algebraic method is the use of a scaling formula [23,28,31], which is given by the adjoint-symmetry/symmetry formula. This applies only to DE systems that admit a scaling symmetry, but it has recently been extended to general DE systems by incorporating a dimensional analysis method as shown in Reference [29]. In particular, with the use of this dimensional-scaling method, the construction of conserved currents becomes completely algebraic.

Therefore, the general method just outlined provides a completely algorithmic computational way to derive all local conservation laws for any given DE system. In particular, there is no need to resort to any special methods or ansatz (such as the “abc” technique [42], partial Lagrangians [43], “nonlinear self-adjointness” [9,12,14,15,17], undetermined coefficients [44]), which at best just yield a subset of all of the local conservation laws admitted by a DE system or just apply to restricted classes of DE systems.

Example: The semilinear wave Equation (63) can be expected to admit conserved currents that depend nonlinearly on  $u_t$  and  $u_x$ , in addition to the previous conserved currents in Table 3, all of which have linear dependence on  $u_t$  and  $u_x$ . The multipliers (100) for the latter conserved currents have the form  $Q_C(t, x)$ . Conserved currents that depend nonlinearly on  $u_t$  and  $u_x$  will arise from multipliers  $Q_C(t, x, u, u_t, u_x)$  that have explicit dependence on  $u_t$  and  $u_x$ . It is straightforward to find all such multipliers by using Maple to set up and solve the multiplier Equation (113), which splits with respect to the variables  $u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{ttx}, u_{txx}, u_{xxx}$ , giving an overdetermined linear system. Alternatively, the multiplier Equation (113) can be split instead into the two terms  $\delta_{Q_C}^* F$  and  $\delta_F^* Q_C$ , which provides a direct connection between multipliers and adjoint-symmetries. In particular, the first term in the multiplier Equation (113) consists of

$$\delta_{Q_C}^* F = D_t^2 Q_C - D_x^2 Q_C - b D_t Q_C - c D_x Q_C + (b' u_t + c' u_x + m') Q_C = R_{Q_C}(F) \quad (129)$$

where the operator  $R_{Q_C}$  is found to be given by

$$R_Q = \frac{\partial Q_C}{\partial u_t} D_t F + \frac{\partial Q_C}{\partial u_x} D_x F + \frac{\partial^2 Q_C}{\partial u_t \partial u_t} F + \frac{\partial Q_C}{\partial u} - 2b \frac{\partial Q_C}{\partial u_t} + 2u_t \frac{\partial^2 Q_C}{\partial u \partial u_t} + 2u_{tx} \frac{\partial^2 Q_C}{\partial u_x \partial u_t} + 2(u_{xx} - bu_t - cu_x - d) \frac{\partial^2 Q_C}{\partial u_t \partial u_t} \quad (130)$$

for multipliers  $Q_C(t, x, u, u_t, u_x)$ , through  $u_{tt} = u_{xx} - b(u)u_t - c(u)u_x - m(u)$ . The second term in the multiplier Equation (113) is given by

$$\delta_F^* Q_C = \frac{\partial Q_C}{\partial u} F - D_t \left( \frac{\partial Q_C}{\partial u_t} F \right) - D_x \left( \frac{\partial Q_C}{\partial u_x} F \right) = - \frac{\partial Q_C}{\partial u_t} D_t F - \frac{\partial Q_C}{\partial u_x} D_x F + E_u(Q) F \quad (131)$$

where

$$E_u(Q) = - \frac{\partial^2 Q_C}{\partial u_t \partial u_t} F + \frac{\partial Q_C}{\partial u} - u_t \frac{\partial^2 Q_C}{\partial u \partial u_t} - u_x \frac{\partial^2 Q_C}{\partial u \partial u_x} - 2u_{tx} \frac{\partial^2 Q_C}{\partial u_x \partial u_t} + (bu_t + cu_x + d - 2u_{xx}) \frac{\partial^2 Q_C}{\partial u_t \partial u_t} \quad (132)$$

for multipliers  $Q_C(t, x, u, u_t, u_x)$ . On the solution space  $\mathcal{E}$  of the wave Equation (63), the terms (131) vanish, while the other terms (129) reduce to the adjoint-symmetry Equation (67). Off of the solution

space  $\mathcal{E}$ , these terms (129) and (131) become a linear combination of  $F, D_t F, D_x F$ , whose coefficients must vanish separately. This splitting is found to yield a single Helmholtz-type equation

$$2 \frac{\partial Q_C}{\partial u} - b \frac{\partial Q_C}{\partial u_t} + u_t \frac{\partial^2 Q_C}{\partial u \partial u_t} - u_x \frac{\partial^2 Q_C}{\partial u \partial u_x} - (bu_t + cu_x + d) \frac{\partial^2 Q_C}{\partial u_t \partial u_x} = 0. \tag{133}$$

Taken together, this Helmholtz-type Equation (133) and the adjoint-symmetry Equation (67) constitute the determining system (109)–(111) for finding all first-order multipliers  $Q_C(t, x, u, u_t, u_x)$  admitted by the wave Equation (63).

The most computationally effective way to solve Equations (133) and (67) in the determining system is by changing variables from  $t, x, u, u_t, u_x$  to  $\mu = \frac{1}{2}(t + x), \nu = \frac{1}{2}(t - x), u, u_\mu = u_t + u_x, u_\nu = u_t - u_x$ , based on null coordinates for the wave Equation (63). In these new variables, the general solution of the determining system consists of three distinct cases (as obtained using the Maple package ‘rifsimp’), after the nonlinearity and homogeneity conditions (71) are imposed on  $b(u), c(u), d(u)$ . The resulting multipliers, after merging cases, are shown in Table 5. Each multiplier determines a non-trivial conserved current through the characteristic Equation (118). These conserved currents ( $\hat{C}^t, \hat{C}^x$ ) can be derived in terms of the multipliers  $Q_C(t, x, u, u_t, u_x)$  in the same way discussed previously for lowest-order multipliers. The results are shown in Table 6.

Table 5. First-order multipliers.

Conditions	$Q_C$
$\frac{2m_1 m_2}{m - m_1} = \frac{4m_1}{b \pm c} = \int (b \mp c) du$	$\frac{2m_1 + (b \pm c)(u_t \pm u_x)}{2m + (b \pm c)(u_t \pm u_x)}$
$m = (m_1 + \frac{1}{4} \int (b - c) du)(b + c),$ $(1 - \gamma)b = (1 + \gamma)c$	$\frac{((1 - \gamma)u_t + (1 + \gamma)u_x)(b^2 - c^2)}{((b + c)(u_t + u_x) + 2m)((b - c)(u_t - u_x)2m)}$

Table 6. First-order conserved currents.

Conditions	$\hat{C}^t, \hat{C}^x$
$\frac{2m_1 m_2}{m - m_1} = \frac{4m_1}{b \pm c} = \int (b \mp c) du$	$\gamma \ln \left( \frac{b \pm c}{2m_1 + (b \pm c)(\gamma + u_t \pm u_x)} \right) + u_t + \frac{1}{2} \int (b \pm c) du,$ $\mp \gamma \ln \left( \frac{b \pm c}{2m_1 + (b \pm c)(\gamma + u_t \pm u_x)} \right) - u_x + \frac{1}{2} \int (c \pm b) du + \gamma x$
$m = (m_1 + \frac{1}{4} \int (b - c) du)(b + c),$ $(1 - \gamma)b = (1 + \gamma)c$	$\ln \left( \frac{(\gamma \int (b + c) du + 2(u_t - u_x) + m_1)^{\frac{1}{\gamma}}}{\gamma \int (b + c) du + 2(u_t + u_x) + m_1} \right),$ $\ln \left( (\gamma \int (b + c) du + 2(u_t - u_x) + m_1)^{\frac{1}{\gamma}} \right)$ $\times (\gamma \int (b + c) du + 2(u_t + u_x) + m_1)$

### 5. Concluding Remarks

The conservation law theorem stated by Ibragimov in References [9,12] for “nonlinear self-adjoint” DEs and subsequent extensions of this theorem in References [14,15,17] are not new. In its most general form, this theorem is simply a re-writing of a standard formula [2–4] that uses a pair consisting of a symmetry and an adjoint-symmetry to produce a conservation law through a well-known Fréchet derivative identity [2,3,24,29,31]. Unfortunately, no references to prior literature are provided in Ibragimov’s papers, which may give the impression that the results are original. One aspect that is novel is the derivation of the formula by using an auxiliary Lagrangian, although it does not in any way simplify either the formula or its content. Moreover, the condition of “nonlinear self-adjointness” is nothing but a re-writing of the condition that a DE system admits an adjoint-symmetry [4,29], and this condition automatically holds for any DE system that admits a local conservation law.

The present paper shows how the symmetry/adjoint-symmetry formula is directly connected to the action of symmetries on conservation laws, which explains a number of major drawbacks in trying to use the formula and, hence, in applying Ibragimov's theorem, as a method to generate conservation laws. In particular, the formula can generate trivial conservation laws and does not always yield all non-trivial conservation laws unless the symmetry action on the set of these conservation laws is transitive, which cannot be known until all conservation laws have been found.

A broader point, which is more important, is that there is a completely general method [29,31] using adjoint-symmetries [2–4,26,27] to find all local conservation laws for any given DE system. This method is a kind of adjoint version of the standard Lie method to find all local symmetries. The method is algorithmic [29], and the required computations are no more difficult than the computations used to find local symmetries.

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## References

- Noether, E. Invariante variations probleme. *Transp. Theory Stat. Phys* **1971**, *1*, 186–207.
- Caviglia, G. Symmetry transformations, isovectors, and conservation laws. *J. Math. Phys.* **1986**, *27*, 972–978.
- Lunev, F.A. An analogue of the Noether theorem for non-Noether and nonlocal symmetries. *Theor. Math. Phys.* **1991**, *84*, 816–820.
- Anco, S.C.; Bluman, G. Direct construction of conservation laws from field equations. *Phys. Rev. Lett.* **1997**, *78*, 2869–2873.
- Blaszak, M. *Multi-Hamiltonian Theory of Dynamical Systems*; Springer: Heidelberg, Germany, 1998.
- Krasil'shchik, I.S.; Vinogradov, A.M. (Eds.) *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*; Translations of Mathematical Monographs 182; American Mathematical Society: Providence, RI, USA, 1999.
- Verbotevsky, A. Notes on the horizontal cohomology. In *Secondary Calculus and Cohomological Physics*; Contemporary Mathematics 219; American Mathematical Society: Providence, RI, USA, 1997.
- Zharinov, V.V. *Lecture Notes on Geometrical Aspects of Partial Differential Equations*; Series on Soviet and East European Mathematics 9; World Scientific: River Edge, NJ, USA, 1992.
- Ibragimov, N.H. A new conservation theorem. *J. Math. Anal. Appl.* **2007**, *333*, 311–328.
- Ibragimov, N.H. Quasi self-adjoint differential equations. *Arch. ALGA* **2007**, *4*, 55–60.
- Ibragimov, N.H. Nonlinear self-adjointness in constructing conservation laws. *Arch. ALGA* **2010**, *7/8*, 1–86.
- Ibragimov, N.H. Nonlinear self-adjointness and conservation laws. *J. Phys. A Math. Theor.* **2011**, *44*, 432002–432009.
- Galiakberova, L.R.; Ibragimov, N.H. Nonlinear self-adjointness of the Krichever-Novikov equation. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 361–363.
- Gandarias, M.L. Weak self-adjoint differential equations. *J. Phys. A Math. Theor.* **2011**, *44*, 262001–262006.
- Gandarias, M.L. Nonlinear self-adjointness through differential substitutions. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 3523–3528.
- Zhang, Z.-Y. On the existence of conservation law multiplier for partial differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *20*, 338–351.
- Zhang, Z.-Y.; Xie, L. Adjoint symmetry and conservation law of nonlinear diffusion equations with convection and source terms. *Nonlinear Anal. Real World Appl.* **2016**, *32*, 301–313.
- Freire, I.L. New classes of nonlinearly self-adjoint evolution equations of third- and fifth-order. *Commun. Nonlin. Sci. Numer. Simul.* **2013**, *18*, 493–499.
- Gandarias, M.L. Conservation laws for some equations that admit compacton solutions induced by a non-convex convection. *J. Math. Anal. Appl.* **2015**, *420*, 695–702.



20. Ibragimov, N.H.; Torrisi, M.; Tracina, R. Quasi self-adjoint nonlinear wave equations. *J. Phys. A Math. Theor.* **2010**, *43*, doi:10.1088/1751-8113/43/44/442001.
21. Ibragimov, N.H.; Torrisi, M.; Tracina, R. Self-adjointness and conservation laws of a generalized Burgers equation. *J. Phys. A Math. Theor.* **2011**, *44*, 145201–145205.
22. Freire, I.L.; Sampaio, J.C.S. Nonlinear self-adjointness of a generalized fifth-order KdV equation. *J. Phys. A Math. Theor.* **2012**, *44*, 032001–032007.
23. Anco, S.C. Symmetry properties of conservation laws. *Int. J. Mod. Phys. B* **2016**, *30*, doi:10.1142/S0217979216400038.
24. Olver, P. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1986.
25. Martinez Alonso, L. On the Noether map. *Lett. Math. Phys.* **1979**, *3*, 419–424.
26. Anco, S.C.; Bluman, G. Direct construction method for conservation laws of partial differential equations. I. Examples of conservation law classifications, *Eur. J. Appl. Math.* **2002**, *13*, 545–566.
27. Anco, S.C.; Bluman, G. Direct construction method for conservation laws of partial differential equations. II. General treatment, *Eur. J. Appl. Math.* **2002**, *13*, 567–585.
28. Anco, S.C. Conservation laws of scaling-invariant field equations. *J. Phys. A Math. Gen.* **2003**, *36*, 8623–8638.
29. Anco, S.C. Generalization of Noether's theorem in modern form to non-variational partial differential equations. In *Recent progress and Modern Challenges in Applied Mathematics, Modeling and Computational Science*; Fields Institute Communications 79; Springer: New York, NY, USA, 2017.
30. *CRC Handbook of Lie Group Analysis of Differential Equations, Volume I: Symmetries, Exact Solutions, and Conservation Laws*; Ibragimov, N.H., Ed.; CRC Press: Boca Raton, FL, USA, 1994.
31. Bluman, G.; Cheviakov, A.; Anco, S.C. *Applications of Symmetry Methods to Partial Differential Equations*; Springer Applied Mathematics Series 168; Springer: New York, NY, USA, 2010.
32. Bluman, G.; Anco, S.C. *Symmetry and Integration Methods for Differential Equations*; Springer Applied Mathematics Series 154; Springer: New York, NY, USA, 2002.
33. Khamitova, R.S. The structure of a group and the basis of conservation laws. *Theoret. Math. Phys.* **1982**, *52*, 244–251.
34. Anco, S.C. Brock University, St. Catharines, ON, Canada. Unpublished work, 2017.
35. Anco, S.C.; Kara, A.H. Symmetry invariance of conservation laws of partial differential equations. *Eur. J. Appl. Math.* **2017**, in press.
36. Anco, S.C.; Bluman, G. Derivation of conservation laws from nonlocal symmetries of differential equations. *J. Math. Phys.* **1996**, *37*, 2361–2375.
37. Mikhailov, A.V.; Shabat, A.B.; Sokolov, V.V. Symmetry approach to classification of integrable equations. In *What Is Integrability*; Springer: Berlin, Germany, 1999.
38. Duzhin, S.V.; Tsujishita, T. Conservation laws of the BBM equation. *J. Phys. A* **1984**, *17*, 3267–3276.
39. Anco, S.C.; Pohjanpelto, J. Classification of local conservation laws of Maxwell's equations. *Acta Appl. Math.* **2001**, *69*, 285–327.
40. Anco, S.C.; Pohjanpelto, J. Conserved currents of massless spin s fields. *Proc. R. Soc.* **2003**, *459*, 1215–1239.
41. Wolf, T. A comparison of four approaches to the calculation of conservation laws. *Eur. J. Appl. Math.* **2002**, *13*, 129–152.
42. Morawetz, C. Variations on conservation laws for the wave equation. *Bull. Am. Math. Soc.* **2000**, *37*, 141–154.
43. Kara, A.H.; Mahomed, F.M. Noether-type symmetries and conservation laws via partial Lagrangians. *Nonlinear Dyn.* **2006**, *45*, 367–383.
44. Poole, D.; Hereman, W. Symbolic computation of conservation laws for nonlinear partial differential equations in multiple space dimensions. *J. Symb. Comput.* **2011**, *46*, 1355–1377.

