Additional Conserved Quantities of Weber's Force Law?

Alan Aversa*

Received January 2, 2021

Abstract

Weber's force law, derived from Ampère's force law for electrical currents, gives the electrodynamic force between two moving charges. It reduces to Coulomb's law in the static case, but in general it is dependent on the relative velocities and accelerations of the interacting charges. Although it conserves linear and anguler momentum and obeys Newton's 3rd Law, the Runge-Lenz vector is not conserved for it (thus it can explain Mercury's perihelion shift). We investigate whether there are additional symmetries and, by Noether-like theorems, additional conserved quantities of Weber's force law.

*ORCID iD, ResearchGate, Google Scholar, homepage, alan_aversa@lavabit.com, AMDG.

1 Introduction

Ampère's force law [1, 15] expresses the force between electrical current elements $I_1 d\vec{\ell_1}$ and $I_2 d\vec{\ell_2}$ a distance r_{12} apart:

$$I_1 I_2 \frac{\hat{r}_{12}}{r_{12}^2} \left[2(d\vec{\ell_1} \cdot d\vec{\ell_2}) - 3(\hat{r}_{12} \cdot d\vec{\ell_1})(\hat{r}_{12} \cdot d\vec{\ell_2}) \right] . \tag{1}$$

From this law Weber derived his relative velocity- and acceleration-dependent electrodynamic force between two moving charges [1, 28, 29]; expressed as a system of nonlinear ordinary differential equations (ODEs), it is:

$$\ddot{x}_i = \frac{q_1 q_2 x_i}{r^3} \left[1 - \frac{\zeta}{2c^2} \dot{r}^2 + \frac{\zeta}{c^2} r \ddot{r} \right] , \tag{2}$$

where q_1, q_2 are the interacting charges; $i = 1, 2, 3$; over-dots are time derivatives; x_i the spatial coordinates of the charge; $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$; c the speed of light; and ζ a dimensionless quantity that determines how "Weberian" the force is ($\zeta = 0$ is the Coulomb case); and c is the speed of light *in vacuo*.

Weber's force law is derivable from Neumann's velocity-dependent potential [2, 478]:

$$\frac{q_1 q_2}{r} \left[1 + \frac{\zeta}{2c^2} \dot{r}^2 \right] . \tag{3}$$

It is repulsive or attractive, depending on the magnitudes of the relative velocities or accelerations; Weber used this fact in his planetary model of the atom [4] to explain—a half-century before Bohr and Rutherford and without any need for a new, nuclear force—how positively charged nucleons can exist in close proximity without being repulsed.

More recently, Frauenfelder and Weber [16] use a Weber Hamiltonian to derive the fine-structure energy levels of a hydrogen atom.

They also show how, according to Carl Neumann, Weber's force can be derived from retarded potentials [14, 148-149; cf. 23, 24].

Weber's force law obeys Newton's 3rd Law.[1] It conserves linear and angular momentum and energy [3, 63-7]. It explains Mercury's perihelion shift [2, 480], so the Runge-Lenz vector is not conserved. But does it have other conservation laws?

2 Symmetry Finding

Sophus Lie originally intended his transformation groups to be a powerful tool for solving differential equations [21, 22], such as by reducing their order. Previously, Seegers [26] used elliptic functions to solve 2, but we investigate whether symmetry methods are more powerful.

A differential equation admits a group of point transformations if it remains invariant under the operation of the transformation given by the group generators [27, 93-94; 5, 46]:

$$X_n = \xi_n(t, q_i)\frac{\partial}{\partial t} + \eta_n^a(t, q_i)\frac{\partial}{\partial q^a}, \tag{4}$$

where $a = 1, 2, 3$ in the case of 2, corresponding to the x, y, z coordinates. Lie showed that $1 \leq n \leq 8$ [22].

The determining equations [20, 202] — partial differential equations (PDEs) that have to be solved to find the symmetry group corresponding to a system of, in this case, second-order differential equations

$$\ddot{q}^a = \omega^a(t, q^i, \dot{q}^i), \tag{5}$$

where $a, i = 1, \ldots, N$, the number of equations; $_{,t} = \frac{\partial}{\partial t}$; and $_{,c} = \frac{\partial}{\partial q^c}$ — are [27, 95]:

[1] cf. 8 for how Ampère's force law obeys Newton's 3rd Law

$$
\begin{aligned}
\xi\omega^a_{,t} + \eta^b\omega^a_{,b} + \left(\eta^b_{,t} + \dot{q}^c\eta^b_{,c} - \dot{q}^b\xi_{,t} + \dot{q}^b\dot{q}^c\xi_{,c}\right)\frac{\partial\omega^a}{\partial\dot{q}^b} \\
+2\omega^a\left(\xi_{,t} + \dot{q}^b\xi_{,b}\right) + \omega^b\left(\dot{q}^a\xi_{,b} - \eta^a_{,b}\right) \\
+\dot{q}^a\dot{q}^b\dot{q}^c\xi_{,bc} + 2\dot{q}^a\dot{q}^c\xi_{,tc} - \dot{q}^c\dot{q}^b\eta^a_{,bc} + \dot{q}^a\xi_{,tt} - 2\dot{q}^b\eta^a_{,tb} - \eta^a_{,tt} = 0.
\end{aligned}
\tag{6}
$$

Recommended by Hereman [18, 19] as the best symmetry-finding software, we use the Maple package GeM[2] [9–13] to solve for the ξ_n and η^a_n of admitted generators 4 of 2:

```
restart;
read("gem32_12.mpl");

gem_decl_vars(indeps=[t], deps=[x(t),y(t),z(t)],
freeconst=[c,k,zeta]);

r := t -> sqrt(x(t)^2+y(t)^2+z(t)^2);
rp := t -> diff(r(t),t);
rpp := t -> diff(rp(t),t);
fac := t -> k/r(t)^3
(1 - zeta/(2c^2)r(t)^2 + zeta/c^2r(t)rpp(t));

#Coulomb problem:
#zeta := 0;

gem_decl_eqs([
diff(x(t),t$2) = x(t)fac(t),
diff(y(t),t$2) = y(t)fac(t),
diff(z(t),t$2) = z(t)fac(t)],
solve_for=[diff(x(t),t$2),diff(y(t),t$2),diff(z(t),t$2)]);

#This step uses over 6GB of memory:
det_eqs:=gem_symm_det_eqs([t,x(t),y(t),z(t)]);
```

GeM finds 60 determining equations. Solving them with Maple's built-in PDE solver,

[2] https://math.usask.ca/~shevyakov/gem/

```
sym_components:=gem_symm_components();
symm_sol:=pdsolve(det_eqs):
gem_output_symm(symm_sol);
```

GeM outputs these admitted generators:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t} \\
X_2 &= -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x} \\
X_3 &= -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} \\
X_4 &= -y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y},
\end{aligned} \tag{7}
$$

which are time and rotation symmetries [7, 476]; however, when $\zeta \neq 0$, `pdsolve` does not find the dilation symmetry

$$
X_5 = \frac{2}{3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) + t\frac{\partial}{\partial t}. \tag{8}
$$

This is strange because the determining equations 6 depend on ζ, so it would seem that 8 would depend on ζ, too.

We also tried solving for higher-order (Lie-Bäcklund) symmetries by making the ξ_n and η_n^a depend not only on t, x, y, z but also on $\dot{x}, \dot{y}, \dot{z}$, by changing

```
det_eqs:=gem_symm_det_eqs([t,x(t),y(t),z(t)]);
```

to

```
det_eqs := gem_symm_det_eqs([t, x(t), y(t), z(t),
diff(x(t), t), diff(y(t), t), diff(z(t), t)])
```

After running for about a day, Maple did not find any solutions for additional symmetries.

2.1 Conservation laws

We also tried to find conservation laws [6, 38-120, 363-4] using the following GeM functions:

```
det_eqs:=gem_conslaw_det_eqs([t,x(t),y(t),z(t)]);
CL_multipliers:=gem_conslaw_multipliers();
multipliers_sol:=pdsolve(det_eqs);
gem_get_CL_fluxes(multipliers_sol);
```

But Maple was unable to find any conservation laws, unless we initially set $\zeta = 0$, the Coulomb case.

3 Conclusion

Weber's force law might have additional symmetries and conservation laws, but we were not able to find them. Perhaps a new mathematics, like Lie-Santilli theory [17], is necessary to make finding solutions to the nonlinear ODE 2 more tractable. Santilli [25, 9:18] seems to allude to the possibility that isomathematics can linearize [cf. 20, 214-8] all(?) nonlinear DEs.

Other future possibilities include applying Weber's force law to cold fusion; cf. the lectures by Indrani B. Das Sarma and Simone Beghella Bartoli at the EPR Debates 2020 conference.

References

[1] André-Marie Ampère. *Ampère's electrodynamics: analysis of the meaning and evolution of Ampère's force between current elements, together with a complete translation of his master-piece: Theory of electrodynamic phenomena, uniquely deduced*

from experience. Apeiron, Montreal, 2015. ISBN 978-1-987980-03-5. URL https://www.ifi.unicamp.br/~assis/Amperes-Electrodynamics.pdf.

[2] André K. T. Assis. *Relational Mechanics and Implementation of Mach's Principle with Weber's Gravitational Force*. Apeiron, Montréal, March 2014. ISBN 978-0-9920456-3-0. URL https://www.ifi.unicamp.br/~assis/Relational-Mechanics-Mach-Weber.pdf.

[3] André Koch Torres Assis. *Weber's Electrodynamics*. Springer Netherlands, Dordrecht, 1994. ISBN 978-90-481-4471-6 978-94-017-3670-1. doi: 10.1007/978-94-017-3670-1. URL http://link.springer.com/10.1007/978-94-017-3670-1.

[4] Gudrun Wiederkehr Karl Heinrich Wolfschmidt Gudrun Andre Koch Torres Assis, Wolfschmidt. *Weber's Planetary Model of the Atom*. tredition, Hamburg, 2011. ISBN 978-3-8424-0241-6 3-8424-0241-4. URL http://nbn-resolving.de/urn:nbn:de:101:1-20110709227.

[5] George W. Bluman and Stephen C. Anco. *Symmetry and Integration Methods for Differential Equations*, volume 154 of *Applied Mathematical Sciences*. Springer New York, New York, NY, 2002. ISBN 978-0-387-98654-8 978-0-387-21649-2. doi: 10.1007/b97380. URL http://link.springer.com/10.1007/b97380.

[6] George W. Bluman, Alexei F. Cheviakov, and Stephen C. Anco. *Applications of Symmetry Methods to Partial Differential Equations*, volume 168 of *Applied Mathematical Sciences*. Springer New York, New York, NY, 2010. ISBN 978-0-387-98612-8 978-0-387-68028-6. doi: 10.1007/978-0-387-68028-6. URL http://link.springer.com/10.1007/978-0-387-68028-6.

[7] Brian Cantwell. *Introduction to symmetry analysis*. Cambridge University Press, Cambridge; New York, N.Y., 2002. ISBN 978-0-521-77183-2 978-0-521-77740-7. OCLC: 774401187.

[8] J. P. M. C Chaib and F. M. S. Lima. Resuming Ampère's experimental investigation of the validity of Newton's third law in electrodynamics. *Annales de la Fondation Louis de Broglie*, 45(1), 2020. URL `https://aflb.minesparis.psl.eu/AFLB-451/aflb451m890.htm`.

[9] Alexei F Cheviakov. GeM software package for computation of symmetries and conservation laws of differential equations. *Computer physics communications*, 176(1): 48–61, 2007. URL `http://math.usask.ca/~shevyakov/publ/papers/afs_gem_cpc.pdf`. Publisher: Elsevier.

[10] Alexei F Cheviakov. Computation of fluxes of conservation laws. *Journal of Engineering Mathematics*, 66(1-3):153–173, 2010. URL `http://math.usask.ca/~shevyakov/publ/papers/Cflux_v31_for_Archive.pdf`. Publisher: Springer.

[11] Alexei F Cheviakov. Symbolic computation of local symmetries of nonlinear and linear partial and ordinary differential equations. *Mathematics in Computer Science*, 4(2-3): 203–222, 2010. URL `http://math.usask.ca/~shevyakov/publ/papers/mcs_v13.pdf`. Publisher: Springer.

[12] Alexei F. Cheviakov. Description Document GeM Symbolic Software Package (Version 32.12 and Higher) for General Symmetry and Conservation Law Analysis of Differential Equations. Technical report, April 2015. URL `https://math.usask.ca/~shevyakov/publ/papers/`

2014-Chev-Symbolic%20Computation%20of%20Nonlocal%20Symmetries-Springer.pdf.

[13] Alexei F. Cheviakov. Symbolic computation of equivalence transformations and parameter reduction for nonlinear physical models. *Computer Physics Communications*, 220(Supplement C): 56 – 73, 2017. ISSN 0010-4655. doi: https://doi.org/10.1016/j.cpc.2017.06.013. URL http://www.sciencedirect.com/science/article/pii/S0010465517301959.

[14] Pierre Maurice Marie Duhem. *The Electric Theories of J. Clerk Maxwell*, volume 314 of *Boston Studies in the Philosophy and History of Science*. Springer International Publishing, Cham, 2015. ISBN 978-3-319-18514-9 978-3-319-18515-6. URL http://link.springer.com/10.1007/978-3-319-18515-6.

[15] Pierre Maurice Marie Duhem. *Ampère's Force Law: A Modern Introduction*. September 2018. URL https://isidore.co/calibre/get/PDF/6855.

[16] Urs Frauenfelder and Joa Weber. The fine structure of Weber's hydrogen atom: Bohr–Sommerfeld approach. *Zeitschrift für angewandte Mathematik und Physik*, 70(4):105, June 2019. ISSN 1420-9039. doi: 10.1007/s00033-019-1149-4. URL https://doi.org/10.1007/s00033-019-1149-4.

[17] Raúl M. Falcón Ganfornina and Juan Núñez Valdés. Mathematical Foundations of Santilli Isotopies. *Algebras, Groups, and Geometries*, 32:135–308, 2015. URL https://www.researchgate.net/publication/276269128_Mathematical_Foundations_of_Santilli_Isotopies.

[18] Willy Hereman. Software, November 2020. URL https://inside.mines.edu/~whereman/software.html.

[19] Willy Hereman. Weber's velocity-dependent force law, December 2020.

[20] Nail H Ibragimov. *A Practical Course in Differential Equations and Mathematical Modeling*. Higher Education Press Limited Company, BeiJing, 2009. ISBN 978-7-89423-622-7. URL `https://www.overdrive.com/media/2592474`. OCLC: 960759820.

[21] Marius Sophus Lie, Albert Victor Bäcklund, and L. V Ovsyannikov. *Lie Group Analysis: Classical Heritage*. Alga, Karlskrona, Sweeden, 2004. ISBN 978-91-7295-996-5. URL `http://www.diva-portal.org/smash/record.jsf?pid=diva2%3A837093&dswid=-6030`. OCLC: 861058198.

[22] Sophus Lie and Georg Wilhelm Scheffers. *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*. B.G. Teubner, Leipzig, 1891. URL `https://goobi.tib.eu:443/viewer/image/1013269691/1`. OCLC: 1022269687.

[23] Carl Neumann. Theoria nova phaenomenis electricis applicanda. *Annali di Matematica Pura ed Applicata*, 2(1):120–128, August 1868. ISSN 0373-3114, 1618-1891. doi: 10.1007/BF02419606. URL `http://link.springer.com/article/10.1007/BF02419606`.

[24] Dr Carl Neumann. Die Principien der Elektrodynamik. *Mathematische Annalen*, 17(3):400–434, September 1880. ISSN 0025-5831, 1432-1807. doi: 10.1007/BF01446235. URL `http://link.springer.com/article/10.1007/BF01446235`.

[25] Ruggero Maria Santilli. EPR Concluding Debates Part 1 FINAL, September 2020. URL `https://youtu.be/I3zgWE8z8Cw`.

[26] Karl Seegers. *De motu perturbationibusque planetarum secundum legem electrodynamicam Weberianam solem ambientium*. PhD thesis, Dieterich, Gottingae, 1864. URL http://digital.bib-bvb.de/webclient/DeliveryManager?custom_att_2=simple_viewer&pid=3560257. OCLC: 36264226.

[27] Hans Stephani. *Differential Equations: Their Solution Using Symmetries*. Cambridge University Press, Cambridge, GBR, 2011. ISBN 978-0-511-59994-1. URL http://public.eblib.com/choice/publicfullrecord.aspx?p=4641505. OCLC: 958550273.

[28] Wilhelm Weber. On the Measurement of Electro-dynamic Forces. *Electrodynamische Maassbestimmungen Annalen der Physik und Chemie (Poggendorff's Annalen)*, 73(2):193–240, January 1848. URL http://www.sizes.com/library/classics/Weber1.pdf.

[29] Wilhelm Weber. *Determinations of Electrodynamic Measure: Concerning a Universal Law of Electrical Action*. 21st Century Science & Technology, August 2008. URL https://www.21stcenturysciencetech.com/Articles2007/Weber_1846.pdf.