Mechanics, cosmology and Mach's principle

Hanno Essén

KTH Mechanics, Stockholm SE-100 44, Sweden

E-mail: hanno@mech.kth.se

Abstract
It is pointed out that recent cosmological findings seem to support the view
that the mass/energy distribution of the universe defines the Newtonian inertial
frames, as originally suggested by Mach. The background concepts of inertial
frame, Newton's second law and fictitious forces are clarified. A precise
definition of Mach's principle is suggested. Then, an approximation to general
relativity discovered by Einstein, Infeld and Hoffmann is used and it is found
that this precise formulation of Mach's principle is realized provided the
mass/energy density of the universe has a specific value. This value turns
out to be twice the critical density. The implications of this approximate result
are put into context.

1. Introduction

In 2011, the final report from the Gravity Probe B experiment was published by Everitt *et al*
[1]. This satellite experiment verified for the first time the frame dragging prediction of general
relativity, thereby corroborating one aspect of Mach's principle. This principle states that the
inertial frames of classical mechanics are defined as being those that do not accelerate with
respect to the average mass/energy distribution of the universe. Since the early work on Mach's
principle by Bondi [2, 3], Sciama [4–6], Dicke [7], Lynden-Bell [8], and others, cosmology has
progressed considerably. Cosmic microwave background radiation has been discovered and
studied in detail. Dark matter has been found to dominate over normal matter in the universe.
The recent discovery of the acceleration of the Hubble expansion by the groups of Perlmutter
and of Schmidt and Riess (Nobel prize 2011) has given new observational information on
the mass/energy distribution of the universe. In particular, it is now clear that matter, dark
matter and dark energy together represent a mass/energy density close to the critical density
of cosmology [9, 10]. It seems appropriate to reassess the status of Mach's principle in view
of these empirical advances in recent decades.

After a brief survey of the most relevant literature, I first discuss Newton's second law and
its interpretation. The concept of inertial frame and the related ideas about real and fictitious

forces are elucidated. This part can be understood by undergraduates who have studied vector mechanics and I try to give a deeper motivation for the various terms that appear in this law. Then, I demonstrate how this law arises by approximation from general relativistic equations of motion, and how this seems to support Mach's principle as a consequence. This part relies on the Einstein–Infeld–Hoffmann (EIH) Lagrangian formulation and requires that the students have been exposed to analytical mechanics.

Mach's principle has been subject to much discussion and speculation in the literature. Some more recent studies can be found in the volume edited by Barbour and Pfister [11] and in Mashhoon *et al* [12], where the gravitomagnetic analogy is studied in detail. Many points of view are of a philosophical and metaphysical nature, but here I concentrate on empirical aspects. There are also quite a few accounts in the pedagogical literature, e.g., [13]. Interesting discussions can be found in the textbooks by Berry [14], Ciufolini and Wheeler [15], Peacock [16] and, most recently, by Cheng [9]. Frame dragging and its relation to Mach's principle and general relativity have been discussed by Grøn [17, 18], Grøn and Eriksen [19], Harris [20], Holstein [21], Hughes [22], Lynden-Bell *et al* [23], Martín *et al* [24], Nightingale [25, 26] and Vető [27, 28]. It has been pointed out that general relativity does not, in itself, imply Mach's principle, the counter example being Gödel's solution [29] to Einstein's equations. In this solution, however, time travel is possible. There have been speculations that banning time travel will restrict us to solutions that obey Mach's principle [30], but these matters are still far from clear.

2. Inertial frames, real and fictitious forces

Newton's second law for a particle is now normally written in the form

$$m\boldsymbol{a} = \boldsymbol{F}. \tag{1}$$

Here, m is (inertial) mass, which is determined by means of a common balance and a reference mass (unit of mass). The acceleration $\boldsymbol{a}$ is a purely kinematic quantity which is determined by recording the positions with respect to time relative to a chosen *reference frame*. This requires the choice of origin, axes, measuring rods and clocks (reliable periodic phenomena). The force $\boldsymbol{F}$ then turns out to be a quantity which is the cause of the acceleration. All known forces decay at least as the inverse distance squared so that forces have a local origin, i.e. they arise from *local sources* near the moving particle and they obey Newton's third law of action and reaction. The success of Newtonian mechanics comes largely from the fact that there is a limited catalogue of forces. We have very accurate mathematical models for electromagnetic and gravitational forces. All other forces of macroscopic importance are material contact forces (normal force, dry friction, pressure, viscosity, drag, force from elastic deformation, pull from a string, etc) for which there are often reasonably accurate mathematical models, albeit of a limited range of validity.

In the paragraph above, I have emphasized the words reference frame and local source. In order for the above theory to work, the reference frame must be the so-called inertial, or non-accelerated. We can use accelerated frames, however, if we know their translational acceleration $\boldsymbol{a}_{\mathrm{f}}$ and their angular velocity $\boldsymbol{\omega}$ about an axis through the origin. The appropriate equation of motion is then

$$m\boldsymbol{a}^* = \boldsymbol{F} + \boldsymbol{F}^*, \tag{2}$$

where

$$\boldsymbol{F}^* = -m[\boldsymbol{a}_{\mathrm{f}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}^*) + 2\,\boldsymbol{\omega} \times \boldsymbol{v}^* + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}^*]. \tag{3}$$

Here, r^*, v^* and a^* are position, velocity and acceleration with respect to the accelerating system, while $\dot{\omega}$ is the time derivative of the angular velocity vector. The new force F^* on the right-hand side of (2) is the vector sum of the so-called fictitious forces (3). These forces do not have local sources and Newton's third law does not apply to them. The crucial questions are now the following. Acceleration relative to what? Which are the inertial frames? How are they found?

When it comes to rotation, these questions are fairly easy to answer. In a non-rotating reference frame, the fixed stars (or even better, the distant galaxies) have fixed directions. Since the Earth rotates with respect to such a frame, the fictitious forces containing the angular velocity ω are necessary to get the correct motion of, e.g., a Foucault pendulum. Also the fact that rotating astronomical bodies are flattened is well accounted for by the fictitious centrifugal force $-m\,\omega \times (\omega \times r^*)$. There is a tiny difference between the angular velocity of the Earth relative to the fixed stars and the angular velocity appearing in these equations that was discovered by Gravity Probe B [1]. The inertial frames near the Earth rotate slightly because the Earth rotates, according to general relativity. Today there is no observational evidence that our universe is a rotating Gödel universe [29].

The question is much more complicated when it comes to translational acceleration. When studying motion in a laboratory on the Earth, we are used to including the gravitational force mg arising from our planet. If we denote the non-gravitational force on our particle by K, we then normally write down the equation of motion

$$ma = K + mg. \tag{4}$$

A critical reader may now note that the particle, in fact, must also be affected by the gravitational forces from the Sun, the Moon, the galaxy and so on. Let us denote the acceleration that these gravitational forces would impart to our particle by $g_f(r)$ at position r. To obtain an accurate equation of motion mg_f must then be added on the right-hand side of (4).

We now realize, however, that the reference frame at rest on the Earth is not really inertial (even neglecting rotation). The Earth is freely falling in the external gravitational fields from bodies other than itself. This acceleration will vary with position, but not that much. Let us choose the value at the origin and put $g_f(0) = a_f$. The true equation of motion for a particle on the (non-rotating) freely falling Earth is then

$$ma^* = K + mg + mg_f(r) - ma_f \approx K + mg. \tag{5}$$

The approximation can be made since $g_f(r) \approx a_f$ for r-values of interest. The fact that there is an exact equality only at a point gives rise to tidal effects. The Earth falls freely in the field of the Sun and the Moon, but since the Earth is extended all points are not subject to the same acceleration $a_f = g_f(0)$.

To summarize, when working in a laboratory on the Earth I am in fact working in an accelerated reference frame that accelerates in such a way that gravitational forces $mg_f(r)$ from other bodies than the Earth itself are transformed away by a fictitious force $F^* = -ma_f = -mg_f(0)$. The usual equation of motion (4) thus works quite well, but a occurring in it is in fact relative to an accelerated reference frame and is thus really a^*.

The precise acceleration of the Earth relative to the universe as a whole is quite difficult to measure. The phenomenon of Doppler shift, however, makes it possible to find the velocity of the Earth with respect to the cosmic microwave background (CMB) quite accurately. Such measurements reveal that a constant velocity with respect to the CMB seems to correspond to inertial frames. This possibility of identifying the rest frame of the CMB (the frame in which the radiation is as isotropic as possible) with the frame in which the mass/energy of the universe as a whole is at rest constitutes an observational verification of Mach's principle (see [9, section 10.5.4]).

3. Newton's second law and Einstein's equations

Mach's principle was one of the inspirations behind Einstein's work on general relativity but the precise connection is still not clear. Einstein discusses the connection in *The Meaning of Relativity* [31] and outlines how the cosmic mass/energy density influences the equations of motion of a particle (on pages 100–102). A detailed derivation of these equations is given by Harris [20].

Let us make Mach's principle more precise. Assume that the equation of motion of a particle is of the form

$$m(\boldsymbol{a} - \boldsymbol{g}_{\mathrm{u}}) = \boldsymbol{F}, \tag{6}$$

where $\boldsymbol{g}_{\mathrm{u}}$ is the acceleration of the universe as a whole. Then, only the acceleration relative to the universe as a whole is what matters in the equation of motion. I now proceed to show that this at least is a possibility.

I will approach this problem from the point of view of the EIH equations of motion [32–34]. Fock found the Lagrangian that yields these equations [35] and this Lagrangian is also derived and discussed in [36–38]. The Lagrangian is given by

$$L = L_0 + L_1 + L_2, \tag{7}$$

where

$$L_0 = T_0 - V_0 = \sum_a \frac{1}{2} m_a \boldsymbol{v}_a^2 + \frac{1}{2} \sum_a \sum_{b \neq a} \frac{G m_a m_b}{r_{ab}} \tag{8}$$

and

$$L_1 = \frac{1}{4c^2} \sum_a \sum_{b \neq a} \frac{G m_a m_b}{r_{ab}} \left[3\left(\boldsymbol{v}_a^2 + \boldsymbol{v}_b^2\right) - 7 \boldsymbol{v}_a \cdot \boldsymbol{v}_b - \frac{(\boldsymbol{v}_a \cdot \boldsymbol{r}_{ab})(\boldsymbol{v}_b \cdot \boldsymbol{r}_{ab})}{r_{ab}^2} \right]. \tag{9}$$

L_2 contains the first relativistic correction, i.e. $\sim (v/c)^2$, to the classical kinetic energy T_0, and higher order corrections to the gravitational interaction $\sim G^2$. These terms are not of interest here since they will not influence the inertia of a slow particle. Here, $\boldsymbol{r}_{ab} = \boldsymbol{r}_b - \boldsymbol{r}_a$ is the vector from particle a to particle b, and $r_{ab} = |\boldsymbol{r}_{ab}|$.

The equation of motion for particle 1 is given by the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{v}_1} = \frac{\partial L}{\partial \boldsymbol{r}_1} \;\Leftrightarrow\; \dot{\boldsymbol{p}}_1 = \boldsymbol{F}_1. \tag{10}$$

All terms involving accelerations will occur on the left-hand side here, so this is what we need to calculate. Calculation gives

$$\frac{\partial L}{\partial \boldsymbol{v}_1} = m_1 \boldsymbol{v}_1 + \frac{G m_1}{2c^2} \sum_{b=2}^{N} \frac{m_b}{r_{1b}} \left[6 \boldsymbol{v}_1 - 7 \boldsymbol{v}_b - \frac{\boldsymbol{r}_{1b}(\boldsymbol{v}_b \cdot \boldsymbol{r}_{1b})}{r_{1b}^2} \right], \tag{11}$$

for the so-called generalized momentum $\boldsymbol{p}_1 = \partial L / \partial \boldsymbol{v}_1$. Now assume that particle 1 is at the origin in a homogeneous isotropic expanding universe of density ρ, and with Hubble parameter H. The particles m_b are then replaced by the mass elements $\rho \, \mathrm{d}V$ of position $\boldsymbol{r}$ and velocity $\boldsymbol{v} = H\boldsymbol{r} + \boldsymbol{u}$. Here, $\boldsymbol{u}$ is an overall velocity of the universe relative to the origin. We can then replace the sum in (11) with an integral and obtain

$$\boldsymbol{p}_1 = m_1 \boldsymbol{v}_1 + \frac{G m_1}{2c^2} \int \frac{\rho}{r} \left[6 \boldsymbol{v}_1 - 7(H\boldsymbol{r} + \boldsymbol{u}) - \frac{\boldsymbol{r}(Hr^2 + \boldsymbol{u} \cdot \boldsymbol{r})}{r^2} \right] \mathrm{d}V. \tag{12}$$

We now calculate the integral on the right-hand side.

We introduce spherical coordinates (r, φ, θ) and perform the integration over the visible universe. At the radius R of the visible universe, the Hubble expansion leads to

recession at the speed of light, i.e. $HR = c$. The volume element in spherical coordinates is $dV = r^2 \sin\theta \, dr \, d\varphi \, d\theta$. Without loss of generality, we assume that $\boldsymbol{u} = u\boldsymbol{e}_z$. Since $\boldsymbol{r} = r(\sin\theta\, \boldsymbol{e}_\varphi + \cos\theta\, \boldsymbol{e}_z)$, where $\boldsymbol{e}_\varphi = \cos\varphi\, \boldsymbol{e}_x + \sin\varphi\, \boldsymbol{e}_y$, the scalar product term becomes

$$\boldsymbol{r}(\boldsymbol{u} \cdot \boldsymbol{r}) = r^2 u(\sin\theta\, \boldsymbol{e}_\varphi + \cos\theta\, \boldsymbol{e}_z)\cos\theta. \tag{13}$$

The integrations over the sphere of radius R will make the terms involving H vanish for symmetry reasons, since these are multiplied by $\boldsymbol{r}$. Nothing depends on the angle φ in the integral so the term multiplying $\boldsymbol{e}_\varphi$ also vanishes. Two different integrals then remain to calculate: first,

$$\int \frac{\rho \, dV}{r} = 4\pi\rho \int_0^R r \, dr = 2\pi\rho R^2 \tag{14}$$

and, then,

$$\int \frac{\rho \cos^2\theta \, dV}{r} = 2\pi\rho \int_0^R r \, dr \int_0^\pi \cos^2\theta \, \sin\theta \, d\theta = \frac{2}{3}\pi\rho R^2, \tag{15}$$

due to the scalar product term.

This gives us the result

$$\boldsymbol{p}_1 = m_1\left[\left(1 + 6\frac{G\pi\rho R^2}{c^2}\right)\boldsymbol{v}_1 - \left(\frac{22}{3}\frac{G\pi\rho R^2}{c^2}\right)\boldsymbol{u}\right]. \tag{16}$$

In order to understand this, the meaning of the quantity $G\pi\rho R^2/c^2$, which we can rewrite as

$$\sigma \equiv \frac{G\pi\rho R^2}{c^2} = \frac{G\pi}{H^2}\rho, \tag{17}$$

using $R = c/H$, must be investigated.

Studying cosmology using general relativity and the assumption of an expanding homogeneous, isotropic universe, one finds that there is a specific mass/energy density ρ_c that makes (three-dimensional) space flat [9, 10]. This density is called the critical density and is given by

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{18}$$

It is interesting to note that the critical density corresponds to the mass M of the universe inside the Hubble radius $R = c/H$ being such that the Hubble radius is equal to the Schwarzschild radius $R = 2GM/c^2$.

Comparing (18) with (17), we see that

$$\sigma = \frac{3}{8}\frac{\rho}{\rho_c} \equiv \frac{3}{8}\Omega, \tag{19}$$

where Ω is the standard notation in cosmology for the ratio of the density to the critical density. The generalized momentum (16) of particle 1 now becomes

$$\boldsymbol{p}_1 = m_1\left[\left(1 + \frac{9}{4}\Omega\right)\boldsymbol{v}_1 - \left(\frac{11}{4}\Omega\right)\boldsymbol{u}\right]. \tag{20}$$

Returning to my formulation of Mach's principle in (6), it is seen to be realized with this $\dot{\boldsymbol{p}}_1$ if $\Omega = 2$. For this value of the density ratio, the generalized momentum is

$$\boldsymbol{p}_1 = m_1\frac{11}{2}(\boldsymbol{v}_1 - \boldsymbol{u}) \implies \dot{\boldsymbol{p}}_1 = m_1\frac{11}{2}(\boldsymbol{a}_1 - \boldsymbol{g}_u). \tag{21}$$

That is, for $\Omega = 2$, the acceleration in Newton's second law is relative to the acceleration $\dot{\boldsymbol{u}} = \boldsymbol{g}_u$ of the universe as a whole. One also notes that the 'bare' mass m_1 has been 'renormalized' to[1] $m = 11m_1/2$.

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[1] If this renormalization is not permissible, one can interpret equation (20) requiring that all of the kinetic energy is due to interaction. Then, the $m_1\boldsymbol{v}_1$-term in $\boldsymbol{p}_1$ vanishes and one concludes that the actual mass is $m = m_1 9\Omega/4$. Mach's principle would then require that the second term also has a factor 9/4 instead of the 11/4 obtained here.

4. Conclusions

We see that while forces, arising from $\boldsymbol{F}_1 = \partial L/\partial \boldsymbol{r}_1$ in (10), decrease at least as r^{-2}, the inertial terms in $\boldsymbol{p}_1$ decrease only as r^{-1}. Consequently, inertia has an intrinsic non-local nature. It is thus difficult to investigate by local measurements—the main reason that these matters remain obscure and intimately connected to cosmology. This should be an important insight of these investigations.

In [28], Vető has found that the fictitious Coriolis force $-2m\,\boldsymbol{\omega} \times \boldsymbol{v}^*$ can be understood as being due to the gravitomagnetic field from the rest of the universe if $\Omega = 1$. Support for such a standpoint also comes from investigations by Martín *et al* [24] where numerical Ω-values near 1 are found. In both these works, however, linearized forms of general relativity were used that neglect the gravitational back-reaction of the accelerating particle on the background universe, whereas the EIH formalism used here retains back-reaction effects to linear order in the mass m_1. This makes the results of [24, 28] comparatively unreliable.

In conclusion, I have elucidated Mach's principle and found that a precise formulation of the principle can be consistent with the EIH approximation to general relativity if the density of the universe is twice the critical density. This is in qualitative agreement with other investigations that find that Mach's principle requires the density parameter to be of order of magnitude unity. Already, Berry [14, p 39] in his 1976 book found that simple estimates required that the density of the universe should be $\rho_{\text{Berry}} = H^2/(2\pi G) = 4\rho_c/3$ to obey Mach's principle. At that time the observed and inferred amount of galactic matter was only 4% of this value. Now we definitely know that the order of magnitude of the mass/energy density of the universe is such that Mach's principle is physically viable.

Acknowledgments

References

[1] Everitt C W F *et al* 2011 Gravity probe B: final results of a space experiment to test general relativity *Phys. Rev. Lett.* **106** 221101
[2] Bondi H 1952 *Cosmology* (Cambridge: Cambridge University Press)
[3] Bondi H 1962 Physics and cosmology *Observatory* **82** 133–43
[4] Sciama D W 1959 *The Unity of the Universe* (London: Faber and Faber)
[5] Sciama D W 1963 Retarded potentials and the expansion of the universe *Proc. R. Soc.* A **273** 484–95
[6] Sciama D W, Waylen P C and Gilman R C 1969 Generally covariant integral formulation of Einstein's field equations *Phys. Rev.* **187** 1762–6
[7] Dicke R H 1963 Cosmology, Mach's principle and relativity *Am. J. Phys.* **31** 500–9
[8] Lynden-Bell D 1967 On the origins of space–time and inertia *Mon. Not. R. Astron. Soc.* **135** 413–28 http://adsabs.harvard.edu/full/1967MNRAS.135..413L
[9] Cheng T-P 2010 *Relativity, Gravitation and Cosmology—A Basic Introduction* 2nd edn (Oxford: Oxford University Press)
[10] Jordan T F 2004 Cosmology calculations almost without general relativity arXiv:astro-ph/0309756v4
[11] Barbour J B and Pfister H (ed) 1995 *Mach's Principle—From Newton's Bucket to Quantum Gravity* (Boston, MA: Birkhäuser)
[12] Mashhoon B, Gronwald F and Lichtenegger H I M 2001 Gravitomagnetism and the clock effect *Gyros, Clocks, Interferometers...: Testing Relativistic Gravity in Space (Lecture Notes in Physics* vol 562*)* ed C Lämmerzahl, C W F Everitt and F W Hehl (Berlin: Springer) pp 83–108
[13] Zylbersztajn A 1994 Newton's absolute space, Mach's principle and the possible reality of fictitious forces *Eur. J. Phys.* **15** 1–8
[14] Berry M 1976 *Principles of Cosmology and Gravitation* (Cambridge: Cambridge University Press)

[15] Ciufolini I and Wheeler J A 1995 *Gravitation and Inertia* (Princeton, NJ: Princeton University Press)
[16] Peacock J A 1999 *Cosmological Physics* (Cambridge: Cambridge University Press)
[17] Grøn Ø and Eriksen E 1989 Translational inertial dragging *Gen. Rel. Grav.* **21** 105–24
[18] Hartman H I and Nissim-Sabat C 2009 Reply to 'The principle of relativity and inertial dragging' *Am. J. Phys.* **77** 381–2
[19] Grøn Ø G 2009 The principle of relativity and inertial dragging *Am. J. Phys.* **77** 373–80
[20] Harris E G 1991 Analogy between general relativity and electromagnetism for slowly moving particles in weak gravitational fields *Am. J. Phys.* **59** 421–5
[21] Holstein B R 2001 Gyroscope precession and general relativity *Am. J. Phys.* **69** 1248–56
[22] Hughes R J 1992 On Feynman's proof of the Maxwell equations *Am. J. Phys.* **60** 301–6
[23] Lynden-Bell D, Bicák J and Katz J 1999 On accelerated inertial frames in gravity and electromagnetism *Ann. Phys.* **271** 1–22
[24] Martín J, Rañada A F and Tiemblo A 2007 On Mach's principle: inertia as gravitation arXiv:gr-qc/0703141v1
[25] Nightingale J D 1977 Specific physical consequences of Mach's principle *Am. J. Phys.* **45** 376–9
[26] Nightingale J D and Ray J R 1978 Comment on 'Specific physical consequences of Mach's principle' *Am. J. Phys.* **46** 586
[27] Vető B 2010 Gravity probe B experiment and gravitomagnetism *Eur. J. Phys.* **31** 1123–30
[28] Vető B 2011 Gravitomagnetic field of the universe and Coriolis force on the rotating Earth *Eur. J. Phys.* **32** 1323–9
[29] Rindler W 2009 Gödel, Einstein, Mach, Gamow, and Lanczos: Gödel's remarkable excursion into cosmology *Am. J. Phys.* **77** 498–510
[30] Laurent B 1980 private communication
[31] Einstein A 1955 *The Meaning of Relativity* 5th edn (Princeton, NJ: Princeton University Press)
[32] Einstein A, Infeld L and Hoffmann B 1938 The gravitational equations and the problem of motion *Ann. Math.* **39** 65–100
[33] Eddington A and Clark G L 1938 The problem of n bodies in general relativity theory *Proc. R. Soc.* A **166** 465–75
[34] Infeld L and Plebański J 1960 *Motion and Relativity* (Oxford: Pergamon)
[35] Fock V 1959 *The Theory of Space Time and Gravitation* (London: Pergamon)
[36] Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields* 4th edn (Oxford: Pergamon)
[37] Brumberg V 2009 On the derivation of EIH (Einstein–Infeld–Hoffmann) equations of motion from the linearized metric of general relativity theory *Celest. Mech. Dyn. Astron.* **99** 245–52
[38] Kennedy F J 1972 Approximately relativistic interactions *Am. J. Phys.* **40** 63–74