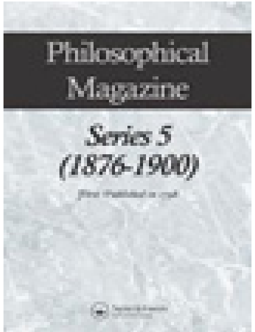

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XXXV. *On the Employment of the Electrodynamic Potential for the Determination of the Ponderomotive and Electromotive Forces.* *By* R. Clausius*.

§ 1.

In order to represent in a convenient manner the electrodynamic forces between moved particles of electricity and the mechanical work performed by them it is well known that the electrodynamic potential can be employed, which facilitates the calculations for these forces in like manner as the electrostatic potential does for the electrostatic forces. Its signification is the same as that of the electrostatic potential; for as the latter is defined by the statement that the work done during a movement of the particles of electricity by the electrostatic forces is equal to the simultaneous diminution of the electrostatic potential, so also the electrodynamic potential is defined by saying that the work done by the electrodynamic forces is equal to the diminution of the electrodynamic potential. In its form, however, the electrodynamic differs essentially from the electrostatic potential by this—that it comprises not only the coordinates, but also the components of the velocity of the electric particles; and with this is, at the same time, connected a difference in the procedure by means of which the force-components are to be derived from it.

If, now, we wish to determine with the help of the electrodynamic potential the forces which a *galvanic current* (which may be in motion and variable) exerts upon a moved particle of electricity, we cannot in general construct the former by simply combining, for each current-element, the two potential-expressions referring to the positive and negative electricity present in the respective element of the conductor in an algebraic sum and then treating the current-element as a whole, but must rather consider each of the two quantities of electricity separately, since the question is not merely what state of motion they have in the conductor-element which concerns us, but also how the state of their motion changes on their passing from this element into the adjacent one, which takes place differently for the two electricities. Of course the formulæ are thereby somewhat complicated. In certain cases, however, especially in that in which the current whose action upon a moved particle of electricity we wish to determine is

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closed, the thing is simplified in this way—that, besides the intensity of the current, we have only to consider the position and direction of the current-elements, without taking separately into consideration the two electricities present in them. Thereby we then arrive at formulæ of extraordinary simplicity, which afford great facilities for the determination of the ponderomotive and electromotive forces, and bring into clear view the entire field of mathematical developments having reference thereto.

These formulæ I will take leave to develop in the sequel—and not merely from the electrodynamic fundamental law advanced by me, but also from those of Riemann and Weber. It will be seen that the results of this formulation corresponding to the three laws differ from each other only by solitary, easily determinable terms, and hence can very conveniently be compared with one another.

§ 2.

Let a moved quantity of electricity, the amount of which does not affect the question, and which we will therefore take as a unit of electricity, be at the time t in the point x, y, z and have the velocity-components $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. Given, further, a galvanic current s', which may likewise be in motion. For the sake of simplicity we will provisionally assume that the current is linear, since this involves no important limitation, as a current which is not linear can always be supposed to be divided into an infinite number of infinitely thin current-threads which can be regarded as linear currents.

Let us now contemplate in the conductor of the current, first, a single particle of the flowing electricity: this has a double motion—first, the motion of the current in the conductor, and, secondly, the motion of the conductor itself. In order to distinguish from one another the variations of the quantities which come into consideration occasioned by these two motions, we will, in like manner as I have already done in a previous investigation*, introduce the following method of designation. The coordinates of a fixed point in the conductor we consider simply as functions of the time t; while for the determination of the coordinates of the particle of electricity flowing in the conductor we take as an auxiliary a

* "Ueber die Behandlung der zwischen linearen Strömen und Leitern stattfindenden ponderomotorischen und electromotorischen Kräfte nach dem electrodynamischen Grundgesetze," *Verhandl. des naturhist. Vereins der preuss. Rheinl. u. Westf.* vol. xxxiii. 1876; Wied. *Ann.* vol. i., and Clausius, *Mechan. Wärmetheorie*, Bd. ii. Abschn. x.

second variable, which defines the situation of the particle in the conductor—namely, the distance s' (measured on the curve of the conductor) of the particle from any initial point. Accordingly each coordinate of the particle is to be regarded as a function of t, while, again, s' itself can be considered a function of t. If, then, x', y', z' are the coordinates of the electricity-particle at the time t, the complete differential coefficient of each of those coordinates with respect to t divides into two terms containing the partial differential coefficients with respect to t and s', so that we obtain for each coordinate an equation of the following form—

$$\frac{dx'}{dt}=\frac{\partial x'}{dt}+\frac{\partial x'}{\partial s'}\frac{ds'}{dt}.$$

For the differential coefficient $\frac{ds'}{dt}$, which represents the velocity of the current, we will introduce a simple symbol; we will denote by c' the velocity of the flow of the positive electricity, and by $-c'_1$ that of the negative electricity, while we then remain at liberty, according to the special assumption we make respecting the behaviour of the two electricities, either to consider the quantities c' and c'_1 equal the one to the other, or to put one of them $=0$, or to ascribe to them any values different from one another. With the aid of this notation we get, instead of the preceding equation, the two following, which refer to the positive and negative electricities:—

$$\left.\begin{aligned}\frac{dx'}{dt}&=\frac{\partial x'}{\partial t}+c'\frac{\partial x'}{\partial s'},\\ \frac{dx'}{dt}&=\frac{\partial x'}{\partial t}-c'_1\frac{\partial x'}{\partial s'}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In a contingent second differentiation with respect to t, we should have to take into account that the quantities c' and c'_1 are again to be treated as functions of t and s', because at a fixed point of the conductor the current-velocity can vary with the time if the intensity of the current is variable, and also because at a fixed time the current-velocity can be different at different points of the conductor if the conductor has not everywhere an equal cross section and like quality.

The distance r between the particle of current-electricity in the conductor s' and the unit of electricity in the point x, y, z is likewise to be regarded as a function of t and s'; and the complete differential coefficients of r with respect to t are therefore to be formed in the following manner for the positive and negative current-electricites:—

$$\left.\begin{aligned}\frac{dr}{dt}&=\frac{\partial r}{\partial t}+c'\frac{\partial r}{ds'},\\ \frac{dr}{dt}&=\frac{\partial r}{\partial t}-c'_1\frac{\partial r}{\partial s'}.\end{aligned}\right\}\quad\ldots\ldots\quad(2)$$

In these the partial differential coefficient $\frac{\partial r}{\partial t}$ comprises the two changes undergone by r, on the one hand through the motion of the unit of electricity, and on the other through the motion of the element ds' of the conductor containing the particle of electricity; while $\frac{\partial r}{\partial s'}$ refers to the change produced in r by the current-motion of the electricity-particle which takes place in the conductor.

Employing this method of notation, the x components of the force which a current-element ds' exerts upon the moved unit of electricity may now be determined, first, according to the fundamental electrodynamic law advanced by me, because it is the most convenient for the working and furnishes the simplest expressions, to which, in order to obtain the expressions corresponding to the two other fundamental laws, certain terms must then be added.

§ 3.

According to my fundamental law the x component of the force which one moved particle e of electricity undergoes from another, e', is represented by the formula

$$ee'\left\{\frac{\partial\frac{1}{r}}{\partial x}\left[-1+k\left(\frac{dx}{dt}\frac{dx'}{dt}+\frac{dy}{dt}\frac{dy'}{dt}+\frac{dz}{dt}\frac{dz'}{dt}\right)\right]-k\frac{d}{dt}\left(\frac{1}{r}\frac{dx'}{dt}\right)\right\},$$

which, if we signify a sum of three terms alike in form, which refer to the three coordinate-directions, by writing only the term referring to the x direction and prefixing to it the symbol of summation, can be written somewhat shorter thus—

$$ee'\left[\frac{\partial\frac{1}{r}}{\partial x}\left(-1+k\Sigma\frac{dx}{dt}\frac{dx'}{dt}\right)-k\frac{d}{dt}\left(\frac{1}{r}\frac{dx'}{dt}\right)\right].$$

We now take, in the point x', y', z', a current-element ds' in which the quantity of positive electricity $h'ds'$ flows with the velocity c', and the negative $-h'ds'$ with the velocity $-c'_1$; and we will first determine the x components of that force which the positive electricity $h'ds'$ exerts upon the above-mentioned moved unit of electricity in the point x, y, z. For that purpose we have to substitute in the preceding expression 1 and $h'ds$ for e and e', by which we get

$$ds'\left[h'\frac{\partial\frac{1}{r}}{\partial x}\left(-1+k\Sigma\frac{dx}{dt}\frac{dx'}{dt}\right)-kh'\frac{d}{dt}\left(\frac{1}{r}\frac{dx'}{dt}\right)\right].$$

The product

$$h'\frac{d}{dt}\left(\frac{1}{r}\frac{dx'}{dt}\right)$$

we can bring into another form; and as the corresponding transformation has frequently to be employed in other cases also, we will presently work it out somewhat more generally. Let F be any quantity which, in the way stated of those referring to the positive current-electricity in the preceding section, depends on t and s'; then, in accordance with (1) and (2), we can write

$$\frac{d\mathrm{F}}{dt}=\frac{\partial\mathrm{F}}{\partial t}+c'\frac{\partial\mathrm{F}}{\partial s};$$

or, after multiplication by h',

$$h'\frac{d\mathrm{F}}{dt}=h'\frac{\partial\mathrm{F}}{\partial t}+h'c'\frac{\partial\mathrm{F}}{\partial s};$$

and this can be transformed into

$$h'\frac{d\mathrm{F}}{dt}=h'\frac{\partial\mathrm{F}}{\partial t}+\frac{\partial(h'c'\mathrm{F})}{\partial s'}-\mathrm{F}\frac{\partial(h'c')}{\partial s'}.$$

Herein another differential coefficient can be substituted for $\frac{\partial(h'c')}{\partial s'}$. The element ds' of the conductor is bounded by two cross sections corresponding to the lengths of arc s' and s' ds'. The two quantities of electricity which flow through these cross sections during the time dt, and of which the first passes into and the other passes out of the element ds', are represented by

$$h'c'dt \text{ and } \left(h'c'+\frac{\partial(h'c')}{\partial s'}ds'\right)dt;$$

and hence follows that the increase which takes place during the time dt, of the quantity of positive electricity in ds', is represented by

$$-\frac{\partial(h'c')}{\partial s'}ds'\,dt.$$

But the same increase can, on the other hand, be also denoted by

$$\frac{\partial h'}{\partial t} ds' \, dt;$$

and we consequently obtain the equation

$$\frac{\partial h'}{\partial t} = -\frac{\partial (h'c')}{\partial s'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Thereby the above equation is changed into

$$h' \frac{d\mathrm{F}}{dt} = h' \frac{\partial \mathrm{F}}{\partial t} + \frac{\partial (h'c'\mathrm{F})}{\partial s'} + \mathrm{F} \frac{\partial h'}{\partial t},$$

or, after contracting the first and last terms on the right-hand side, into

$$h' \frac{d\mathrm{F}}{dt} = \frac{\partial (h'\mathrm{F})}{\partial t} + \frac{\partial (h'c'\mathrm{F})}{\partial s'}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Returning now to the expression of the x component of the force exerted by the quantity $h'ds'$ of positive electricity upon the unit of electricity, and applying the preceding method of transformation to the term $h' \frac{d}{dt}\left(\frac{1}{r} \frac{dx'}{dt}\right)$, in which $\frac{1}{r} \frac{dx'}{dt}$ is the quantity that was previously denoted generally by F, the expression changes into

$$ds' \left[h' \frac{\partial \frac{1}{r}}{\partial x} \left(-1 + k\Sigma \frac{dx}{dt} \frac{dx'}{dt} \right) - k \frac{\partial}{\partial t} \left(\frac{h'}{r} \frac{dx'}{dt} \right) - k \frac{\partial}{\partial s'} \left(\frac{h'c'}{r} \frac{dx'}{dt} \right) \right].$$

The differential coefficient $\frac{dx'}{dt}$ in this may, lastly, pursuant to (1) be resolved into its two parts; the expression then takes the form

$$ds' \left\{ h' \frac{\partial \frac{1}{r}}{\partial x} \left[-1 + k\Sigma \frac{dx}{dt} \left(\frac{\partial x'}{\partial t} + c' \frac{\partial x'}{\partial s'} \right) \right] - k \frac{\partial}{\partial t} \left(\frac{h'}{r} \frac{\partial x'}{\partial t} + \frac{h'c'}{r} \frac{\partial x}{\partial s} \right. \right.$$
$$\left. \left. - k \frac{\partial}{\partial s'} \left(\frac{h'c'}{r} \frac{\partial x'}{\partial t} + \frac{h'c'^2}{r} \frac{\partial x'}{\partial s'} \right) \right\}.$$

We can now express in a corresponding manner also the x components of that force which the quantity $-h'ds'$ of negative electricity (whose current-velocity is $-c'_1$) contained in the element ds' exerts upon the unit of electricity. To this end we have to substitute $-h'$ for h', and $-c'_1$ for c', in the preceding expression, by which we get

$$ds'\left\{-h'\frac{\partial\frac{1}{r}}{\partial x}\left[-1+k\Sigma\frac{dx}{dt}\left(\frac{\partial x'}{\partial t}-c'_1\frac{\partial x'}{\partial s'}\right)\right]\right.$$
$$\left.-k\frac{\partial}{\partial t}\left(-\frac{h'}{r}\frac{\partial x'}{\partial t}+\frac{h'c'_1}{r}\frac{\partial x'}{\partial s'}\right)-k\frac{\partial}{\partial s'}\left(\frac{h'c'_1}{r}\frac{\partial x'}{\partial t}-\frac{h'c'^2_1}{r}\frac{\partial x'}{\partial s'}\right)\right\}.$$

The sum of these two expressions *represents the* x *component of the total force exerted by the current-element* ds' *upon the unit of electricity.* On forming that sum several terms cancel one another, and others admit of simplification from the fact that for the product $h'(c'+c'_1)$ the symbol i', which signifies the intensity of the current in ds', can be substituted, whence it at the same time follows that the product $h'(c'^2-c'^2_1)$, which can also be written in the form $h'(c'+c'_1)(c'-c'_1)$, can be replaced by $i'(c'-c'_1)$. Hence, if we denote by $\mathfrak{r}ds'$ the x component of the force which the current-element ds' exerts upon the unit of electricity, we get the equation

$$\mathfrak{r}=k\left[i'\frac{\partial\frac{1}{r}}{\partial x}\Sigma\frac{dx}{dt}\frac{\partial x'}{\partial s'}-\frac{\partial}{\partial t}\left(\frac{i'}{r}\frac{\partial x'}{\partial s'}\right)-\frac{\partial}{\partial s'}\left(\frac{i'}{r}\frac{\partial x'}{\partial t}+\frac{i'(c'-c'_1)}{r}\frac{\partial x'}{\partial s'}\right)\right]. \quad (5)$$

§ 4.

Riemann's fundamental law may now be treated in the same manner; and this is very easy in connexion with the foregoing.

The x component of the force which a moved electricity-particle e suffers from a moved electricity-particle e' is expressed, according to Riemann, by the formula

$$ee'\left\{\frac{\partial\frac{1}{r}}{\partial x}\left[-1-\frac{k}{2}\Sigma\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)^2\right]+k\frac{d}{dt}\left[\frac{1}{r}\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)\right]\right\}.$$

This formula can also be written as follows—

$$ee'\left[\frac{\partial\frac{1}{r}}{\partial x}\left(-1+k\Sigma\frac{dx}{dt}\frac{dx'}{dt}\right)-k\frac{d}{dt}\left(\frac{1}{r}\frac{dx'}{dt}\right)\right]$$
$$+ee'k\left\{-\frac{1}{2}\frac{\partial\frac{1}{r}}{\partial x}\Sigma\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dx'}{dt}\right)^2\right]+\frac{d}{dt}\left(\frac{1}{r}\frac{dx}{dt}\right)\right\}.$$

The first term of this expression agrees perfectly with the expression which according to my fundamental law represents the force-component in question; we can therefore use for this term the developments already carried out in the preceding section, and need only carry out the developments for the second term.

To determine the force exerted by a current-element ds' upon a moved unit of electricity, let us consider in the element first, again, the positive electricity $h'ds'$, which flows with the velocity c'. In order to express for this electricity the portion of the force-component which corresponds to the second term of the preceding expression, we have to substitute in it 1 and $h'\,ds'$ for e and e', whereby we obtain

$$kds'\left\{-\frac{h'}{2}\frac{\partial\frac{1}{r}}{\partial x}\Sigma\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dx'}{dt}\right)^2\right]+h'\frac{d}{dt}\left(\frac{1}{r}\frac{dx}{dt}\right)\right\}.$$

In this we put, in accordance with (1) and (2),

$$\frac{dx'}{dt}=\frac{\partial x'}{\partial t}+c'\frac{\partial x'}{\partial s'},$$

$$\frac{d}{dt}\left(\frac{1}{r}\frac{dx}{dt}\right)=\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{dx}{dt}\right)+c'\frac{\partial}{\partial s'}\left(\frac{1}{r}\frac{dx}{dt}\right),$$

by which the expression is changed into

$$kds'\left\{-\frac{h'}{2}\frac{\partial\frac{1}{r}}{\partial x}\Sigma\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{\partial x'}{\partial t}\right)^2+2c'\frac{\partial x'}{\partial t}\frac{\partial x'}{\partial s'}+c'^2\left(\frac{\partial x'}{\partial s'}\right)^2\right]\right.$$
$$\left.+h'\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{dx}{dt}\right)+h'c'\frac{\partial}{\partial s'}\left(\frac{1}{r}\frac{dx}{dt}\right)\right\}.$$

The corresponding expression for the negative electricity $-h'ds'$, which flows with the velocity $-c'_1$, is

$$kds'\left\{\frac{h'}{2}\frac{\partial\frac{1}{r}}{\partial x}\Sigma\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{\partial x'}{\partial t}\right)^2-2c'_1\frac{\partial x'}{\partial t}\frac{\partial x'}{\partial s'}+c'^2_1\left(\frac{\partial x'}{\partial s'}\right)^2\right]\right.$$
$$\left.-h'\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{dx}{dt}\right)+h'c'_1\frac{\partial}{\partial s'}\left(\frac{1}{r}\frac{dx}{dt}\right)\right\}.$$

By addition of these two expressions we get

$$kds'\left\{-i'\frac{\partial\frac{1}{r}}{dx}\Sigma\left[\frac{\partial x'}{\partial t}\frac{\partial x'}{\partial s'}+\frac{c'-c'_1}{2}\left(\frac{\partial x'}{\partial s'}\right)^2\right]+i'\frac{\partial}{\partial s'}\left(\frac{1}{r}\frac{dx}{dt}\right)\right\},$$

for which, on account of the self-evident equation

$$\Sigma\left(\frac{\partial x'}{\partial s'}\right)^2=\left(\frac{\partial x'}{\partial s'}\right)^2+\left(\frac{\partial y'}{\partial s'}\right)^2+\left(\frac{\partial z'}{\partial s'}\right)^2=1,$$

and because i' is indepndent of s' and hence in the last term can be put with the rest under the differentiation-symbol, can be also written

$$kds'\left[-i'\frac{\partial\frac{1}{r}}{\partial x}\left(\Sigma\frac{\partial x'}{\partial t'}\frac{\partial x'}{\partial s'}+\frac{c'-c'_1}{2}\right)+\frac{\partial}{\partial s'}\left(\frac{i'}{r}\frac{dx}{dt}\right)\right].$$

This is the constituent resulting from the *second* term of the above expression of the x component, of the force which the current-element ds' exerts upon a moved unit of electricity according to Riemann's fundamental law. The constituent resulting from the *first* term agrees, as already stated, with the value that holds good according to my fundamental law, of the force-component which we have denoted by $\mathfrak{x}ds'$ and determined in the preceding section. Hence, if we denote the total value of the force-component according to Riemann's fundamental law by $\mathfrak{x}_1 ds'$, we get

$$\mathfrak{x}_1=\mathfrak{x}+k\left[-i'\frac{\partial\frac{1}{r}}{\partial x}\left(\Sigma\frac{\partial x'}{\partial t}\frac{\partial x'}{\partial s'}+\frac{c'-c'_1}{2}\right)+\frac{\partial}{\partial s'}\left(\frac{i'}{r}\frac{dx}{dt}\right)\right]. \quad (6)$$

§ 5.

Now, thirdly, Weber's fundamental law must be treated in the same manner.

According to this law, between two moved particles of electricity e and e' a repulsion takes place the intensity of which is

$$\frac{ee'}{r^2}\left[1-\frac{k}{2}\left(\frac{dr}{dt}\right)^2+kr\frac{d^2r}{dt^2}\right]$$

and from this, by multiplication with $\frac{x-x'}{r}$, we obtain the x component of the force which the particle e suffers, thus—

$$ee'\frac{x-x'}{r^3}\left[1-\frac{k}{2}\left(\frac{dr}{dt}\right)^2+kr\frac{d^2r}{dt^2}\right].$$

In applying this expression to the quantity of electricity $h'ds'$ flowing in the current-element ds' with the velocity c', and to the moved unit of electricity, we have again first to replace e and e' by 1 and $h'ds'$. We will then, in accordance with (4) make the following transformation,

$$h'\frac{d^2r}{dt^2}=h'\frac{d}{dt}\left(\frac{dr}{dt}\right)=\frac{\partial}{\partial t}\left(h'\frac{dr}{dt}\right)+\frac{\partial}{\partial s'}\left(h'c'\frac{dr}{dt}\right),$$

and, besides, put everywhere

$$\frac{dr}{dt}=\frac{\partial r}{\partial t}+c'\frac{\partial r}{\partial s'}.$$

Then we get

$$ds'\frac{x-x'}{r^3}\left\{h'-\frac{k}{2}\left[h'\left(\frac{\partial r}{\partial t}\right)^2+2h'c'\frac{\partial r}{dt}\frac{\partial r}{\partial s'}+h'c'^2\left(\frac{\partial r}{\partial s'}\right)^2\right]\right.$$
$$\left.+kr\frac{\partial}{\partial t}\left(h'\frac{\partial r}{\partial t}+h'c'\frac{\partial r}{\partial s'}\right)+kr\frac{\partial}{\partial s'}\left(h'c'\frac{\partial r}{\partial t}+h'c'^2\frac{\partial r}{\partial s'}\right)\right\}.$$

Just so we obtain for the negative electricity $-h'ds'$, flowing with the velocity $-c'_1$,

$$ds'\frac{x-x'}{r^3}\left\{-h'-\frac{k}{2}\left[-h'\left(\frac{\partial r}{\partial t}\right)^2+2h'c'_1\frac{\partial r}{\partial t}\frac{\partial r}{\partial s'}-h'c'^2_1\left(\frac{\partial r}{\partial s'}\right)^2\right]\right.$$
$$\left.+kr\frac{\partial}{\partial t}\left(-h'\frac{\partial r}{\partial t}+h'c'_1\frac{\partial r}{\partial s'}\right)+kr\frac{\partial}{\partial s'}\left(h'c'_1\frac{\partial r}{\partial t}-h'c'^2_1\frac{\partial r}{\partial s'}\right)\right\}.$$

The sum of these two expressions represents the x component of the force which the entire current-element ds' must, according to Weber's fundamental law, exert upon the unit of electricity. If this is denoted by $\mathfrak{x}_2 ds'$, then we have

$$\mathfrak{x}_2=k\frac{x-x'}{r^3}\left\{-i'\frac{\partial r}{\partial t}\frac{\partial r}{\partial s'}+r\frac{\partial}{\partial t}\left(i'\frac{\partial r}{\partial s'}\right)-\frac{1}{2}i'(c'-c'_1)\left(\frac{\partial r}{\partial s'}\right)^2\right.$$
$$\left.+r\frac{\partial}{\partial s'}\left[i'\frac{\partial r}{\partial t}+i'(c'-c'_1)\frac{\partial r}{\partial s'}\right]\right\}. \quad (7)$$

This expression of $\mathfrak{x}_2$ can, like the above expression of $\mathfrak{x}_1$, be brought into such a form as to appear as the sum of $\mathfrak{x}$ and some superadded terms. For that purpose we will divide the preceding equation by k, then carry out on the right-hand side the suggested multiplication by $\frac{x-x'}{r^3}$, and at the same time resolve some of the terms. Above the resulting terms we will place numbers, in order to be able afterwards to designate them simply by the numbers:—

$$\frac{\mathfrak{x}_2}{k}=\overset{1}{-i'\frac{x-x'}{r^3}\frac{\partial r}{\partial t}\frac{\partial r}{\partial s'}}+\overset{2}{\frac{x-x'}{r^2}\frac{\partial}{\partial t}\left(i'\frac{\partial r}{\partial s'}\right)}-\overset{3}{\frac{i'(c'-c'_1)}{2}\frac{x-x'}{r^3}\left(\frac{\partial r}{\partial s'}\right)^2}$$
$$+\overset{4}{i'\frac{x-x'}{r^2}\frac{\partial^2 r}{\partial t\partial s'}}+\overset{5}{\frac{x-x'}{r^2}\frac{\partial}{\partial s'}\left[i'(c'-c'_1)\frac{\partial r}{\partial s'}\right]}.$$

In a similar manner we will treat the expression of $\mathfrak{x}$ given in equation (5); but at the same time we will besides transform the first term separately. We can, namely, put

$$\frac{\partial^2(r^8)}{\partial t\,\partial s'}=2\frac{\partial r}{\partial t}\frac{\partial r}{\partial s'}+2r\frac{\partial^2 r}{\partial t\,\partial s'};$$

and simultaneously we get from $r^2=\Sigma(x-x')^2$

$$\frac{\partial^2(r^2)}{\partial t\,\partial s'}=-2\Sigma\frac{dx}{dt}\frac{\partial x'}{\partial s'}-2\frac{\partial}{\partial s'}\Sigma(x-x')\frac{\partial x'}{\partial t}.$$

From the combination of these two equations results

$$\Sigma\frac{dx}{dt}\frac{\partial x'}{\partial s'}=-\frac{\partial r}{\partial t}\frac{\partial r}{\partial s'}-r\frac{\partial^2 r}{\partial t\,\partial s'}-\frac{\partial}{\partial s'}\Sigma(x-x')\frac{\partial x'}{\partial t}.$$

The algebraic sum standing here on the right-hand side we will insert in equation (5) for $\Sigma\frac{dx}{dt}\frac{\partial x'}{\partial s'}$. At the same time we will reverse all the signs of this equation, so that, after division by k, it will determine the quantity $-\frac{\mathfrak{x}}{k}$, in the following manner:—

$$-\frac{\mathfrak{x}}{k}=\overset{6}{-i'\frac{x-x'}{r^3}\frac{\partial r}{\partial t}\frac{\partial r}{\partial s'}}\overset{7}{-i'\frac{x-x'}{r^2}\frac{\partial^2 r}{\partial t\,\partial s'}}\overset{8}{-i'\frac{x-x'}{r^3}\frac{\partial}{\partial s'}\Sigma(x-x')\frac{dx'}{dt}}$$

$$\overset{9}{+\frac{\partial}{\partial t}\left(\frac{i'}{r}\right)\frac{\partial x'}{\partial s'}}\overset{10}{+\frac{i'}{r}\frac{\partial^2 x'}{\partial t\,\partial s'}}\overset{11}{+\frac{\partial}{\partial s'}\left(\frac{i'}{r}\frac{\partial x'}{\partial t}\right)}\overset{12}{+\frac{\partial}{\partial s'}\left(\frac{i'(c'-c'_1)}{r}\frac{\partial x'}{\partial s'}\right)}.$$

The twelve terms occurring in these two expressions form together the expression of $\frac{\mathfrak{x}_2-\mathfrak{x}}{k}$; and now it requires to be brought into a form as simple as possible and suitable for the further calculations; this can be done by suitable grouping of the terms. We obtain, namely, if we indicate the terms briefly by their numbers:—

$$4+7=0;$$

$$1+6+2+9=-\frac{\partial}{\partial s'}\left[(x-x')\frac{\partial}{\partial t}\left(\frac{i'}{r}\right)\right];$$

$$8+10=\frac{\partial}{\partial x}\left[\frac{i'}{r}\frac{\partial}{\partial s'}\Sigma(x-x')\frac{\partial x'}{\partial t}\right];$$

$$3+5+12=-\frac{\partial}{\partial x}\left[\frac{i'(c'-c'_1)}{2r}\left(\frac{\partial r}{\partial s'}\right)^2\right]-\frac{\partial}{\partial s'}\left[i'(c'-c'_1)\frac{\partial^2 r}{\partial x\,\partial s'}\right].$$

Herewith all the terms except the 11th, which will have to be taken into consideration separately, are brought into calculation; and hence we get, on the whole,

$$\frac{\mathfrak{x}_2-\mathfrak{x}}{k}=\frac{\partial}{\partial x}\left[\frac{i'}{r}\frac{\partial}{\partial s'}\Sigma(x-x')\frac{\partial x'}{\partial t}\right]-\frac{\partial}{\partial x}\left[\frac{i'(c'-c'_1)}{2r}\left(\frac{\partial r}{\partial s'}\right)^2\right]$$

$$-\frac{\partial}{\partial s'}\left[(x-x')\frac{\partial}{\partial t}\left(\frac{i'}{r}\right)\right]+\frac{\partial}{\partial s'}\left(\frac{i'}{r}\frac{\partial x'}{\partial t}\right)-\frac{\partial}{\partial s'}\left[i'(c'-c'_1)\frac{\partial^2 r}{\partial x\,\partial s'}\right];$$

and since all the terms herein are differential coefficients with respect to x or s', they can be collected into two differential coefficients. From this equation we get the sought expression of $\mathfrak{x}_2$, namely

$$\mathfrak{x}_2=\mathfrak{x}+k\frac{\partial}{\partial x}\left[\frac{i'}{r}\frac{\partial}{\partial s'}\Sigma(x-x')\frac{\partial x'}{\partial t}-\frac{i'(c'-c'_1)}{2r}\left(\frac{\partial r}{\partial s'}\right)^2\right]$$

$$-k\frac{\partial}{\partial s'}\left[(x-x')\frac{\partial}{\partial t}\left(\frac{i'}{r}\right)-\frac{i'}{r}\frac{\partial x'}{\partial t}+i'(c'-c'_1)\frac{\partial^2 r}{\partial x\, ds'}\right]. \quad (8)$$

§ 6.

In the three preceding sections the x component of the force exerted by a current-element ds' upon a moved unit of electricity is deduced from the three fundamental laws. In each of the three expressions (5), (6), and (8) there is a term which is a differential coefficient with respect to s', and which therefore vanishes in the integration over a *closed* current s'. Hence the force exerted by a closed current, or even by a system of closed currents, is represented by expressions of simplified form, which we will now consider more closely.

We start from the expression given in equation (5). When we multiply this by ds' and then integrate it over a closed current or a system of closed currents, we obtain the x component of that force which, according to my fundamental law, the current, or system of currents, must exert on a moved unit of electricity. These components being denoted by $\mathfrak{X}$, we get

$$\mathfrak{X}=k\int i'\frac{\partial\frac{1}{r}}{\partial x}\Sigma\frac{dx}{dt}\frac{\partial x'}{\partial s'}\,ds'-k\int\frac{\partial}{\partial t}\left(\frac{i'}{r}\frac{\partial x'}{\partial s'}\right)ds'. \quad . \quad . \quad (9)$$

In this equation it is tacitly presupposed that the length of the closed conductor s' remains unchanged, so that those elements ds' which at a given time form the closed conductor form it also during the succeeding time, and no element enters or leaves it. In reality, however, cases may occur in which the length of the conductor changes—for instance, when at a place a sliding of two parts of the conductor on one another takes place, causing parts of the conductor which were previously outside of the circuit to be afterwards within it, or *vice versâ.* In the parts which in this process are added the current commences; in those withdrawn it ceases; and by this alteration of the current-intensity in particular parts of the conductor a force is conditioned, which must also be taken into account. It is true that, on account of the great velocity with which the commencing and cessation of the current are accomplished, the parts of the conductor in which at any moment they take place are very small; but the differential coefficient $\frac{\partial i'}{\partial t}$ for them is very great, and through this the corresponding portion of the force may take a considerable value. The question now is, how this part of the force can be also expressed in the formula.

We will choose the places where the entry and exit of parts of the conductor take place as the initial and final points respectively of the closed conductor s', so that a newly entering piece of conductor is annexed exactly at the end of the conductor. If s'_1 denote the length of the conductor at the time t, the element of conductor added during the time-element will be represented by $\frac{ds'_1}{dt}dt$. If, further, the very short time requisite for the production of the current in a conductor-piece entering the closed circuit be denoted by τ, then will, during the lengthening of the conductor, a piece at the end of it of the length $\frac{ds'_1}{dt}\tau$ be that in which the origination of the current takes place. This origination is an increase, taking place during the time τ, from 0 to the value i' prevailing for the rest of the conduction. The mean value of the differential coefficient $\frac{\partial i}{\partial t}$ in this piece during the time τ is consequently $=\frac{i'}{\tau}$; and so we can represent the corresponding mean value of the differential coefficient $\frac{\partial}{\partial t}\left(\frac{i'}{r}\frac{\partial x'}{\partial s'}\right)$ by $\frac{1}{\tau}\left(\frac{i'}{r}\frac{\partial x'}{\partial s'}\right)_1$, wherein the index 1 put to the brackets is to indicate that the quantities r and $\frac{\partial x'}{\partial s'}$ within the brackets have the values belonging to s'_1.

Now, in order to bring likewise into calculation in our formula the origination of the current in this small piece of conductor, we have to add to the second integral in the formula, which if we write it with the limits has the form

$$\int_0^{s'_1}\frac{\partial}{\partial t}\left(\frac{i'}{r}\frac{\partial x'}{\partial s'}\right)ds',$$

a quantity which is the product of the just-determined mean differential coefficient and the length of the piece of conductor in question; thus

$$\frac{1}{\tau}\left(\frac{i'}{r}\frac{\partial x'}{\partial s'}\right)_1 \frac{ds'_1}{dt}\tau = \left(\frac{i'}{r}\frac{\partial x'}{\partial s'}\right)_1 \frac{ds'_1}{dt}.$$

Consequently we have to put in the place of the preceding integral the following sum—

$$\int_0^{s'_1} \frac{\partial}{\partial t}\left(\frac{i'}{r}\frac{\partial x'}{\partial s'}\right) ds' + \left(\frac{i'}{r}\frac{\partial x'}{\partial s'}\right)_1 \frac{ds'_1}{dt}.$$

But this sum is nothing else but the differential coefficient, taken with respect to t, of the integral

$$\int_0^{s'_1} \frac{i'}{r}\frac{\partial x'}{\partial s'} ds',$$

if therein not only the quantity under the integral-symbols but also the upper limit s'_1 be regarded as a function of t. The alteration to be undertaken with the above integral consists therefore only in this, that the differentiation there indicated under the integral-symbol is to be indicated before it. Moreover it must be remarked that the integral extended over the whole of the closed circuit is not, like one referred to a single conductor-element, to be looked on as a function of t and s', but as a function of t only, and that hence, in indicating the differentiation, d can be employed in this case instead of ∂, so that the expression will be

$$\frac{d}{dt}\int_0^{s'_1} \frac{i'}{r}\frac{\partial x'}{\partial s'} ds'.$$

Accordingly equation (9), when account is taken of the circumstance that the length of the conductor can change, changes into the following, in which we will now, for simplicity, omit the limits of the integral (the adding of which was expedient for the preceding consideration), because, after it has once been said that all the integrals are to be extended over the entire closed conductor, they are understood:—

$$\mathfrak{X} = k\frac{\partial}{\partial x}\int \frac{i'}{r}\Sigma\frac{dx}{dt}\frac{\partial x'}{\partial s'} ds' - k\frac{d}{dt}\int \frac{i'}{r}\frac{\partial x'}{\partial s'} ds'. \quad . \quad . \quad (10)$$

In the same way, denoting by $\mathfrak{X}_1$ and $\mathfrak{X}_2$ those values which the same force-component must take according to Riemann's and Weber's fundamental laws, we obtain from equations (6) and (8) the following equations:—

$$\mathfrak{X}_1=\mathfrak{X}-k\frac{\partial}{\partial x}\int\frac{i'}{r}\left(\Sigma\,\frac{\partial x'}{\partial t}\frac{\partial x'}{\partial s'}+\frac{c'-c'_1}{2}\right)ds'; \quad . \quad . \quad . \quad (11)$$

$$[\mathfrak{X}_2=\mathfrak{X}+k\frac{\partial}{\partial x}\int\frac{i'}{r}\left[\frac{\partial}{\partial s'}\,\Sigma(x-x')\frac{\partial x'}{\partial t}-\frac{c'-c'_1}{2}\left(\frac{\partial r}{\partial s'}\right)^2\right]ds'. (12)$$

Precisely corresponding expressions to those here derived for the x component of the force of course holds good also for y and z components.

§ 7.

The three force-components referring to the three directions of coordinates can now, in the manner discussed in § 1, be traced back to one quantity, from which they can be derived by differentiation. This is the electrodynamic potential of the closed current or system of currents upon the moved unit of electricity existing in the point x, y, z. Now, as with the forces which are independent of the motion that potential of a given agent which has reference to *a unit* of the same agent *supposed to be concentrated in a point* is by Green named the *potential function*, we will here also introduce the same distinction, and call the electrodynamic potential of a closed current or current-system, *so far as it refers to a unit of electricity supposed concentrated in a point*, the electrodynamic *potential function*.

This electrodynamic potential function is distinguished (as was mentioned in § 1), even externally, from Green's potential function, which refers to forces that are independent of the motion. It contains, namely, not merely the coordinates x, y, z of the unit of electricity, but also their velocity-components $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. Further, the operation by means of which the force-components are to be derived from the electrodynamic potential function is the same operation as that to which, according to Lagrange, the *vis viva*, expressed in universal coordinates, is to be submitted in the derivation of the components of the force. For if the electrodynamic potential function be denoted by Π, and the x component of the force by $\mathfrak{X}$, then the following equation can be formed:—

$$\mathfrak{X}=\frac{\partial\Pi}{\partial x}-\frac{d}{dt}\left(\frac{\partial\Pi}{\partial\frac{dx}{dt}}\right). \quad . \quad . \quad . \quad . \quad (13)$$

We have now to construct the forms of the potential function of a closed current corresponding to the three fundamental laws.

According to my fundamental law the electrodynamic potential of two quantities e and e' of electricity, supposed to be concentrated in points, is represented by

$$k\frac{ee'}{r}\Sigma\frac{dx}{dt}\frac{dx'}{dt}.$$

If in employing this formula we put for e the unit of electricity, and for e' successively the two quantities $h'ds'$ and $-h'ds'$ of electricity contained in a current-element ds', and in regard to the velocity-components of the latter take into account that they flow in the conductor in opposite directions with the velocities c' and c'_1, while they have in common any motion of the conductor, and if we then form the sum of these two expressions, putting $h'(c'+c'_1)=i'$, and, lastly, integrate this sum over the closed current, we get

$$\Pi=k\int\frac{i'}{r}\Sigma\frac{dx}{dt}\frac{\partial x'}{\partial s'}ds'. \quad . \quad . \quad . \quad . \quad (14)$$

If this expression of Π be now inserted in equation (13), we obtain from it in fact the value for $\mathfrak{X}$ determined by equation (10).

Since the velocity-components $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$, occurring in the expression of Π, are independent of the quantity s', with respect to which the integration is to be performed, we can put them outside of the integral-symbol and then give to the expression the following form:—

$$\Pi=k\Sigma\frac{dx}{dt}\int\frac{i'}{r}\frac{\partial x'}{\partial s'}ds'. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The sum here indicated contains three integrals, which differ from each other only by this—that in them either $\frac{\partial x'}{\partial s'}$, or $\frac{\partial y'}{\partial s'}$, or $\frac{\partial z'}{ds'}$ occurs. These three integrals, together with the factor k, we will, for brevity, represent by simple symbols, putting

$$\mathrm{H}_x=k\int\frac{i'}{r}\frac{\partial x'}{\partial s'}ds', \quad \mathrm{H}_y=k\int\frac{i'}{r}\frac{\partial y'}{\partial s'}ds', \quad \mathrm{H}_z=k\int\frac{i'}{r}\frac{\partial z'}{\partial s'}ds'. \quad (16)$$

Then we get

$$\Pi=\mathrm{H}_x\frac{dx}{dt}+\mathrm{H}\ \frac{dy}{dt}+\mathrm{H}_z\frac{dz}{dt}, \quad . \quad . \quad . \quad (17)$$

or, the sign of summation being employed,

$$\Pi = \Sigma \mathrm{H}_x \frac{dx}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17\,\text{A})$$

This changes equation (13) into

$$\mathfrak{X} = \frac{\partial \mathrm{H}_x}{\partial x}\frac{dx}{dt} + \frac{\partial \mathrm{H}_y}{\partial x}\frac{dy}{dt} + \frac{\partial \mathrm{H}^z}{\partial x}\frac{dz}{dt} - \frac{d\mathrm{H}_x}{dt}, \quad . \quad (18)$$

or, with the aid of the symbol of summation,

$$\mathfrak{X} = \frac{\partial}{\partial x}\Sigma \mathrm{H}_x \frac{dx}{dt} - \frac{d\mathrm{H}_x}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18\,\text{A})$$

According to the fundamental laws of Riemann and Weber, the electrodynamic potential of two moved quantities of electricity e and e', supposed concentrated in points, upon each other is represented by the expressions

$$-\frac{k}{2}\frac{ee'}{r}\Sigma\left(\frac{dx}{dt} - \frac{dx'}{dt}\right)^2,$$

$$-\frac{k}{2}\frac{ee'}{r}\left(\frac{dr}{dt}\right)^2.$$

From these are obtained for the potential of a closed current s' upon a unit of electricity, consequently for the *potential function* of the closed current, which according to these laws may be denoted by Π_1 and Π_2, the expressions :—

$$\Pi_1 = k\int\frac{i'}{r}\left[\Sigma\left(\frac{dx}{dt} - \frac{\partial x'}{\partial t}\right)\frac{\partial x'}{\partial s'} - \frac{c' - c'_1}{2}\right]ds'; \quad . \quad (19)$$

$$\Pi_1 = -k\int\frac{i'}{r}\left[\frac{\partial r}{\partial t}\frac{\partial r}{\partial s'} + \frac{c' - c'_1}{2}\left(\frac{\partial r}{\partial s'}\right)^2\right]ds'. \quad . \quad . \quad (20)$$

The latter expression can be transformed in the following manner. From

$$r^2 = \Sigma(x - x')^2$$

is obtained

$$r\frac{\partial r}{dt} = \Sigma(x - x')\left(\frac{dx}{dt} - \frac{\partial x'}{\partial t}\right)$$

$$= \Sigma(x - x')\frac{dx}{dt} - \Sigma(x - x')\frac{\partial x'}{\partial t};$$

and from this we get, further, by differentiation with respect to s',

$$\frac{\partial r}{\partial t}\frac{\partial r}{\partial s'} + r\frac{\partial^2 r}{\partial t\,\partial s'} = -\Sigma\frac{dx}{dt}\frac{\partial x'}{\partial s'} - \frac{\partial}{\partial s'}\Sigma(x - x')\frac{\partial x'}{\partial t},$$

and consequently

$$\frac{1}{r}\frac{\partial r}{\partial t}\frac{\partial r}{\partial s'}=-\frac{1}{r}\Sigma\frac{dx}{dt}\frac{\partial x'}{\partial s'}-\frac{1}{r}\frac{\partial}{\partial s'}\Sigma(x-x')\frac{\partial x'}{\partial t}-\frac{\partial^2 r}{\partial t\,\partial s'}.$$

Putting now, in equation (20), for $\frac{1}{r}\frac{\partial r}{\partial t}\frac{\partial r}{\partial s}$, the expression here found, whose last term gives 0 in the integration, we obtain

$$\Pi_2=k\int\frac{i'}{r}\left[\Sigma\frac{dx}{dt}\frac{\partial x'}{\partial s'}+\frac{\partial}{\partial s'}\Sigma(x-x')\frac{\partial x'}{\partial t}-\frac{c'-c'_1}{2}\left(\frac{\partial r}{\partial s'}\right)^2\right]ds'. \quad (21)$$

In the two expressions (19) and (21) of Π_1 and Π_2 the first term arising on the resolution of the brackets agrees with the expression of Π given under (14); hence we can write:

$$\Pi_1=\Pi-k\int\frac{i'}{r}\left(\Sigma\frac{\partial x'}{\partial t}\frac{\partial x'}{\partial s'}+\frac{c'-c'_1}{2}\right)ds', \quad . \quad . \quad . \quad . \quad (22)$$

$$\Pi_2=\Pi+k\int\frac{i'}{r}\left[\frac{\partial}{\partial s'}\Sigma(x-x')\frac{\partial x'}{\partial t}-\frac{c'-c'_1}{2}\left(\frac{\partial r}{\partial s'}\right)^2\right]ds'. \quad (23)$$

If we now form, corresponding with equations (13), the equations

$$\mathfrak{X}_1=\frac{\partial \Pi_1}{\partial x}-\frac{d}{dt}\left(\frac{\partial \Pi_1}{\partial \frac{dx}{dt}}\right), \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$\mathfrak{X}_2=\frac{\partial \Pi_2}{\partial x}-\frac{d}{dt}\left(\frac{\partial \Pi_2}{\partial \frac{dx}{dt}}\right), \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and if, in these, for Π_1 and Π_2 we employ the previously given expressions, in which the terms added to Π do not contain the velocity-components $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$, and hence give 0 on differentiation with respect to these quantities, we obtain for $\mathfrak{X}_1$ and $\mathfrak{X}_2$ the expressions given under (11) and (12).

For abbreviation, simple symbols may be brought in for those additional terms independent of $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$, by putting

$$G_1=-k\int\frac{i'}{r}\Sigma\frac{\partial x'}{\partial t}\frac{\partial x'}{\partial s'}+\frac{c'-c'_1}{2}\Big)ds', \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$G_2=k\int\frac{i'}{r}\left[\frac{\partial}{\partial s'}\Sigma(x-x')\frac{\partial x'}{\partial t}-\frac{c'-c'_1}{2}\left(\frac{\partial r}{\partial s'}\right)^2\right]ds'. \quad . \quad (27)$$

The result is

$$\Pi_1 = \Pi + G_1, \quad \ldots \ldots \quad (28)$$

$$\Pi_2 = \Pi + G_2, \quad \ldots \ldots \quad (29)$$

whereby equations (24) and (25) are changed into the following,

$$\mathfrak{X}_1 = \frac{\partial(\Pi + G_1)}{\partial x} - \frac{d}{dt}\left(\frac{\partial \Pi}{\partial \frac{dx}{dt}}\right), \quad \ldots \quad (30)$$

$$\mathfrak{X}_2 = \frac{\partial(\Pi + G_2)}{\partial x} - \frac{d}{dt}\left(\frac{\partial \Pi}{\partial \frac{dx}{dt}}\right), \quad \ldots \quad (31)$$

which in conjunction with (13) are very convenient for comparison of the results of the three fundamental laws.

The electrodynamic potential function of a closed current (or system of currents) above introduced, and denoted, in its three forms corresponding to the three fundamental laws by Π, Π_1, and Π_2, is readily perceived to be very different from that potential function of which the differential coefficients occur already in Ampère's theory of the ponderomotive forces, and which in a previously published analysis* I named the *magnetic* potential function of the closed current, and denoted by P. This latter is obtained when, in the well-known manner, two magnetic surfaces are imagined to be substituted for the closed current, and then Green's potential function is formed for the quantities of magnetism present on those surfaces; accordingly its obvious signification is that it represents by its differential coefficients with respect to x, y, and z, taken negatively, the components which fall into the directions of the coordinates, of that force which the closed current exerts upon a unit of *magnetism* conceived as situated in the point x, y, z. It can only serve indirectly, and with the aid of special theoretical considerations, for the determination of the ponderomotive force exerted upon a *current-element* and of the electromotive force induced in it. The electrodynamic potential, on the contrary, which can be used directly for the determination of the force exerted upon a moved *unit of electricity*, needs only to be applied to the electricity in the conductor in order at once to determine the ponderomotive and the electromotive force.

* *Die mechanische Behandlung der Electricität,* Abschnitt VIII. p. 211.

§ 8.

Now, in order to deduce from the preceding formulæ the *ponderomotive* force exerted upon a current-element by a closed current, we first form from the potential function the potentials of the closed current upon the two electricities flowing in the current-element. From these we get, by the above-stated operation, the components, in any one direction (*e. g.* the x direction), of the forces which the two electricities undergo; and the sum of these two components is the respective force-component which refers to the entire current-element.

Suppose, then, given, in the point x, y, z, an element of current ds, in which the electricities hds and $-hds$ flow in opposite directions with the velocities c and c_1. Now, by my fundamental law we first employ for the potential function the value

$$\Pi = \Sigma \mathrm{H}_x \frac{dx}{dt},$$

given in equation (17 A), and obtain for the quantity hds of positive electricity:—

$$\text{Potential} = hds \Sigma \mathrm{H}_x \frac{dx}{dt},$$

$$\text{Force-component} = hds \left(\frac{\partial}{\partial x} \Sigma \mathrm{H}_x \frac{dx}{dt} - \frac{d\mathrm{H}_x}{dt} \right).$$

In these we must put

$$\frac{dx}{dt} = \frac{\partial x}{dt} + c \frac{\partial x}{\partial s},$$

$$\frac{d\mathrm{H}_x}{dt} = \frac{\partial \mathrm{H}_x}{\partial t} + c \frac{\partial \mathrm{H}_x}{\partial s},$$

by which the expressions are changed into:—

$$\text{Potential} = hds \Sigma \mathrm{H}_x \left(\frac{\partial x}{\partial t} + c \frac{\partial x}{\partial s} \right),$$

$$\text{Force-comp.} = hds \left[\frac{\partial}{\partial x} \Sigma \mathrm{H}_x \left(\frac{\partial x}{\partial t} + c \frac{\partial x}{\partial s} \right) - \frac{\partial \mathrm{H}_x}{\partial t} - c \frac{\partial \mathrm{H}_x}{\partial s} \right].$$

In precisely the same way we get for the quantity $-hds$ of negative electricity (for which we must bring into use the velocity of flow $-c_1$):—

$$\text{Potential} = -hds \Sigma \mathrm{H}_x \left(\frac{\partial x}{\partial t} - c_1 \frac{\partial x}{\partial s} \right),$$

Force-component

$$= -hds \frac{\partial}{\partial x} \Sigma \mathrm{H}_x \left(\frac{\partial x}{\partial t} - c_1 \frac{\partial x}{\partial s} \right) - \frac{\partial \mathrm{H}_x}{\partial t} + c_1 \frac{\partial \mathrm{H}_x}{\partial s} \Big].$$

If we now add the expressions referring to the two electricities, we get for the total current-element ds:—

$$\text{Potential} = hds(c+c_1)\Sigma \mathrm{H}_x \frac{\partial x}{\partial s},$$

$$\text{Force-component} = hds(c+c_1)\left(\frac{\partial}{\partial x}\Sigma \mathrm{H}_x \frac{\partial x}{\partial s} - \frac{\partial \mathrm{H}_x}{\partial s}\right),$$

or, if i denotes the product $h(c+c_1)$, which signifies the current-intensity in ds,

$$\text{Potential} = ids\Sigma \mathrm{H}_x \frac{\partial x}{\partial s},$$

$$\text{Force-component} = ids\left(\frac{\partial}{\partial x}\Sigma \mathrm{H}_x \frac{\partial x}{\partial s} - \frac{\partial \mathrm{H}_x}{\partial s}\right).$$

We will now denote the potential of the closed current upon the current-element ds by $\mathrm{U}ds$, and the x component of the force undergone by the current-element by Ξds; we have then, for the determination of U, taking also into account equations (16), to put

$$\mathrm{U} = i\Sigma \mathrm{H}_x \frac{\partial x}{\partial s} = ki\int \frac{i'}{r}\Sigma \frac{\partial x}{\partial s}\frac{\partial x'}{\partial s'}ds'; \quad . \quad . \quad . \quad (32)$$

and regarding this quantity U as a function of x, y, z, $\frac{\partial x}{\partial s}$, $\frac{\partial y}{\partial s}$, $\frac{\partial z}{\partial s}$, we can give to the expression of Ξ the following form:—

$$\Xi = \frac{\partial \mathrm{U}}{\partial x} - \frac{\partial}{\partial s}\left(\frac{\partial \mathrm{U}}{\partial \frac{\partial x}{\partial s}}\right). \quad . \quad . \quad . \quad . \quad (33)$$

If, instead of the potential function Π corresponding to my fundamental law, the potential-function $\Pi_1 = \Pi + \mathrm{G}_1$ or $\Pi_2 = \Pi + \mathrm{G}_2$, corresponding to the fundamental laws of Riemann or Weber, be employed, we have only to take also into account separately the additional term G_1 or G_2. But this, since it is independent of the velocity-components $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, is equal for both the electricities flowing in ds, and hence, after multiplication by hds and $-hds$, is cancelled in the addition. Accordingly, in regard to the potential of a closed current upon a current-element, and in regard to the ponderomotive force exerted by a closed current upon an element of the current, there is no difference between the three laws; equations (32) and (33) hold in all three cases*.

* I will here incidentally remark that if we had been treating only of the ponderomotive force, and not at the same time of the electromotive

§ 9.

We now turn to the determination of the *electromotive* force which is induced in an element of the conductor by a closed current or system of currents.

For it we have only to determine the component of the force

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force also, the consideration of the subject could have been simplified. That is to say, we obtain for the ponderomotive force, even with single current-elements acting upon one another, expressions which contain not the velocities of the positive and the negative electricity as quantities to be separately dealt with, but only the current-intensity on the whole. According to my fundamental law the expressions for this case have the very same form as for the case in which the current exerting the force is closed. If the potential of the two current-elements ds and ds' upon each other is denoted by $u\,ds\,ds'$, and the x component of the force which ds suffers from ds' by $\xi ds\,ds'$, then we can put

$$u=k\frac{ii'}{r}\Sigma\frac{\partial x}{\partial s}\frac{\partial x'}{\partial s'},$$

$$\xi=\frac{\partial u}{\partial x}-\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial\frac{\partial x}{\partial s}}\right).$$

According to Riemann's fundamental law the same expression holds good for the potential; but the operation to be employed for the derivation of the force-component is somewhat more complicated, namely

$$\xi_1=\frac{\partial u}{\partial x}-\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial\frac{\partial x}{\partial s}}\right)+\frac{\partial}{\partial s'}\left(\frac{\partial u}{\partial\frac{\partial x'}{\partial s'}}\right).$$

Finally, according to Weber's fundamental law, for the potential, which in this case may be denoted by $u_2 ds\,ds'$, the equation

$$u_2=-k\frac{ii'}{r}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}=kii'\left(\frac{1}{r}\Sigma\frac{\partial x}{\partial s}\frac{\partial x'}{\partial s'}+\frac{\partial^2 r}{\partial s\,\partial s'}\right)$$

is valid; and for the derivation of the force-component the same operation as with Riemann's law is to be employed, namely

$$\xi_2=\frac{\partial u_2}{\partial x}-\frac{\partial}{\partial s}\left(\frac{\partial u_2}{\partial\frac{\partial x}{\partial s}}\right)+\frac{\partial}{\partial s'}\left(\frac{\partial u_2}{\partial\frac{\partial x'}{\partial s'}}\right).$$

According to this the ponderomotive force can be deduced from the potential of each two current-elements upon one another; but this potential, notwithstanding its partially correspondent form, is clearly to be distinguished from the quantity which is obtained when, of Neumann's potential of two closed currents upon one another, the part corresponding to two single current-elements ds and ds' is taken. For Neumann's potential is the *magnetic* potential, and consequently a potential of the same sort as Green's, while the thing in question here is the *electrodynamic* potential, on which account also an operation quite other than with Green's potential is requisite in order to derive the force-components.

exerted in the direction of the conductor-element by the current or system of currents upon a unit of electricity (to which we can ascribe any velocity of flow, c, we please) imagined in the conductor-element. The force-components falling into the directions of the coordinates are, acccording to our previous notation, to be represented by $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$; and, in correspondence with this, we will denote by $\mathfrak{S}$ the force-component falling into the direction of the element ds, therefore into the s direction. We have then to put

$$\mathfrak{S}=\mathfrak{X}\frac{\partial x}{\partial s}-\mathfrak{Y}\frac{\partial y}{\partial s}+\mathfrak{Z}\frac{\partial z}{\partial s}=\Sigma\mathfrak{X}\frac{\partial x}{\partial s}. \quad . \quad . \quad (34)$$

In this we must now insert for the quantities $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ their values resulting from the three fundamental laws.

According to my fundamental law, in accordance with (13) we can put

$$\mathfrak{X}=\frac{\partial \Pi}{\partial x}-\frac{d}{dt}\left(\frac{\partial \Pi}{\partial \frac{dx}{dt}}\right);$$

and consequently

$$\mathfrak{S}=\Sigma\frac{\partial \Pi}{\partial x}\frac{\partial x}{\partial s}-\Sigma\frac{\partial x}{\partial s}\frac{d}{dt}\left(\frac{\partial \Pi}{\partial \frac{dx}{dt}}\right).$$

If herein we use for Π the expression given under (17), namely

$$\Pi=\mathrm{H}_x\frac{dx}{dt}+\mathrm{H}_y\frac{dy}{dt}+\mathrm{H}_z\frac{dz}{dt},$$

we can put, if we wish to write all the terms singly,

$$\begin{aligned}\Sigma\frac{\partial \Pi}{\partial x}\frac{\partial x}{\partial s}=&\frac{dx}{dt}\left(\frac{\partial \mathrm{H}_x}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial \mathrm{H}_x}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial \mathrm{H}_x}{\partial z}\frac{\partial z}{\partial s}\right)\\&+\frac{dy}{dt}\left(\frac{\partial \mathrm{H}_y}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial \mathrm{H}_y}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial \mathrm{H}_y}{\partial z}\frac{\partial z}{\partial s}\right)\\&+\frac{dz}{dt}\left(\frac{\partial \mathrm{H}_z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial \mathrm{H}_z}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial \mathrm{H}_z}{\partial z}\frac{\partial z}{\partial s}\right).\end{aligned}$$

Now, since the quantities H_x, H_y, and H_z depend on s only inasmuch as the coordinates occurring in them, x, y, z, of the unit of electricity, are dependent on s, the three sums in brackets represent the differential coefficients of the three quantities with respect to s; and hence we can write:—

$$\Sigma\frac{\partial \Pi}{\partial x}\frac{\partial x}{\partial s}=\frac{dx}{dt}\frac{\partial \mathrm{H}_x}{\partial s}+\frac{dy}{dt}\frac{\partial \mathrm{H}_y}{\partial s}+\frac{dz}{dt}\frac{\partial \mathrm{H}_z}{\partial s},$$

or, now bringing in again the symbol of summation on the right-hand side,

$$\Sigma\frac{\partial \Pi}{\partial x}\frac{\partial x}{\partial s}=\Sigma\frac{\partial \mathrm{H}_x}{\partial s}\frac{dx}{dt}.$$

Accordingly the above equation for $\mathfrak{S}$ passes into

$$\mathfrak{S}=\Sigma\frac{\partial \mathrm{H}_x}{\partial s}\frac{dx}{dt}-\Sigma\frac{\partial x}{\partial s}\frac{d\mathrm{H}_x}{dt}. \quad . \quad . \quad . \quad (35)$$

As, then, the unit of electricity has a double motion, namely the motion of the conductor-element and the flowing motion which takes place in the conductor-element with the velocity c, we will, in correspondence with the notation we have previously employed, put

$$\frac{dx}{dt}=\frac{\partial x}{\partial t}+c\frac{\partial x}{\partial s},$$

$$\frac{d\mathrm{H}_x}{dt}=\frac{\partial \mathrm{H}_x}{\partial t}+c\frac{\partial \mathrm{H}_x}{\partial s},$$

in which the differentiation indicated by $\frac{\partial}{\partial t}$ refers to the variations which are independent of the flowing motion of the unit of electricity. We thus obtain

$$\mathfrak{S}=\Sigma\frac{\partial \mathrm{H}_x}{\partial s}\left(\frac{\partial x}{\partial t}+c\frac{\partial x}{\partial s}\right)-\Sigma\frac{\partial x}{\partial s}\left(\frac{\partial \mathrm{H}_x}{\partial t}+c\frac{\partial \mathrm{H}_x}{\partial s}\right).$$

Here the terms containing the factor c cancel one another, and there remains

$$\mathfrak{S}=\Sigma\frac{\partial \mathrm{H}_x}{\partial s}\frac{\partial x}{\partial t}-\Sigma\frac{\partial \mathrm{H}_x}{\partial t}\frac{\partial x}{\partial s}. \quad . \quad . \quad . \quad . \quad (36)$$

To this expression of $\mathfrak{S}$ we can give a somewhat different form by adding the quantity

$$\Sigma \mathrm{H}_x\frac{\partial^2 x}{\partial t\,\partial s},$$

positive to the first term, and negative to the second term. The two terms then become differential coefficients with respect to s and t, and we get

$$\mathfrak{S}=\frac{\partial}{\partial s}\Sigma \mathrm{H}_x\frac{\partial x}{\partial t}-\frac{\partial}{\partial t}\Sigma \mathrm{H}_x\frac{\partial x}{\partial s}. \quad . \quad . \quad . \quad (37)$$

Finally, if in this we put for H_x and the two other quantities contained in the sums (H_y and H_z) their values determined by equations (16), we get

$$\mathfrak{S}=k\frac{\partial}{\partial s}\int\frac{i'}{r}\Sigma\frac{\partial x}{\partial t}\frac{\partial x'}{\partial s'}ds'-k\frac{\partial}{\partial t}\int\frac{i'}{r}\Sigma\frac{\partial x}{\partial s}\frac{\partial x'}{\partial s'}ds'. \quad . \quad (38)$$

This is the most convenient form of the expression of $\mathfrak{S}$ resulting from my fundamental law; and the product $\mathfrak{S}ds$ is the electromotive force induced in a conductor-element ds by a closed current or system of currents.

To obtain the corresponding expressions for Riemann and Weber's fundamental laws, we need only in the formulæ (28) and (29), representing the potential function, to take separately into consideration the added terms G_1 and G_2, which do not contain the velocity-components $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, and therefore are to be differentiated only with respect to x, y, z. As we can now again form for G_1 the equation

$$\frac{\partial G_1}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial G_1}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial G_1}{\partial z}\frac{\partial z}{\partial s}=\frac{\partial G_1}{\partial s},$$

and for G_2 the corresponding equation, we obtain, denoting the electromotive force according to Riemann and Weber's fundamental laws by $\mathfrak{S}_1$ and $\mathfrak{S}_2$:—

$$\mathfrak{S}_1=\mathfrak{S}+\frac{\partial G_1}{\partial s}, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

$$\mathfrak{S}_2=\mathfrak{S}+\frac{\partial G_2}{\partial s}. \quad . \quad . \quad . \quad . \quad . \quad (40)$$

These expressions represent very clearly the difference between the electromotive forces resulting from the three fundamental laws.

From the developments carried out in the last two sections it will, I think, be sufficiently apparent how much the introduction of the electrodynamic potential function of closed currents contributes to giving to the entire department of electrodynamics with which we are concerned a uniform character—the knowledge of that one quantity being sufficient, without any accessory assumption, for the derivation of every thing further by simple analytical operations.