TRANSFORMATION GROUPS APPLIED TO ORDINARY DIFFERENTIAL EQUATIONS.

By Dr. James M. Page, Cobham, Va.

We propose in the following article to show in as elementary a manner as possible how Transformation Groups may be utilized in integrating certain differential equations of the first order. We shall limit ourselves to two variables; but the most important of the results may be extended at once to n variables.

The matter is, for the most part, Lie's; and it was obtained chiefly from the writer's notes on Lie's lectures in Leipzig in 1887–8.

It was shown in a previous artice (Annals, Vol. VIII, No. 4) that, if

$$\omega(x, y) = \text{const.} \tag{1}$$

be any family of ∞^1 curves in the plane; and if there is given an infinitesimal transformation of the form

$$Uf \equiv \xi(x, y) \frac{\partial f}{\partial x} + \eta(x, y) \frac{\partial f}{\partial y},$$

then the curves (1) are said to *admit of* the transformation Uf, when, and only when,

$$U(\omega) \equiv \Omega(\omega). \tag{2}$$

The criterion (2) holds also for the group of one member, or the G_1, which is *generated* by Uf; thus it is said that the family of curves (1) *admits of* the G_1, Uf, when the equation (2) is satisfied. It was also shown that

$$U(\omega) \equiv \xi \frac{\partial \omega}{\partial x} + \eta \frac{\partial \omega}{\partial y};$$

and if this expression becomes identically zero, the curves (1) are called the *loci* of the G_1, Uf.

The family of ∞^1 curves (1) is, of course, also represented by a differential equation of the first order, of the form

$$X(x, y)\, dy - Y(x, y)\, dx = 0, \tag{3}$$

of which $\omega(x, y)$ is the integral.

Hence, ω must satisfy a linear partial differential equation of the first order, of the form

$$X \frac{\partial \omega}{\partial x} + Y \frac{\partial \omega}{\partial y} = 0;$$

and every function of ω, $\Phi(\omega)$, must also satisfy this equation, and be an integral of (3).

If we perform the transformation Uf on $\Phi(\omega)$, we find

$$U(\Phi) \equiv \frac{d\Phi}{d\omega} \cdot U(\omega) \equiv \frac{d\Phi}{d\omega} \cdot \Omega(\omega) .$$

If the curves (1) are not the *loci* of the G_1, Uf,—that is, if $\Omega(\omega)$ is not zero,—we can evidently always choose Φ as a function of u in such a manner that

$$\frac{d\Phi}{d\omega} \cdot \Omega(\omega) = 1 .$$

Let us suppose Φ so chosen; then we have,

$$X \frac{\partial \Phi}{\partial x} + Y \frac{\partial \Phi}{\partial y} = 0 ,$$

$$U(\Phi) \equiv \xi \frac{\partial \Phi}{\partial x} + \eta \frac{\partial \Phi}{\partial y} = 1 .$$

Hence,

$$\frac{\partial \Phi}{\partial x} = - \frac{Y}{X\eta - Y\xi}, \quad \frac{\partial \Phi}{\partial y} = - \frac{X}{X\eta - Y\xi} .$$

Then

$$d\Phi \equiv \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy = \frac{Xd\eta - Ydx}{X\eta - Y\xi} .$$

The last expression must be a *complete differential;* that is,

$$U \equiv \frac{1}{Y\eta - Y\xi}$$

must be an integrating factor of (3).

A differential equation is said to *admit of* a transformation, when, after carrying out the transformation, the differential equation preserves in the new variables its original form, with the exception of a factor which may be canceled. Thus the equation

$$X(x, y)\, dy - Y(x, y)\, dx = 0 \qquad (3)$$

is said to admit of the transformation

$$x_1 = \varphi(x, y) , \quad y_1 = \psi(x, y) ,$$

if, in the new variables, (3) assumes the form

$$\rho(x_1, y_1) \{X(x_1, y_1)\, dx_1 - Y(x_1, y_1)\, dy_1\} = 0 ,$$

ρ being any function of x_1, y_1.

It is quite easy to show by a rigid proof the almost obvious fact that a differential equation (3) will admit of the infinitesimal transformation Uf,—or, as we may say, of the G_1, Uf,—when and only when the family of the ∞^1 integral curves of (3) admits of Uf.

Hence we find the important result:

If a given differential equation of the first order in x and y,

$$X(x, y)\, dy - Y(x, y)\, dx = 0\,;$$

admits of the G_1, Uf, where the infinitesimal transformation of the G_1 has the general form

$$Uf \equiv \xi(x, y) \frac{\partial f}{\partial x} + \eta(x, y) \frac{\partial f}{\partial y}\,,$$

then

$$U \equiv \frac{1}{X\eta - Y\xi}$$

is an integrating factor of the differential equation.

The condition must be fulfilled here that $X\eta - Y\xi$ is not identically 0. It can be readily seen that if $X\eta - Y\xi \equiv 0$, then the curves (1) are the *loci* of the G_1, Uf; and we say in this case that the transformation Uf is *trivial* as regards the differential equation, since it tells us nothing new.

It is, of course, necessary to develop a practical criterion to tell *when* a given differential equation (3) will admit of a given infinitesimal transformation, Uf.

If the differential equation be taken in the form

$$Xdy - Ydx = 0\,, \tag{3}$$

then to find an integral of (3) is the same problem as to find a solution of the linear partial differential equation,

$$Af \equiv X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} = 0\,.$$

If, thus, $\omega(x, y)$ is an integral of (3),

$$A(\omega) \equiv X\frac{\partial \omega}{\partial x} + Y\frac{\partial \omega}{\partial y} \equiv 0\,.$$

Since (3) admits, by hypothesis, of Uf,

$$U(\omega) \equiv \Omega(\omega)\,.$$

We shall write, as usual,

$$U(Af) - A(Uf) \equiv \{U(X) - A(\xi)\}\frac{\partial f}{\partial x} + \{U(Y) - A(\eta)\}\frac{\partial f}{\partial y}\,;$$

and if we put $f \equiv \omega$ in this identity, remembering that

$$U\big(A(\omega)\big) \equiv U(0) \equiv 0\,,$$

$$A\big(U(\omega)\big) \equiv A\big(\Omega(\omega)\big) \equiv \frac{d\Omega}{d\omega}\,.\,A(\omega) \equiv 0\,,$$

we find

$$\{U(X) - A(\xi)\}\frac{\partial\omega}{\partial x} + \{U(Y) - A(\eta)\}\frac{\partial\omega}{\partial y} \equiv 0\,.$$

Also, since

$$X\frac{\partial\omega}{\partial x} + Y\frac{\partial\omega}{\partial y} \equiv 0\,;$$

from the last two identities

$$\frac{U(X) - A(\xi)}{X} \equiv \frac{U(Y) - A(\eta)}{Y}\,.$$

If we call the value of these fractions $\lambda(x, y)$, this identity gives

$$U(X) - A(\xi) \equiv \lambda\,.\,X, \quad U(Y) - A(\eta) \equiv \lambda\,.\,Y\,.$$

Hence, for all values of f,

$$\{U(X) - A(\xi)\}\frac{\partial f}{\partial x} + \{U(Y) - A(\eta)\}\frac{\partial f}{\partial y} \equiv \lambda(x, y)\left\{X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y}\right\},$$

or, as it may be written,

$$U(Af) - A(Uf) \equiv \lambda(x, y)\,Af\,. \tag{4}$$

This is the condition that the differential equation (3) should admit of Uf. If, conversely, a condition (4) holds, the differential equation (3) must admit of Uf.

For, if ω be an integral of (3), $A(\omega) \equiv U\big(A(\omega)\big) \equiv 0$; or, from (4) :

$$A\big(U(\omega)\big) \equiv \xi\frac{\partial U(\omega)}{\partial x} + \eta\frac{\partial U(\omega)}{\partial y} \equiv 0\,.$$

The last equation shows that $U(\omega)$ is a function of ω alone. Hence the integral curves of (3), that is, (3) itself will admit of Uf, if the condition (4) holds. Hence,

The differential equation

$$X(x, y)\,dy - Y(x, y)\,dx = 0$$

will admit then and only then of the G_1, Uf, when

$$U(Af) - A(Uf) \equiv \lambda(x, y)\,Af\,,$$

where

$$Af \equiv X(x, y) \frac{\partial f}{\partial x} + Y(x, y) \frac{\partial f}{\partial y}.$$

Let us illustrate this criterion by an example. As we know from geometrical considerations, the family of ∞^1 circles with equal radii,

$$(x - a)^2 + y^2 - r^2 = 0,$$

will admit of a translation along the x-axis. If we form the differential equation of these circles by the usual method, we find

$$ydy + \sqrt{r^2 - y^2}\, dx = 0.$$

In this case, therefore,

$$Af \equiv y \frac{\partial f}{\partial x} - \sqrt{r^2 - y^2} \frac{\partial f}{\partial y}.$$

The infinitesimal translation along the x-axis has the form

$$Uf \equiv \frac{\partial f}{\partial x}.$$

Hence, may be verified at once that

$$U(Af) - A(Uf) \equiv 0;$$

that is, since λ can be zero in (4), the criterion holds.

If the differential equation (3) happens to admit of *two* known infinitesimal transformations,

$$U_1 f \equiv \xi_1 \frac{\partial f}{\partial x} + \eta_1 \frac{\partial f}{\partial y}, \qquad U_2 f \equiv \xi_2 \frac{\partial f}{\partial x} + \eta_2 \frac{\partial f}{\partial y},$$

of which neither is *trivial*, then *two* integrating factors of (3) are known,

$$U_1 \equiv \frac{1}{X\eta_1 - Y\xi_1}, \qquad U_2 \equiv \frac{1}{X\eta_2 - Y\xi_2}.$$

It is a well-known theorem of the ordinary text-books on differential equations, that if U_1 and U_2 are two integrating factors of (3), then *the ratio* $U_1 : U_2$ *is either an integral of* (3), *or it is a constant.* When, therefore, (3) admits of *two* known infinitesimal transformations, one can very often find the integral of (3), without even a quadrature, by mere algebraic operations.

As an example, it can be easily verified that the differential equation

$$dy - (x - \sqrt{x^2 - 2y})\, dx = 0,$$

admits of

$$U_1 f \equiv \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y}, \qquad U_2 f \equiv x \frac{\partial f}{\partial x} + 2y \frac{\partial f}{\partial y}.$$

Therefore the quotient $U_1 : U_2$ has the form

$$\frac{2y - (x - \sqrt{x^2 - 2y})\,x}{x - (x - \sqrt{x^2 - 2y})} \equiv x - \sqrt{x^2 - 2y}\,,$$

and this is the integral of the above differential equation.

Our next object will be to show how to find all families of ∞^1 curves,—that is, all differential equations of the first order,—in the plane, which are invariant under a given G_1, Uf.

If the ∞^1 finite transformations of the G_1 be performed upon any curve in the plane which is not a *locus* of the G_1, a family of ∞^1 new curves will be obtained. Since the ∞^1 transformations form a G_1, a little reflection will show that this family of ∞^1 curves must, as a whole, be invariant, and no curve of the family can be a *locus* of the G_1.

If

$$\Omega\,(x_1,\, y_1) = 0$$

be the curve (taken for convenience in the variables x_1, y_1) upon which the ∞^1 transformations

$$x_1 = \varphi\,(x, y, a)\,, \quad y_1 = \psi\,(x, y, a)\,,$$

of the G_1 are performed, then the resulting invariant family will have the form

$$\Omega\,\big(\varphi\,(x, y, a)\,,\ \psi\,(x, y, a)\big) = 0\,.$$

To find the general form of the invariant differential equation of the first order, it is only necessary to eliminate the parameter a from the equations

$$\Omega\,(\varphi, \psi) = 0\,, \quad d\Omega\,(\varphi, \psi) = 0\,. \qquad (5)$$

The differential equation of the ∞^1 *loci* of the G_1 will not be included in the general form of the invariant differential equation. Since, however, every G_1, according to the rule for integrating factors, is *trivial* as regards the differential equation of its own *loci*, it is a matter of no importance to us that we cannot obtain the differential equation of the *loci* from (5). It would be easy to show that when the equations to the finite transformations of a G_1 are given, the differential equation to the loci of the G_1, and its integral, can be found by processes which involve only differentiation and algebraic operations. In fact, this integral is the function which on a former occasion was called the *Invariant* of the G_1.*

The differential equations of the first order which are integrable by the methods of the ordinary text-books, are all such classes of differential equations as admit of certain G_1, as we shall show by the following elementary examples :—

* See Annals, Vol. VIII, No. 4, "Transformation Groups."

1°. Suppose we wish to find all differential equations of the first order, which are invariant under the G_1 of all translations along the x-axis.

The finite transformations of the G_1 have the form

$$x_1 = x + a\,, \quad y_1 = y\,, \quad a = \text{const.} \tag{6}$$

In order to find all invariant families of ∞^1 curves, we must perform (6) upon some curve in the plane. If the equation to this curve does not contain x, it will evidently be a *locus* of the G_1; and the equation to the *loci* can be written

$$y = \text{const.},$$

with the invariant differential equation

$$dy = 0\,.$$

On the other hand, if the equation to the curve with which we begin really contains x, it may be written in the form

$$x - \varphi(y) = 0\,.$$

By means of (6), this curve is transformed into the ∞^1 curves,

$$x - \varphi(y) = \text{const.}$$

The corresponding differential equation of the first order is

$$1 - \varphi'(y)\,.\,y' = 0\,. \tag{7}$$

Hence, *all differential equations of the first order, which are free of x, admit of the G_1 of all translations along the x-axis.*

The form (7), as is obvious, does not include the differential equation to the *loci*.

The infinitesimal transformation of the G_1 has the form

$$Uf \equiv \frac{\partial f}{\partial x}\,.$$

If (7) be written in the form

$$F(y)\,dy - dx = 0\,,$$

the rule for an integrating factor gives, in this case, 1. This is, of course, as it should be, since the left hand side of the last equation is already a complete differential.

2°. As a second example, let us find the differential equations of the first order which are invariant under the G_1 of the *affine* transformations in the plane.

The finite transformations of this G_1 have the form

$$x_1 = ax\,, \quad y_1 = y\,,$$

and the infinitesimal transformation is $x\,\dfrac{\partial f}{\partial x}$.

If the equation of the curve, upon which the ∞^1 finite transformations of the G_1 are performed, be free of x, it may be taken in the form

$$y = \text{const.}$$

If the constant in this equation be given ∞^1 different values, we evidently obtain the family of ∞^1 *loci* of the G_1, with the invariant differential equation

$$dy = 0 .$$

If the original curve contains x, it may be given the form

$$x - \varphi(y) = 0 .$$

Here φ cannot be zero; for every point on the line $x = 0$ is absolutely invariant. By the transformations the ∞^1 curves

$$\frac{x}{a} - \varphi(y) = 0$$

are obtained, or

$$\frac{\varphi(y)}{x} = \text{const.},$$

with the invariant differential equation

$$x\varphi'(y)\,dy - \varphi(y)\,dx = 0 .$$

If φ be a constant, this form gives the invariant differential equation

$$dx = 0 .$$

If φ be an arbitrary function which really contains y, the general differential equation of the first order which is invariant under the G_1 of affine transformations may evidently be written in the form

$$xdy - F(y)\,dx = 0 .$$

The rule gives the obvious integrating factor

$$\frac{1}{xF(y)},$$

for this differential equation.

3°. For a third example, let us take the G_1 of similitudinous transformations

$$x_1 = ax , \quad y_1 = ay , \tag{8}$$

with the infinitesimal transformation

$$Uf \equiv x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} .$$

We know that the loci of the G_1 are the lines

$$\frac{y}{x} = \text{const.},$$

and we find thus the invariant differential equation

$$y' = \frac{x}{y}.$$

Let us perform all of the transformations (8) on a curve of the form

$$x - \varphi\left[\frac{y}{x}\right] = 0.$$

This curve must not be a *locus* of the G_1, and hence φ is not zero. If φ be a constant, we find the invariant differential equation

$$dx = 0.$$

If φ really contains $\frac{y}{x}$, we find, in the usual manner, the invariant family

$$ax - \varphi\left[\frac{y}{x}\right] = 0,$$

with the differential equation

$$\varphi\left[\frac{y}{x}\right] dx - \varphi'\left[\frac{y}{x}\right]\left[dy - \frac{y}{x}\,dx\right] = 0.$$

This may be written in the form

$$y' = F\left[\frac{y}{x}\right], \tag{9}$$

and we see that *the general homogeneous differential equation of the first order admits of the G_1 of the similitudinous transformations.*

$F\left[\frac{y}{x}\right]$ in (9) can have any value except $\frac{y}{x}$, (since φ is not 0). Hence the rule gives the integrating factor

$$M \equiv \frac{1}{y - xF\left[\frac{y}{x}\right]}.$$

To shorten the work in this example the curve

$$x - \varphi\left[\frac{y}{x}\right] = 0$$

is chosen in a somewhat artificial manner; but it may be readily verified that the same results would have been arrived at if we had started with a curve of the form

$$x - \varphi(y) = 0 .$$

4°. If we make use of polar coordinates, the G_1 of rotations around the origin is given by the equations

$$r_1 = r , \quad \varphi_1 = \varphi + a , \tag{10}$$

where a is the amplitude of the rotation. We may choose

$$\omega(r_1, \varphi_1) = 0 \tag{11}$$

as the curve to begin with; and by (10) we obtain from (11) the invariant family

$$\omega(r, \varphi + a) = 0 . \tag{12}$$

If ω in (11) does not contain φ_1, it is evident that (12) will not contain $\varphi + a$; in other words,

$$r = \text{const.},$$

or, in rectangular coordinates,

$$x^2 + y^2 = \text{const.}$$

is the invariant family of *loci*. This gives the invariant differential equation

$$xdx + ydy = 0 .$$

If, however, (11) really contains φ_1, (12) may be solved in the form

$$\varphi - f(r) = \text{const.}$$

That is, in rectangular coordinates, the invariant family of curves will have the form:

$$\tan^{-1}\frac{y}{x} - \Phi(x^2 + y^2) = \text{const.}$$

This gives the differential equation

$$\frac{xy' - y}{x + yy'} = F(x^2 + y^2) . \tag{13}$$

An infinitesimal rotation around the origin has the form

$$Uf \equiv -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} .$$

Hence, an integrating factor of (13) must have the form

$$M \equiv \frac{1}{(x - yF)x + (y + xF)y} \equiv \frac{1}{x^2 + y^2} .$$

5°. It may be easily verified that the equations

$$x_1 = x , \quad y_1 = y + a \cdot \varphi(x) , \tag{14}$$

define a G_1. The family of the ∞^1 invariant *loci* evidently are defined by the invariant differential equation

$$dx = 0 .$$

Let us perform the transformations (14) upon a curve of the form

$$y - \psi(x) = 0 .$$

This gives the invariant family

$$y - a \, . \, \varphi(x) - \psi(x) = 0 ,$$

with the differential equation

$$dy - \left\{ \frac{y - \psi(x)}{\varphi(x)} \, . \, \varphi'(x) + \psi'(x) \right\} dx = 0 .$$

If, now, we write

$$\Phi(x) = \frac{\varphi'(x)}{\varphi(x)} ,$$

$$\Psi(x) = \psi'(x) - \psi(x) \, . \, \frac{\varphi'(x)}{\varphi(x)} ,$$

the differential equation becomes

$$y' - \Phi(x) \, . \, y - \Psi(x) = 0 . \tag{15}$$

This is the general *linear* differential equation of the first order. We have

$$\varphi(x) = e^{\int \Phi(x)\, dx} ,$$

and hence (15) admits of the G_1

$$x_1 = y , \quad y_1 = y + a \, . \, e^{\int \Phi(x)\, dx} .$$

The infinitesimal transformation of this G_1 has the form

$$e^{\int \Phi(x)\, dx} \, . \, \frac{\partial f}{\partial y} ;$$

and we find as integrating factor of the general linear differential equation of the first order (15)

$$M = \frac{1}{e^{\int \Phi(x)\, dx}} .$$

These examples, which could be multiplied indefinitely, serve to show that the classes of differential equations of the first order which were integrated by the old methods can all be defined as admitting of the ∞^1 transformations of a certain G_1, and the integrating factor sought can be immediately given.

As we know, each infinitesimal transformation in the plane generates a G_1, and each G_1 has its invariant differential equations of the first order, which can be found by the above methods. All such differential equations are, of course, immediately integrable, and it would be both interesting and useful to tabulate the simplest of these classes of integrable differential equations.