First Integrals in the Discrete Variational Calculus

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Abstract

The intent of this paper is to show that first integrals of the discrete Euler equation can be determined explicitly by investigating the invariance properties of the discrete Lagrangian. The result obtained is a discrete analog of the classical theorem of E. Noether in the Calculus of Variations.

1. Introduction

Due to its application to optimization and engineering problems, and because of recent emphases on numerical methods, the discrete calculus of variations has become an important mathematical discipline (Cadzow [1]) in the analysis of discrete systems, i.e., systems governed by difference equations. Briefly, the central problem in the discrete calculus of variations is to determine a finite sequence $r_{M-1}, r_M, \dots, r_N$ of real numbers for which the sum

$$J\{r_n\} = \sum_{n=M}^{N} F(n, r_n, Lr_n), \qquad Lr_n = r_{n-1} \tag{1.1}$$

is extremal, where the given function $F(x, y, z)$ is assumed to be continuously differentiable. A necessary condition for J to have an extremum for a given sequence $\{r_n\}$, $n = M-1, M, \dots, N$ is that $\{r_n\}$ satisfy the second-order difference equation

$$F_y(n, r_n, r_{n-1}) + F_z(n+1, r_{n+1}, r_n) = 0 \qquad n = M, \dots, N-1 \tag{1.2}$$

where we have denoted

$$F_y(n, r_n, r_{n-1}) = \left.\frac{\partial F(x, y, z)}{\partial y}\right|_{x=n,\, y=r_n,\, z=r_{n-1}}$$

and

$$F_z(n+1, r_{n+1}, r_n) = \left.\frac{\partial F(x, y, z)}{\partial z}\right|_{x=n+1,\, y=r_{n+1},\, z=r_n}.$$

Equation (1.2), which is in general nonlinear, is called the discrete Euler equation because of its similarity to the classical Euler equation of the Calculus of Variations. Its derivation has been carried out (see Cadzow [1]) by treating J as an ordinary function of the $N-M+1$ variables $r_{M-1}, r_M, \dots, r_N$ and applying the usual rules of the Calculus, namely

$$\frac{\partial J}{\partial r_n} = 0, \qquad n = M-1, M, \dots, N.$$

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In the present paper, however, an attempt is made to derive the discrete Euler equation via a variational approach and thereby generate the proper boundary terms which are necessary for obtaining conservation theorems, i.e., first integrals of the discrete Euler equation, from a study of the invariance properties of F. In addition, the variational approach affords a discussion of higher order discrete problems and their integrals. The catalytic factor in formulating this approach is a discrete version of Lagrange's identity for difference operators and their adjoints. In Section 2 some essential facts from the finite difference calculus are reviewed, while in Section 3 the first order discrete variational problem is discussed. A theorem similar to the classical Noether theorem on invariant variational problems is presented in Section 4, thereby generating first integrals of the discrete Euler equation. Section 5 contains a discussion of a discrete problem in which the Lagrangian depends on a vector sequence $\{r_n^1, r_n^2, \ldots, r_n^K\}$. First integrals for these problems are also obtained from an invariance assumption. In Section 6 the problem in which the Lagrangian depends on higher order operators L^q, where $L^q r_n = r_{n-q}$, is investigated.

Instead of using the lag operator L, it is of course possible to formulate (1.1) in terms of the difference operator Δ defined by $\Delta r_n = r_n - r_{n-1}$. Although the latter offers better analogy with the continuous case, the use of the operator L pays off in simpler computations, especially in the formulation of Lagrange's identity.

2. Preliminary Remarks on Difference Operators

An essential role in obtaining a discrete variational formalism is played by the discrete version of Lagrange's identity for operators and their adjoints. A recent monograph by Miller [3] discusses adjoint difference operators in detail, so we will review in this section only the essential definitions and results which are required for our investigations of the one dimensional problem.

We shall consider a p-th order linear difference operator R defined by

$$R = a_0(n)\,L^0 + a_1(n)\,L^1 + \cdots + a_p(n)\,L^p \tag{2.1}$$

where L is the lag operator defined by

$$L^q r_n = r_{n-q}.$$

The coefficients $a_i(n)$, $0 \leqslant i \leqslant p$, are assumed to be defined on $I_{p+m} = \{p+m, p+m+1, \ldots\}$ where $m \in I = \{\text{integers}\}$. It is also assumed that

$$a_0(n)\,a_p(n) \neq 0, \qquad n \in I_{p+m} \tag{2.2}$$

in order that R be of order p. The adjoint operator R^* is defined in analogy with the theory of ordinary linear differential operators.

DEFINITION (2.1). If R is given by (2.1), then the adjoint R^* of the operator R is

$$R^* = a_0^*(n) L^0 + a_1^*(n) L^1 + \cdots + a_p^*(n) L^p \tag{2.3}$$

where $a_i^*(n) = a_{p-i}(n+p-i)$, $0 \leqslant i \leqslant p$, $n \in I_{p+m}$.

Clearly, R^* is also of the p-th order since, from (2.2),

$$a_0^*(n)\, a_p^*(n) = a_p(n+p)\, a_0(n) \neq 0, \qquad n \in I_{p+m}.$$

Lagrange's identity follows directly.

THEOREM 2.1. *Let R and R^* be defined by* (2.2) *and* (2.3) *and let t_n and s_n have domain I_m. Then*

$$s_n R t_n - t_n R^* s_{n+p} = \Delta B(t_n, s_n), \qquad n \in I_{m+p} \tag{2.4}$$

where

$$B(t_n, s_n) = - \sum_{i=1}^{p} \sum_{j=1}^{i} s_{n-j+i} t_{n-j} a_i(n-j+i) \tag{2.5}$$

and Δ is the difference operator, $\Delta r_n = r_{n-1} - r_n$.

Equation (2.4) is Lagrange's identity and $B(t_n, s_n)$ is the discrete bilinear concomitant. Equation (2.4) will provide a mechanism analogous to the essential integration by parts in the classical calculus of variations. That is, it will provide a method by which the variational derivatives can be isolated while at the same time certain boundary terms are generated. These variational derivatives and boundary terms will lead to discrete equations of motion and first integrals, respectively.

3. The Fundamental Variational Equation

The simplest problem in the discrete variational calculus is to select from among all finite sequences $r_{M-1}, r_M, \ldots, r_N$ the one which extremizes the sum

$$J\{r_n\} = \sum_{n=M}^{N} F(n, r_n, Lr_n) \tag{3.1}$$

where F is a given continuously differentiable function. Certain boundary conditions may be required, e.g.,

$$r_{M-1} = a \qquad r_N = b \tag{3.2}$$

in a fixed endpoint problem.

The approach to the discrete problem is similar to that of the continuous problem. We embed the sequence in a one-parameter family of sequences and calculate the derivative of J, or which is the same, of F along this family. More precisely, we define the variation δr_n of r_n by

$$\delta r_n = \varepsilon \varrho_n, \qquad n = M-1, \quad M, \ldots, N$$

where ε is a parameter and ϱ_n is a sequence of real numbers. The resulting variation of F is then defined to be the linear part in ε of the increment

$$F(n, \bar{r}_n, L\bar{r}_n) - F(n, r_n, Lr_n)$$

where

$$\bar{r}_n = r_n + \delta r_n .$$

If we denote this variation by $\delta F(\delta r_n)$, then

$$\delta F(\delta r_n) = \left.\frac{\partial F(n, \bar{r}_n, L\bar{r}_n)}{\partial \varepsilon}\right|_{\varepsilon=0} . \tag{3.3}$$

From (3.3), it is not difficult to show, by the chain rule and by the fact that the operations L and δ commute, that

$$\delta F(\delta r_n) = F_y(n, r_n, r_{n-1})\,\delta r_n + F_z(n, r_n, r_{n-1})\,L\,\delta r_n . \tag{3.4}$$

A procedure analogous to the integration by parts in the classical calculus of variations can be performed on Equation (3.4) using the results mentioned in Section 2. We notice that $\delta F(\delta r_n)$ in (3.4) can be written

$$\delta F(\delta r_n) = R(\delta r_n) \tag{3.5}$$

where R is the difference operator

$$R = F_y(n, r_n, r_{n-1})\,L^0 + F_z(n, r_n, r_{n-1})\,L^1 . \tag{3.6}$$

According to Definition (2.1), the adjoint of R is given by

$$R^* = F_z(n+1, r_{n+1}, r_n)\,L^0 + F_y(n, r_n, r_{n-1})\,L^1 . \tag{3.7}$$

Therefore, by Lagrange's identity (2.4) with $s_n = 1$ and $t_n = \delta r_n$, equation (3.4) can be written

$$R(\delta r_n) = \delta r_n R^*(1) + \Delta B(\delta r_n, 1) \tag{3.8}$$

where

$$B(\delta r_n, 1) = -\,\delta r_{n-1} F_z(n, r_n, r_{n-1}) . \tag{3.9}$$

Thus, from (3.5), (3.7), (3.8), and (3.9) we conclude that

$$\delta F(\delta r_n) = [F_y(n, r_n, r_{n-1}) + F_z(n+1, r_{n+1}, r_n)]\,\delta r_n + \Delta[-\,\delta r_{n-1} F_z(n, r_n, r_{n-1})] . \tag{3.10}$$

This is the fundamental variational equation for F. The expressions

$$\psi_n \equiv F_y(r, r_n, r_{n-1}) + F_z(n+1, r_{n+1}, r_n) \tag{3.11}$$

$n = M-1, \ldots, N$, will be called the discrete variational derivative.

In order to obtain the discrete Euler equations, which represent a necessary condition for J to be extremal, we define the variation of J by

$$\delta J(\delta r_n) = \sum_{n=M-1}^{N} \delta F(\delta r_n).$$

Using (3.10) and (3.11), we obtain

$$\delta J(\delta r_n) = \sum_{n=M-1}^{N} \psi_n \delta r_n + \delta r_{M-1} F_z(M, r_M, r_{M-1}) - \delta r_{N-1} F_z(N, r_N, r_{N-1}).$$

By requiring that δJ vanish for all δr_n, it follows that

$$\psi_n = 0, \quad n = M-1, \ldots, n. \tag{3.12}$$

These equations represent the Euler equations for the discrete problem defined by (3.1). If $\psi_n = 0$ for all n, then the condition that $\delta J = 0$ takes the form

$$\delta r_{M-1} F_z(M, r_M, r_{M-1}) - \delta r_{N-1} F_z(N, r_N, r_{N-1}) = 0,$$

from which it follows that

$$F_z(M, r_M, r_{M-1}) = 0, \quad F_z(N, r_N, r_{N-1}) = 0 \tag{3.13}$$

since the δr_n are arbitrary.

Analogous to the continuous case, equations (3.13) are called the natural boundary conditions. In a problem with varied endpoints, i.e., where r_{M-1} and r_N are not fixed, conditions (3.13) serve to determine the two arbitrary parameters which arise in the general solution of the second order difference equation, equation (3.12). In the case where both endpoints are fixed, the conditions given by (3.2) are used to select the free parameters in the solution of the discrete Euler equation. On the other hand, if only one endpoint is fixed, e.g., $r_{M-1} = a$, then the boundary conditions become

$$r_{M-1} = a \quad \text{and} \quad F_z(N, r_N, r_{N-1}) = 0.$$

4. First Integrals

In the classical calculus of variations, E. Noether [4] in 1918 showed that if the variational integral is invariant under a π parameter infinitesimal group of transformations, then π combinations of the variational derivatives can be written as divergences. This is the well-known Noether theorem. Under the assumption that the equations of motion hold true, the π identities reduce to conservation laws, or expressions which are constant on the extremals. These conservation laws, which are first integrals of the equations of motion, follow therefore only from the invariance properties of the variational integrals. (Recent accounts of Noether's work can be found in Logan [2] or Rund [5].)

For discrete systems, similar conclusions can be drawn. In this section it is shown that a systematic procedure for the establishment of first integrals of the discrete Euler equation can be developed from a direct study of the invariance properties the discrete Lagrangian $F(n, r_n, Lr_n)$.

Let $u(n, r_n)$ be a sequence depending upon n and r_n, $n = M, \dots, N-1$.

DEFINITION 4.1. The discrete Lagrangian $F(n, r_n, Lr_n)$ is difference-invariant with respect to the infinitesimal transformation

$$\bar{r}_n = r_n + \varepsilon u(n, r_n), \qquad n = M, \dots, N-1 \tag{4.1}$$

if there exists a sequence $v(n, r_n)$, $n = M, \dots, N-1$ such that

$$\delta F(\varepsilon u(n, r_n)) = \varepsilon \Delta v(n, r_n) \tag{4.2}$$

for each n, where δF is given by (3.10).

The following theorem, which is analogous to the classical Noether theorem, is valid. Actually, the following theorem is more similar to a statement about conservation laws, which in the continuous case follows as a corollary to the Noether theorem.

THEOREM 4.1. *If $F(n, r_n, Lr_n)$ is difference-invariant with respect to the transformation* (4.1), *and if $\psi_n = 0$, $n = M, \dots, N-1$, then*

$$\left.\begin{array}{c} u(n-1, r_{n-1}) F_z(n, r_n, r_{n-1}) + v(n, r_n) = \text{constant} \\ n = M, \dots, N-1 . \end{array}\right\} \tag{4.3}$$

Proof. From (4.2) and (3.10),

$$\psi_n \varepsilon u(n, r_n) + \Delta[-\varepsilon u(n-1, r_{n-1}) F_z(n, r_n, r_{n-1})] = \varepsilon \Delta v(n, r_n).$$

Simplifying and using the hypothesis $\psi_n = 0$, one obtains

$$\Delta(u(n-1, r_{n-1}) F_z(n, r_n, r_{n-1}) + v(n, r_n)) = 0,$$

whence (4.3) holds. This completes the proof.

Equation (4.3), which is a first-order difference relation, represents a first integral of the second-order discrete Euler equation given by (1.2). We now illustrate this method by the following example. Let

$$J = \sum_{n=1}^{N} \tfrac{1}{2} m_n (r_n - r_{n-1})^2 \tag{4.4}$$

where $m_1, \dots, m_N$ are given constants. From (1.2), the discrete Euler equation is the second-order difference equation

$$m_n (r_n - r_{n-1}) - m_{n+1}(r_{n+1} - r_n) = 0, \qquad n = 1, \dots, N-1 \tag{4.5}$$

which, if desired, can be solved by classical techniques or the z-transformation method. On the other hand, direct verification shows that

$$F(n, r_n, Lr_n) = \tfrac{1}{2} m_n (r_n - Lr_n)^2$$

is difference-invariant (with $v \equiv 0$) with respect to the infinitesimal translation

$$\bar{r}_n = r_n + \varepsilon .$$

Therefore, by Theorem 4.1, we immediately obtain a first integral of (4.5),

$$F_z(n, r_n, r_{n-1}) = \text{constant}$$

or

$$-m_n(r_n - r_{n-1}) = \text{constant}.$$

Due to the discrete nature of the problem in the variable n, it is not possible to vary n continuously, and consequently we have not obtained the 'energy' integrals as in the classical, continuous calculus of variations, but only the 'momentum' integrals. In addition, we note that $v(n, r_n)$ in the invariance assumption of Definition (4.1) could also depend upon r_{n-1} and not affect the order of the first integral. However, the sequence $u(n, r_n)$ in the invariance transformation given by equation (4.1) can depend only upon n and r_n; otherwise the first integral would not be one degree less than that of the Euler equation.

5. A More General Problem

In this section we discuss a discrete variational problem in which the Lagrangian depends on several 'discrete' functions $r_n^1, r_n^2, \ldots, r_n^K$, More precisely, the problem is to select a vector sequence $\{r_n^1, r_n^2, \ldots, r_n^K\}$, n $= M-1, \ldots, N$ for which the sum

$$J = \sum_{n=M}^{N} F(n, r_n^1, \ldots, r_n^K, Lr_n^1, \ldots, Lr_n^K) \tag{5.1}$$

is extremal, where the given function $F(x, y_1, \ldots, y_K, z_1, \ldots, z_K)$ is continuously differentiable. After determining the discrete Euler equations for this problem, we derive first integrals by again investigating the invariance properties of F.

For simplicity, we denote

$$r_n' = \{r_n^1, r_n^2, \ldots, r_n^K\}$$

and

$$Lr_n' = \{Lr_n^1, Lr_n^2, \ldots, Lr_n^K\}.$$

Proceeding in a fashion similar to Section 3, we define the variation of F with respect to the infinitesimal transformation

$$\bar{r}_n' = r_n' + \delta r_n', \quad \delta r_n' = \varepsilon \varrho_n'$$

as

$$\delta F(\delta r_n') = \left.\frac{\partial F(n, \bar{r}_n', L\bar{r}_n')}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

Using the commutativity of L and δ and the chain rule for derivatives, we obtain

$$\delta F(\delta r_n') = \sum_{A=1}^{K} [F_{y_A}(n, r_n', Lr_n')\,\delta r_n^A + F_{z_A}(n, r_n', Lr_n')\,L\delta r_n^A].$$

Applying Lagrange's identity (2.4) to each term in the above sum, we get

$$\left.\begin{aligned} \delta F(\delta r_n') = &\sum_{A=1}^{K} [F_{y_A}(n, r_n', r_{n-1}') + F_{z_A}(n+1, r_{n+1}', r_n')]\,\delta r_n^A \\ &+ \sum_{A=1}^{K} \Delta[-\delta r_{n-1}^A F_{z_A}(n, r_n', r_{n-1}')]. \end{aligned}\right\} \tag{5.2}$$

The variation δJ of the sum J given by (5.1) can be calculated as in Section 3, and by imposing the requirement that δJ vanish for all δr_n^A, the following set of discrete Euler equations for the variational problem can be obtained:

$$\psi_{n,A} \equiv F_{y_A}(n, r_n', r_{n-1}') + F_{z_A}(n+1, r_{n+1}', r_n') = 0 \tag{5.3}$$

where $A=1, 2, \ldots, K$; $n=M-1, \ldots, N$. As before, first integrals of this system of difference equations can be obtained if the Lagrangian F is invariant under certain transformations. In the present case, because of the large number of variables in F, it is possible to have invariance under infinitesimal transformations having several parameters.

Let $u_\alpha^A(n, r_n')$, $A=1, 2, \ldots, K$; $\alpha=1, \ldots, \Gamma$ be a sequence depending upon n and r_n', $n=M-1, \ldots, N$, and let ε^α, $\alpha=1, \ldots, \Gamma$ be essential parameters. In the following we will assume the summation convention of summing over repeated indices on different levels.

DEFINITION 5.1. The Lagrangian $F(n, r_n', Lr_n')$ is difference-invariant with respect to the infinitesimal transformations

$$\bar{r}_n^A = r_n^A + \varepsilon^\alpha u_\alpha^A(n, r_n') \tag{5.4}$$

if there exists a vector sequence $v_\alpha(n, r_n')$, $\alpha=1, \ldots, \Gamma$ such that

$$\delta F = \varepsilon^\alpha \Delta v_\alpha(n, r_n') \tag{5.5}$$

for each n.

The following theorem gives conditions under which explicit first integrals of (5.3) can be obtained.

THEOREM 5.1. *If $F(n, r_n', Lr_n')$ is difference-invariant with respect to the Γ-para-*

meter infinitesimal transformations given by (5.4), *and if* $\psi_{n,A}=0$ *for all n and A, then*

$$u_\alpha^A(n-1, r'_{n-1})\, F_{z_A}(n, r'_n, r'_{n-1}) + v_\alpha(n, r'_n) = \text{constant} \tag{5.6}$$

$\alpha=1,\ldots,\Gamma$.

From (5.5) and (5.2), it follows that

$$\psi_{n,A}\varepsilon^\alpha u_\alpha^A + \Delta\left[-\varepsilon^\alpha u_\alpha^A(n-1, r'_{n-1})\, F_{z_A}(n, r'_n, r'_{n-1})\right] = \varepsilon^\alpha \Delta v_\alpha(n, r'_n).$$

Since $\psi_{n,A}=0$ and the parameters ε^α are independent, we conclude that

$$\Delta\left[u_\alpha^A(n-1, r'_{n-1})\, F_{z_A}(n, r'_n, r'_{n-1}) + v_\alpha(n, r'_n)\right] = 0.$$

Therefore (5.6) holds true, and this completes the proof of the theorem. The expressions (5.6) represent Γ first integrals of the system of difference equations $\psi_{n,A}=0$.

6. Higher Order Discrete Problems

In this section we deal with a discrete Lagrangian

$$F(n, r_n, Lr_n, L^2 r_n, \ldots, L^p r_n) \tag{6.1}$$

where L is the lag operator and $F(x, z_0, z_1, \ldots, z_p)$ is a continuously differentiable function of the $p+2$ variables $x, z_0, z_1, \ldots, z_p$. This case is analogous to the continuous variational problem in which the Lagrangian depends upon a function and its derivatives up to the p-th order. We shall show that the discrete Euler equation corresponding to (6.1) is a $2p$-th order finite difference equation, and we shall derive an integral of the equation from an invariance principle. Although the problem discussed in this section is a special case of the one in Section 5, it is sufficiently different to merit its own discussion. Moreover, the higher order problem especially reveals the important role played by Lagrange's identity. The range of the index n in the following discussion is omitted; we assume that the range is large enough so that the problem has meaning.

The variation of the Lagrangian (6.1) with respect to an infinitesimal transformation

$$\bar{r}_n = r_n + \delta r_n$$

is defined as in Section 3. It is not difficult to show that

$$\delta F(\delta r_n) = F_{z_k}(n, r_n, Lr_n, \ldots, L^p r_n)\, \delta L^k r_n \tag{6.2}$$

where the summation convention is assumed for $k=0, 1, \ldots, p$. Since $\delta L\ r_n = L\,\delta r_n$ for all k, it follows that Equation (6.2) can be written as an operator R acting upon the variations δr_n as

$$\delta F(\delta r_n) = R(\delta r_n) \tag{6.3}$$

where

$$R = F_{z_k}(n, r_n, Lr_n, \ldots, L^p r_n)\, L^k.$$

According to Definition (2.1), the adjoint R^* of R is given by

$$R^* = F_{z_{p-k}}(n + p - k, r_{n+p-k}, r_{n+p-k-1}, \ldots, r_{n-k})\, L^k. \tag{6.4}$$

Therefore, by Lagrange's identity with $s_n = 1$ and $t_n = \delta r_n$, we conclude that

$$R(\delta r_n) = \delta r_n R^*(1) + \Delta B(\delta r_n, 1) \tag{6.5}$$

where

$$B(\delta r_n, 1) = -\sum_{i=1}^{p} \sum_{j=1}^{i} \delta r_{n-j} F_{z_i}(n - j + i, r_{n-j+i}, r_{n-j+i-1}, \ldots, r_{n-j+i-p}). \tag{6.6}$$

Consequently, the requirement that $\delta F(\delta r_n)$ vanish for all δr_n implies from equation (6.5) that $R^*(1) = 0$, or

$$\Phi_n \equiv \sum_{k=0}^{p} F_{z_{p-k}}(n + p - k, r_{n+p-k}, r_{n+p-k-1}, \ldots, r_{n-k}) = 0. \tag{6.7}$$

Equation (6.7) is the discrete Euler equation of order $2p$ for the p-th order discrete variational problem.

We now give an invariance procedure for determining an integral of equation (6.7).

DEFINITION 6.1. The function F given by (6.1) is a difference-invariant with respect to the infinitesimal transformation

$$\bar{r}_n = r_n + \varepsilon u(n, r_n) \tag{6.8}$$

if there exists a sequence $v(n, r_n, r_{n-1}, \ldots, r_{n-p})$ for which

$$\delta F(\varepsilon u(n, r_n)) = \varepsilon \Delta v(n, r_n, \ldots, r_{n-p}). \tag{6.9}$$

From equations (6.5) through (6.7), it follows that equation (6.9) can be written

$$\varepsilon \Phi_n u(n, r_n) + \Delta B(\varepsilon u(n, r_n), 1) = \varepsilon \Delta v(n, r_n, \ldots, r_{n-p}). \tag{6.10}$$

If $\Phi_n = 0$, then equation (6.10) becomes

$$\begin{aligned} &v(n, r_n, r_{n-1}, \ldots, r_{n-p}) \\ &\quad + \sum_{i=1}^{p} \sum_{j=1}^{i} u(n - j, r_{n-j}) F_{z_i}(n - j + 1, r_{n+j-1}, r_{n-j+1-1}, \ldots, r_{n-j+i-p}) \\ &\quad = \text{constant}. \end{aligned} \tag{6.11}$$

Consequently, we can state the following theorem.

THEOREM 6.1. *If F is difference-invariant with respect to the infinitesimal transformation*

$$\bar{r}_n = r_n + \varepsilon u(n, r_n)$$

then equation (6.11) *is an integral of equation* (6.7).

The first integral represented by equation (6.11) is of order $2p-1$. For this fact to remain true, it is essential that the sequence $u(n, r_n)$ in the invariance transformation depend only upon n and r_n, and not on $r_{n-1}, r_{n-2}, \ldots$. Otherwise, the first integral would have an order greather than or equal to the order of the Euler equation.

An analysis of higher dimensional discrete problems is straightforward and will not be discussed in this paper. For two dimensions, the fundamental problem can be briefly described as follows: Let $\{r_{m,n}\}$ be a sequence depending upon two indicies m and n. Further, define the lag operators L_m and L_n by

$$L_m r_{m,n} = r_{m-1,n}, \qquad L_n r_{m,n} = r_{m,n-1}.$$

If $F(x, y, u, v, w)$ is a given continuously differentiable function, then the problem is to select a sequence $r_{M_1-1,N_1}, r_{M_1,N_1-1}, \ldots, r_{M_2,N_2}$ for which the sum

$$J\{r_{m,n}\} = \sum_{m=M_1}^{M_2} \sum_{n=N_1}^{N_2} F(m, n, r_{m,n}, L_m r_{m,n}, L_n r_{m,n})$$

is extremal. In this case, the necessary condition takes the form of a partial difference equation, and first integrals can be generated from the invariance properties of F.

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