Foundations of Physics, Vol. 19, No. 6, 1989

The Meaning of Time in the Theory of Relativity and "Einstein's Later View of the Twin Paradox"

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Received August 24, 1987; revised June 9, 1988

*The purpose of the present paper is to reply to a misleading paper by M. Sachs entitled "Einstein's later view of the Twin Paradox" (TP) (*Found. Phys. ***15***, 977 (1985)). There, by selecting some passages from Einstein's papers, he tried to convince the reader that Einstein changed his mind regarding the asymmetric aging of the twins on different motions. Also Sachs insinuates that he presented several years ago "convincing mathematical arguments" proving that the theory of relativity does not predict asymmetrical aging in the TP. Here we give a definitive treatment to the clocks problem showing that Sachs' "convincing mathematical arguments" are non sequitur. Also, by properly quoting Einstein, we show that his later view of the TP coincides with the one derived from the rigorous theory of time developed in this paper.*

1. INTRODUCTION

Mendel Sachs paper[1] with the title "Einstein's later view of the twin paradox" is misleading in (at least) two aspects:

(i) It uses certain quotations from Einstein's papers[2,3] to suggest to the reader that Einstein abandoned his earlier view that if two identical standard clocks meet in a point x_1 in space-time and are synchronized and then follows different world-lines that meet again in a second point $x_2 \neq x_1$, then they will not in general show identical times at x_2.

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0015-9018/89/0600-0705$06.00/0

(ii) It gives to reader the impression that the arguments presented in an old Sachs paper[4] are correct within general relativity and that in Ref. 5 he contested the "majority view" in a legitimate manner.

In what follows, we show that both (i) and (ii) are non sequitur. To this end, we introduce in Secs. 2 and 3 several concepts necessary in order to understand the meaning of time in the theory of relativity.

In Sec. 2 we present the fundaments of the theory and the standard clock postulate (SCP).

We introduce in Sec. 3 the notion of reference frames in a Lorentzian manifold and the notion of the coordinate system naturally adapted to a given reference frame. We classify the reference frames according to their synchronizability. This classification is extremely important, for it shows in which conditions the time-like coordinate x^0 has the meaning of time as measured by standard clocks at rest in a given reference frame.

In Sec. 4 we present a rigourous mathematical treatment of the clocks problem (no paradox, of course), which is independent of the introduction of charts in the space-time manifold. All observers in all reference frames must then agree with the result.

In Sec. 5, we discuss the old Sachs paper[4] and show explicitly that Sachs calculation are not in accord with the theory of relativity.

In Sec. 6, we present a small selection of passages from Einstein's "Autobiographical Notes"[1] which endorse the theory of time presented in this paper and show very clearly that the Sachs paper[1] is ill conceived. The Appendix contains the derivation of the fundamental anti-Minkowski inequality used in Sec. 4 as well as several important results related to the linear algebra of Minkowski space, which should be very well known.

2. THE SPACE-TIME OF THE THEORY OF RELATIVITY AND THE CHRONOMETRIC HYPOTHESIS

The most important feature of the theory of relativity is the hypothesis that the collection of all possible happenings, i.e., all possible events constituting space-time, i.e., $V_4 = (M, g, D)$ is a connected 4-dimensional oriented and time oriented Lorentzian manifold (M, g) together with the Levi-Civita connection D of g on M.[6,7] The events in $U \subset M$ in a particular chart of a given atlas have coordinates (x^0, x^1, x^2, x^3), x^0 is called time-like coordinate, and the x^i, $i = 1, 2, 3$ are called space-like coordinates. These labels *do not necessarily have* a metrical meaning.

The metric of the manifold (in a coordinate basis) is

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$$
$$g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = g_{\nu\mu} \tag{1}$$

$g(\partial/\partial x^\mu, \partial/\partial x^\nu)$ being calculated, of course, for each $x \in M$ in M_x, the tangent space to M at x. [The properties of the vectors at M_x (Minkowski space) are studied in the Appendix.] Now, tangent space magnitudes defined by the metric are related to magnitudes on the manifold by the following definition.

Let $I \subset \mathbb{R}$ be an interval on the real line and $\Gamma: I \to M$ a map. We suppose that Γ is a C^0, piecewise C^1 curve in M. We denote the inclusion function $I \to \mathbb{R}$ by u, and the distinguished vector field on I by d/du. For each $u \in I$, $\Gamma_* u$ denotes the tangent vectors at $\Gamma_u \in M$; thus

$$\Gamma_* u = \left[\Gamma_*\left(\frac{d}{du}\right)\right](u) \in M_{\Gamma_u}$$

Finally, the path length between points $x_1 = \Gamma(a)$, $x_2 = \Gamma(b)$, $a, b \in I$, $x_1, x_2 \in M$ along the curve [curves are classified as timelike, lightlike and spacelike when (for all $u \in I$) $g(\Gamma_* u, \Gamma_* u) > 0$, $g(\Gamma_* u, \Gamma_* u) = 0$, $g(\Gamma_* u, \Gamma_* u) < 0$, respectively]. $\Gamma: I \to M$, and such that $g(\Gamma_* u, \Gamma_* u)$ has the same sign at all points along Γu, is the quantity

$$\int_a^b du\, [|g(\Gamma_* u, \Gamma_* u)|]^{1/2} \tag{2}$$

Observe now that taking the point $\Gamma(a)$ as a reference point, we can use Eq. (2) to define the function

$$s: \Gamma(I) \to R \qquad \text{by} \quad s(u) = \int_a^u [|g(\Gamma_* u', \Gamma_* u')|]^{1/2}\, du' \tag{3}$$

With Eq. (3) we can calculate the derivative ds/du. We have

$$\frac{ds}{du} = [|g(\Gamma_* u, \Gamma_* u)|]^{1/2} = \left[\left| g_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du} \right|\right]^{1/2} \tag{4}$$

From Eq. (4), old text books on differential geometry and general relativity infer the equation.

$$(ds)^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{5}$$

which is supposed to represent the square of the length of the "infinitesimal" arc determined by the coordinate displacement

$$x^\mu(a) \to x^\mu(a) + \frac{dx^\mu}{du}(a)\varepsilon$$

where ε is an "infinitesimal" and $a \in I$.

The abusive and noncareful use of Eq. (5) has produced many incorrect interpretations in the theory of relativity, as we will see in what follows.

Now, given a time-like curve γ: $R \supset I \to M$, any event $e \in \gamma(I)$ separates all other events in two disjoint classes, the past and the future (see Appendix). The theory models an observer as

Definition 1. An observer in V_4 is a future-pointing time-like curve γ: $R \supset I \to M$ by $I \ni u \to e \in \gamma(I) \subset M$, and such that $g(\gamma_*, \gamma_*) = 1$.

We now introduce

Postulate I. (standard clock postulate) (SCP). Let γ be an observer, then there exist *standard clocks* that "can be carried by γ" and such that they register (in γ) *proper-time*, i.e., the inclusion parameter u of the definition of observer.[8]

It seems that atomic-clocks are standard clocks,[9] but see also.[10]

Definition 2. A reference frame Q in V_4 is a time-like vector field such that each of its integral lines is an observer.

This definition is due to R. Sachs and H. Wu.[6] B. O'Neill[11] calls observer fields the reference frames fields.

Given $U \subset M$ where Q is defined, there are an infinity of charts (coordinate systems) $\langle x^\mu \rangle$: $U \to \mathbb{R}^4$ of the maximal oriented atlas of M. We have the following definition.

Definition 3. A chart in $U \subset M$ is said to be a naturally adapted coordinate system to a reference frame Q (nacs/Q) if in the natural coordinate base of $T_x U (x \in U)$ associated with the chart the space-like components of Q are null.

3. THE MEANING OF THE TIME-LIKE COORDINATE x^0

Old treatments of the clocks problem involve at least two reference frames Q and Q', each one containing a standard clock at rest. A (nacs/Q),

$\langle x^\mu \rangle$, and a (nacs/Q'), $\langle x'^\mu \rangle$ are also used. For $U \subset M$ where both Q and Q' are defined we have the coordinate transformation $\langle x^\mu \rangle \to \langle x'^\mu \rangle$. In particular, we have $x'^\circ = f^\circ(x^\circ, x^1, x^2, x^3)$, relating the time-like coordinate of an event $e \in U \subset M$ in Q' with the time-like and space-like coordinates of the same $e \in U \subset M$ in Q.

Now—*and this point is crucial*—we must find an answer to the following question: Given an arbitrary reference frame Q, when does there exist a (nacs/Q) such that (for $U \subset M$ where Q is defined) x° has the meaning of proper time as determined by standard clocks at rest in Q and synchronized by Einstein's method? [Einstein's synchronization method is given by Definition 6.]

Let be $\alpha = g(Q, \quad)$. We have

Definition 3 (Ref. 6).

(i) Q is locally synchronizable if and only if $d\alpha \wedge \alpha = 0$.

(ii) Q is locally proper-time synchronizable if and only if $d\alpha = 0$.

(iii) Q is synchronizable if there are mappings $f\colon M \to R$ and $x^\circ\colon M \to \mathbb{R}$, such that $f > 0$ and $\alpha = f\, dx^\circ$.

(iv) Q is proper-time synchronizable if and only if $\alpha = dx^\circ$.

It is clear that (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii) and the reciprocals are valid only locally.

Definition 4. When Q is synchronizable (proper-time synchronizable) whatever function x° as in Definition 3 is called a time function (proper-time function).

When there exists a time function, it obviously is not unique. If there exists a proper-time function, we have $du = \gamma^*(dx^\circ)$, $\forall \gamma \in Q$, and $\gamma\colon \mathbb{R} \supset I \to M$, u being the inclusion function of the curve γ.

When Q is synchronizable, all hypersurfaces of the time function x° are orthogonal to Q, being then orthogonal to all observers in Q. These hypersurfaces are space-like. In this case, we say that the observers in Q can separate M in time $\times$ space. When M is contractible (i.e., $\pi_1(M) = 0$), we have, using the reciprocal of Poincaré's lemma.[12]

Proposition 1. If $V_4 = (M, g, D)$ is contractible and $d\alpha = 0$, $\alpha = g(Q, \quad)$, $g(Q, Q) = +1$, then observes in Q can separated M in time $\times$ space.

When Proposition 1 holds true, there exists $x^\circ\colon M \to \mathbb{R}$, such that $\alpha = dx^\circ$, and it is possible to give to the time-like coordinate x° the meaning of proper-time as measured by standard clocks at rest in Q and synchronized a l'Einstein. This statement will be proved later.

Let α be a 1-form field ($\alpha = g(Q, \quad)$) such that $\alpha \neq 0$, $d\alpha \neq 0$, $\forall x \in U \subset M$. We then have the following questions: when does it exist a function $f\colon U \to \mathbb{R}$ such that $df \neq 0$ $\forall x \in U$ and such that the hypersurfaces of the type

$$N = \{x \in U \mid f(x) = \text{constant}, df(x) \neq 0\}$$

are integral manifolds of α?

It can be shown(12) that a necessary and sufficient condition for the existence of this f is the Frobenius condition

$$d\alpha \wedge \alpha = 0 \tag{6}$$

In this case $\alpha = g\, df$, where $g\colon U \to R$ is a nonvanishing in U. We then have:

Proposition 2. Let (M, g, D) be a Lorentzian manifold. If there exists in $U \subset M$ a reference frame Q such that $d\alpha \wedge \alpha = 0$, $\alpha = g(Q, \quad)$, then Q can separate locally M in time $\times$ space.

Proposition 2 justifies then Definition 3(i).

We now introduce the concept of synchronization of clocks necessary in order to physically justify the definitions of this section. We need to introduce:

Postulate II. (Light Axiom). Let (λ, m_λ) be a photon, i.e., $m_\lambda = 0$ and $\lambda\colon \mathbb{R} \supseteq I \to M$ a null curve. Then, in any Lorentzian manifold $V_4 \equiv (M, g, D)$, the world line of any photon λ is a null geodesic.

We have the following:

Proposition 3. Let $\gamma\colon \mathbb{R} \supset I \to M$ be an observer in (M, g, D). Suppose that $u_e \in I$ is given. Then there exists an open interval $E \subset I$, $u_e \in E$ and an open neighborhood V of $e = \gamma u_e$ such that $\forall e' \in V - \gamma E$; there exist u_e, and u_{e_2} and a light signal λ from e' to $e_2 = \gamma u_{e_2}$, and a light signal λ' from $e_1 = \gamma u_{e_1}$ to e; u_{e_1}, u_{e_2}, λ, λ' are unique.

The proof can be found in Ref. 6.

Let Z be a reference frame in $V_4 = (M, g, D)$, μ_s its flux and $\gamma\colon \mathbb{R} \supset I \to M$ an integral curve of Z.

Definition 5. An infinitesimally nearby observer is a vector field $W\colon I \to TM$ which is Lie-parallel with respect to Z and such that for $u \in I$ there is a neighborhood ε on u, a neighborhood μ of γu, and a vector field V on μ such that $\mathscr{L}_Z V = 0$ and $W = V \circ \gamma$ on ε.

The reason for calling W an infinitesimally nearby observer is the following: Let $\langle x^\mu \rangle : u \to \mathbb{R}^4$ and $W|_\varepsilon = a^\mu(\partial_\mu \circ \gamma|_\varepsilon)$. We may write $U = \{(x^\circ, x^1, x^2, x^3) \mid |x^\mu| < \varepsilon \; \forall \mu\}$ and assume that $\gamma_u = (u, \circ, \circ, \circ)$, since in U we can always choose $Z|_U = \partial/\partial x^\circ$.[13] There is a congruence of integral curves of Z determined by

$$(u, t) \mapsto (u + a^\circ x^\circ, a^1 x^\circ, a^2 x^\circ, a^3 x^\circ)$$

and $x^\circ = 0$ gives $\gamma|_\varepsilon$, and x° times an appropriate constant gives another curve of the congruence in U where the parametrization given by $\langle x^\mu \rangle$ holds. Now, when $W|_\varepsilon$ and $Z \circ \gamma$ are linearly independent, different curves have distinct images. This family uniquely determines $W|_\varepsilon$ as its *transversal vector field*, i.e.,

$$(Wf)(u) = \left[\left(\frac{\partial}{\partial x^\circ} \right) \{ f(u + a^\circ x^\circ, a^1 x^\circ, a^2 x^\circ, a^3 x^\circ) \} \right]_{x^\circ = 0}$$

for each $f: U \to R$ and $u \in \varepsilon$. Conversely, once $W|_\varepsilon$ is given, the family is determined up to first order in x°, in the sense of a Taylor expansion in x°.

Now, let Q be a reference frame in (M, g, D), and let γ and γ' be two "infinitesimal nearby" observers of Q. Suppose that γ' contains e' of Proposition 3. (Fig. 1.)

According to Postulate I, the observers in γ and γ' can order all the events in their respective world lines. We write $e_1 < e < e_2$ to indicate that according to γ the event e is later than e_1 and e_2 is later than e.

The problem of the synchronization of clocks is as follows: which event e in the world-line γ is simultaneous to the event e' in γ'?

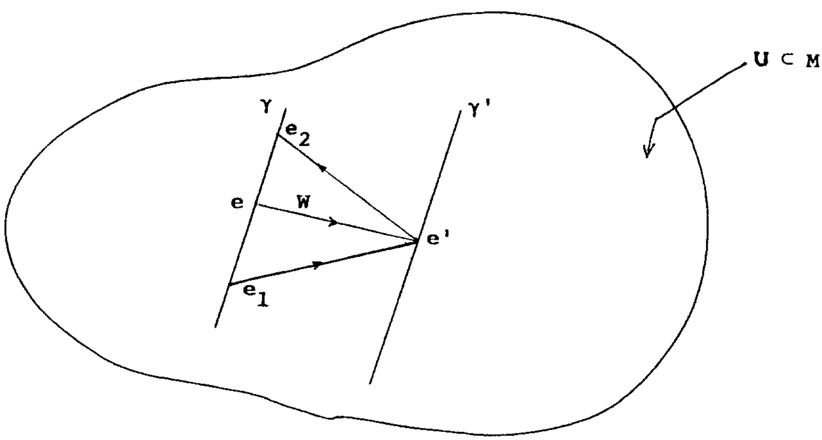

Fig. 1. Nearby integral curves of Q.

The answer to this question depends on a *definition*. Intuitively we consider that the event e' simultaneous to e must not be causally related to e, i.e., there must be no causal curve [a causal curve is a mapping $\gamma: I \to M$ ($I \subseteq \mathbb{R}$) such that $g(\gamma_*, \gamma_*) \geqslant 0$ $\forall x = \gamma u \in \gamma(I)$] connecting e' to e. Now, Proposition 3 and the definition of a nearby observer imply that there is no causal curve connecting e' to e if $e_1 < e < e_2$. Let $\langle x^\mu \rangle: M \supset U \to \mathbb{R}^4$ be a local chart of the maximal oriented atlas of M naturally adapted to Q, and let

$$\begin{aligned} e_1 &= (x^\circ_{e_1}, 0, 0, 0); \qquad & e_2 &= (x^\circ_{e_2}, 0, 0, 0) \\ e &= (x^\circ_{e}, 0, 0, 0); \qquad & e' &= (x^\circ_{e'}, \Delta x^1, \Delta x^2, \Delta x^3) \end{aligned} \tag{7}$$

be the space-time coordinates of the respective events. We have

Definition 6. The event e in γ simultaneous to the event e' in γ' is the one such that its time-like coordinate is given by

$$x^\circ_e = x^\circ_{e_1} + \tfrac{1}{2}[(x^\circ_{e'} - x^\circ_{e_1}) + (x^\circ_{e_2} - x^\circ_{e'})] \tag{8}$$

Physically, the synchronization procedure given by Eq. (8) [Einstein's method] means that the observer at γ proceeds as follows: (i) At e_1 he sends a light signal λ' to e' in γ', (ii) the signal is immediatly reflected back through the path λ and arrives at γ at the event e_2.

Calling $\Delta x^\circ_1 = x^\circ_{e'} - x^\circ_{e_1}$, $\Delta x^\circ_2 = x^\circ_{e_2} - x^\circ_{e'}$, and taking into account that the lengths of the arcs $e_1 e'$ in λ' and $e' e_2$ in λ can be represented by the vectors $W_{\lambda'} = (-\Delta x^\circ_1, \Delta x^1, \Delta x^2, \Delta x^3)$ and $W_{\lambda'} = (\Delta x^\circ_2, \Delta x^1, \Delta x^2, \Delta x^3)$ and since $g(W_\lambda, W_\lambda) = g(W_{\lambda'}, W_{\lambda'}) = 0$, we get

$$x^\circ_e = x^\circ_{e'} + \frac{g_{\circ i}}{g_{\circ\circ}} \Delta x^i \neq x^\circ_{e'} \tag{9}$$

This equation permits the synchronization of two infinitesimal nearby clocks in γ and γ' at rest in the reference frame Q and in the local chart $\langle x^\mu \rangle: M \supset U \to \mathbb{R}^4$.

We also observe that an unique synchronization of all clocks at rest in Q in the region $U \subset M$ is possible only if there exists a (nacs/Q) such that in this coordinate system $g_{i\circ} = 0$, $\forall x \in U$. When Q is proper-time synchronizable, i.e., $\alpha = dx^\circ$, $\alpha = g(Q, \quad)$ and $x^\circ: M \to R$ then there exists a local chart where $Q = \partial/\partial x^\circ$ and $g_{\circ\circ} = 1$ $\forall x \in M$. As all level surfaces of the function x° are orthogonal to Q (and then orthogonal to all observers in Q), the spatial coordinates x^i are such that $g(\partial/\partial x^\circ, \partial/\partial x^i) = g_{ij} = 0$ $\forall x \in M$.

Given an arbitrary reference frame Z in $U \subset M$, in general there does not exist a (nacs/Z) such that $g_{oo} = 1$ and $g_{io} = 0$ $\forall x \in U$. This is the reason why we classified Z according to Definition 3. Now, suppose that Z is an arbitrary reference frame in $U \subseteq M$ and $\langle x^\mu \rangle$ a (nacs/Z). Let Q be another frame defined also in $U \subseteq M$ which is proper-time synchronizable, and let $\langle \tilde{x}^\mu \rangle$ be a (nacs/Q). The coordinate transformation $\langle x^\mu \rangle \to \langle \tilde{x}^\mu \rangle$ must then satisfy

$$
\begin{aligned}
g^{\mu\nu}(x) \frac{\partial \tilde{x}^o}{\partial x^\mu} \frac{\partial \tilde{x}^o}{\partial x^\nu} &= \tilde{g}^{oo}(\tilde{x}) = 1 \\
g^{\mu\nu}(x) \frac{\partial \tilde{x}^i}{\partial x^\mu} \frac{\partial \tilde{x}^o}{\partial x^\nu} &= g^{io}(\tilde{x}) = 0 \\
g^{\mu\nu}(x) \frac{\partial \tilde{x}^i}{\partial x^\mu} \frac{\partial \tilde{x}^j}{\partial x^\nu} &= \tilde{g}^{ij}(\tilde{x})
\end{aligned}
\tag{10}
$$

where $g^{\mu\nu}(x)\, g_{\nu\alpha}(x) = \delta^\mu_\alpha$ $\forall x \in U \subset M$. We put $\tilde{g} = g^{\mu\nu} e_\mu \otimes e_\nu$.

Equations (10) have the form of the relativistic Hamilton–Jacobi equation for a free-particle. It can be very hard to find solutions to these equations, and the interested reader can consult Ref. 14.

To finish, we must comment on a basic point.[15] The reference frames Q introduced earlier are mathematical instruments. This means that a given frame does not need to have a material support in all points of the world manifold. An example will illustrate this point. Let $V_4 = (M, g, D)$ be a flat Lorentzian manifold, namely Minkowski space-time. Let $i = \partial/\partial t$ be an inertial frame [inertial frames, which exist only in a flat manifold, are such that $Di = 0$. Then $d\alpha \wedge \alpha = 0$, $\alpha = g(i, \quad)$] defined of course for all $x \in M$. Let now (t, r, ϕ, z) be the cylindrical coordinates naturally adapted to i. Then g is

$$
g = dt \otimes dt - dr \otimes dr - r^2\, d\phi \otimes d\phi - dz \otimes dz \tag{11}
$$

Let

$$
Q = (1 - \omega^2 r^2)^{-1/2} \frac{\partial}{\partial t} + \omega (1 - \omega^2 r^2)^{-1/2} \frac{\partial}{\partial \phi} \tag{12}
$$

be a reference frame defined in $U \equiv (-\infty < t < \infty;\ 0 < r < 1/\omega;\ 0 \leqslant \phi < 2\pi,\ -\infty < z < \infty)$ $(U \subset M)$.

Then

$$
\alpha = g(Q, \quad) = (1 - \omega^2 r^2)^{-1/2}\, dt - \omega r^2 (1 - \omega^2 r^2)\, d\phi \tag{13}
$$

and

$$d\alpha \wedge \alpha = \frac{-2\omega r^2}{(1-\omega^2 r^2)^{1/2}} dt \wedge dr \wedge d\phi \neq 0 \tag{14}$$

The rotation vector[7] associated to Q is $\Omega = \tilde{g}(*(d\alpha \wedge \alpha), \quad) = \omega(1-\omega^2 r^2)^{-1/2}\, \partial/\partial z$. This means that Q is rotating with constant angular velocity ω relative to the z axis of i. Now Q can be materialized in $U \subset M$ by a solid rotating disc, but it is obvious that in U, i cannot have material support.

The reference frame Q defined by Eq. (12) is also an example where there does not exist a (nacs/Q) such that the time-like coordinate of the system can have the meaning of proper-time registered by standard clocks at rest in Q, for all $x \in U$.

4. SOLUTION OF THE CLOCK PROBLEM

We are now prepared to discuss the "clocks problem," unfortunately known as the clock paradox. As the problem first arose in the special theory of relativity we will first discuss the clock problem in the case where D is a flat connection. In this case, the manifold M is an affine vector manifold, known as Minkowski space-time (see Appendix). Due to this fact, it is possible to present a coordinate-free treatment of the clock problem.

Let there Γ_1, $\vec{\Gamma}_2$, and $\overleftarrow{\Gamma}_2$ be three timelike and straight lines in M, as in Fig. 2. Γ_1 and $\vec{\Gamma}_2$ has x_i as a common point and Γ_1, $\overleftarrow{\Gamma}_2$ has x_f as a common point and $\vec{\Gamma}_2$ and $\overleftarrow{\Gamma}_2$ has x_m as a common point. Γ_1 represents the path of a standard clock called ① and $\Gamma_2 = \vec{\Gamma}_2 + \overleftarrow{\Gamma}_2$ represents the path of a standard clock called ②. Now, according to Eq. (2), the proper time registered by clock ① between the events x_j and x_f is given by $T_{①} = \|x_f - x_i\|$, i.e., the norm of the vector $x_f - x_i \in M$. The proper time registered by clock ② is given by $T_{②} = \|x_m - x_i\| + \|x_f - x_m\|$. Now, according to the fundamental *anti-Minkowski inequality*, valid for time-like vectors in the same class (Appendix, Prop. 9), we have

$$\|x_f - x_i\| \geqslant \|x_m - x_i\| + \|x_f - x_m\| \tag{15}$$

and thus $T_{①} \geqslant T_{②}$.

This result is an intrinsic consequence of the mathematical model of the theory of relativity. All observers in all reference frames in M (inertial or not) must agree with the validity of the result $T_{①} \geqslant T_{②}$.

We observe that path Γ_1 is a geodesic path between x_f and x_i, as can be trivially proved. We also can prove the following theorem, which is valid in a general Lorentzian manifold (i.e., D does not need to be flat).

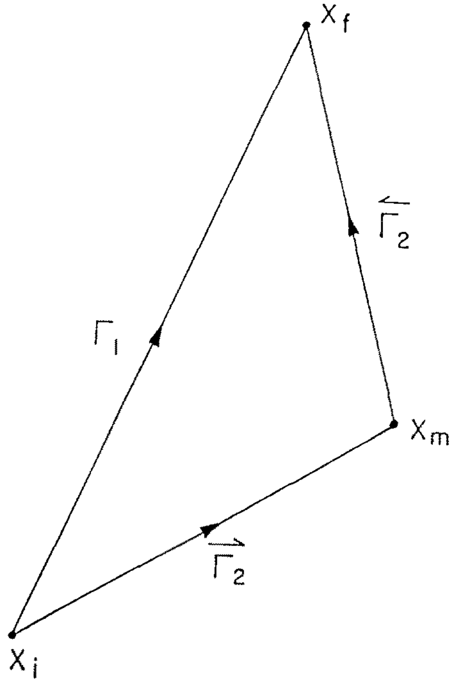

Fig. 2. Paths of two clocks ① and ② that are synchronized at x_1 and meet again at x_f in M.

Theorem. Among all timelike curves in $V_4 = (M, g, D)$ passing through the points $x_i = \Gamma(a)$, $x_f = \Gamma(b)$ the integral in Eq. (2) is a maximum when Γ is a timelike geodesic.

If the reader fails to construct the proof of is theorem he can consult Ref. 16.

To end this section, we mention that, in the case of the special theory of relativity, we can give a proof of the nonexistence of a Lorentz-invariant clock, i.e., a clock such that it, when in motion relative to an inertial frame S, does not lag behind relative to a series of clocks synchronized à l'Einstein in S. Indeed, in Ref. 17 it is proved that the existence of one such clock implies the breakdown of Lorentz invariance.

5. THE "SQUARE ROOT" OF g

In this section we give the promised proof that the "solution of the clock paradox" offered in the Sachs paper[4] is not in accord with the theory of relativity.

To start, let $V_4=(M, g, D)$ as in Sec. 2 be a model of the space-time manifold. In a coordinate basis, $g=g_{\mu\nu}\,dx^\mu\otimes dx^\nu$ [Eq. (2)]. We now ask if g can be "factored" as the tensor product of two 1-forms ω_1 and ω_2, i.e., if we can write

$$g=\omega_1\otimes\omega_2 \tag{16}$$

The answer to this question is "yes," and we can exhibit easily two solutions where ω_1 and ω_2 are Clifford-valued 1-forms:

(i) $\omega_1=\omega_2=\gamma_\mu(x)\,dx^\mu$, and the $\gamma_\mu(x)$ satisfy

$$\gamma_\mu(x)\,\gamma_\nu(x)+\gamma_\nu(x)\,\gamma_\mu(x)=2g_{\mu\nu}(x) \tag{17}$$

γ_μ are then the generators of the local Clifford algebra $\mathbb{R}_{1,3}$ of space-time.[18,19]

(ii) $\omega_1=\tilde{\omega}_2$, where $\omega_1=q_\alpha(x)\,dx^\alpha$ and $\omega_2=q_\beta(x)\,dx^\beta$, and where q_α are the generators of the quaternion field, $\tilde{q}_\beta$ is the quaternion conjugate field, and we have

$$q_\alpha(x)\,\tilde{q}_\beta(x)+q_\beta(x)\,\tilde{q}_\alpha(x)=2g_{\alpha\beta}(x) \tag{18}$$

Choosing the solution given by Eq. (9), we ask now what conditions the q_α fields must satisfy in order for $d\omega=0$. This is the condition for no "clock paradox" if "time" were to be associated with the quaternion-valued function of Eq. (21). The solution is that q_α must obey the Cauchy–Riemann-like identity

$$\frac{\partial q_\alpha}{\partial x^\beta}-\frac{\partial q_\beta}{\partial x^\alpha}=0 \tag{19}$$

So, if Eq. (19) is satisfied, we can write

$$\omega=d\mathbf{s} \tag{20}$$

where $\mathbf{s}$: $M\to H$ is a quaternion-valued function defined in the space-time manifold. We have

$$\mathbf{s}=\int\omega \tag{21}$$

independent of the path.

We can now understand what happened in Mendel Sachs papers.[4,20,21] Using the Neanderthal notation of Eq. (5), viz., $(ds)^2=g_{\alpha\beta}\,dx^\alpha\,dx^\beta$, he concluded that it is possible to write $ds=d\mathbf{s}$. This is obviously impossible, since $\mathbf{s}: M \to H$ is a quaternion-valued function, whereas Eq. (5) defines only the real-valued function $s: \Gamma(I) \to R$ given by Eq. (3). It is impossible to extend s as defined in Eq. (3) to a function defined on all of the space-time manifold.

6. EINSTEIN'S TRUE VIEW CONCERNING THE CLOCK'S PROBLEM

In this section we present a small selection of passages (A, B, C) from Einstein's "Autobiographical Notes" together with some comments that show very clearly that Sachs' paper[1] is ill conceived. Indeed Einstein said:

(A) "A clock at rest relative to the system of inertia defines a local time. The local times of all space points taken together are the time, which belongs to the selected system of inertia, if a means is given to set these clocks relative to each other."

(B) "The presupposition of the existence (in principle) of (ideal, viz., perfect) measuring rods and clocks is not independent of each other; since a light signal, which is reflected back and forth between the ends of a rigid rod, constitutes an ideal clock, provided that the postulate of the constancy of the light-velocity in vacuum does not lead to contradictions.

This paradox may then be formulated as follows. According to the rules of connection, used in classical physics, of the spatial coordinates and of the time of events in the transition from one inertial system to another, the two assumptions of

(1) the constancy of the light velocity,

(2) the independence of the laws (thus specially also of the law of the constancy of the light velocity) of the choice of the inertial system (principle of special relativity)

are mutually incompatible (despite the fact that both taken separately are based on experience).

The insight which is fundamental for the special theory of relativity is this: The assumptions (1) and (2) are compatible if relations of a new type ("Lorentz-transformation") are postulated for the conversion of coordinates and the times of events. With the given physical interpretation of coordinates and time, this is by no means merely a conventional step, *but implies certain hypotheses concerning the actual behavior of moving measuring-rods and clocks, which can be experimentally validated or disproved.*" (Our italics.)

Our comments concerning passages A and B here are as follows: In an inertial system i_1 standard clocks at rest read directly the time-like coordinate x°, (see Sec. 3) if the clocks are synchronized à l'Einstein in order to obtain a one-way velocity of light which is isotropic.[13] The Lorentz-transformations between two inertial frames are not physical cause-effect relations but implies certain hypothesis concerning the actual behavior of moving measuring-rods and clocks, which can be experimentally validated or disproved.

The hypothesis concerning the behavior of clocks is the one introduced in Sec. 2, viz., that there exist standard (or ideal) clocks which measure proper-time, i.e., the integral given by Eq. (2) when Γ is time-like. (Postulate I.)

If real clocks do not satisfy the standard clock postulate then the Lorentz-transformations could not give the relation between the times x° in i_1 and $x^{\circ\prime}$ in i_2 (i_1, $i=1, 2$ being inertial frames) where x° and $x^{\circ\prime}$ are measured by standard clocks at rest, respectively in i_1 and i_2 and synchronized à l'Einstein.

(C) "One is struck [by the fact] that the theory (except for the four-dimensional space) introduces two kinds of physical things, i.e., (1) measuring rods and clocks, (2) all other things, e.g., the electro-magnetic field, the material point, etc. This, in a certain sense is inconsistent; strictly speaking measuring rods and clocks would have to be represented as solutions of the basic equations (objects consisting of moving atomic configurations), not, as it were, as theoretically self-sufficient entities. However, the procedure justifies itself because it was clear from the very beginning that the postulates of the theory are not strong enough to deduce from them sufficiently complete equations for physical events sufficiently free from arbitrariness, in order to base upon such a foundation a theory of measuring rods and clocks. If one did not wish to forego a physical interpretation of the coordinates in general (*something which, in itself, could be possible*), it was better to permit such inconsistency—with the obligation, however, of eliminating it at a later stage of the theory." (Italics ours.)

In passage C Einstein remind us that it cannot be an accident that standard clocks register the time defined by Eq. (3). This basic fact must be explained as an adjustment to the field in which the clocks are embedded as test bodies. This needs, of course, a detailed theory of matter, which unfortunately does not yet exist. In C Einstein also remind us that in an arbitrary reference frame Q in a Lorentzian manifold the coordinate labels

(x°, x^1, x^2, x^3) of a particular chart valid for $U \subset M$ of the (nacs/Q) (as defined in Sec. 3) does not have a metrical meaning in general. This means in particular that in general x° is not the time registered by standard clocks at rest in Q. Indeed, this is the case since the standard clocks register the time given by Eq. (3)! Passage C is in complete agreement with the theory of time developed in this paper.

7. CONCLUSIONS

In this paper we presented a rigorous theory of the meaning of time in the theory of relativity. The material is not completely original, since it can be found scattered in the literature.[6,8,15] However, we think that our work can be of utility for all readers that have yet some doubt concerning the twin paradox. The paper also demonstrates that Sachs' treatment of the twin paradox[1,20,21] is non sequitur and that Einstein never wrote a single line which endorses Sachs' misleading point of view.

APPENDIX

The objective of this appendix is to prove the anti-Minkowski triangle inequality, used in Sec. 4. We take the opportunity to present some results related to the linear algebra of Minkowski space, which should be very well known. The tangent space to any point of the space time manifold is the Minkowski space, and its precise definition is:

Definition 1. Minkowski space M is a 4-dimensional vector space over the real field with a Lorentzian inner product, that is, we can associate the metric tensor to the matrix $\mathrm{diag}(1, -1, -1, -1)$ in one orthonormal basis.

Definition 2. Let be $v \in M$, then we say that v is spacelike if $v^2 < 0$ or $v = 0$, that v is lightlike if $v^2 = 0$ and $v \neq 0$, and that v is timelike if $v^2 > 0$.

Definition 3. Let $S \subset M$ be a subspace. We say that S is spacelike if all its vectors are spacelike, that S is lightlike if it contains a lightlike vector but no timelike vector, and that S is timelike if it contains a timelike vector.

We note that the definitions establish that a subspace $S \subset M$ is spacelike or lightlike or timelike. We are going to prove some propositions that will permit us to understand the linear algebra of M.

Proposition 1. A subspace S is timelike if and only if its orthogonal complement $S^{\perp}$ is spacelike.

Proof. Let $S \subset M$ be a timelike subspace, then there exists a timelike vector $v_{\circ} \in S$, and we define $e_{\circ} = \|v_{\circ}\|^{-1} v_{\circ}(\|v_{\circ}\| = (|v_{\circ}^2|)^{1/2})$ and add to it other three vectors e_i $(i = 1, 2, 3)$ in such a way to construct an orthonormal basis $\{e_{\mu}\}$ $(\mu = 0, 1, 2, 3)$ with $e_{\mu} \cdot e_{\nu} = \eta_{\mu\nu}$, $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Then $S^{\perp} \subset \text{span}[e_1, e_2, e_3]$ (the space generated by e_1, e_2, e_3) and $S^{\perp}$ is spacelike (we define $u \cdot v \equiv g(u, v)$).

Conversely (we note that $(S^{\perp})^{\perp} = S$), if S is space-like, we have $M = S \oplus S^{\perp}$ (direct sum), and, for any time-like vector $v \in M$, we have $v = v' + v''$ with $v' \in S$ and $v'' \in S^{\perp}$. Then we have $v'' \cdot v' = v \cdot v - v'' \cdot v'' < 0$ and therefore v'' is time-like and $S^{\perp}$ is time-like.

Proposition 2. Let $S \subset M$ be a light-like subspace. Then its orthogonal complement $S^{\perp}$ is lightlike and $S \cap S^{\perp} \neq \{0\}$.

Proof. If S is lightlike, $S^{\perp}$ cannot be timelike or spacelike by Prop. 1, then $S^{\perp}$ is light-like. There is a light vector $n \in S$, but there is not any time-like vector belonging to S. Then $\forall a \in \mathbb{R}$, $\forall s \in S$, $(s + an)(s + an) = s \cdot s + 2as \cdot n \leqslant 0$, $\forall a \in \mathbb{R}$; therefore $s \cdot n = 0$ $\forall s \in S$ and $n \in S^{\perp}$ $\therefore$ $S^{\perp} \cap S \neq \{0\}$.

Proposition 3. Two light-like vectors n_1, $n_2 \in M$ are orthonormal if and only if they are proportional.

Proof. Let $v \in M$ be a time-like vector. By Prop. 2, we have $v \cdot n_1 \neq 0$, $v \cdot n_2 \neq 0$; then there exists $\alpha \in \mathbb{R}$, $\alpha \neq 0$, such that $v \cdot (n_1 + \alpha n_2) = 0$, and, by Prop. 1, $n_1 + \alpha n_2$ is space-like. But $(n_1 + \alpha n_2)^2 = 2\alpha n_1 \cdot n_2$, thus, if n_1, n_2 are orthogonal, $n_1 \cdot n_2 = 0$. Therefore, $n_1 + \alpha n_2 = 0$ and n_1, n_2 are proportional. Conversely, if n_1 and n_2 are proportional, we have $n_1 = \beta n_2$ and $n_1 \cdot n_2 = \beta (n_2)^2 = 0$.

Proposition 4. There are only two orthogonal space-like vectors that also are orthogonal to a light-like vector.

Proof. Let $n \in M$ be light-like and s_1, $s_2 \in M$ space-like such that $s_1 \cdot s_2 = 0$. We define a basis $\{e_{\mu}\}$ with $-e_{\circ}^2 = e_i^2 = -1$ in such a way that $s_1 = (0, s_1^1, 0, 0)$ and $s_2 = (0, 0, s_2^2, 0)$ in this basis. If $s_1 \cdot n = s_2 \cdot n = 0$, we have that $n = (n^{\circ}, 0, 0, n^3)$ with $(n^{\circ})^2 = (n^3)^2 \neq 0$. Another space-like vector orthogonal to s_1 and s_2 must have the form $s_3 = (s_3^1, 0, 0, s_3^3)$, with $(s_3^3)^2 > (s_3^1)^2$; then $s_3 \cdot n = n^3 s_3^3 \neq 0$.

Proposition 5. The unique way to construct an orthonormal basis for M is with one time-like vector and three spacelike vectors.

Proof. If we have a light-like vector in our basis, we have no timelike vector by Prop. 1. Then we must have only light-like and space-like vectors in our basis. By Prop. 3, orthogonal light vectors are proportional, and we must have only one light-like vector in our basis; the other three must be space-like, which is an absurd by Prop. 4.

Therefore, we only have timelike and space-like vectors in the basis in the proportion 1:3 in order for the signature to be -2.

Now we are going to show that we can divide the set $\tau \subset M$ of all time-like vectors into two disjoint subsets τ^+ and τ^-, which we identify, by convention which future and past.

Proposition 6. The relation $u \uparrow v$, defined by $u \uparrow v$ if and only if $u \cdot v > 0$, is an equivalence relation for time-like vectors, and this relation divides τ into two disjoint equivalence classes τ^+ and τ^-.

Proof. First we take $e_\circ$, $e_\circ^2 = 1$ as the future time-like direction. We add to it three space-like vectors e_i, $e_i^2 = -1$ in such a way as to construct an orthonormal basis $\{e_\mu\}$. If $u, v \in \tau$, we have $u = (u^\circ, u^1, u^2, u^3)$ and $v = (v^\circ, v^1, v^2, v^3)$ in this basis with $(u^\circ)^2 > \sum_i (u^i)^2$, $(v^\circ)^2 > \sum_i (v^i)^2$; therefore $u \cdot v = u^\circ v^\circ - \sum_i u^i v^i$. Then we observe that, if u°, v° have the same signal, $u \cdot v > 0$; if not, $u \cdot v < 0$, since, by Schwarz inequality in $\mathbb{R}^3$, we have

$$\left(\sum_i (u_i)^2\right)^{1/2} \left(\sum_i (v^i)^2\right)^{1/2} \geqslant \sum_i u^i v^i \therefore |u^\circ|\,|v^\circ| > \sum_i u^i v^i$$

Therefore $\uparrow$ is obviously an equivalence relation that divides τ into two equivalence classes: τ^+, such that, if $u' \in \tau^+$, $u \cdot e_0 > 0$, and τ^-, such that $u \in \tau^- \Rightarrow u \cdot e_\circ < 0$. We call τ^+ the future component of τ and τ^- the past component, and we note that this definition depends of the choice of a future time-like direction.

Proposition 7. τ^+ (and τ^-) are convex sets, that is, given $u, v \in \tau^+$, $a \in (0, \infty)$, $b \in [0, \infty)$, then $w = au + bv \in \tau^+$.

Proof. u, $v \in \tau^+ \Leftrightarrow u \cdot e_\circ > 0$, $v \cdot e_\circ > 0$, where $e_\circ$ as fixed future timelike direction, therefore, $w \cdot e_\circ = au \cdot e_\circ + bv \cdot e_\circ > 0$ and $w \in \tau^+$.

At this point we are prepared to derive the anti-Minkowski inequality. First we are going to show the anti-Schwarz inequality.

Proposition 8. Let u, $v \in M$ be time-like vectors. Then we have for them the anti-Schwarz inequality, that is, $|u \cdot v| \geqslant \|u\| \, \|v\|$ and the equality only occurs if u, v are proportional.

Proof. We choose an orthonormal basis $\{\mathbf{e}_\alpha\}$ with $e_\circ = \|u\|^{-1} u$, such that $u = (u^\circ, 0, 0, 0)$ in this basis and $v = (v^\circ, v^1, v^2, v^3)$ with $(v^\circ)^2 > \sum_i (v^i)^2$. Then we have

$$\|u\| = |u^\circ|, \qquad \|v\| = \left((v^\circ)^2 - \sum_i (v^i)^2 \right)^{1/2} \leqslant |v^\circ|,$$

$$|u \cdot v| = |u^\circ| \, |v^\circ| \geqslant \|u\| \, \|v\|$$

If the equality is satisfied we have $\|v\| = |v^\circ|$, and therefore

$$v = (v^\circ, 0, 0, 0)$$

being proportional to u.

Proposition 9. Let u, $v \in \tau^+$, then we have for u, v the anti-Minkowski inequality, that is,

$$\|u + v\| \geqslant \|u\| + \|v\|$$

Proof. We note that, by Prop. 6, $u \cdot v > 0$, and, using Prop. 8, we have

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v \geqslant \|u\|^2 + \|v\|^2 + 2 \, \|u\| \, \|v\|$$

$$= (\|u\| + \|v\|)^2 \Rightarrow \|u + v\| \geqslant \|u\| + \|v\|$$

To complete this appendix we are going to study the possibility of any Schwarz-like or Minkowski-like inequality for spacelike vectors $u, v \in M$.

Proposition 10. Let u, $v \in M$ be spacelike vectors such that span $[u, v]$ is spacelike, then the usual Schwarz inequality $|u \cdot v| \leqslant \|u\| \, \|v\|$ obtains, and so does the usual Minkowski inequality $\|u + v\| \leqslant \|u\| + \|v\|$. If the equalites are satisfied, then u and v are proportional.

Proof. If the span $[u, v]$ is spacelike, then it has an orthonormal basis $\{e_1, e_2\}$ with $e_1^2 = e_2^2 = -1$ such that $u = (u^1, 0)$ and $v = (v^1, v^2)$ in this basis. Then

$$\|u\| = |u^1|, \quad \|v\| = ((v^1)^2 + (v^2)^2)^{1/2} \geqslant |v^1|, \quad |u \cdot v| = |u^1| \, |v^1| \leqslant \|u\| \, \|v\|$$

If the equality holds, then $v^1 = 0$, and u, v are proportional.

We therefore have

$$\|u+v\|^2 = -(u+v)^2 = -u^2 - v^2 - 2u\cdot v$$
$$= \|u\|^2 + \|v\|^2 - 2u\cdot v \leqslant \|u\|^2 + \|v\|^2 + 2\,\|u\|\,\|v\|$$
$$= (\|u\| + \|v\|)^2 \;\therefore\; \|u+v\| \leqslant \|u\| + \|v\|$$

The equality holds only if $-u\cdot v = |u\cdot v| = \|u\|\,\|v\|$; then u, v are proportional.

Proposition 11. Let u, $v\in M$ be spacelike vectors such that the span $[u, v]$ is timelike, then the anti-Schwarz inequality $|u\cdot v| \geqslant \|u\|\,\|v\|$ holds; if the equality holds, then u, v are proportional. If $u\cdot v \leqslant 0$, that is, u, v have different "time directions," and more, if $u+v$ is spacelike; we then have the anti-Minkowski inequality, $\|u+v\| \geqslant \|u\| + \|v\|$, and the equality only holds if u, v are proportional.

Proof. If span $[u, v]$ is timelike, we choose an orthonormal basis $\{e_\circ, e_1\}$ for it with $e_\circ^2 = e_1^2 = 1$ and such that, in this basis, $u = (0, u^1)$ and $v = (v^\circ, v^1)$ with $|v^\circ| < |v^1|$. Then we have $\|u\| = |u^1|$, $\|v\| = [(v^1)^2 - (v^\circ)^2]^{1/2} \leqslant |v^1|$, and $|u\cdot v| = |u^1|\,|v^1| \geqslant \|u\|\,\|v\|$. If the equality holds, $v^\circ = 0$ and u, v are proportional.

If $u\cdot v \leqslant 0$, we have $|u\cdot v| = -u\cdot v$; if also $u+v$ is spacelike, we have

$$\|u+v\|^2 = -(u+v)^2 = -u^2 - v^2 - 2u\cdot v = \|u\|^2 + \|v\|^2 + 2\,|u\cdot v|$$
$$\geqslant \|u\|^2 + \|v\|^2 + 2\,\|u\|\,\|v\| = (\|u\| + \|v\|)^2$$

Therefore,

$$\|u+v\| \geqslant \|u\| + \|v\|$$

The equality only holds if $-u\cdot v = |u\cdot v| = \|u\|\,\|v\|$; then u, v are proportional.

ACKNOWLEDGMENTS

We thank Prof. E. Recami, V. L. Figueiredo, and M. E. F. Scanavini for useful discussions. We are grateful to CNPq, CNR and FAPESP for financial support.

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