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THE DIFFERENCE BETWEEN NEWTONIAN AND RELATIVISTIC FORCES

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This paper discusses a new turn in the 148-year old electrodynamic force law controversy between the 1822 Ampère force law of the Newtonian electrodynamics and Grassmann's 1845 law which has become the electrodynamic force law of relativistic electromagnetism. Faced with the infallibility of Ampère's empirical law, defenders of relativity theory now argue that Ampère's law is "equivalent" to the relativistic law. This paper demonstrates that, far from being equivalent, the laws require two different mechanics of solid bodies, disagree on internally generated stresses, and predict different force distributions.

Key words: Newtonian electrodynamics, special relativity, Ampère's force law, Lorentz force.

1. THE FORCE LAW CONTROVERSY

In 1822 Ampère [1] proposed the first law for the interaction force $\Delta\mathbf{F}_{mn}$ between two metallic current elements $i_m\mathbf{dm}$ and $i_n\mathbf{dn}$. In practical importance, this force law, or any substitute of it, is second only to Newton's universal law of gravitation. Following Newton's example, Ampère based his law entirely on experimental findings. Both laws are therefore infallible, so long as nature does not change course with time.

With r_{mn} being the distance between two metallic current elements, the most useful form of Ampère's formula has proved to be

$$\Delta F_{mn} = -i_m i_n (dm \cdot dn / r_{mn}^2)(2\cos\varepsilon - 3\cos\alpha \ \cos\beta), \tag{1}$$

where ε is the angle of inclination between the elements and the angles α and β are the inclinations of the elements to the distance vector. The negative sign in Eq. (1) stands for attraction. Ampère's law is in every respect a Newtonian law based on simultaneous mutual attraction or repulsion, and it does not allow the interaction of an element with itself.

Ampère also proved that the interaction between two current elements always reduces to a two-dimensional problem, as in Eq. (1), because the interaction force between perpendicular elements located in the same plane is zero.

In 1845 Grassmann [2] suggested a non-reciprocal interaction law for two metallic current elements which has to be stated by two formulas for the unequal forces $\Delta \mathbf{F}_m$ and $\Delta \mathbf{F}_n$ on the elements. The most useful form of Grassmann's law are the vector equations

$$\Delta \boldsymbol{F}_m = (i_m i_n / r_{mn}^2) \boldsymbol{dm} \times (\boldsymbol{dn} \times \mathbf{1}_{rn}), \tag{2}$$

$$\Delta \boldsymbol{F}_n = (i_m i_n / r_{mn}^2) \boldsymbol{dn} \times (\boldsymbol{dm} \times 1_{rm}), \tag{3}$$

where $\mathbf{1}_{rm}$ and $\mathbf{1}_{rn}$ are unit distance vectors pointing to $\mathbf{dm}$ and $\mathbf{dn}$, respectively.

Following Ampère's lead, Grassmann assumed that Eqs. (2) and (3) represented instantaneous actions at a distance. Of the two electrodynamic force laws, only Ampère's agreed with Newton's third law. This meant, as will be explained later, that the Newtonian electrodynamics, which flowed from Coulomb's and Ampère's laws, did not apply to Grassmann's law.

When Lorentz re-interpreted Grassmann's law in terms of field contact actions, the name of the law was changed to the "Biot-Savart law," since Biot and Savart had shown how to calculate the magnetic field strength at a current element due to another element. In this way the Grassmann force law ended up in the magnetic component of the Lorentz force.

Beginning with Grassmann himself, numerous physicists [3] have argued that the two electrodynamic laws should lead to different experimental consequences and, therefore, experiment would prove which is right and which is wrong. As the Lorentz force is an essential part of the special theory of relativity, both the Lorentz force and Grassmann's formula were, of course, covariant under Lorentz transformations. Ampère's law, on the other hand, was Galilean covariant, as all of Newtonian physics must be.

Many experiments to test the validity of Ampère's law were performed over a period of almost 170 years [3], but we find that the force law controversy is as lively as ever [4] at the end of the twentieth century. It is reminiscent of the 2000-year hiatus over Aristotle's teaching that heavy bodies fall faster than light bodies. The simple experiment of dropping two unequal coins from the palm of the hand decided the issue, but scholars were adamant not to abandon the dogma laid down in scripture.

It was certainly not Aristotle's fault. He stated [5]:

> "I say apparently, for the actual facts are not yet sufficiently made out. Should further research ever discover them, we must yield to their guidance rather than to that of theory; for theories must be abandoned, unless their teachings tally with the undisputable results of observation."

On similar grounds, the defense of present-day physics dogma cannot be laid at the feet of Einstein who, during the last year of his life, wrote [6]:

> "I consider it quite possible that physics cannot be based on the field concept, i. e., on continuous structures. In that case nothing remains of my entire castle in the air, gravity theory included, and the rest of physics."

2. CLAIMS OF EQUIVALENCE

Faced with the infallibility of Ampère's empirical law, the defenders of our modern textbooks [4, 7-9] fell back on a mathematical argument which they claim shows that there is really no difference between the Newtonian and the relativistic electrodynamics of metals.

In vector notation, Eq. (1) may also be expressed by two terms, as follows:

$$\begin{aligned}\Delta \boldsymbol{F}_m = & -\mathbf{1}_{rn} i_m i_n (d\boldsymbol{m} \cdot d\boldsymbol{n}/r_{mn}^2)(\cos\varepsilon - 3\cos\alpha\ \cos\beta) \\ & -\mathbf{1}_{rn} i_m i_n (d\boldsymbol{m} \cdot d\boldsymbol{n}/r_{mn}^2)\cos\varepsilon,\end{aligned} \tag{4}$$

The triple vector product of Eq. (2) may be resolved to give

$$\Delta \boldsymbol{F}_m = i_m i_n (dn\, dm/r_{mn}^2)\cos\alpha_n - \mathbf{1}_m i_m i_n (dm \cdot dn/r_{mn}^2)\cos\varepsilon, \tag{5}$$

where α_n is the inclination of $\mathbf{dn}$ to $\mathbf{1}_m$.

Comparing Eqs. (4) and (5), it is evident that both force laws contain an identical Newtonian attraction-repulsion term, which is the second term in both equations. But the remaining two terms are very different, not only in magnitude but also in direction. The first term of Eq. (4) is also a Newtonian force. The first term of (5), however, is neither attraction nor repulsion. It clearly is the relativistic component of the Grassmann or Biot-Savart law. This is the stark difference between the Newtonian and relativistic force laws.

Whittaker [10], and many others, pointed out that in a closed-path integration the first terms of both laws vanish because they are perfect differentials. This is the essence of gauge invariance. For the force on a current element by a closed circuit, therefore, both laws give the same force. This fact has misled Jolly [7], Ternan [8], Christodoulides [9], and Robson and Sethian [4], when they claim the two force laws are equivalent.

The important fact is that the loop integration reduces the relativistic force law to a Newtonian force law. At this point it should be recognized that the relativistic Grassmann-Lorentz force law of Eqs. (2) and (3) give rise to a relativistic mechanics (statics) which has nothing to do with the common high-speed phenomena of the special theory of relativity. Also, in this connection, it is frequently overlooked that Maxwell's equations require special relativity to explain Faraday's law of induction, which is another low-speed relativistic effect.

Then the defenders of the relativistic force law make the point that metallic currents always flow in closed circuits and, therefore, practical applications always require the closed-loop integration. Field theory requires this integration for another reason. A current element responds to the total magnetic influence of all other current elements and cannot distinguish the field of one element from the field of another.

Clearly this philosophy does not apply to Ampère's force law, which is not compatible with a magnetic field and does distinguish between the interacting elements by Newton's third law. Hence we should expect the laws to make differing predictions about experiments. The difference emerges clearly when one performs a stress analysis in metallic conductors.

3. NEWTONIAN AND RELATIVISTIC STRESSES

A definition of Newtonian stress in a solid body was provided by Slater and Frank [11]; it reads:

> "To specify such a (stress) force, we imagine a surface element *da* to be drawn somewhere in the body, with a normal **n**. The material on either side of *da* exerts a force on the material on the other side; thus this force is a push normal to the surface if there is a pressure in the body, it is a tension if that is the form of stress, or it may be a shearing force, tangential to the surface."

Because of the importance of this definition, it may be helpful to consider another wording of it [12]:

> "A stress is a force per unit area with which the part of the medium on one side of an imaginary surface acts on the part of the other side."

An important corollary of this definition of Newtonian stress is that the interaction of two elements of matter on the same side of the stress surface makes no contribution to the stress at this surface.

Newtonian stress analysis appears to be no longer in the physics curriculum, but the subject is fully covered in engineering textbooks. The stress is felt by the atomic bonds which intersect the imaginary stress surface. The atoms themselves are not torn apart or compressed. Therefore, calculations of forces *on* atoms will not reveal stress. What has to be calculated is the force *between* atoms. This is to say, stress is the result of Newtonian action and reaction forces bridging the stress surface.

When stress is internally generated by mutual interaction forces between atoms, rather than applied external forces on the body as a whole, we have to specify the interaction force by a formula which complies with Newton's third law. The two important electromagnetic forces which fulfill this condition are Coulomb and Ampère forces.

Consider first an example which involves Coulomb's law. This concerns a dielectric string charged along its length with additional electrons. Two electrons of charge *e* and distance r_{mn} repel each other with the force

$$\Delta F_e = ke^2/r_{mn}^2. \tag{6}$$

It is immediately obvious that the string will find itself in tension everywhere, except at the ends.

From the Newtonian definition of stress, the tension T at some surface which intersects the string is

$$T = ke^2 \sum_{m=1}^{x} \sum_{n=1}^{y} (1/r_{mn}^2), \tag{7}$$

where the electrons on one side of the stress surface are labeled 1, 2, ..., m, ..., x; and on the other side they are labeled 1, 2, ..., n, ..., y. Equation (7) does not give the sum of the force densities on one side of the stress surface, because the force density calculations would include interactions between electrons on the same side of the stress surface. Equation (7) also ensures that internally generated Newtonian stresses do not exert a net force on the string as a whole. Hence the charged string will not move in any direction, but lie still and stretched on the laboratory bench.

A thin wire carrying a steady dc current will behave like the dielectric string, if it is subject to Ampère's force law. That is to say, all the current elements in the wire then repel each other. This creates Ampère tension in the wire [3].

If the Lorentz force law is now applied to this wire, it will not predict tension, because the Lorentz forces must be transverse to the wire axis, and, in any case, the magnetic field strength at every current element is zero. In the wire example we have found an instance in which the two force laws do make opposite predictions and are not equivalent to each other. A more dramatic difference between the two force laws can hardly be expected. Experiments have been performed to check the prediction of Ampère tension [3], but we will not consider them here, because the purpose of the present paper is merely to demonstrate that the two laws are not equivalent.

4. REACTION FORCES IN A RECTANGULAR CIRCUIT

Critics of the wire example correctly argue that a complete circuit has to be considered, because forces due to the remaining parts of the circuit may cancel the tension. To meet this criticism, the closed rectangular circuit ABCD of Fig. 1 will now be analyzed. This circuit is assumed to

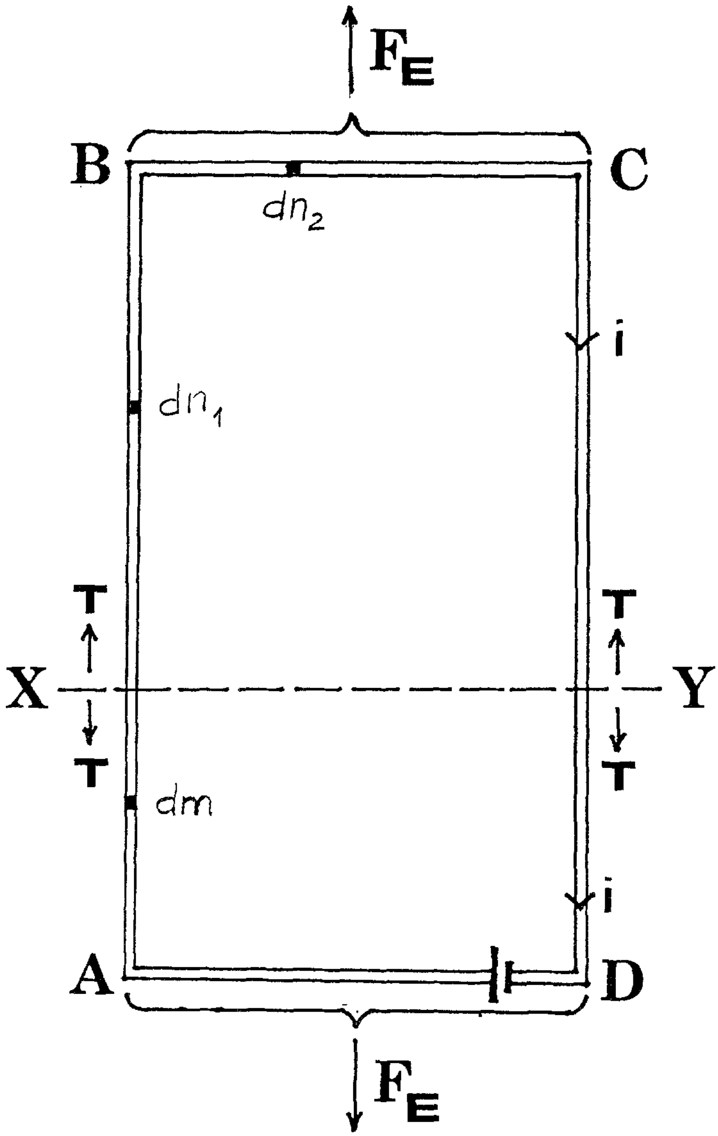

Fig. 1. Electrodynamic forces in a rectangular circuit.

carry a steady current i and stands in a vertical plane. X-Y is a horizontal surface which cuts the circuit in two parts, which are then electrically reconnected by thin liquid mercury films. The tension 2T-2T in this surface is balanced by the upward force F_E on XBCY and the downward force F_E on YDAX. F_E is an experimentally determined force which has been measured for various rectangular circuits [3,13]. Since the circuit is known not to lift itself as a whole, we must have

$$F_E = 2T. \tag{8}$$

Surprisingly, certain calculations with Grassmann's law lead to Eq. (8). This is one more reason why it has been argued that the two force laws are equivalent, and there is no controversy. In other words, the Grassmann law, notwithstanding its appearance in Eqs. (2) and (3), is compatible with Newton's third law and Newtonian mechanics, and there arises no need for a relativistic mechanics.

Let $\Delta \mathbf{F}_R$ stand for the relativistic (Grassmann) interaction of two current elements, while $\Delta \mathbf{F}_N$ represents the Newtonian (Ampère) interaction of the same pair of elements, so that

$$\Delta F_R \neq \Delta F_N. \tag{9}$$

In relativistic electromagnetism, two current elements on the same straight line exert no force on each other. In all other cases the Grassmann force is perpendicular to the current element on which it acts. Hence no relativistic force exists between **dm** and **dn** of AB of Fig. 1, which could contribute to the tension T-T. One might expect, therefore, that the whole of the tension is the result of interactions between AD and BC. Calculations show, however, that

$$\left| \sum_A^D \sum_B^C \Delta F_R \right| \ll F_E, \tag{10}$$

or the calculated force is much smaller than the measured force. The inescapable conclusion from this is that Grassmann's law, after all, is not compatible with the Newtonian mechanics. As Grassmann's law leads to the measured force F_E, it must obey a different relativistic mechanics.

In relativistic mechanics it has to be assumed that the magnetic field at any current element is due to the circuit as a whole. When this is taken into account, the correct result

$$\left| \sum_{ABCD} \sum_{XBCY} \Delta F_R \right| = F_E \tag{11}$$

is obtained. It is as if a current element generates a magnetic field strength at another element, and then absolves itself from any responsibility with regard to the reaction force. In this way the current element i $d\mathbf{n}_1$ of Fig. 1 produces a magnetic field at i$d\mathbf{n}_2$, and this field then generates a lift force $\Delta\mathbf{F}_R$ on i$d\mathbf{n}_2$, which contributes to F_E, without an elemental reaction force. This is the relativistic "self-force mechanism," because $d\mathbf{n}_1$ and $d\mathbf{n}_2$ are integral parts of the solid body XBCY cut from the rectangular circuit and electrically reconnected across the surface X-Y by two thin liquid mercury films. Roper [13] actually performed measurements of F_E with liquid mercury films at X and Y.

From this it has to be concluded that, in order to arrive at the correct experimental result of Eq. (8), calculations with the Grassmann formula must invoke the relativistic mechanics of self-forces. Hence the possible agreement of the two force laws on a particular prediction does not eliminate the need for the relativistic mechanics.

It does not follow that when using the two mechanics in their appropriate spheres of validity, they will always agree on the outcome of a specific experiment. For example, they demand different force distributions in the rectangular circuit of Fig. 1. This can be shown as follows:

Using the Newtonian electrodynamics with Ampère's force law, F_E has to be calculated according to the Slater-Frank rule,

$$F_E = \left| \sum_{XADY} \sum_{XBCY} \Delta F_N \right| . \tag{12}$$

While performing this summation, step by step, it will be found that most of F_E exists in the form of two longitudinal forces which have their seats near X and Y. In contrast to this, the Grassmann formulas place all of the lift force F_E in the top branch BC of the circuit. In one case the body XBCY is pushed upward from below, and in the other case it is pulled up by a force on the uppermost part of the body. There exist experiments which can distinguish between the two force distributions [3].

The step-by-step summation of Eq. (12) also reveals that the tension T-T in AB and CD does not disappear when a complete circuit is considered. Hence the earlier straight wire example was sufficient to prove that Ampère's law predicts tension in any straight wire section. The force law controversy can, therefore, be resolved by experiments which demonstrate the existence of Ampère tension and the force distribution predicted by Ampère's law for the rectangular circuit of Fig. 1.

The difference between Newtonian and relativistic forces is discussed at length in a recent book [14].

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