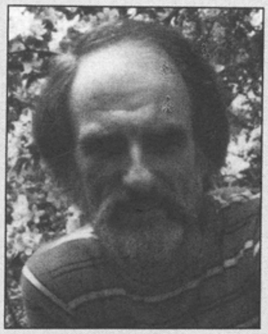
The Birth of Lie's Theory of Groups

Thomas Hawkins

In 1865 when Sophus Lie (1842–1899) completed his studies at the University of Christiania (now Oslo), Norway, he had no idea he was destined to become a mathematician. He had done well, but not brilliantly, in all subjects and was toying with the idea of becoming an observational astronomer. He even gave lectures on the subject in the student union. He had a real talent for explaining the geometry of the heavens. To support himself financially while in this state of career indecision he gave private instruction in mathematics. In this connection, Lie began to read the geometrical works by Poncelet, Chasles, and above all Plücker. Inspired by his reading, he did some original mathematical research on the real representation of imaginary quantities in projective geometry, a portion of which was accepted for publication by one of the leading mathematics journals of the time — Crelle's journal in Berlin. On the basis of this experience, he decided to devote himself to geometrical research, to become a mathematician. He was 26 years old. Five years later, during the fall of 1873, Lie made a second fateful decision: to devote himself to the enormous task of creating a theory of continuous transformation groups — a task that meant doing mathematics of a quite different sort from the geometrical work that had occupied him in his first years of mathematical research, 1869–1871 — a task that ended by occupying most of his creative mathematical energies for the remainder of his career.

The purpose of this article is to explain how Lie was led from the one decision to the other. To accomplish this, I have to immerse you in the mathematical world inhabited by Lie, a world that is quite different from the one to which you are accustomed. The first two sections of this article concern the early geometrical work of Lie (1869–1871), done in close contact with Felix Klein. It was from this work that the ideas emerged which served to redirect Lie's researches. The third section briefly discusses the years 1872–1873, when the theory of first-order PDEs, particularly in the form given to it by the work of Jacobi, provided what turned out to be a fertile context for the development of the group-related ideas that had emerged during Lie's geometrical investigations. One of the reasons the history of mathematics is fascinating is that it provides insight into the dynamics by means of which ideas and concepts from diverse mathematical theories combine, in remarkable, often unexpected ways, to give birth to entirely new mathematical theories. In the course of showing you how Lie was led from his deci-



Thomas Hawkins

Thomas Hawkins has devoted his professional life to the history of mathematics — theories of integration, the origins of matrix theory, and the early years of group theory and representation theory. After getting his doctorate at the University of Wisconsin in mathematics and history of science, he taught at Swarthmore and, since 1972, at Boston University. His many visiting lectures have included the Collège de France and the 1986 International Congress of Mathematicians. His recreations include swimming, hiking, bicycling, and cross-country skiing — but above all bicycling.

sion to become a mathematician to his decision to create a new and far-reaching mathematical theory, I also hope to give you some feeling for the manner in which concepts, theorems, ideas, and viewpoints from 19th-century algebra, geometry, and analysis played roles in this creative process.

Tetrahedral Line Complexes

On the basis of the mathematical creativity exhibited by Lie in his essay on the real representation of imaginaries, he was granted a stipend to leave Norway and to travel to various centers. His first stop was Berlin, where the mathematical scene was dominated by Kummer and Kronecker and, above all, Weierstrass. Berlin was certainly one of the world's foremost centers of mathematics in 1869. However, the mathematics and the spirit in which it was done did not appeal to Lie, who found it too analytical. In Berlin he met another visitor who felt as he did. His name was Felix Klein (1849–1925). He was just 20 years old — 7 years younger than Lie — although he already had his doctorate. Klein had been Plücker's student, and after his mentor's untimely death in 1868, he had edited Plücker's lectures on line geometry for publication. Lie had, in fact, studied these lectures, and when they met in Berlin, they were both actively engaged in line-geometric research. So Klein and Lie enjoyed each other's mathematical company. They were self-styled "Synthesists" in the midst of analysts and arithmeticians.

Although Lie and Klein had interests in common, their backgrounds and personalities were quite different. Klein had a relatively solid mathematical education — first at Bonn under Plücker and then in Göttingen, where Alfred Clebsch (1833–1872) had attracted a circle of students, including, for example, Max Noether. Clebsch had been trained in Jacobian analysis and mathematical physics at the University of Königsberg. As we shall see in the third section, his contributions to Jacobi's theory of PDEs in the 1860s were to influence Lie. By the time Klein met Clebsch, however, he was working on algebraic geometry and the theory of invariants. His unexpected death from diphtheria at age 39 was a blow to Klein, who had great admiration for him.

Klein enjoyed learning about the work of others because he wanted to "understand" it from his own point of view and to place it within a more encompassing, conceptually unified picture. By contrast, Lie had a limited mathematical background (and no doctorate yet). He tended to focus rather exclusively on developing his own (highly original) ideas and became interested in the work of others only when it was clear it was relevant to his own interests. This difference in personalities dictated the way they related to each other: Lie developed his ideas, explained them to Klein, Klein reacted, Lie (sometimes) responded to Klein's reactions, and so on.

When Klein and Lie met in Berlin, Lie was studying what were known as tetrahedral line complexes. Line complexes were a basic object of study in line geometry, a geometry in which lines rather than points are taken as the basic objects. Thus, lines rather than points are coordinatized. Given a line $\ell \subset \mathbf{P}^3(\mathbf{C})$, if $(x_1, \ldots, x_4)$ and $(y_1, \ldots, y_4)$ are the homogeneous coordinates of two points on ℓ, set $p_{ij} = x_i y_j - x_j y_i$. Then $p_{12}, p_{13}, p_{14}, p_{23}, p_{42}$, and p_{34} are the six homogeneous Plücker coordinates of ℓ. They satisfy the relation

$$\Omega = p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

In terms of line coordinates, a line complex of degree d is a set of lines $\ell = (p_{12}, \ldots, p_{42})$ whose line coordinates p_{ij} satisfy a homogeneous polynomial equation of degree d (in addition to $\Omega = 0$). From an abstract viewpoint, line complexes are simply projective varieties in $\mathbf{P}^5(\mathbf{C})$, but in the mid-19th century the fact that the homogeneous coordinates corresponded to lines was always in view, and attention was focused on particular types of line complexes and their special geometrical properties.

The tetrahedral line complex studied by Lie was a special second-degree complex, which may be defined geometrically as follows. Let Δ denote a tetrahedron with faces determined by planes $\pi_1, \ldots, \pi_4$, and let k be a fixed constant. The totality $\mathcal{T}$ of all lines $\ell \subset \mathbf{CP}^3$ such that the cross ratio of $p_i = \ell \cap \pi_i$, $i = 1, \ldots, 4$, is k is called a tetrahedral complex. Other mathematicians had considered the geometry associated to a tetrahedral complex before Lie, but Lie's approach was totally original. Consider the set **G** of all projective transformations of space which fix the vertices of Δ. (Lie himself did not introduce notation such as **G**; he simply spoke of the transformations in **G** as the transformations of the tetrahedron.) Then for any fixed line ℓ_0, the set $\mathcal{T}$ of all lines $T[\ell_0]$ such that $T \in \mathbf{G}$ is a tetrahedral complex. In effect, a tetrahedral complex for Lie was the orbit of some fixed line under the transformation of **G**. An idea of what the transformations comprising **G** are like can be obtained by choosing for Δ the tetrahedron with vertices at the origin and the points at infinity on the coordinate axes. Then the $T \in \mathbf{G}$ are given in Cartesian coordinates by

$$x' = \lambda x, \qquad y' = \mu y, \qquad z' = \nu z. \tag{1}$$

In his study of the geometry of tetrahedral complexes, Lie took advantage of the following properties of the transformations of **G**: If $T_1, T_2 \in \mathbf{G}$: (a) $T_1 \circ T_2 \in \mathbf{G}$; (b) $T_1 \circ T_2 = T_2 \circ T_1$; (c) for "general" $p, q \in \mathbf{P}^3(\mathbf{C})$, a unique $T \in \mathbf{G}$ exists such that $T(p) = q$; (d) there is a threefold infinity (∞^3) of transformations in **G**. For us today, the fact that **G** is a group stands out. We see Lie using the commutativity and simple transitivity of **G**. But in 1869, the theory of groups was not a part of the basic mathematics known to all active mathematicians. In 1869, "group theory" meant the theory of finite permutation groups, which was known to have a nice application to algebraic equations as shown by Galois. However, relatively few mathematicians were actively interested in this subject.