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Geometric Proof of Lie's Linearization Theorem

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Abstract. In 1883, S. Lie found the general form of all second-order ordinary differential equations transformable to the linear equation by a change of variables and proved that their solution reduces to integration of a linear third-order ordinary differential equation. He showed that the linearizable equations are at most cubic in the first-order derivative and described a general procedure for constructing linearizing transformations by using an over-determined system of four equations. We present here a simple geometric proof of the theorem, known as Lie's linearization test, stating that the compatibility of Lie's four auxiliary equations furnishes a necessary and sufficient condition for linearization.

Key words: Christoffel's symbols, geodesic flow, Lie's linearization test, Riemann's tensor

1. Introduction

S. Lie, in his general theory of integration of ordinary differential equations admitting a group of transformations, proved *inter alia* ([1], Section 1) that if a non-linear equation of second order $y'' = f(x, y, y')$ is transformable to a linear equation by a change of variables x, y, its integration requires only quadratures and solution of a linear third-order ordinary differential equation.

As a first step, Lie showed that the linearizable second-order equations are at most cubic in the first derivative, i.e. belong to the family of equations of the form

$$y'' + F_3(x, y)y'^3 + F_2(x, y)y'^2 + F_1(x, y)y' + F(x, y) = 0. \tag{1}$$

Furthermore, he found that the following over-determined system of four equations is compatible for linearizable equations (see [1], Section 1, Equations (3) and (4)):

$$\begin{aligned}
\frac{\partial w}{\partial x} &= zw - FF_3 - \frac{1}{3}\frac{\partial F_1}{\partial y} + \frac{2}{3}\frac{\partial F_2}{\partial x},\\
\frac{\partial w}{\partial y} &= -w^2 + F_2 w + F_3 z + \frac{\partial F_3}{\partial x} - F_1 F_3,\\
\frac{\partial z}{\partial x} &= z^2 - Fw - F_1 z + \frac{\partial F}{\partial y} + FF_2,\\
\frac{\partial z}{\partial y} &= -zw + FF_3 - \frac{1}{3}\frac{\partial F_2}{\partial x} + \frac{2}{3}\frac{\partial F_1}{\partial y},
\end{aligned} \tag{2}$$

and used this system as a basis for the theoretical construction of linearizing transformations. Note that the compatibility conditions of the system (2) are provided by the following equations:

$$3(F_3)_{xx} - 2(F_2)_{xy} + (F_1)_{yy} = 3(F_1 F_3)_x - 3(FF_3)_y - \left(F_2^2\right)_x - 3F_3 F_y + F_2(F_1)_y,$$
$$3F_{yy} - 2(F_1)_{xy} + (F_2)_{xx} = 3(FF_3)_x - 3(FF_2)_y + \left(F_1^2\right)_y + 3F(F_3)_x - F_1(F_2)_x,$$

where the subscripts x and y denote differentiations with respect to x and y, respectively.

Finally, he proved that Equation (1) is linearizable if and only if the over-determined system (2) is compatible (the statement is formulated in [1], Note 1, see p. 423 in Vol. 5 of his *Gessammelte Abhandlundgen*). *Lie's linearization test* is indeed simple and convenient in practice. Consider the following examples (see also [2], Section 12.3).

Example 1. The equation

$$y'' + F(x, y) = 0$$

has the form (1) with $F_3 = F_2 = F_1 = 0$. The linearization test yields $F_{yy} = 0$. Hence, the equation $y'' = F(x, y)$ cannot be linearized unless it is already linear.

Example 2. The equations

$$y'' - \frac{1}{x}(y' + y'^3) = 0$$

and

$$y'' + \frac{1}{x}(y' + y'^3) = 0$$

also have the form (1). Their coefficients are $F_3 = F_1 = -1/x$, $F_2 = F = 0$ and $F_3 = F_1 = 1/x$, $F_2 = F = 0$, respectively. The linearization test shows that the first equation is linearizable, whereas the second one is not.

2. Outline of Lie's Approach

Recall that any linear equation of the second order

$$y'' + a(x)y' + b(x)y = 0$$

can be reduced, by a change of variables, to the simplest form

$$\frac{d^2u}{dt^2} = 0. \tag{3}$$

Therefore, all linearizable equations $y'' = f(x, y, y')$ are obtained from Equation (3) by an arbitrary change of variables

$$t = \phi(x, y), \quad u = \psi(x, y). \tag{4}$$

In the new variables x and $y = y(x)$ defined by (4), Equation (3) takes the form

$$y'' + Ay'^{\,3} + (B + 2w)\,y'^{\,2} + (P + 2z)\,y' + Q = 0, \tag{5}$$

where (see, e.g. [2], Section 12.3)

$$A = \frac{\phi_y\psi_{yy} - \psi_y\phi_{yy}}{\phi_x\psi_y - \phi_y\psi_x}, \qquad B = \frac{\phi_x\psi_{yy} - \psi_x\phi_{yy}}{\phi_x\psi_y - \phi_y\psi_x}, \qquad w = \frac{\phi_y\psi_{xy} - \psi_y\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x},$$
$$P = \frac{\phi_y\psi_{xx} - \psi_y\phi_{xx}}{\phi_x\psi_y - \phi_y\psi_x}, \qquad Q = \frac{\phi_x\psi_{xx} - \psi_x\phi_{xx}}{\phi_x\psi_y - \phi_y\psi_x}, \qquad z = \frac{\phi_x\psi_{xy} - \psi_x\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x}. \tag{6}$$

Equation (5) takes the form (1) after writing

$$A = F_3(x, y), \quad B + 2w = F_2(x, y), \quad P + 2z = F_1(x, y), \quad Q = F(x, y). \tag{7}$$

Thus, any linearizable equation has the form (1) with the coefficients $F_3(x, y)$, $F_2(x, y)$, $F_1(x, y)$, $F_1(x, y)$ defined by (7) and (6).

Consider now an arbitrary equation of the form (1). According to the above calculations, it is linearizable if and only if Equations (7) hold, i.e. if the system

$$\begin{aligned}
&\phi_y\psi_{yy} - \psi_y\phi_{yy} = F_3(x, y)\,(\phi_x\psi_y - \phi_y\psi_x),\\
&\phi_x\psi_{yy} - \psi_x\phi_{yy} + 2(\phi_y\psi_{xy} - \psi_y\phi_{xy}) = F_2(x, y)\,(\phi_x\psi_y - \phi_y\psi_x),\\
&\phi_y\psi_{xx} - \psi_y\phi_{xx} + 2(\phi_x\psi_{xy} - \psi_x\phi_{xy}) = F_1(x, y)\,(\phi_x\psi_y - \phi_y\psi_x),\\
&\phi_x\psi_{xx} - \psi_x\phi_{xx} = F(x, y)\,(\phi_x\psi_y - \phi_y\psi_x),
\end{aligned} \tag{8}$$

with given coefficients $F_3(x, y), \ldots, F(x, y)$ and two unknown functions, ϕ and ψ, is compatible.

To summarize: equation (1) is linearizable if and only if its coefficients $F_3(x, y)$, $F_2(x, y)$, $F_1(x, y)$ and $F(x, y)$ are such that the over-determined system (8) is compatible. Provided that the system (8) is compatible, its integration furnishes a transformation (4) of the corresponding equation (1) to the linear equation (3).

Thus, one has to find primarily compatibility conditions for the over-determined system of non-linear equations (8). Lie's crucial observation is that the combinations

$$A, \quad B + 2w, \quad P + 2z, \quad Q$$

of the quantities (6) are differential invariants of the general projective transformation

$$\overline{\phi} = \frac{L_2\,\psi + M_2\,\phi + N_2}{L\,\psi + M\,\phi + N}, \qquad \overline{\psi} = \frac{L_1\,\psi + M_1\,\phi + N_1}{L\,\psi + M\,\phi + N} \tag{9}$$

of ϕ and ψ, where $L, M, \ldots, M_2, N_2$ are arbitrary constants. Using this observation, he found four relations connecting the six quantities (6) and their first-order derivatives with respect to x and y. Then, eliminating A, B, P, Q by means of Equations (7), he arrived at Equations (2), thus proving that compatibility of Equations (2) is necessary for Equation (1) to be linearizable.

To prove that compatibility of Equations (2) is sufficient for linearization, Lie used the following reasoning. The projective invariance hints that Equations (2) can be linearized by introducing the homogeneous projective coordinates

$$w = \frac{\widetilde{w}}{v}, \qquad z = \frac{\tilde{z}}{v} \cdot$$

Furthermore, Lie noticed that the resulting linear system belongs to a special type of linear systems that can be reduced, by his general theory, to a linear third-order ordinary differential equation, provided that the coefficients F_3, F_2, F_1, F of Equation (1) satisfy the compatibility conditions of the system (2). Thus, the quantities w and z can be found by solving a linear third-order ordinary differential equation.

Lie's further observation is that Equations (2) hold when one replaces B, w, Q, z by

$$B_1 = \frac{\phi_x\psi_{yy} - \psi_x\phi_{yy}}{\phi_x\psi_y - \phi_y\psi_x} - 2\frac{\phi_y}{\phi}, \qquad w_1 = \frac{\phi_y\psi_{xy} - \psi_y\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x} + \frac{\phi_y}{\phi},$$

$$Q_1 = \frac{\phi_x\psi_{xx} - \psi_x\phi_{xx}}{\phi_x\psi_y - \phi_y\psi_x} + 2\frac{\phi_x}{\phi}, \qquad z_1 = \frac{\phi_x\psi_{xy} - \psi_x\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x} - \frac{\phi_x}{\phi}$$

as well as by

$$B_2 = \frac{\phi_x\psi_{yy} - \psi_x\phi_{yy}}{\phi_x\psi_y - \phi_y\psi_x} - 2\frac{\psi_y}{\psi}, \qquad w_2 = \frac{\phi_y\psi_{xy} - \psi_y\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x} + \frac{\psi_y}{\psi},$$

$$Q_2 = \frac{\phi_x\psi_{xx} - \psi_x\phi_{xx}}{\phi_x\psi_y - \phi_y\psi_x} + 2\frac{\psi_x}{\psi}, \qquad z_2 = \frac{\phi_x\psi_{xy} - \psi_x\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x} - \frac{\psi_x}{\psi} \cdot$$

Therefore, one can find the quantities w_1 and z_1, as well as w_2 and z_2, by solving the previous linear third-order ordinary differential equation. In consequence, one obtains

$$\frac{\phi_x}{\phi} = z - z_1, \quad \frac{\phi_y}{\phi} = w_1 - w,$$

$$\frac{\psi_x}{\phi} = z - z_2, \quad \frac{\psi_y}{\phi} = w_2 - w.$$

The quadrature provides the solution of the non-linear system (8), and hence completes the determination of a linearizing transformation (4).

3. Alternative Proof of the Linearization Test

We now formulate Lie's linearization test as follows and provide its alternative proof.

Theorem. *A necessary and sufficient condition that the equation*

$$y'' + F_3(x, y)y'^{\,3} + F_2(x, y)y'^{\,2} + F_1(x, y)y' + F(x, y) = 0 \tag{10}$$

be linearizable is that its coefficients F_3, F_2, F_1, F satisfy the equations

$$3(F_3)_{xx} - 2(F_2)_{xy} + (F_1)_{yy} = 3(F_1F_3)_x - 3(FF_3)_y - \left(F_2^2\right)_x - 3F_3F_y + F_2(F_1)_y,$$

$$3F_{yy} - 2(F_1)_{xy} + (F_2)_{xx} = 3(FF_3)_x - 3(FF_2)_y + \left(F_1^2\right)_y + 3F(F_3)_x - F_1(F_2)_x. \tag{11}$$

Proof. Equations (11) provide necessary conditions for linearizable equations. Indeed, the changes of variables (4) are equivalence transformations of the set of all equations of the form (11). In other words, Equation (10) is merely permuted among themselves by any change of variables (4). Furthermore, it is proved in [3] that the infinite group of equivalence transformations (4) has an invariant system comprising precisely by the equations (11). Note that Equations (11) are satisfied for the linear equation $y'' = 0$. It follows from the invariance that Equations (11) hold for all Equations (10) obtained from $y'' = 0$ by the changes of variables (4).

Let us prove that Equations (11) provide sufficient conditions for Equation (10) to be linearizable. We consider plane curves given in a parametric form:

$$x = x(t), \qquad y = y(t), \tag{12}$$

set $y(t) = u(x(t)),\ u' = du/dx$ and represent Equation (10) in the form

$$u'' + F_3 u'^3 + F_2 u'^2 + F_1 u' + F = 0. \tag{13}$$

Then, denoting

$$\dot{x} = \frac{dx}{dt}, \qquad \dot{y} = \frac{dy}{dt},$$

we have:

$$\dot{y} = u'\dot{x}, \qquad \ddot{y} = u''\dot{x}^2 + u'\ddot{x}, \qquad \dot{x}^3 u'' = \dot{x}\ddot{y} - \dot{y}\ddot{x},$$

and

$$\begin{aligned}&\dot{x}^3\left[u'' + F_3(x,y)u'^3 + F_2(x,y)u'^2 + F_1(x,y)u' + F(x,y)\right]\\ &\quad = \dot{x}\left[\ddot{y} + \alpha\dot{y}^2 + \gamma\dot{x}\dot{y} + F\dot{x}^2\right] - \dot{y}\left[\ddot{x} - (a\dot{y}^2 + \beta\dot{x}\dot{y} + \delta\dot{x}^2)\right],\end{aligned} \tag{14}$$

where

$$\alpha + \beta = F_2, \qquad \gamma + \delta = F_1. \tag{15}$$

Hence, one can consider the projection of Equation (13) onto the (x, y) plane to obtain the geodesic flow:

$$\ddot{x}^i + \Gamma^i_{kl}\dot{x}^k\dot{x}^l = 0, \quad i = 1, 2, \tag{16}$$

where $x^1 = x$, $x^2 = y$, and the Christoffel symbols have the form

$$\begin{aligned}&\Gamma^1_{11} = -\delta, \qquad \Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{2}\beta, \qquad \Gamma^1_{22} = -F_3,\\ &\Gamma^2_{11} = F, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}\gamma, \qquad \Gamma^2_{22} = \alpha.\end{aligned} \tag{17}$$

To prove the theorem, it suffices to show that the curves (12) can be straightened out. In other words, we have to show that if Equations (11) are satisfied, we can annul the Cristoffel symbols (12) by a proper change of variables x and y. It is possible if the Riemann tensor

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{ij}}{\partial x^k} + \Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk}$$

associated with (17) vanishes. Computation shows that it has the components

$$
\begin{aligned}
R^1_{112} &= -\frac{1}{2}\,\beta_x + \delta_y + FF_3 - \frac{1}{4}\,\beta\gamma,\\
R^1_{212} &= \frac{1}{2}\,\beta_y - (F_3)_x - \frac{1}{4}\,\beta^2 + F_3\,\delta + \frac{1}{2}\,F_3\,\gamma - \frac{1}{2}\,\alpha\,\beta,\\
R^2_{112} &= \frac{1}{2}\,\gamma_x - F_y + \frac{1}{4}\,\gamma^2 - \alpha F + \frac{1}{2}\,\gamma\,\delta - \frac{1}{2}\,\beta F,\\
R^2_{212} &= -\frac{1}{2}\,\gamma_y + \alpha_x - FF_3 + \frac{1}{4}\,\beta\,\gamma.
\end{aligned}
\tag{18}
$$

Thus, the equations

$$
R^l_{ijk} = 0 \tag{19}
$$

provide 4 first-order partial differential equations for eight quantities $F_3,\ F_2,\ F_1,\ F,\ \alpha,\ \beta,\ \gamma$ and δ. Besides, these quantities are related by two conditions (15). Substituting $\alpha = F_2 - \beta,\ \delta = F_1 - \gamma$ in (18), solving Equations (19) with respect to the derivatives of β and γ, and denoting $\beta/2 = w$ and $\gamma/2 = z$ one arrives at Lie's equations (2) compatibility of which is guaranteed by Equations (11). $\square$

References

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