

Symmetry Analysis of
Differential Equations
with *Mathematica*[®]

Gerd Baumann

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Dedicated to my beloved one and only

Preface

The purpose of this book is to provide the reader with a comprehensive introduction to the applications of symmetry analysis to ordinary and partial differential equations. The theoretical background of physics is illustrated by modern methods of computer algebra. The presentation of the material in the book is based on *Mathematica* 3.0 notebooks. The entire printed version of this book is available on the accompanying CD. The text is presented in such a way that the reader can interact with the calculations and experiment with the models and methods. Also contained on the CD is a package called *MathLie*—in honor of Sophus Lie—carrying out the calculations automatically. The application of symmetry analysis to problems from physics, mathematics, and engineering is demonstrated by many examples.

The study of symmetries of differential equations is an old subject. Thanks to Sophus Lie we today have available to us important information on the behavior of differential equations. Symmetries can be used to find exact solutions. Symmetries can be applied to verify and to develop numerical schemes. They can provide conservation laws for differential equations. The theory presented here is based on Lie, containing improvements and generalizations made by later mathematicians who rediscovered and used Lie's work. The presentation of Lie's theory in connection with *Mathematica* is novel and vitalizes an old theory. The extensive symbolic calculations necessary under Lie's theory are supported by *MathLie*, a package written in *Mathematica*.

Each chapter of the present book includes theoretical considerations and practical applications of *MathLie* and *Mathematica*. The *Mathematica* examples range from simple definitions to complete notebooks discussing specific problems. The examples include definitions of general derivatives, derivations and solutions of determining equations, drop formations in liquids, and the first atomic explosion.

The end of a definition and a theorem in the text is indicated by \square . The end of an example is indicated by \square . On the CD, *MathLie* and *Mathematica* notations in the text are denoted by the color dark red. *Mathematica* input is given in red while the output is in blue.

I wish to express my gratitude to Peter Olver, Willy Hereman, and Mike Mezzino for reading the manuscript. My appreciation goes to Gerda Göler and Joachim Engelmann for proofreading the text. I also acknowledge contributions by Gernot Haager, Gerald Landhäußer, and Ronald Schmid.

Any suggestions and comments related to the book or to *MathLie* are most appreciated. Please send your e-mail to Gerd.Baumann@physik.uni-ulm.de or visit my home page at <http://www.physik.uni-ulm.de/math/gbaumann/bau.html>.

Gerd Baumann

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