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Induction of correct centrifugal force in a rotating mass shell

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Abstract. Mach's idea of relativity of rotation is confirmed for a shell-type model of the universe by showing that flat geometry in rotating coordinates, realising correct Coriolis and centrifugal forces, can be continuously connected through a rotating mass shell with not exactly spherical shape and latitude-dependent mass density to an asymptotically Minkowskian outside metric. The corresponding solutions of Einstein's field equations are given to second order in the angular velocity ω but it is plausible that the problem has a solution to any order of ω.

1. Introduction

It is a very old idea [1–3] that rotation should have only relative meaning in physics, and that it should be impossible to decide, in principle, whether an observer rotates relative to the fixed stars or all the stars and galaxies in the universe rotate relative to him. As is well known, this idea was of essential importance for Einstein in building up general relativity, and Einstein was also the first to consider a thin mass shell as a substitute for all the stars in order to probe this idea by concrete calculation. In a virtually unknown and hardly accessible publication [4], he found by ingenious Gedanken experiments, within the framework of a preliminary scalar gravitation theory, that indeed an accelerated mass shell exerts inertial forces on test particles near the centre of the shell. His result for the 'centrifugal force' differs from the result in final general relativity only by a factor $\frac{15}{8}$.

Within general relativity, Thirring demonstrated in his classic paper [5] that an infinitely thin, rotating mass shell (mass M, radius R, angular velocity ω) exerts a Coriolis-type force $\boldsymbol{K}_{\mathrm{C}} = -(8Mm/3R)(\boldsymbol{\omega}\times\boldsymbol{v})$ on test particles (mass m, velocity $\boldsymbol{v}$) near the centre of the shell, if the approximations $M/R \ll 1$ and $\omega R \ll 1$ are made, and units with $G=c=1$ are used. Thirring obtained an additional force $\boldsymbol{K}_{\mathrm{Z}} = -(4Mm/15R)[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})+2(\boldsymbol{\omega}\cdot\boldsymbol{r})\boldsymbol{\omega}]$ in order ω^2 which he interpreted as centrifugal force, although it also has an axial component and cannot be made zero in the same rotating frame in which the Coriolis force vanishes. In 1923, Lanczos [6] noticed that Thirring's paper suffered from an inconsistency, because his solutions violated the local energy-momentum conservation law $T^{\mu}{}_{\nu;\mu}=0$ in order ω^2, since he had neglected any stresses in the rotating mass shell. However the correction of this error, which was not done fully until 1955 [7], did not produce a correct centrifugal force inside the rotating shell, even if one allowed for a latitude-dependent mass density on the

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shell. Therefore, it became more and more probable and convincing that the original ideas and hopes of Mach and Einstein are not fully realisable in general relativity [8], even more so when it was argued [9] that a gravitationally induced centrifugal force should not be of order $M\omega^2$, as it is in Thirring's work, but of order $M^2\omega^2$.

In this paper, we prove, however, that for given parameters M, R and ω, there exists exactly one quasi-spherical rotating mass shell which induces flat geometry in its whole interior and therefore correct Coriolis and centrifugal forces and no additional spurious forces. In this way, Mach's ideas on the relativity of rotation (not the whole so-called Mach principle, as stated by Einstein [10]!) are materialised in general relativity as completely as one could ever hope within the model of a shell-type sky. Our work can be seen in continuation of the classical paper of Brill and Cohen [11], which admittedly is the most important positive contribution to Mach's question hitherto. Whereas this paper was able to confirm Mach's idea in the case of a mass shell only to order ω (for the Coriolis force), we have succeeded in extending the positive results of Brill and Cohen to order ω^2 (for the centrifugal force), the essential point for this success being that we allow for a deviation from the spherical form of the shell and for a latitude-dependent mass density. Like Brill and Cohen, we perform all calculations exactly in M so that there is no debate as to whether the centrifugal force is of order $M\omega^2$ or $M^2\omega^2$. Furthermore, from our procedure it can be seen that it is plausible that the problem has a unique solution in any order of ω.

The only other systems, for which an induction of correct Coriolis and centrifugal forces could be established up to now, are rotating cylindrical mass shells. The cylindrical symmetry of these models forbids, however—in contrast to our model—any analogy or extension to some realistic model of the universe. Nevertheless, we will return to cylindrical mass shells, and their relation to our model, in § 4.

2. Field equations, and solutions in zeroth and first order in ω

The mathematical problem, which we have to solve, can be stated very simply as follows. Take the general stationary, axisymmetric and asymptotically Minkowskian vacuum metric outside some axially symmetric, finite region of space. Take flat geometry inside this region using coordinates which are rotating relative to the asymptotic coordinates. Is it then possible to connect both solutions of Einstein's vacuum field equations at the boundary (which should be spherical in the limit $\omega \to 0$) in such a way, that the metric is continuous, and the discontinuities of its orthogonal derivatives produce a δ-type energy-momentum tensor $T^\mu{}_\nu$, which represents a rigidly rotating mass shell?

For the main part of our calculations, we use isotropic coordinates $t = x^0$, $r = x^1$, $\theta = x^2$, $\phi = x^3$. If in stationary, axisymmetric systems $T_{\mu\nu}$ has a block form with $T_{01} = T_{02} = T_{31} = T_{32} = 0$, as is true for rigidly rotating mass shells, $g_{\mu\nu}$ has, according to the Kundt–Trümper theorem [12], the same block form with $g_{\mu\nu} = g_{\mu\nu}(r, \theta)$. Furthermore $g_{\mu\nu}$ can be diagonalised in the (1, 2) space. For our centrifugal problem, the following form of the metric turns out to be optimally adapted (and superior to similar forms, used in the literature [13])

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -e^{2U}\, dt^2 + e^{-2U}[e^{2K}(dr^2 + r^2\, d\theta^2) + W^2(d\phi - \omega A\, dt)^2]. \tag{2.1}$$

Then, the flat metric inside the shell is simply given by $U, K, A =$ constant, $W = e^K r \sin\theta$, and the geodesic equations, formed with these metric functions, coincide with the

Newtonian equations of motion under Coriolis and centrifugal force (with angular velocity ωA). The Minkowski metric in the asymptotic region is realised by $U=K=A=0$, $W=r\sin\theta$. If the metric (2.1) is 'produced' by a matter model with constant angular velocity ω, the metric is invariant under the transformation $(\omega, t)\to(-\omega, -t)$, so that all metric functions (which are independent of t) are even in ω, for example $U=\overset{0}{U}+\omega^2\overset{2}{U}+\ldots$.

With the useful abbreviations $t^\mu{}_\nu = 8\pi \exp[2(K-U)]T^\mu{}_\nu$, $D=\partial^2/\partial r^2 + r^{-1}\,\partial/\partial r + r^{-2}\,\partial^2/\partial\theta^2$, and $W_1=\partial W/\partial r$, $W_2=\partial W/\partial\theta$ etc, the field equations for our metric read

$$t^1{}_1 = W^{-1}(r^{-1}W_1 + r^{-2}W_{22}) - U_1{}^2 + r^{-2}U_2{}^2 + W^{-1}(K_1W_1 - r^{-2}K_2W_2) + \tfrac{1}{4}\omega^2W^2\,e^{-4U}(A_1{}^2 - r^{-2}A_2{}^2), \tag{2.2}$$

$$t^2{}_2 = W^{-1}W_{11} + U_1{}^2 - r^{-2}U_2{}^2 - W^{-1}(K_1W_1 - r^{-2}K_2W_2) - \tfrac{1}{4}\omega^2W^2\,e^{-4U}(A_1{}^2 - r^{-2}A_2{}^2), \tag{2.3}$$

$$t^1{}_2 = -W^{-1}W_{12} + r^{-1}W^{-1}W_2 - 2U_1U_2 + W^{-1}(K_1W_2 + K_2W_1) + \tfrac{1}{2}\omega^2W^2\,e^{-4U}A_1A_2, \tag{2.4}$$

$$t^0{}_0 + \omega A t^0{}_3 = -2DU + DK + W^{-1}DW + U_1{}^2 + r^{-2}U_2{}^2 - 2W^{-1}(U_1W_1 + r^{-2}U_2W_2) + \tfrac{1}{4}\omega^2W^2\,e^{-4U}(A_1{}^2 + r^{-2}A_2{}^2), \tag{2.5}$$

$$t^3{}_3 - \omega A t^0{}_3 = DK + U_1{}^2 + r^{-2}U_2{}^2 - \tfrac{3}{4}\omega^2W^2\,e^{-4U}(A_1{}^2 + r^{-2}A_2{}^2), \tag{2.6}$$

$$t^0{}_3 = -\tfrac{1}{2}\omega W^2\,e^{-4U}[DA + 3W^{-1}(A_1W_1 + r^{-2}A_2W_2) - 4(U_1A_1 + r^{-2}U_2A_2)]. \tag{2.7}$$

By linear combination, two simpler equations can be formed

$$t^1{}_1 + t^2{}_2 = W^{-1}DW, \tag{2.8}$$

$$\tfrac{1}{2}(t^1{}_1 + t^2{}_2 + t^3{}_3 - t^0{}_0 - 2\omega A t^0{}_3) = DU + W^{-1}(U_1W_1 + r^{-2}U_2W_2) - \tfrac{1}{2}\omega^2W^2\,e^{-4U}(A_1{}^2 + r^{-2}A_2{}^2). \tag{2.9}$$

Our general procedure for solving these equations as power series in ω goes now as follows. We first solve (2.8) for W in the vacuum region outside the mass shell with correct asymptotic behaviour, insert this into (2.9), and solve for U. Here, the last term of (2.9) is zero or known from the lower-order solutions. With W and U known, we solve equations (2.3) and (2.4) for K, and (2.7) for A. With these solutions, the remaining equations (2.2), (2.5) and (2.6) are automatically fulfilled in the vacuum regions.

In zeroth order ω^0, the metric is static and spherically symmetric, since we demanded a spherical mass shell in the limit $\omega\to 0$; therefore, the solutions of equations (2.2)-(2.9) are simply the Schwarzschild metric (with $\alpha = M/2$):

$$\overset{0}{W} = e^{\overset{0}{K}} r\sin\theta$$

$$\overset{0}{U} = \begin{cases} \log[(r-\alpha)/(r+\alpha)] & \text{for } r>R \\ \log[(R-\alpha)/(R+\alpha)] & \text{for } r\leqslant R \end{cases}$$

$$\overset{0}{K} = \begin{cases} \log[(r^2-\alpha^2)/r^2] & \text{for } r>R \\ \log[(R^2-\alpha^2)/R^2] & \text{for } r\leqslant R \end{cases} \tag{2.10}$$

The discontinuities in the derivatives of these functions at $r=R$ produce through

(2.2)-(2.6) a δ-type energy-momentum tensor $t^{\mu}{}_{\nu} = \tau^{\mu}{}_{\nu}\delta(r-R)$ with

$$\overset{0}{\tau}{}^{1}{}_{1} = \overset{0}{\tau}{}^{1}{}_{2} = 0, \tag{2.11}$$

i.e. no radial stresses, as is to be expected for a spherical shell;

$$\overset{0}{\tau}{}^{2}{}_{2} = \overset{0}{\tau}{}^{3}{}_{3} = 2\alpha^2/R(R^2-\alpha^2); \qquad \overset{0}{\tau}{}^{0}{}_{0} = -4\alpha/R(R+\alpha). \tag{2.12}$$

The singularities of the surface stresses for $R = \alpha$ are also to be expected, because $R = \alpha = M/2$ just marks the Schwarzschild radius. For $R < \alpha$ the shell matter can no more withstand the stresses, and, in violation of our stationarity condition, suffers a gravitational collapse.

In order ω^1, equation (2.7) for $\overset{0}{A}$ has to be solved outside the mass shell. Inserting $\overset{0}{U}$ and $\overset{0}{W}$ from (2.10) and noticing that in order ω^1 the shell is still spherical, and therefore $\overset{0}{A}$ independent of θ, this equation reads

$$\overset{0}{A}_{11} + 2r^{-1}(r^2-\alpha^2)^{-1}(2r^2-4r\alpha+\alpha^2)\overset{0}{A}_{1} = 0,$$

and has the solution

$$\overset{0}{A} = \begin{cases} \lambda r^3/(r+\alpha)^6 & \text{for } r > R \\ \lambda R^3/(R+\alpha)^6 & \text{for } r \leq R \end{cases} \tag{2.13}$$

with some integration constant λ. The discontinuity in $\overset{0}{A}_1$ produces by (2.7) the result

$$\overset{1}{\tau}{}^{0}{}_{3} = \tfrac{3}{2}\lambda\omega(R^2-\alpha^2)^{-1}\sin^2\theta. \tag{2.14}$$

The integration constant λ is fixed by demanding that the energy-momentum tensor $T^{\mu}{}_{\nu}$, given by (2.12) and (2.14), really represents a rigidly rotating body, i.e. that the timelike eigenvector of $T^{\mu}{}_{\nu}u^{\nu} = -\rho u^{\mu}$ has the form $u^{\mu} = (u^0, 0, 0, \omega u^0)$ with constant ω, ρ being the invariant mass density

$$\lambda = 4\alpha(2R-\alpha)(R+\alpha)^5 R^{-3}(3R-\alpha)^{-1}. \tag{2.15}$$

Herewith, the dragging coefficient inside the shell takes the form

$$\overset{0}{A}(r \leq R) = 4\alpha(2R-\alpha)(R+\alpha)^{-1}(3R-\alpha)^{-1}. \tag{2.16}$$

In the limit $R \gg \alpha$, (2.16) coincides with Thirring's result. In the other extreme case $R \to \alpha$, one gets $\overset{0}{A}(r \leq R) = 1$, and therefore total dragging of the inertial frames, resp. total screening of the asymptotic Minkowski metric by a compact rotating shell. This result can be interpreted as full realisation (to order ω!) of Mach's idea concerning the relativity of rotation. The results (2.11)-(2.16) have, in a different notation, already been derived by Brill and Cohen [11].

3. Solutions of the field equations in order ω^2

Whereas the framework of Brill and Cohen is not naturally generalisable to higher orders of ω, where the mass shell is expected to deviate from a sphere, our procedure, following equations (2.2)-(2.9), works without much change in all orders of ω, and also for not exactly spherical mass shells; only the algebra gets more complicated. In the following, we give some of the details in order ω^2, which is of main interest in connection with the centrifugal force.

In order ω^2, the solutions (2.10) and (2.13) operate as source terms in the field equations (2.2)-(2.9). It is therefore clear that $\overset{2}{U}$, $\overset{2}{K}$, $\overset{2}{W}/\sin\theta$, and $\overset{2}{A}$, expanded in Legendre polynomials $P_l(\cos\theta)$, contain only P_0 and P_2, similar to Hartle's work [14] on slowly rotating stars. (The uneven Legendre polynomials are missing due to the equatorial symmetry of the problem.) The metric function $\overset{2}{W}$ outside the mass shell is found as a solution of equation (2.8), for which $W/\sin\theta$ does not contain higher 'angular momenta' than $l=2$:

$$\overset{2}{W}=\beta_0 r^{-1}\sin\theta+\beta_2 r^{-3}\sin 3\theta. \tag{3.1}$$

We insert this and the zero-order results (2.10) and (2.13) into (2.9). Since $\overset{2}{U}$ has to have the form

$$\overset{2}{U}=g(r)+h(r)P_2(\cos\theta), \tag{3.2}$$

(2.9) splits into two differential equations:

$$g_{11}+2r(r^2-\alpha^2)^{-1}g_1=4\alpha\beta_0 r(r^2-\alpha^2)^{-3} \\ +\tfrac{4}{3}\alpha\beta_2 r^{-3}(2r^2-\alpha^2)(r^2-\alpha^2)^{-3}+3\lambda^2r^2(r+\alpha)^{-8}, \tag{3.3}$$

$$h_{11}+2r(r^2-\alpha^2)^{-1}h_1-6r^{-2}h=\tfrac{32}{3}\alpha\beta_2 r^{-3}(2r^2-\alpha^2)(r^2-\alpha^2)^{-3}-3\lambda^2r^2(r+\alpha)^{-8}. \tag{3.4}$$

The asymptotically decreasing homogeneous solution $g_h(r)=\log[(r-\alpha)/(r+\alpha)]$ is the Schwarzschild solution from (2.10), the homogeneous solution h_h is found either by imaginative analogy to this solution, or systematically by intermediate expansion in α:

$$h_h(r)=(r^2+\tfrac{2}{3}\alpha^2+\alpha^4r^{-2})\log[(r-\alpha)/(r+\alpha)]+2\alpha r^{-1}(r^2+\alpha^2). \tag{3.5}$$

Inhomogeneous solutions of (3.3) and (3.4) are again found either by analogy to the inhomogeneities or in a systematic but laborious way by the method of variation of constants. The complete, asymptotically decreasing solution $\overset{2}{U}$ is given by

$$g(r)=\beta_0\alpha^{-1}r(r^2-\alpha^2)^{-1}+\tfrac{1}{3}\beta_2\alpha^{-3}r^{-1}(3r^2-2\alpha^2)(r^2-\alpha^2)^{-1} \\ -\tfrac{1}{192}\lambda^2\alpha^{-3}(r+\alpha)^{-6}[3r(r^2+3r\alpha+\alpha^2)^2+13r^3\alpha^2]+\gamma_0 g_h(r), \tag{3.6}$$

$$h(r)=\tfrac{8}{3}\beta_2\alpha^{-1}r^{-1}(r^2-\alpha^2)^{-1}+\tfrac{1}{12}\lambda^2\alpha^{-1}r^3(r+\alpha)^{-6}+\gamma_2 h_h(r), \tag{3.7}$$

with integration constants γ_0, γ_2. In order that the total mass M of the shell is not changed in order ω^2, $\overset{2}{U}$ has to fall off faster than r^{-1} for $r\to\infty$, which requires

$$\gamma_0=\tfrac{1}{2}\beta_0\alpha^{-2}+\tfrac{1}{2}\beta_2\alpha^{-4}-\tfrac{1}{128}\lambda^2\alpha^{-4}. \tag{3.8}$$

In order to find the solution $\overset{2}{K}$, we first consider equation (2.4). After inserting $\overset{2}{W}$, $\overset{2}{U}$, and $\overset{0}{A}_2=0$, and taking into account the 'angular momentum cutoff' in the form

$$\overset{2}{K}=k(r)+l(r)\sin^2\theta, \tag{3.9}$$

(2.4) breaks up into two equations

$$k_1=-2r(r^2-\alpha^2)^{-2}[\beta_0+3\beta_2r^{-4}(2r^2-\alpha^2)], \tag{3.10}$$

$$(r^2-\alpha^2)l_1+2r^{-1}(r^2+\alpha^2)l=24\beta_2 r^{-3}(2r^2-\alpha^2)(r^2-\alpha^2)^{-1}-12\alpha h(r). \tag{3.11}$$

They have the solutions

$$k(r)=\beta_0(r^2-\alpha^2)^{-1}+3\beta_2 r^{-2}(r^2-\alpha^2)^{-1}, \tag{3.12}$$

$$\begin{aligned} l(r)=&-2\beta_2 r^{-2}(4r^2-3\alpha^2)(r^2-\alpha^2)^{-2}+\tfrac{1}{2}\lambda^2 r^4(r+\alpha)^{-6}(r-\alpha)^{-2} \\ &-4\alpha\gamma_2\{2\alpha(r^4+\alpha^4)(r^2-\alpha^2)^{-2}+r^{-1}(r^2+\alpha^2)\log[(r-\alpha)/(r+\alpha)]\} \\ &+\delta_2 r^2(r^2-\alpha^2)^{-2}, \end{aligned} \tag{3.13}$$

with an integration constant δ_2. In order that this function $\overset{2}{K}$ also fulfils equations (2.2) and (2.3) (which are equivalent, since (2.8) is already solved) in order ω^2, the following relations between the integration constants have to be fulfilled

$$\gamma_2=-\tfrac{3}{8}\alpha^{-2}\gamma_0, \qquad \delta_2=-3\beta_0-\beta_2\alpha^{-2}+\tfrac{1}{64}\lambda^2\alpha^{-2}. \tag{3.14}$$

After having integrated all essential field equations in order ω^2 outside the mass shell, we have to perform the continuous connection of $\overset{2}{U}$, $\overset{2}{K}$, and $\overset{2}{W}$ to the inside constants $\overset{2}{U}_0$, $\overset{2}{K}_0$, resp. to $\overset{2}{W}=[(R^2-\alpha^2)/R^2]\overset{2}{K}_0 r\sin\theta$. This is, however, possible only if we allow for a deviation of the mass shell from the spherical shape. Due to the fact that in order ω^2 no higher 'angular momenta' than $l=2$ occur, we can make the following ansatz for the shell geometry

$$r_S=R(1+\omega^2 f\sin^2\theta), \tag{3.15}$$

where f is some constant, resp. some function of the physical parameters R and α of our system. The continuity conditions for the two 'angular momentum' parts of $\overset{2}{W}$ at $r=r_S$ now read

$$\beta_0=(R^2-\alpha^2)\overset{2}{K}_0-\tfrac{3}{2}\alpha^2 f, \tag{3.16}$$

$$\beta_2=\tfrac{1}{2}R^2\alpha^2 f. \tag{3.17}$$

The corresponding continuity conditions for $\overset{2}{U}$ are

$$\overset{2}{U}_0=g(R)+h(R), \tag{3.18}$$

$$[2\alpha R/(R^2-\alpha^2)]f-\tfrac{3}{2}h(R)=0. \tag{3.19}$$

The θ independent term of $\overset{2}{K}$ is automatically continuous at $r=r_S$ if $\overset{2}{W}$ is continuous, because for $r\to\infty$ and $r<R$, W and K are connected via $W=e^K r\sin\theta$, and, furthermore, K is calculated from the differential equations (2.3) and (2.4) which are of first order in K (the solution (3.12) contains no new integration constant!). The continuity condition for the 'quadrupole term' of $\overset{2}{K}$ reads

$$l(R)+[(R^2+\alpha^2)/(R^2-\alpha^2)]f=f. \tag{3.20}$$

After insertion of β_0 and β_2 from (3.16) and (3.17), equations (3.19) and (3.20) are inhomogeneous linear equations for the 'constants' $\overset{2}{K}_0$ and f, which have unique solutions (and would be in conflict with each other for $f=0$, i.e. for a spherical mass shell!). The 'constants' β_0, β_2 and $\overset{2}{U}_0$ are finally given by equations (3.16)-(3.18). All these expressions depend, however, in a quite complicated way on the physical parameters R and $\alpha=M/2$, and we give only the explicit result for f, which is of primary interest from the physical point of view. With the abbreviation $x = R/\alpha$, the 'ellipticity parameter' of our rotating mass shell is given by

$$\frac{f}{R^2} = -\frac{16(x+1)^4(2x-1)^2}{3x^4(3x-1)^2} \times \left(\frac{2x+(x^2+1)\log[(x-1)/(x+1)]}{2x(x^2+1)+(x^4+\frac{2}{3}x^2+1)\log[(x-1)/(x+1)]} - \frac{3(x^2+6x+1)}{32x^2} \right). \tag{3.21}$$

The graph of $-f/R^2$ is shown in the physical region $x \geqslant 1$ in figure 1, and will be discussed in § 4. The dependence of the 'constants' $\overset{2}{U}_0/R^2$ and $\overset{2}{K}_0/R^2$ on x is mathematically similar to formula (3.21). Both expressions go to zero (from negative values) for $x \to \infty$, and diverge linearly to $+\infty$ for $x \to 1$.

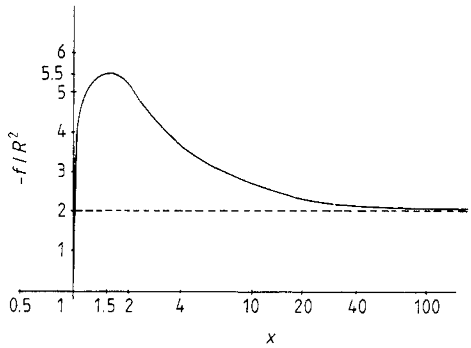

Figure 1. The function $-f/R^2$ from equation (3.21), describing the 'ellipticity' of the rotating mass shell, as a function of $x = R/\alpha = 2R/M$, in the physical region $x \geqslant 1$.

4. Discussion of results

We should like to start with a discussion of the 'ellipticity parameter' f in its dependence on $x = R/\alpha$, for which at least three properties are remarkable.

(a) f is negative for all $x > 1$, so that the rotating shell with flat inside geometry has (suprisingly?) to have prolate shape. The invariant equatorial circumference is smaller than a polar circumference by an amount $-\omega^2\pi R^{-1}(R+\alpha)^2 f$. We think however that a deeper discussion or 'understanding' of this property is not necessary because for realistic models of the universe with extended matter and non-Minkowskian 'asymptotics' the problem does not pose itself in this form. For the same reason, we do not want to go into a discussion of the maximum of $-f/R^2$ near $x = 1.5$.

(b) f/R^2 reaches the non-zero value -2 for $x \to \infty$ in accordance with the fact that Thirring and others could not get the correct centrifugal force with spherical mass shells.

(c) For $x \to 1$, i.e. compact, nearly collapsing shells, f goes to zero. This can be compared with a paper by de la Cruz and Israel [15], who investigated (to order ω^2), whether rotating mass shells can be the source of Kerr's metric. They obtained the result that sphericity and rigid rotation of the shell, and flat inside geometry can only be reached in the limit $R \to \alpha$. In this limit, our constants have the values $\beta_0 = 64\alpha^4$, $\beta_2 = \gamma_0 = \gamma_2 = 0$, $\delta_2 = -128\alpha^4$, and the resulting metric coincides with Kerr's metric to order ω^2, as Robinson's theorem [16] demands. (The transformation to Boyer-Lindquist coordinates is given by $r_{\rm BL}=r^{-1}(r+\alpha)^2-\frac{1}{4}a^2r^{-1}$ with $a=-\frac{1}{4}\lambda\alpha^{-1}\omega$.) For $R>\alpha$, our outside metric differs however from Kerr's metric, which should be of no surprise, since the Kerr metric does not seem to be a natural vacuum solution outside of rigidly rotating, non-collapsed bodies.

We now come to the energy-momentum tensor $\overset{2}{T}{}^{\mu}{}_{\nu}$ within the shell, i.e. the correction terms of order ω^2 to the results (2.12). All components $\overset{2}{T}{}^{\mu}{}_{\nu}$ can be calculated as discontinuities in the orthogonal derivatives of the metric functions with respect to the shell, according to equations (2.2)-(2.6). For the details of these calculations (which we do not give here), we found it advantageous to turn over to a radial coordinate $\bar{r}=r(1-\omega^2\tilde{f}(r)\sin^2\theta)$ (cf equation (3.15)), with $\tilde{f}(R)=f$, and $\lim_{r\to\infty}\tilde{f}(r)=0$, in which the shell looks spherical also in order ω^2, and the 'radial' stress components $\overset{2}{T}{}^{\bar{r}}{}_{\bar{r}}$ and $\overset{2}{T}{}^{\bar{r}}{}_{\theta}$ vanish. Strictly speaking, only in this coordinate the zeroth-order metric is a good approximation to the second-order metric in whole spacetime, also at $\bar{r}=R$ [14]. Instead of giving the components $\overset{2}{T}{}^{\mu}{}_{\nu}$, we prefer to consider the non-trivial eigenvalues λ_i of the equation $T^{\mu}{}_{\nu}u_i^{\nu}=\lambda_i u_i^{\mu}$: $\lambda_1=-\rho$ (invariant mass density); $\lambda_2=p_\theta$; $\lambda_3=p_\phi$. Since in order ω^2 only 'angular momenta' $l=0$ and $l=2$ contribute, we can write (for $i=1,2,3$)

$$\overset{2}{\lambda}_i=\overset{0}{\lambda}_i(S_i+Q_i\sin^2\theta)\delta(r-R-\omega^2Rf\sin^2\theta) \tag{4.1}$$

with $\overset{0}{\lambda}_1=-\overset{0}{\rho}=-\alpha R^3/2\pi(R+\alpha)^5$; $\overset{0}{\lambda}_2=\overset{0}{\lambda}_3=\alpha^2R^3/4\pi(R+\alpha)^5(R-\alpha)$ (cf equation (2.12)). For the 'quadrupole terms' we obtain

$$Q_1/R^2=2f/R^2+\tfrac{9}{2}(1+x^{-1})^4(2-x^{-1})(x-1)^2(3x-1)^{-2}, \tag{4.2}$$

$$Q_2/R^2=(3x-1)(x+1)^{-1}f/R^2, \tag{4.3}$$

$$Q_3/R^2=(5x+1)(x+1)^{-1}f/R^2-9x(1+x^{-1})^4(2-x^{-1})(x-1)^2(3x-1)^{-2}. \tag{4.4}$$

All these terms are again negative for all $x>1$. Q_1/R^2 and Q_2/R^2 look very similar to f/R^2 in figure 1. The formal singularity of Q_3/R^2 for $x\to\infty$ comes about only by factoring out the term $\overset{0}{\lambda}_3$ in (4.1), and expresses the simple physical fact that the azimuthal stress, compensating the centrifugal force, is of order M, whereas the static azimuthal stress $\overset{0}{\lambda}_3$ is 'only' of order M^2. The 'monopole terms' S_i are not easily expressible by f/R^2. Furthermore, all S_i are singular at $R=\alpha$, as have been the 'constants' $\overset{2}{U}_0$ and $\overset{2}{K}_0$. This shows that, at least for some properties of our rotating mass shell, the expansion in powers of ω breaks down for $R\to\alpha$, which calls for exact solutions of Einstein's equations, in order to understand all the physical phenomena occurring in the collapse of rotating bodies.

We should like to add a remark concerning more realistic, non-shell-type models of rotating 'skies', which might produce correct Coriolis and centrifugal forces. In extension of earlier work by Cohen and Brill [17] and Lausberg [18] to higher orders of ω along the lines of our procedure, it might be possible to confirm Mach's ideas also in a somewhat more realistic cosmological model. It should however be kept in mind that there are limits to such a 'realism'. By definition there is only one universe, and the question of Mach does not mean that this universe will rotate, but he poses 'only' the question of principle of whether a hypothetical rotating universe would induce centrifugal forces.

As announced in the introduction, we will now come to some remarks concerning rotating hollow cylinders, a topic, to which some contributions have been published recently which might be of interest in connection with out results. Since the metric inside a rotating hollow cylinder is automatically flat [19], and since this metric is in general in rotation relative to the asymptotic inertial frames, there is automatically an induction of correct Coriolis and centrifugal forces inside a rotating cylinder. One great advantage of cylindrical mass shells is that Einstein's equations can be solved exactly [20, 21], so that all details of the problem and its physical implications can be studied even for relativistic velocities. In the case of simple equations of state, Einstein's equations can be solved exactly even for rotating cylinders of finite thickness [22]. It should however be seen that this advantage in mathematical simplicity has to be paid for by a great deficiency in physical reality of all cylindrical models. Furthermore, the following differences between cylindrical models and spatially finite models are worth mentioning.

(a) Since the metric outside a rotating cylinder is static [23], there exists no Lense-Thirring effect for such models, i.e. no dragging of the axes of gyroscopes outside the cylinder relative to the infinitely far 'fixed stars'.

(b) Since stationary and cylindrically symmetric metrics depend only on one significant coordinate, and, since the flatness inside the cylinder represents no additional restriction, the energy-momentum tensor of a cylindrical mass shell has more 'degrees of freedom' than the energy-momentum tensor of a spatially finite mass shell with flat interior. For instance, besides the invariant mass density ρ the stress p_z in axial direction can also be freely given for a cylindrical mass shell, and only the azimuthal stress p_ϕ is then determined as a function of ρ, p_z, ω and R [21].

(c) The unrealistic, infinite z extension of cylindrical models has the unpleasant mathematical consequence that a metric which is Minkowskian (in rotating coordinates) inside the cylinder, and is everywhere continuous, cannot have an explicitly Minkowskian limit far away from the axis, a fact which is connected with the well known result that the Newtonian potential for an infinite cylinder diverges logarithmically far away from the axis.

Summing up, we should like to argue that the interpretation of results with cylindrical models as confirmation of Mach's ideas concerning the relativity of rotation in general relativity is much less convincing than it is for the results of Brill and Cohen [11] and for our results.

Finally, as an extension of our positive results for circular acceleration, we should like to advance the hypothesis that in general relativity some type of global equivalence between acceleration fields and gravitational fields is valid, in the following sense. If some large but finite laboratory is in arbitrarily accelerated motion relative to the fixed stars, then all motions of free particles and all physical laws, measured from laboratory axes, are modified by inertial forces. It is argued that exactly the same modified motions and laws can be induced (at least for some time) at all places of a laboratory at rest relative to the fixed stars, by suitable and suitably moving masses outside the laboratory. Mathematically, this implies the hypothesis that there exist continuous (but not analytic) solutions of Einstein's field equations (with matter) with the following boundary conditions: flatness in an arbitrary finite region of spacetime, and asymptotic flatness, but with nearly arbitrary acceleration between the asymptotic and the 'interior' inertial frames.

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