Peirce's Triadic Logic

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Max Fisch and
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Charles S. Peirce is generally recognized as an originator of what is often called the matrix method for constructing the usual two-valued sentential calculus,[1] but credit for extending this method to the case of three-valued calculi is usually given to Jan Łukasiewicz and Emil Post. In discussing the origin of triadic logic, Alfred Tarski asserts the following:

> Łukasiewicz was also the first to define by means of a matrix a system of sentential calculus different from the usual one, namely his three-valued system. This he did in the year 1920. Many-valued systems, defined by matrices, were also known to Post.[2]

> What is called the *three-valued* system of the sentential calculus was constructed by Łukasiewicz in the year 1920 and described in a lecture given to the Polish Philosophical Society in Lwów. A report by the author, giving the content of that lecture fairly thoroughly was published in the journal *Ruch Filozoficzny*, vol. 5 (1920), p. 170 (in Polish).[3]

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*We are indebted to the Department of Philosophy of Harvard University for permission to reproduce three pages from Peirce's Logic Notebook (1865-1909) and to quote from it and from other unpublished manuscripts. We cite them by the numbers assigned to them in the Catalogue which has been prepared by Professor Richard S. Robin. The Logic Notebook (Ms. 339) had been dismembered and its pages scattered, but it has recently been reassembled by Professor Don D. Roberts, and he hopes shortly to publish an account of it. Its pages were not numbered, but he has numbered the leaves in his arrangement of them. In his numbering, the pages we reproduce are 340 verso (Plate 1), 341 verso (Plate 2), and 344 recto (Plate 3).

Alonzo Church expresses essentially the same points in the following way:

Using three truth-values instead of two, and truth-tables in these three truth-values, Łukasiewicz first introduced a three-valued propositional calculus in 1920. He was led to this by ideas about modality, according to which a third truth-value—possibility, or better, contingency—has to be considered in addition to truth and falsehood; but the abstract importance of the new calculus transcends that of any particular associated ideas of this kind. Generalization to a many-valued propositional calculus, with $\nu + 1$ truth-values of which $\mu + 1$ are designated $(1 \leqq \mu < \nu)$, was made by Post in 1921, . . .[4]

Church points out further that 1921 denotes the date of the published version[5] of Post's dissertation of 1920.[6]

Some hitherto unpublished Peircean fragments now give what appears to be conclusive evidence that by February 23, 1909, Peirce had already succeeded in extending to the case of triadic logic the matrix method which he originated for the ordinary two-valued logic and for which he is now given full credit.[7] If our evidence is as conclusive as it seems, this means that Peirce not only originated a matrix method for two-valued logic, but he also originated a matrix method for three-valued logic. In this latter regard, therefore, Peirce anticipated similar work by Łukasiewicz and Post by at least ten years. This does not detract from the value of Łukasiewicz's and Post's contributions to triadic and many-valued logic. They both worked without knowledge of Peirce's earlier results and their findings constitute important additions to Peirce's preliminary investigations. Peirce, however, should be given proper credit for his role in the history of triadic logic which these newly found unpublished fragments from his writings bring to light.

These fragments are unnumbered pages from Peirce's *Logic Notebook* (Ms. 339). They are reproduced here, in the order in which they seem to have been written, as Plates 1, 2 and 3. Peirce's remark that "all this is mighty close to nonsense" suggests that he was not happy with the results as shown in Plate 1. There is, however, no indication that he was not satisfied with the results as shown in Plate 2. Since Peirce concludes at the end of Plate 3 that "Triadic Logic is universally true," it seems safe to assume that he was completely satisfied with the results as shown in Plate 3. Perhaps Plate 1 reveals Peirce's first recorded experiments with triadic

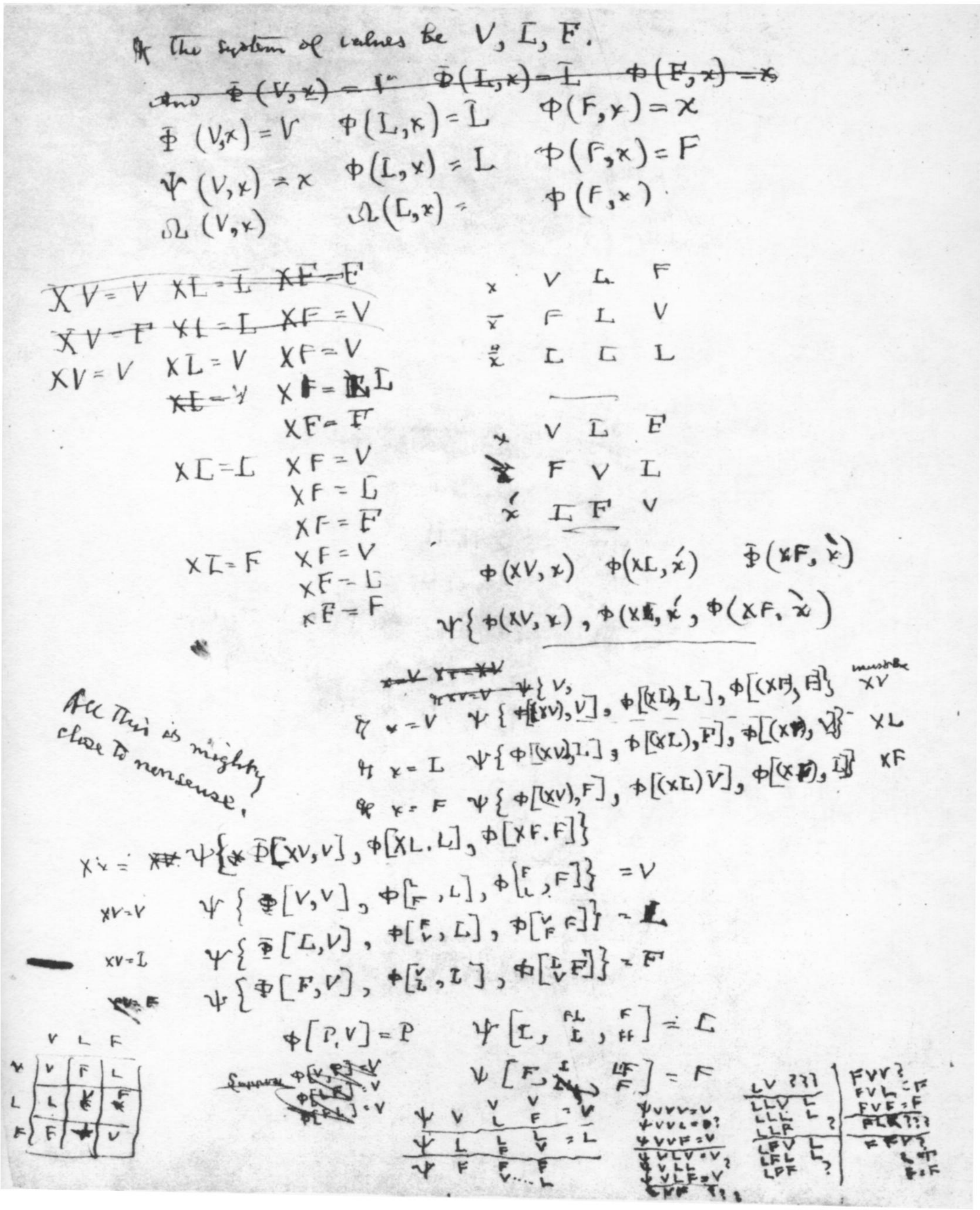
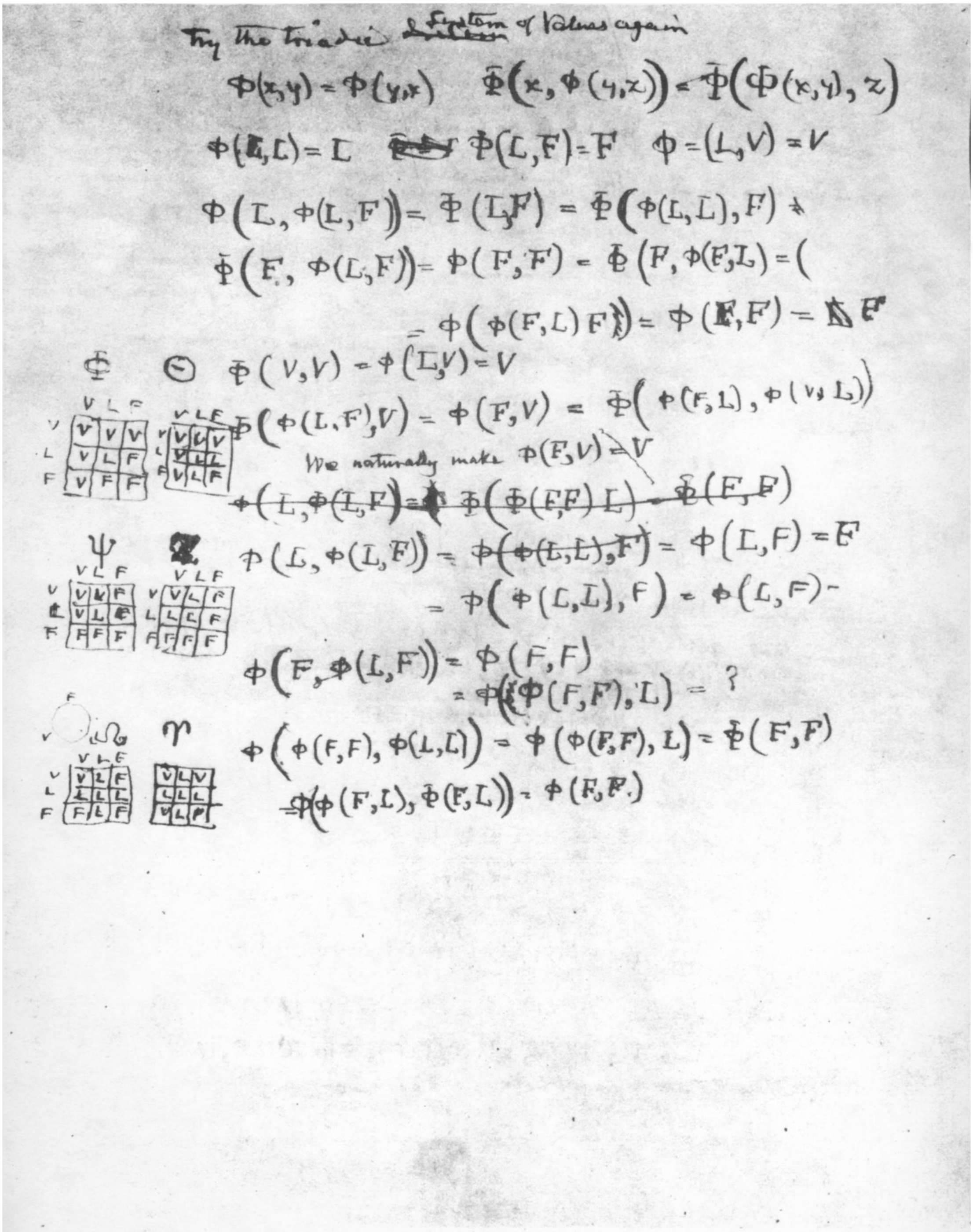


Plate 1. Peirce's Logic Notebook, page 340v.



Try the triadic system of Values again

$\Phi(x,y) = \Phi(y,x) \quad \Phi(x, \Phi(y,z)) = \Phi(\Phi(x,y), z)$

$\Phi(L,L) = L \quad \Phi(L,F) = F \quad \Phi(L,V) = V$

$\Phi(L, \Phi(L,F)) = \Phi(L,F) = \Phi(\Phi(L,L), F)$

$\Phi(F, \Phi(L,F)) = \Phi(F,F) = \Phi(F, \Phi(F,L)) = ($

$= \Phi(\Phi(F,L) F) = \Phi(F,F) = F$

$\Phi(V,V) = \Phi(L,V) = V$

$\Phi(\Phi(L,F),V) = \Phi(F,V) = \Phi(\Phi(F,L), \Phi(V,L))$

We naturally make $\Phi(F,V) = V$

~~$\Phi(L, \Phi(L,F)) = \Phi(\Phi(F,F) L) = \Phi(F,F)$~~

$\Phi(L, \Phi(L,F)) =$ ~~$\Phi(\Phi(L,L),F)$~~ $= \Phi(L,F) = F$

$= \Phi(\Phi(L,L), F) = \Phi(L,F)$

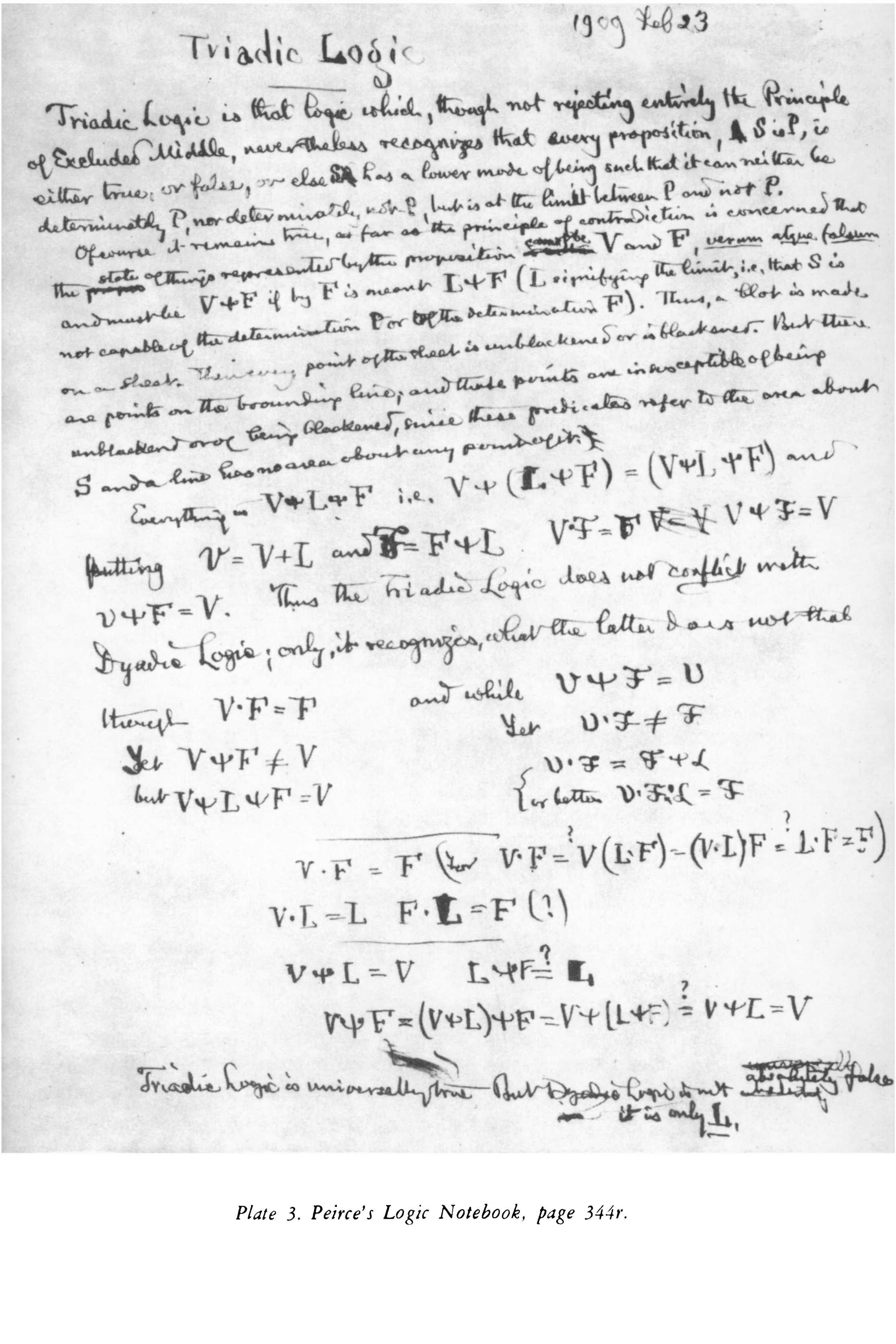
$\Phi(F, \Phi(L,F)) = \Phi(F,F)$

$= \Phi(\Phi(F,F), L) = ?$

$\Phi(\Phi(F,F), \Phi(L,L)) = \Phi(\Phi(F,F), L) = \Phi(F,F)$

$= \Phi(\Phi(F,L), \Phi(F,L)) = \Phi(F,F)$

Plate 2. Peirce's Logic Notebook, page 341v.



1909 Feb 23

Triadic Logic

Triadic Logic is that logic which, though not rejecting entirely the Principle of Excluded Middle, nevertheless recognizes that every proposition, S is P, is either true, or false, or else S has a lower mode of being such that it can neither be determinately P, nor determinately not-P, but is at the limit between P and not P.

Of course it remains true, as far as the principle of contradiction is concerned that the state of things represented by the proposition cannot be V and F, *verum atque falsum* and must be V+F if by F is meant L+F (L signifying the limit, i.e. that S is not capable of the determination P or of the determination F). Thus, a blot is made on a sheet. Then every point of the sheet is unblackened or is blackened. But there are points on the bounding line, and those points are insusceptible of being unblackened or of being blackened, since these predicates refer to the area about S and a line has no area about any point of it.

Everything = V+L+F i.e. V+(L+F) = (V+L+F) and

Putting $\mathfrak{V} = V+L$ and $\mathfrak{F} = F+L$ $\quad V\cdot\mathfrak{F} = F \quad V+\mathfrak{F} = V$

$\mathfrak{V}+F = V$. Thus the Triadic Logic does not conflict with Dyadic Logic; only, it recognizes, what the latter does not that

though $V\cdot F = F$ and while $\mathfrak{V}+\mathfrak{F} = \mathfrak{V}$

yet $V+F \neq V$ yet $\mathfrak{V}\cdot\mathfrak{F} \neq \mathfrak{F}$

but $V+L+F = V$ $\quad \mathfrak{V}\cdot\mathfrak{F} = \mathfrak{F}+L$ {or better $\mathfrak{V}\cdot\mathfrak{F}\cdot L = \mathfrak{F}$

$V\cdot F = F$ (for $V\cdot F \overset{?}{=} V(L\cdot F) - (V\cdot L)F \overset{?}{=} L\cdot F = F$)

$V\cdot L = L \quad F\cdot L = F$ (?)

$V+L = V \quad L+F \overset{?}{=} L$

$V+F = (V+L)+F = V+(L+F) \overset{?}{=} V+L = V$

Triadic Logic is universally true. But Dyadic Logic is not absolutely false — it is only L.

Plate 3. Peirce's Logic Notebook, page 344r.

logic which he did not consider successful. Plate 2 may give results of experiments with triadic logic which Peirce considered successful. Plate 3 may indicate Peirce's conclusions about triadic logic reached after what he considered to be successful experiments with the new kind of logic. In any event, a close examination of Plates 1 and 2 indicates that Peirce was developing triadic logic by means of a matrix method. It is equally clear from Plate 3 that Peirce was willing to conclude that dyadic logic is limited (though "not absolutely false") in a way in which the "universally true" triadic logic is not.

It is interesting to notice that some of Peirce's matrices which appear in Plates 1 and 2 have later been made famous by other logicians working in the area of many-valued logics. Consider the one-place matrices given in Plate 1 under the following correspondences:

V	$\longleftrightarrow$	1	$\longleftrightarrow$	1	$\longleftrightarrow$	t_1	$\longleftrightarrow$	*T*
L	$\longleftrightarrow$	2	$\longleftrightarrow$	½	$\longleftrightarrow$	t_2	$\longleftrightarrow$	*N*
F	$\longleftrightarrow$	3	$\longleftrightarrow$	0	$\longleftrightarrow$	t_3	$\longleftrightarrow$	*F*

Peirce's $\bar{x}$ then corresponds to Łukasiewicz's negation Nx.[8]

Peirce's $\overset{\circ}{\bar{x}}$ corresponds to Słupecki's "tertium function" Tx.[9]

Peirce's $\acute{x}$ and $\grave{x}$ correspond respectively to Post's negations $\sim_3 x$ and $\sim {}^2_3 x$.[10] Applying the same set of correspondences to the two-place matrices which appear in Plate 2, Peirce's operators Θ, Z, T, and Ω are readily associated with well-known matrices used by later logicians. Θ corresponds to Post's alternation[11] $\mathbf{V}_3$ which represents a minimum function and plays a useful role in most systems of logic. Z corresponds to the dual of Θ under Peirce's bar negation (Łukasiewicz's *N*)[12] and represents a maximum function which is as useful as a minimum. Ω corresponds to Bocvar's[13] ∩ which is the same as Kleene's weak conjunction.[14] Finally, T is the dual of Ω under Peirce's bar negation and is the same as Kleene's weak alternation.[15]

Aside from satisfying commutative and associative laws, just what motivated Peirce to introduce the operators Φ and Ψ is not entirely clear. Φ appears to be a slight variation of Θ and Ψ appears to be a similar variation of Z. It is clear that under Peirce's bar negation, Φ and Ψ constitute a dual pair of operators. Regardless of the motivation, however, it was not necessary for Peirce to introduce Φ and Ψ. Without them, he already had sufficient machinery to develop an entirely adequate triadic logic. In fact, from such later results as Post's,[16] it follows that Peirce could have defined all of his operators in terms of Θ and $'$. The set $[\Theta, ']$ defines what is now called a functionally complete logical system.[17] In the present case, this means, among other things, that all possible operators which are definable by matrices of triadic logic can be defined in terms of Θ and $'$ alone. Hence, on purely formal logical grounds, Peirce had every reason for being completely satisfied with the results obtained from what we take to be his successful experiment with triadic logic. This experiment followed Peirce's decision to "try the triadic System of Values again" as indicated at the top of Plate 2. We assume, however, that Peirce's unsuccessful experiment, as shown in Plate 1, was not a total loss since it did yield a fruitful set of one-place operators. Plates 1 and 2 are not sufficiently clear to justify the conclusion that Peirce was fully aware of the adequacy of his triadic logic. The various expansions which appear in the plates suggest that Peirce may have been thinking along lines of functional completeness. If this was not the case, however, it is still a tribute to Peirce's logical intuition that he actually discovered a thoroughly adequate triadic logical system whether he was fully aware of this fact or not.

Plate 3 adds very little to Peirce's formal development of triadic logic, but it does give insight into Peirce's possible motivation for developing such logics and the manner in which he thought of interpreting them. It is clearly indicated that the motivation arises from problems associated with the kind of proposition which "has a lower mode of being such that it can neither be determinately P, nor determinately not-P" — assuming that the proposition in question is of the form S is P. This alone is sufficient to suggest that Church's account of Łukasiewicz's discovery of a three-valued calculus might also be applied to Peirce's discovery of triadic logic, namely: "He was led to this by ideas about modality, according to which a third truth-value . . . has to be considered in addition to truth and falsehood."[18]

The suggestion becomes even more plausible in the light of the fact that just prior to the period when Peirce was developing his triadic logic, he was giving serious consideration to problems of trichotomic or triadic modality. For example, under the date of January 1908, *The Prescott Book* (Ms. 277) deals with modality in terms of the triad "potentiality," "actuality," and "necessitation." Peirce characterizes these as follows:

Potentiality is the absence of Determination (in the usual broad sense) not of a mere negative kind but a positive capacity to be Yea and to be Nay; not ignorance but a state of being. . .
Actuality is the Act which determines the merely possible . . .
Necessitation is the support of Actuality by reason . . .

On August 28, 1908, Peirce records in his *Logic Notebook* (Ms. 339) an account of the co-reality of the three universes: "1) of Ideas, 2) of Occurrences . . ., and 3) of Powers." He then argues that the "mode of being" of an Idea is that of "Real Possibility," that of an Occurrence is "Actuality," and that of a Power is "Real Necessity." Then, in the same *Logic Notebook* (Ms. 339) on December 27, 1908, Peirce lets "1" denote Idea, "2" denote Occurrence, and "3" denote Habit, which is presumably a kind of Power. This same date, December 27, 1908, appears on the recto of the leaf whose verso is reproduced in Plate 1; however, a line has been drawn through the date and just below it there is recorded January 7, 1909. Peirce could have written Plate 1 on December 27, 1908, but in any event, what has been called Peirce's first recorded experiment with triadic logic followed shortly upon his assignment of numerals to modal triads.

That Peirce was still giving serious attention in a similar fashion to problems of triadic modality late in 1910 is apparent from his treatment of the subject in *The Art of Reasoning Elucidated* (Ms. 678, pp. 34-35). The following quotation from this work is a good summary of Peirce's views:

Now, in this respect, a simply assertory proposition differs just half as much from the assertion of a Possibility, or that of a Necessity, as these two differ from each other. For, as we have seen above, that which characterizes and defines an assertion of Possibility is its emancipation from the Principle of Contradiction, while it remains subject to the Principle of Excluded Third;

while that which characterizes and defines an assertion of Necessity is that it remains subject to the Principle of Contradiction, but throws off the yoke of the Principle of Excluded Third; and what characterizes and defines an assertion of Actuality, or simple Existence, is that it acknowledges allegiance to both formulae, and is thus just midway between the two rational "Modals", as the modified forms are called by all the old logicians.

If Peirce's discovery of triadic logic was actually motivated by his consideration of triadic modality, as the evidence suggests, then it is not too difficult to understand the statements regarding interpretations of triadic logic which appear in Plate 3. Essentially, Peirce seems to be saying that triadic logic may be interpreted as a modal logic which is designed to deal with the indeterminacies resulting from that mode of being which Peirce has called "Potentiality" and "Real Possibility." Under such an interpretation, dyadic logic becomes a limiting case of triadic modal logic resulting from removing indeterminacy and being determined entirely by "Actuality."

In this connection, it will be helpful to recall Peirce's analysis of indeterminacy in his "Issues of Pragmaticism." In particular, consider the following quotations:

A sign (under which designation I place every kind of thought, and not alone external signs), that is in any respect objectively indeterminate (i.e., whose object is undetermined by the sign itself) is objectively *general* in so far as it extends to the interpreter the privilege of carrying its determination further. (5.447)

Every utterance naturally leaves the right of further exposition in the utterer; and therefore, in so far as a sign is indeterminate, it is vague, unless it is expressly or by a well-understood convention rendered general. (5.447)

Perhaps a more scientific pair of definitions would be that anything is *general* in so far as the principle of excluded middle does not apply to it and is *vague* in so far as the principle of contradiction does not apply to it. Thus, although it is true that "Any proposition you please, *once you have determined its identity,* is either true or false"; yet *so long as it remains indeterminate and so without identity,* it need neither be true that any proposition you please is true, nor that any proposition you please is false. So likewise, while it is false that "A proposition *whose identity I have determined* is both true and false," yet until it is determinate, it may be true that a proposition is true and that a proposition is false. (5.448)

Such a treatment of indeterminacy appears to leave open the question as to whether it is always possible in principle to remove an indeterminacy. To borrow Peirce's own language, is it always possible to remove an indeterminacy by making use of "the privilege of carrying its determination further"? The answer clearly depends on whether or not a limit exists for "carrying its determination further." Bertrand Russell's criticism[19] of Hugh MacColl's *variables,*[20] which are similar in several respects to Peirce's assertions of possibility, is typical of the view that in principle an indeterminacy can always be removed. The trick, according to Russell, is not to overlook "two relevant and connected distinctions . . ., namely (1) that between a verbal or symbolic expression and what it means, (2) that between a proposition and a propositional function."[21] Hans Reichenbach's analysis of "indeterminate statements" in quantum mechanics will serve to illustrate a view contrary to Russell's. For Reichenbach, indeterminacy is "inherent in the nature of the physical world" and no amount of "interpolation" can remove all indeterminate statements.[22] Another interesting example might be provided by the now famous Gödel undecidable sentences.[23]

In interpreting his triadic logic, did Peirce intend to take a stand on this kind of issue? The black blot illustration of indeterminacy given in Plate 3 is not very helpful in providing an answer. This is especially true when comparisons are made with similar illustrations in the *Grand Logic* (4.127) and "Grounds of Validity of the Laws of Logic" (5.336). However, judging from Peirce's remarks on "Real Possibility" and "Potentiality," it would seem clear that his intention was to side with those who would deny that indeterminacy can always be removed. In particular, "Potentiality" is "a positive capacity to be Yea and to be Nay; not ignorance but a state of being." This suggests that Peirce would have agreed with the later view of Werner Heisenberg to the effect that there is "necessary uncertainty."[24] It is interesting to note that such necessary uncertainties led Heisenberg to reinstate Aristotle's "potentia."[25] A letter to William James[26] gives further evidence that Peirce firmly believed in unavoidable indeterminacy at the time he was writing his remarks on triadic logic as given in Plate 3. The letter is dated March

9, 1909, and thus was written just a short time after Peirce wrote *Triadic Logic.* In the letter, Peirce writes to James that "I hold to my 'tychism' more than ever." Perhaps this unavoidable indeterminacy was one of the principal factors which led Peirce to conclude at the bottom of Plate 3: "Triadic Logic is universally true."

In the March 9, 1909, letter to James, Peirce mentions an earlier draft of "forty sheets" just before indicating that he believed in tychism more than ever. This forty-sheet draft (more exactly, it runs to forty-two numbered pages) has recently been re-assembled. Dated February 26, 1909, just three days after Plate 3, it contains some additional evidence that in Peirce's mind there was an important connection between his version of tychism and triadic logic. For example, in explaining his tychism to James in the forty-sheet draft, Peirce writes the following (pp. 21-22):

> I have long felt that it is a serious defect in existing logic that it takes no heed of the *limit* between two realms. I do not say that the Principle of Excluded Middle is downright *false;* but I *do* say that in every field of thought whatsoever there is an intermediate ground between *positive assertion* and *positive negation* which is just as Real as they. Mathematicians always recognize this, and seek for that limit as the presumable lair of powerful concepts; while metaphysicians and oldfashioned logicians, — the sheep & goat separators, — never recognize this. The recognition does not involve any denial of existing logic, but it involves a great addition to it.

Another very important factor back of Peirce's belief in triadic logic, no doubt, was his cenopythagoreanism (1.351, 8.328, and 1.568). In a letter to Lady Welby dated "1904 Oct. 12," Peirce writes as follows:

> I now come to Thirdness. To me, who have for forty years considered the matter from every point of view that I could discover, the inadequacy of Secondness to cover all that is in our minds is so evident that I scarce know how to begin to persuade any person of it who is not already convinced of it. Yet I see a great many thinkers who are trying to construct a system without putting any thirdness into it. Among them are some of my best friends who acknowledge themselves indebted to me for ideas but have never learned the principal lesson. Very well. It is highly proper that Secondness should be searched to its very bottom. Thus only can the indispensableness and irreducibility of thirdness be made out, although for him who has the mind to grasp it, it is sufficient to say that no branching of a line can result from putting one line on the end of another. (8.331)

Peirce seems to have already reached essentially the same conclusion by 1885 in his "One, Two, Three, Fundamental Categories of Thought and of Nature" (1.369 with an error of transcription corrected). There he writes:

Kant, the King of modern thought, it was who first remarked the frequency in logical analytics of *trichotomies* or threefold distinctions. It really is so; I have tried hard and long to persuade myself that it is only fanciful, but the facts will not countenance that way of disposing of the phenomenon.

If Peirce was as convinced of his cenopythagoreanism in 1885 as he was in 1904 when he wrote Lady Welby, why didn't he develop his triadic logic sooner than he did? In particular, why was it not incorporated within his closely related investigations of "trichotomic mathematics" in the *Minute Logic* of 1902 (4.307-323)? In fact, the work on trichotomic mathematics in the *Minute Logic* suggests so strongly the possibility of triadic logic, it seems surprising that Peirce did not incorporate his work on triadic logic with that on trichotomic mathematics in the *Minute Logic*. The reason appears to be that Peirce had not solved the problem of triadic logic at the time he was working on trichotomic mathematics in the *Minute Logic*. This assumes, as was suggested earlier, that Plate 2 represents Peirce's first successful recorded experiments with triadic logic and that he had not solved the problem of triadic logic to his satisfaction before February 1909. It seems likely, however, that Peirce may have conducted unrecorded thought experiments with triadic logic at least as early as his work on trichotomic mathematics in the *Minute Logic*. He may even have recorded some of these experiments in notebooks referred to in Ms. 339 which have not yet been found. Hence, there are still many unknowns associated with Peirce's development of triadic logic.

It would be interesting to know exactly what circumstances stimulated Peirce's successful solution of the problem of triadic logic. For example, could it have been that Peirce had been following the controversy in *Mind* between Bertrand Russell and Hugh MacColl, prompted by Russell's review[27] of MacColl's book[28] and MacColl's reply?[29] If so, late in 1908 or early in 1909, Peirce might have seen MacColl's note entitled " 'If' and 'Imply' "[30] which had appeared in

January 1908. In this note, MacColl considers the difference between his and Russell's treatment of implication. He indicates that for "nearly thirty years" he has been "vainly trying to convince" logicians of the errors involved in equating "implication" with what is now called "Russell's material implication." MacColl then asks the following question:

Is it too much to hope that this test case will at last open the eyes of logicians to the necessity of accepting my three-fold division of statements (ε, η, θ) with all its consequences?

This three-fold division of statements was introduced quite explicitly in "Def. 2" of MacColl's fifth paper (1896-1897) in the *Proceedings of the London Mathematical Society.*[31] In the "Postscript" of the same paper, MacColl speaks of the great utility of "this logic of *three dimensions* (ε, η, θ),"[32] and points out the fallacy resulting from confounding "*truth* with *certainty,* and *falsehood* with *impossibility.*" Considering MacColl's rejection of Russell's material implication, it is interesting to notice also that MacColl's "Def. 13"[33] gives what is now called "C. I. Lewis's strict implication."

It is beyond the scope of the present paper to investigate in detail all the possible relationships between Peirce's construction of his triadic logic and MacColl's three-dimensional logic. However, it is known that Peirce and MacColl were interested in one another's work for a considerable period of time. In a letter to Peirce dated May 16, 1883, MacColl wrote the following:

It will be a great pleasure indeed to me if you can stay a little while in Boulogne on your way to England. It is not often that I have the opportunity of making the personal acquaintance of my correspondents in logic and mathematics. (L261)

Much later in a draft of a letter dated November 16, 1906, Peirce writes the following to MacColl:

Although my studies in symbolic logic have differed from yours in that my aim has not been to apply the system to the working out of problems, as yours has, but to aid in the study of logic itself, nevertheless I have always thought that you alone, so far as I know, except myself, have understood how the matter ought to be treated by making the *elements* propositions or predicates and not common nouns. (L261)

As far as it is now known, however, there is not sufficient concrete evidence available to make possible an accurate account of the exact relationship between MacColl's work in three-dimensional logic and Peirce's investigations of triadic logic. The solution to this problem will have to await future historical research.

University of Illinois

NOTES

1. For example, see Alfred Tarski, *Logic, Semantics, Metamathematics; papers from 1923 to 1938,* translated by J. H. Woodger (Oxford: Oxford University Press, 1956), p. 40, n. 2. Also, see Alonzo Church, *Introduction to Mathematical Logic* (Princeton: Princeton University Press, 1956), p. 162. In the notes which follow, these works will be referred to simply as "Tarski" and "Church" respectively.

2. Tarski, p. 40, n. 2.

3. Tarski, p. 47, n. 2.

4. Church, p. 162.

5. Emil L. Post, "Introduction to a general theory of elementary propositions," *American Journal of Mathematics,* XLIII (1921), pp. 163-185.

6. Church, p. 162, n. 277.

7. Atwell R. Turquette, "Peirce's icons for deductive logic," *Studies in the Philosophy of Charles Sanders Peirce* (Amherst: The University of Massachusetts Press, 1964), pp. 95-96.

8. Tarski, pp. 47-48. Also see C. I. Lewis and C. H. Langford, *Symbolic Logic* (New York: Dover Publications, 1959), pp. 213-214.

9. Jerzy Słupecki, "Der volle dreiwertige Aussagenkalkül,"*Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie,* Classe III, vol. 29 (1936), pp. 9-11. Also see J. B. Rosser and A. R. Turquette, *Many-valued Logics* (Amsterdam: North-Holland Publishing Company, 1952), chapter II.

10. *Op. cit.* (n. 5 above), p. 180.

11. *Loc. cit.*

12. See Rosser and Turquette, *op. cit.,* pp. 15-16.

13. D. A. Bočvar, "Ob odnom tréhznačnom isčislénii i égo priménénii k analizu paradoksov klassičéskogo rassirénnogo funkcional'nogo isčisléniá," *Matématičéskij sbornik,* vol. 4 (1939), pp. 287-308. Also see Alonzo Church's reviews of Bočvar in *The Journal of Symbolic Logic,* vol. 4 (1939), pp. 98-99, and vol. 5 (1940), p. 119.

14. Stephen C. Kleene, *Introduction to Metamathematics* (New York: D. Van Nostrand Company, 1952), pp. 327-336.

15. *Loc. cit.*

16. *Op. cit.* (note 5 above), pp. 180-181.

17. J. B. Rosser and A. R. Turquette, "Axiom schemes for *M*-valued propositional calculi", *The Journal of Symbolic Logic.* vol. 10 (1945), p. 61, n. 5.

18. See the quotation from Church in the first paragraph of page 72.

19. *Mind,* n.s. XV (1906), pp. 255-260.

20. *Proceedings of the London Mathematical Society,* XXVIII (1897), p. 157.

21. *Mind,* n.s. XV (1906), p. 256.

22. Hans Reichenbach, *Philosophic Foundations of Quantum Mechanics* (Berkeley and Los Angeles: University of California Press, 1944), Parts I and III.

23. For example, see Andrzej Mostowski, *Sentences Undecidable in Formalized Arithmetic: An Exposition of the Theory of Kurt Gödel* (Amsterdam: North-Holland Publishing Company, 1952), pp. 1-13.

24. Werner Heisenberg, *Physics and Philosophy: The Revolution in Modern Science* (New York: Harper and Brothers, 1958), p. 46. Also see Atwell R. Turquette, "Modality, Minimality, and Many-Valuedness," *Acta Philosophica Fennica* fasc. 16 (1963), pp. 261-276.

25. *Op. cit.,* pp. 180-186.

26. Ralph B. Perry, *Thought and Character of William James* (Boston: Little, Brown and Co., 1935), vol. 1, pp. 437-438.

27. *Op. cit.* (n. 19 above).

28. Hugh MacColl, *Symbolic Logic and Its Applications* (London, 1906).

29. *Mind,* n.s. XVI (1907), pp. 470-472.

30. *Mind,* n.s. XVII (1908), pp. 151-152.

31. *Loc. cit.* (n. 20 above).

32. *Ibid.,* pp. 182-183.

33. *Ibid.,* p. 159.