



The fine structure of Weber's hydrogen atom: Bohr–Sommerfeld approach

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Abstract. In this paper, we determine in second order in the fine structure constant the energy levels of Weber's Hamiltonian that admit a quantized torus. Our formula coincides with the formula obtained by Wesley using the Schrödinger equation for Weber's Hamiltonian. We follow the historical approach of Sommerfeld. This shows that Sommerfeld could have discussed the fine structure of the hydrogen atom using Weber's electrodynamics if he had been aware of the at-his-time-already-forgotten theory of Wilhelm Weber (1804–1891).

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Contents

1. Introduction	1
2. Weber's Hamiltonian	3
3. Quantized tori	4
4. Bohr–Sommerfeld quantization	6
Acknowledgements	7
A. Weber's Lagrangian and delayed potentials	8
B. Proton-proton system—Lorentzian metric	10
References	10

1. Introduction

Although Weber's electrodynamics was highly praised by Maxwell [22, p. XI], see also [4, Preface], it was superseded by Maxwell's theory and basically forgotten. In contrast to Maxwell's theory, which is a field theory, Weber's theory is a theory of action-at-a-distance, like Newton's theory. Weber's force law can be used to explain Ampère's law and Faraday's induction law; see [4, Ch. 4 and Ch. 5]. In contrast to the Maxwell–Lorentz theory, Ampère's original law also predicted transverse Ampère forces, in physics actually called longitudinal Ampère forces. These are explained by Weber's force law, too. Interesting experiments about the question of existence of such Ampère forces were carried out by Graneau and Graneau [13]. A touching account of the life of Weber can be found in Wiederkehr's biography of Weber [43].

By quantizing the Coulomb potential, Bohr and Sommerfeld, cf. [25, Ch. II], obtained for the hydrogen atom the following energy levels

$$-\frac{1}{2n^2}, \quad n \in \mathbb{N}, \quad (1.1)$$

in atomic units.¹ Later Schrödinger interpreted these numbers as eigenvalues of his equation, cf. [26]. By taking into account velocity-dependent mass, as suggested by Einstein's theory of relativity, Sommerfeld obtained a more refined formula which also takes into account the angular momentum quantum number ℓ . Sommerfeld's formula [34, 5. Kap. §2 Eq. (24)] is given in second order in the fine structure constant α by

$$-\frac{1}{2n^2} - \frac{\alpha^2}{2n^3\ell} + \frac{3}{8} \frac{\alpha^2}{n^4}, \quad n \in \mathbb{N}, \quad \ell = 1, \dots, n. \quad (1.2)$$

This formula, referred to as the **fine structure of the hydrogen atom**, coincides with the formula one derives in second order in α from Dirac's equation as computed by Darwin [8] and Gordon [11]. We refer to Schweber [32, §1.6] for the historical context of how Dirac discovered his equation.

The mathematical reason why the formula (1.1) of energy levels for the Coulomb potential of the Kepler problem is degenerate in the sense that it is independent of the angular momentum quantum number lies in the fact that the Coulomb problem is super-integrable. Namely, it is not just rotation invariant, but admits further integrals given by the Runge–Lenz vector. Dynamically this translates to the fact that for negative energy, except for collision orbits, all orbits are periodic. In fact, they are given by Kepler ellipses. While for velocity-dependent mass, rotation symmetry is preserved, the additional symmetry from the Runge Lenz vector is broken. Dynamically one sees that orbits are in general not anymore periodic, but given by rosettes as illustrated by Fig. 1. The situation for Weber's Hamiltonian is quite analogous. In fact, Bush [7] pointed out that the shapes of the orbits in both theories coincide. However, note that even when the shapes of the orbits are the same this does not mean that their parametrization or their energy coincides.

In this paper, we compute the energy levels of the Weber rosettes using Sommerfeld's method.

Theorem A. *For Weber's Hamiltonian, the fine structure formula for the hydrogen atom in second order in the fine structure constant α becomes*

$$-\frac{1}{2n^2} - \frac{\alpha^2}{2n^3\ell} + \frac{1}{2} \frac{\alpha^2}{n^4}.$$

Proof. Equation (4.4). □

This formula coincides with the formula Wesley [42, Eq. (100)] obtained using Schrödinger's equation. Note that the difference to Sommerfeld's formula (1.2) involves $\alpha^2 \approx 10^{-5}$ and only lies in the term just involving the main quantum number and not the angular one. Because this term is of much lower order than the Balmer term, i.e., the first term in (1.2), it seems difficult to actually measure the difference.

Outlook—symplectic topology and non-local Floer homology

Weber's Hamiltonian is related to delayed potentials as pointed out by Carl Neumann in 1868. We explain this relation in “Appendix A.” The question how to extend Floer theory to delayed potentials is a topic of active research, see [1–3]. In particular, for delayed potentials Floer's equation is not local and therefore new analytic tools have to be developed inspired by the recent theory of polyfolds due to Hofer, Wysocki, and Zehnder [15]. While to our knowledge, Weber's Hamiltonian was so far unknown to the symplectic community, we hope to open up with this article a new branch of research in symplectic topology. To

¹Differences of these energy levels give rise to the **Rydberg formula**

$$\frac{1}{2n^2} - \frac{1}{2m^2}, \quad n, m \in \mathbb{N}, m > n,$$

that corresponds to the energy of the emitted photon when an electron falls from an excited energy level to a lower one. Historically these differences were first measured in spectroscopy. In 1885, the Swiss school teacher Balmer discovered the formula for the $n = 2$ series, nowadays referred to as the **Balmer series**; cf. [25, p. 163].

our knowledge so far nobody incorporated the spin of the electron in Weber's electrodynamics. In fact, it is even an amazing coincidence that Sommerfeld obtained the same formula for the fine structure of hydrogen as predicted by Dirac's theory. Using semiclassical techniques, Keppeler [17, 18] explains the reason: The Maslov index and the influence of spin mutually canceled out each other—both were not taken into account by Sommerfeld; neither are they in our paper. We expect that the proper incorporation of spin into Weber's electrodynamics requires techniques from non-local Floer homology currently under development.

Lamb and Retherford [19] discovered in 1947 that the spectrum of hydrogen shows an additional small shift not predicted by Dirac's theory. This shift, nowadays referred to as the **Lamb shift**, was a major topic in the Shelter Island Conference and the driving force for the development of Quantum Field Theory and Renormalization Theory; see [32, §4 §5]. An intuitive explanation of the Lamb shift is that vacuum fluctuations cause a small correction of the potential energy close to the nucleus and this small correction then leads to a shift in the spectrum of the hydrogen atom.

In "Appendix A" we explain following Neumann how a retarded Coulomb potential when Taylor approximated up to second order in the fine structure constant leads to Weber's Hamiltonian. Higher-order perturbations lead to perturbations of Weber's Hamiltonian which are most strongly felt close to the nucleus. Whether there is a relation between these higher-order perturbations and vacuum fluctuations is an important topic in the non-local Floer homology under development and its interaction with the semiclassical approach.

The paper [10] explains how Neumann's retarded action functional gives rise to a new viewpoint and new analytic challenges in Floer theory.

2. Weber's Hamiltonian

In this article, we use atomic units to describe a model for the hydrogen atom. There are four **atomic units** which are unity: The electron mass $m_e = 1$, the elementary charge $e = 1$, the reduced Planck constant $\hbar = h/2\pi = 1$, and the Coulomb force constant $k_0 = 1/(4\pi\epsilon_0) = 1$. In particular, Coulomb's Lagrangian and Hamiltonian for an electron ($-e = -1$) attracted by a proton are given by

$$L = \frac{1}{2}|v|^2 + \frac{1}{|q|}, \quad H = \frac{1}{2}|p|^2 - \frac{1}{|q|},$$

respectively. The speed of light is given in atomic units by $c = \frac{1}{\alpha} \approx 137$ where α is Sommerfeld's **fine structure constant**. In his work, Weber used a different constant, namely

$$c_W = \sqrt{2}c.$$

In his famous experiment in 1856, he measured this constant together with Kohlrausch. This experiment was later crucial for Maxwell, because it indicated a strong relationship between electrodynamics and light [40].

The hydrogen atom consists of a (heavy) proton and a (light) electron. We just consider the planar case and suppose that the proton sits at the origin of \mathbb{R}^2 . Polar coordinates $(r, \phi) \in \mathbb{R}_\times^2 := \mathbb{R}^2 \setminus \{0\}$ provide the coordinates $(r, \phi, v_r, v_\phi) \in T\mathbb{R}_\times^2$ and $(r, \phi, p_r, p_\phi) \in T^*\mathbb{R}_\times^2$. Consider the Lagrangian function

$$L_W(r, \phi, v_r, v_\phi) = \underbrace{\frac{1}{2}(v_r^2 + r^2v_\phi^2)}_{=: T_{\text{flat}}} + \underbrace{\frac{1}{r}\left(1 + \frac{v_r^2}{2c^2}\right)}_{=: -S} \quad (2.1)$$

Here T_{flat} is just the kinetic energy in polar coordinates, while S is referred in [4] as the 'Lagrangian energy'. Note that L_W differs from the Coulomb Lagrangian L by the additional term $v_r^2/2c^2r$. This Lagrangian was introduced by Carl Neumann [28] in 1868. We refer to S as **Neumann's potential function**. Its Euler Lagrange equation is precisely the equation studied by Wilhelm Weber [39] twenty years earlier

in 1846. In “Appendix A” we explain how Neumann’s potential function can be obtained as Taylor approximation of a retarded functional.

Observe that the S term in L_W depends on the velocity. Historically this led to a lot of confusion and so Helmholtz [38] and Maxwell [21] doubted for a long time that Weber’s force law complies with conservation of energy; cf. [4, §3.6]. In [35, §384/385], Weber’s theory was even classified under the theories “*pernicious rather than useful.*”

Although in 1871 Weber explained in detail that his force law satisfies the principle of preservation of energy, his article [41] was not mentioned in the translation [36] of Thomson and Tait’s book by Helmholtz and Wertheim which infuriated Zöllner [46, Vorrede] “*Ich wage es zuversichtlich zu behaupten, dass in der ganzen deutschen Literatur nicht ein einziges Lehrbuch anzutreffen sein wird, welches wie jener berühmte §385 des Werkes von THOMSON und TAIT auf dem engen Raume von nur dreissig Zeilen eine solche Fülle von absolutem Nonsens enthält.*” In the new edition of Thomson and Tait [37], the infamous §385 has disappeared.

A symplectic way to see that the disputed preservation of energy holds for Weber’s force law is to rearrange in (2.1) the brackets to obtain

$$\begin{aligned} L_W(r, \phi, v_r, v_\phi) &= \frac{1}{2} \left(1 + \frac{1}{c^2 r} \right) v_r^2 + \frac{1}{2} r^2 v_\phi^2 + \frac{1}{r} \\ &= \frac{1}{2} (g_{rr} v_r^2 + g_{\phi\phi} v_\phi^2) + \frac{1}{r} \\ &= T - V. \end{aligned}$$

The first term in the sum can be interpreted as kinetic energy with respect to a non-flat Riemannian metric g on \mathbb{R}_x^2 , while the second term is minus the (velocity independent) Coulomb potential. In “Appendix B” we explain how in the case of two protons the metric becomes singular at Weber’s critical radius ρ . Outside this critical radius, the metric is Riemannian, while inside it is Lorentzian.

Legendre transformation $\mathcal{L} : T\mathbb{R}_x^2 \rightarrow T^*\mathbb{R}_x^2$ of the mechanical Lagrangian $L_W = T - V$ yields the mechanical Hamiltonian

$$\begin{aligned} H_W(r, \phi, p_r, p_\phi) &= T^* + V \\ &= \frac{1}{2} \left(\frac{1}{g_{rr}} p_r^2 + \frac{1}{g_{\phi\phi}} p_\phi^2 \right) - \frac{1}{r} \\ &= \frac{1}{2} \left(\frac{c^2 r}{c^2 r + 1} p_r^2 + \frac{1}{r^2} p_\phi^2 \right) - \frac{1}{r}. \end{aligned}$$

But this is an autonomous Hamiltonian and any such is preserved along its flow; see, e.g., [27, p. 99]. This explains preservation of energy.

With the help of the fine structure constant $\alpha = \frac{1}{c}$ the Hamiltonian reads

$$H_W(r, \phi, p_r, p_\phi) = \frac{1}{2} \frac{r}{r + \alpha^2} p_r^2 + \frac{1}{2r^2} p_\phi^2 - \frac{1}{r}. \tag{2.2}$$

We refer to H_W as **Weber’s Hamiltonian**.

3. Quantized tori

Weber’s Hamiltonian is completely integrable in the planar case under consideration. Indeed, it is rotational invariant and therefore angular momentum commutes with H_W .

On closed symplectic manifolds the Arnol’d–Liouville theorem, see, e.g., [16, App. A.2], tells us how the manifold gets foliated by invariant tori. Even below the escape threshold, i.e., for negative energy, in the case at hand the tori might have some holes, because of collisions.

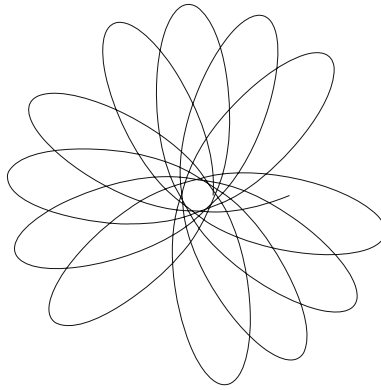


FIG. 1. A rosette ($r = 1 + \kappa \cos \gamma \phi$, eccentricity $0 < \kappa < 1$)

For the Kepler Hamiltonian collisions can be regularized. There exist different regularizations in the literature. For example, Moser showed in [24] that for negative energies $E < 0$ the Kepler flow after regularization can be identified with the geodesic flow on a 2-dimensional sphere. For zero energy $E = 0$, Kepler flow after regularization becomes identified with the geodesic flow on the Euclidean plane, and for positive energy $E > 0$ with the geodesic flow on the hyperbolic plane, as shown by Belbruno [5] and Osipov [31]; see also Milnor [23]. Even older is the regularization by Goursat [12] which was rediscovered independently by Levi-Civita [20] and is nowadays referred to as Levi-Civita regularization. In contrast to Moser regularization, the Levi-Civita regularization is 2:1 and transforms the Kepler flow for negative energy $E < 0$ to the flow of two uncoupled harmonic oscillators.

To our knowledge, so far nobody studied regularization for the Weber Hamiltonian and this would be an interesting project. Nevertheless, one can easily see how the invariant tori look like. Indeed, for negative energy usual orbits, apart from circle orbits and collisions, are given by *rosettes*; see Fig. 1.

In contrast to the Kepler problem, where these orbits are given by Kepler ellipses, the rosettes don't need to be closed and they show a perihel shift; see references in [45, p. 56 footnote 37]. In the case at hand, where the central body is interpreted as a proton, let us replace the expression “perihel shift” by **periproton shift**. If the periproton shift is a rational multiple of 2π , then the rosette finally closes and we obtain a periodic orbit. Hence by rotation invariance of H_W we obtain a circle family of periodic orbits by rotating our closed rosette. But a circle times a circle is a 2-dimensional torus. In this case, the flow on the invariant torus is rational.

If the periproton shift is irrational, the rosette is not closed and we obtain the invariant torus by looking at the closure of the rosette. In this case, the flow on the invariant torus is irrational.

A conceptual explanation what Bohr-Sommerfeld quantization of a completely integrable system is was given by Einstein [9] in 1917; see also [14, §14.1]. The invariant tori are Lagrangian. Hence if we consider the restriction of the Liouville 1-form to the torus T it is closed, and therefore, it defines a class $[\lambda_{\text{can}}] \in H^1(T; \mathbb{R})$ in the first cohomology of the torus. If this class is integer-valued, then we call the torus a **quantized torus**. For some unknown reasons, the electron just likes to stay on quantized tori. Emission occurs if the electron jumps from one quantized torus to another one. In this case, the frequency we observe is given by $\frac{1}{2\pi}$ times the energy difference of the two energy levels on which the quantized tori lie.

Therefore, to understand the spectrum of the electron we have to figure out which energy levels contain a quantized torus. How to do this in practice is the content of Sommerfeld's book [33].

In the next section, we explain how to apply Sommerfeld's calculations to the Weber Hamiltonian.

4. Bohr–Sommerfeld quantization

We quantize the Hamiltonian according to the rules of Bohr and Sommerfeld. Note that H_W does not depend on ϕ , so p_ϕ is a preserved quantity that corresponds to angular momentum. According to Bohr [6] angular momentum has to be quantized: The **angular momentum quantum number** is

$$\frac{1}{2\pi} \int_0^{2\pi} p_\phi \, d\phi = p_\phi =: \ell \in \mathbb{N}. \tag{4.1}$$

Here the first identity holds by preservation of angular momentum. Later Sommerfeld referred to ℓ as the azimuthal quantum number which in his notation was called n . Originally Bohr just considered circular orbits, and therefore, the azimuthal quantum number was enough. In contrast, Sommerfeld [33] allowed more general orbits and imposed a quantization condition on p_r as well, namely

$$\frac{1}{2\pi} \int p_r \, dr =: n_r \in \mathbb{N}_0.$$

This integral has to be interpreted as follows. Note that, because of rotational invariance (independence of ϕ) of the Weber Hamiltonian H_W , orbits in the configuration space are given by rosettes, i.e., the r variable is periodic in time, oscillating between the periproton r_{\min} and the apoproton r_{\max} (closest/farthest point from proton). Therefore

$$\int p_r \, dr = 2 \int_{r_{\min}}^{r_{\max}} p_r \, dr.$$

How to interpret and calculate $\int_{r_{\min}}^{r_{\max}} p_r \, dr$ is illustrated by [34, p.478 Fig. 101]. Using (2.2) and (4.1), we compute for p_r the formula

$$\begin{aligned} p_r &= \sqrt{2 \left(1 + \frac{\alpha^2}{r}\right) \left(H_W + \frac{1}{r} - \frac{\ell^2}{2r^2}\right)} \\ &= \sqrt{2H_W + \frac{2 + 2\alpha^2 H_W}{r} - \frac{\ell^2 - 2\alpha^2}{r^2} - \frac{\ell^2 \alpha^2}{r^3}}. \end{aligned}$$

We abbreviate

$$A = 2H_W, \quad B = 1 + \alpha^2 H_W, \quad C = -\ell^2 + 2\alpha^2 < 0, \quad D_1 = -\ell^2 \alpha^2 \tag{4.2}$$

so that p_r becomes

$$p_r = \sqrt{A + \frac{2B}{r} + \frac{C}{r^2} + \frac{D_1}{r^3}}.$$

As calculated by Sommerfeld [34, p.480 (16)] the integral n_r is given by

$$n_r = -i \left(\sqrt{C} - \frac{B}{\sqrt{A}} - \frac{BD_1}{2C\sqrt{C}} \right)$$

where as explained in [34, p.479] the square root of C has to be taken *negative* imaginary, whereas the one of A *positive*. Plugging in (4.2), we obtain

$$\begin{aligned} n_r &= -i \left(-i\sqrt{\ell^2 - 2\alpha^2} - \frac{1 + \alpha^2 H_W}{\sqrt{2H_W}} + \frac{(1 + \alpha^2 H_W)\ell^2 \alpha^2}{2i(\ell^2 - 2\alpha^2)^{3/2}} \right) \\ &= -\sqrt{\ell^2 - 2\alpha^2} + i \frac{1 + \alpha^2 H_W}{\sqrt{2H_W}} - \frac{(1 + \alpha^2 H_W)\ell^2 \alpha^2}{2(\ell^2 - 2\alpha^2)^{3/2}}. \end{aligned}$$

Taking the two real terms to the left-hand side shows that

$$n_r + \sqrt{\ell^2 - 2\alpha^2} + \frac{(1 + \alpha^2 H_W)\ell^2 \alpha^2}{2(\ell^2 - 2\alpha^2)^{3/2}} = i \frac{1 + \alpha^2 H_W}{\sqrt{2H_W}}.$$

By Taylor expansion of the left-hand side in (α^2) up to first order, we get

$$\begin{aligned} n_r + \sqrt{\ell^2 - 2\alpha^2} + \frac{(1 + \alpha^2 H_W)\ell^2 \alpha^2}{2(\ell^2 - 2\alpha^2)^{3/2}} &\approx n_r + \ell - \frac{\alpha^2}{\ell} + \frac{\ell^2 \alpha^2}{2\ell^3} \\ &= (n_r + \ell) - \frac{\alpha^2}{2\ell}. \end{aligned} \quad (4.3)$$

Plugging this formula into the previous formula and taking squares, we obtain to first order in (α^2) the approximation

$$(n_r + \ell)^2 - \frac{\alpha^2(\ell + n_r)}{\ell} \approx -\frac{1 + 2\alpha^2 H_W}{2H_W} = -\frac{1}{2H_W} - \alpha^2.$$

Rearranging we get

$$\frac{-1}{2H_W} \approx (n_r + \ell)^2 - \frac{\alpha^2 n_r}{\ell}$$

and therefore, expanding again up to first order in (α^2) , we obtain

$$H_W \approx -\frac{1}{2((n_r + \ell)^2 - \frac{\alpha^2 n_r}{\ell})} \approx -\frac{1}{2(n_r + \ell)^2} - \frac{\alpha^2 n_r}{2(n_r + \ell)^4 \ell}.$$

By introducing the **main quantum number**

$$n := n_r + \ell \in \mathbb{N}$$

the previous formula becomes

$$H_W \approx -\frac{1}{2n^2} - \frac{\alpha^2}{2n^3 \ell} + \frac{\alpha^2}{2n^4}. \quad (4.4)$$

The corresponding formula for Sommerfeld's relativistic Hamiltonian is

$$H_S \approx -\frac{1}{2n^2} - \frac{\alpha^2}{2n^3 \ell} + \frac{3\alpha^2}{8n^4}.$$

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A. Weber’s Lagrangian and delayed potentials

In several works, Neumann treated the connection between Weber’s dynamics and delayed potentials, see [28, 29] and [30, Ch. 8]. Neumann explained how Weber’s potential function is related to Hamilton’s principle which in [29] he called “*norma suprema et sacrosancta, nullis exceptionibus obvia*”.

Strictly speaking, a delay potential only makes sense for loops and not for chords. Hence we abbreviate by $\mathcal{L} = C^\infty(S^1, \mathbb{R}_\times^2)$ the free loop space on the punctured plane $\mathbb{R}_\times^2 := \mathbb{R}^2 \setminus \{0\}$. For a potential $V \in C^\infty(\mathbb{R}_\times^2, \mathbb{R})$ and a constant $c_W > 0$, we define three functions

$$\mathcal{S}_{\text{kin}}, \mathcal{S}_{\text{pot}}, \mathcal{S}: \mathcal{L} \rightarrow \mathbb{R}$$

by $\mathcal{S} = \mathcal{S}_{\text{kin}} - \mathcal{S}_{\text{pot}}$ and by

$$\mathcal{S}_{\text{kin}}(q) := \frac{1}{2} \int_0^1 |q'(t)|^2 dt, \quad \mathcal{S}_{\text{pot}}(q) := \int_0^1 V\left(q\left(t - \frac{|q(t)|}{c_W}\right)\right) dt.$$

Physically this means that the potential energy is evaluated at a retarded time. Namely, the position of the proton at the origin has to be transmitted to the electron at speed c_W . It is a strange fact that to obtain Weber’s force this transmission velocity is given by the Weber constant c_W which, as measured by Weber [40], equals $\sqrt{2}c$ where c is the speed of light.

We assume that V only depends on the radial coordinate $V(q) = V(|q|) = V(r)$. Setting $r(t) := |q(t)|$ the functional \mathcal{S}_{pot} becomes a function of $r = r(t)$, still denoted by

$$\mathcal{S}_{\text{pot}}(r) = \int_0^1 V\left(r\left(t - \frac{r(t)}{c_W}\right)\right) dt.$$

We further abbreviate

$$a = \frac{1}{c_W} = \frac{\alpha}{\sqrt{2}}, \quad c_W = \sqrt{2}c,$$

where α is the fine structure constant.

Setting $V_r(a, t) := V(r(t - ar(t)))$, we obtain for the partial derivatives in a the formulas

$$\frac{\partial}{\partial a} V_r(a, t) = -V'(r(t - ar(t)))r'(t - ar(t))r(t)$$

and

$$\begin{aligned} \frac{\partial^2}{\partial a^2} V_r(a, t) &= V''(r(t - ar(t)))r'(t - ar(t))^2 r(t)^2 \\ &\quad + V'(r(t - ar(t)))r''(t - ar(t))r(t)^2. \end{aligned}$$

In particular, at $a = 0$ we get

$$\frac{\partial}{\partial a} V_r(0, t) = -V'(r(t))r'(t)r(t)$$

and

$$\frac{\partial^2}{\partial a^2} V_r(0, t) = V''(r(t))r'(t)^2 r(t)^2 + V'(r(t))r''(t)r(t)^2.$$

We define

$$\mathcal{S}_{\text{pot}}^k(r) := \frac{1}{k!} \int_0^1 \frac{\partial^k}{\partial a^k} V_r(0, t) dt, \quad k \in \mathbb{N}_0.$$

For $k = 0$ this is the unretarded action functional

$$\mathcal{S}_{\text{pot}}^0(r) = \int_0^1 V(r(t)) dt.$$

To see that $\mathcal{S}_{\text{pot}}^1 \equiv 0$ vanishes identically choose a primitive F of the function in one variable $r \mapsto V'(r)r$, that is $F'(r) = V'(r)r$. Indeed, we get that

$$\begin{aligned} \mathcal{S}_{\text{pot}}^1(r) &= - \int_0^1 V'(r(t))r'(t)r(t) dt \\ &= - \int_0^1 F'(r(t))r'(t) dt \\ &= - \int_0^1 \frac{d}{dt} F(r(t)) dt \\ &= 0. \end{aligned}$$

The last equation follows, because $r(t) = |q(t)|$ is periodic. Using integration by parts for the second term in the sum we get that

$$\begin{aligned} \mathcal{S}_{\text{pot}}^2(r) &= \frac{1}{2} \int_0^1 \left(V''(r(t))r'(t)^2 r(t)^2 + \underbrace{r''(t)}_{u'} \underbrace{V'(r(t))r(t)^2}_v \right) dt \\ &= \frac{1}{2} \int_0^1 V''(r(t))r'(t)^2 r(t)^2 dt \\ &\quad - \frac{1}{2} \int_0^1 \underbrace{r'(t)}_u \underbrace{\left(V''(r(t))r'(t)r(t)^2 + 2V'(r(t))r'(t)r(t) \right)}_{v'} dt \\ &= - \int_0^1 V'(r(t))r'(t)^2 r(t) dt. \end{aligned}$$

For the Coulomb potential

$$V(r) = -\frac{1}{r}, \quad V'(r) = \frac{1}{r^2},$$

this simplifies to

$$\mathcal{S}_{\text{pot}}^2(r) = - \int_0^1 \frac{r'(t)^2}{r(t)} dt.$$

Hence Taylor approximation of \mathcal{S}_{pot} to second order in $a = \frac{1}{c_W}$ leads to

$$\begin{aligned} \mathcal{S}_{\text{pot}}^0(r) + \frac{1}{c_W} \mathcal{S}_{\text{pot}}^1(r) + \frac{1}{c_W^2} \mathcal{S}_{\text{pot}}^2(r) &= - \int_0^1 \frac{1}{r(t)} \left(1 + \frac{r'(t)^2}{c_W^2} \right) dt \\ &= \int_0^1 S(r(t)) dt \end{aligned}$$

where

$$S(r) = -\frac{1}{r} \left(1 + \frac{r'^2}{2c^2} \right)$$

is Neumann's potential function, see (2.1), and $2c^2 = c_W^2$.

B. Proton-proton system—Lorentzian metric

For a positive charge influenced by the proton, the Weber force exhibits fascinating properties as well. For simplicity, suppose both charges are protons and set their mass equal to one. In this case, the Lagrangian function is given by

$$L_W(r, \phi, v_r, v_\phi) = \frac{1}{2}(v_r^2 + r^2 v_\phi^2) - \frac{1}{r} \left(1 + \frac{v_r^2}{2c^2} \right).$$

Changing brackets

$$\begin{aligned} L_W(r, \phi, v_r, v_\phi) &= \frac{1}{2} \left(1 - \frac{1}{c^2 r} \right) v_r^2 + \frac{1}{2} r^2 v_\phi^2 - \frac{1}{r} \\ &= \frac{1}{2} (g_{rr} v_r^2 + g_{\phi\phi} v_\phi^2) - \frac{1}{r}. \end{aligned}$$

Now the metric gets singular at **Weber's critical radius**

$$\rho := \frac{1}{c^2} = \alpha^2.$$

Outside Weber's critical radius, the metric is Riemannian, while inside it is Lorentzian. An interesting aspect of Weber's critical radius is that while outside Weber's critical radius the force is repulsive—inside it is *attracting*! This led Weber to predict—40 years before Rutherford's experiments—an atom consisting of a nucleus built of particles of the same charge together with particles of the opposite charge moving around the nucleus like planets. For more informations about Weber's planetary model of the atom see [44, 45].

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