AMERICAN JOURNAL of PHYSICS

Gödel, Einstein, Mach, Gamow, and Lanczos: Gödel's remarkable excursion into cosmology

Wolfgang Rindler

Citation: American Journal of Physics **77**, 498 (2009); doi: 10.1119/1.3086933
View online: http://dx.doi.org/10.1119/1.3086933
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/77/6?ver=pdfcov
Published by the American Association of Physics Teachers

Gödel, Einstein, Mach, Gamow, and Lanczos: Gödel's remarkable excursion into cosmology

Wolfgang Rindler[a)]
Physics Department, University of Texas at Dallas, Mail St. EC3.6, Richardson, Texas 75080-3021

This article is an expanded version of a talk given at the International Symposium Celebrating the 100th Birthday of Kurt Gödel (Vienna, 2006). It seeks to trace the path which led this preeminent mathematical logician to discover one of the famous results of General Relativity, the rotating Gödel Universe. This universe has some remarkable properties, which gave the philosophers plenty to worry about. It allows a person to travel into his own past, with all the ensuing causal paradoxes; it allows no unique temporal ordering of events; and though Gödel's Universe is rigid and infinite, the Foucault pendulum planes everywhere in it rotate in unison, a clear affront to adherents of Mach's Principle. We also discuss some lesser known precursors in the field, who just missed discovering Gödel's universe. While the article gives all the necessary derivations in simplified form (for example, of the metric and its geodesics), much of it should be accessible to the general reader, who can simply skip most of the mathematics.
[DOI: 10.1119/1.3086933]

[Reprinted, with permission, from *Kurt Gödel and the Foundations of Mathematics: Horizons of Truth, edited by Matthias Baaz, Christos H. Papadimitriou, Dana S. Scott, Hilary Putnam, and Charles L. Harper, Jr. (Cambridge U. P., New York, 2009).]*

I. INTRODUCTION

This lecture was originally given in the hope that it might serve an audience not necessarily familiar with the details of Einstein's General Relativity theory, still to appreciate the gist of Gödel's contribution to relativistic cosmology. In this expanded version of the lecture I have now included all the technical details which general readers may wish to skip, but which those familiar with relativity theory might appreciate. In particular, I give an "elementary" derivation of Gödel's metric and of its geodesics.

Gödel's brilliant burst into the world of physics in 1949 came as a surprise to those who knew him "only" as one of the greatest logicians of all time, and thus as a very pure mathematician. But to his colleagues at the Princeton Institute for Advanced Study it was less of a surprise. There he had famously befriended Einstein. And much earlier, before switching over to mathematics, he had even entered the University of Vienna (in 1924) as a physics student and attended lectures by Hans Thirring, one of the earliest protagonists of Einstein's theories. Moreover, though this was not apparent from his published work, Gödel had maintained a lifelong interest in physics, attending the physics seminars at the Institute and keeping abreast of ongoing developments. And then came the crucial trigger: The year 1949 brought Einstein's 70th birthday and Gödel was expected to contribute to the projected Festschrift for his friend. Not for the first time did pressure prove conducive to invention!

What Gödel invented for the occasion[1] was a model-universe consistent with General Relativity, but which nevertheless exhibited two startlingly disturbing features: bulk rotation (but with respect to what, since there is no absolute space in General Relativity?) and travel routes into the past (enabling one to witness or even prevent one's own birth?). Gödel did not claim for his model that it represented the actual universe we live in. He well knew that General Relativity permits much more appropriate models for that. But he nevertheless maintained that if General Relativity permits such strange behavior, then that behavior should be studied in detail. In particular, he urged astronomers to look for evidence of rotation, and philosophers to rethink their ideas of time.

II. A FIRST LOOK AT GÖDEL'S MODEL

For the sake of concreteness, we shall begin by giving a brief preliminary description of Gödel's model-universe. It is based on General Relativity. General Relativity is Einstein's "new" (by now some 90 years old!) theory of gravity, in which Newton's force of gravity is replaced by the curvature of four-dimensional spacetime, and where free matter moves along the natural "rails" of this curved spacetime, namely its geodesics. A geodesic is the closest analog in any curved space to a straight line in flat Euclidean space. For example, if you march as straight as you can on the surface of a sphere, you will follow a great circle, and so great circles are the geodesics of a sphere. Unencumbered by extraneous concepts like absolute space, General Relativity is ideally applicable to whole universes. It determines, for example, how a universe moves under the action of its own gravity.

Our actual universe is, of course, lumpy, containing big blobs of matter, separated by even bigger blobs of apparent emptiness. The exact dynamics of such lumpy systems cannot in practice be analyzed directly. So one studies instead the smoothed out version of actual universes, and one makes the assumption that the dynamics are effectively the same. The smoothed out counterpart of any universe is called its substratum. And not only the lumpiness must be smoothed out, but also the locally irregular motions. The actual galaxies then sit on this (generally expanding) substratum more or less uniformly distributed, and with only relatively small irregular proper motions.

For the standard models of General Relativity, as well as for Gödel's model, these substrata satisfy the so-called Cosmological Principle. This is a hypothesis well supported by the observations. It asserts that the world is regular, and that our place in it, and in fact that of any other galaxy, is not

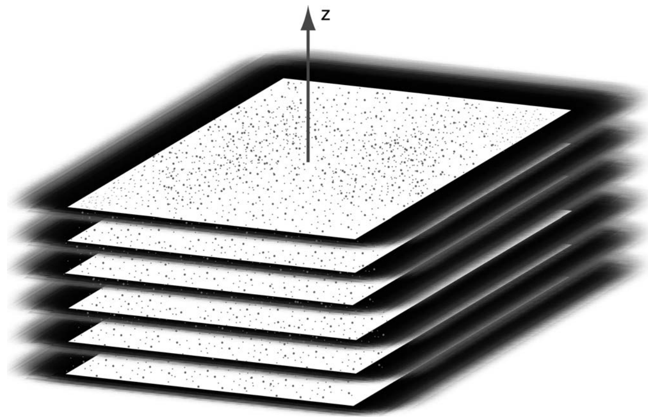


Fig. 1. A spatial map of Gödel's universe; the map flattens the z=const sections into planes.

special. Thus, for the sake of constructing the model, the substratum is assumed to be perfectly homogeneous at all times.

Additionally, our universe is known to expand, and it is commonly believed to have originated in a "big bang" some 14 billion years ago. A realistic substratum must therefore expand. Gödel's model, though homogeneous, ignores this expansion: it is stationary, the same at all times. And there is another difference from the usual models of General Relativity: Gödel's model is not isotropic. Its substratum is somewhat like a homogeneous crystal, having preferred directions at each point. We can picture it (see Fig. 1) as a stack of identical layers, infinite in all directions. Each layer, though we have drawn it as plane, is actually a Lobachewski plane, namely a two-dimensional space of constant negative curvature. (The circumferences of circles, centered anywhere, increase faster than in the Euclidean plane as we go away from the center. It is, of course, one of the features of General Relativity that it permits, and often requires, curvature both in space and in time.) On this layered spatial framework there exists an overall time, indicated by identical standard clocks, one sitting on each galaxy, and all ticking in unison forever. However, somewhat as in Special Relativity (the theory of flat vacuum spacetime), the synchronization of these clocks is not unique, and depends on which clock declares itself the boss.

So far, it all looks fairly harmless. But now come the surprises. Consider an "inertial compass," also called a gyrocompass. This is an instrument containing a number of gyroscopes and having the property of always pointing in the same direction in space. Install such a gyrocompass suitably in a stunt airplane and point it, for example, at the sun. Then fly any number of loops and twists and turns. The gyrocompass ignores them all and keeps steadily pointing at the sun. Now fix such a gyrocompass to every galaxy in Gödel's universe and behold: they all rotate in unison about the normals to the layers. They seem to indicate that the entire universe rotates rigidly in the opposite sense. But relative to what? As we mentioned before, in General Relativity there is no space but the space determined by the universe itself. Gödel laconically comments: "Evidently this state of affairs shows that the inertial field is to a large extent independent of the state of motion of the matter. This contradicts Mach's principle but it does not contradict relativity theory."[2] These two sentences seem to be the sum-total of what Gödel ever said about this paradoxical aspect of his model, and they occur in a lecture he never even published.

Mach's principle, as formulated by Einstein in his early quest for General Relativity, was supposed to explain the mysterious existence of the preferred set of inertial frames against which rotation and acceleration are measured in both Newton's theory and Einstein's Special Relativity. (Newton's absolute space, as an "explanation," had already been repeatedly challenged, most recently by Special Relativity.) Mach's principle says that the local inertial frame, or inertial field, is actively determined by some average of the motions of all the masses in the universe. Einstein had hoped that General Relativity would show in detail how this determination works. But for a number of reasons he later (already in the 1930s) discarded Mach's principle. So the inertial properties of Gödel's model, though paradoxical, were to Einstein not totally unacceptable. Yet Mach's principle has a life of its own, and to its adherents these properties are still considered to be the most troubling feature of Gödel's model.

Now for surprise number two. Consider a large circle in one of the layers of Gödel's substratum. (There is a minimum radius for this to work.) Now travel along this circle with a very large velocity. (Again, a certain minimum velocity is necessary, but it is less than the velocity of light.) And behold: you return to the galaxy from which you started at an earlier time than when you left, as indicated by the local clock. Yet, by your own reckoning, you have aged normally all along the trip. You could now encounter your own father when he was a child, and, if you were wicked, you could kill him, thereby preventing your own birth! That is an awful paradox, and one would hope that nature has ways to prevent spacetimes like Gödel's from actually materializing. (In Special Relativity, where a similar danger lurks, nature prevents it by imposing a universal speed limit, the speed of light.) That hope, indeed, was Einstein's reaction to Gödel's result. Gödel himself, surprisingly perhaps, defended his model on the grounds that it would cost impossible amounts of energy for a space-traveler to accomplish such a journey.[1] Later he granted that one could simply send a light signal, guided by suitably placed mirrors, along a sufficiently large polygonal path to do the same damage, but that the radius would have to be so immense as to render even this procedure impracticable.[2]

III. HOW THE MODEL CAME INTO BEING

Gödel tells us in his Einstein-Festschrift contribution[1] that he was motivated by sympathy for Kant's philosophy of time to invent his model universe. It was to serve as the first counterexample on the cosmic scale to the "objective" view of time, which treats time as an infinity of layers of "now" coming into existence successively. Already in 1905, Einstein in his Special Theory of Relativity had shown this view to be problematic. Indeed, one of the greatest shocks delivered by that theory was the discovery that simultaneity is relative; namely, that the "nows" of different observers correspond to different sets of parallel slices through spacetime, the slices of one observer being inclined to those of another. If the "before now" already exists, and the "after now" exists not yet, existence itself would then be relative to the observer, which Gödel held to be nonsensical. He also pointed out that observers really play no essential role in this argument: the vacuum spacetime of Special Relativity (Minkowski space) simply lacks distinguishing features between alternative parallel time slicings.

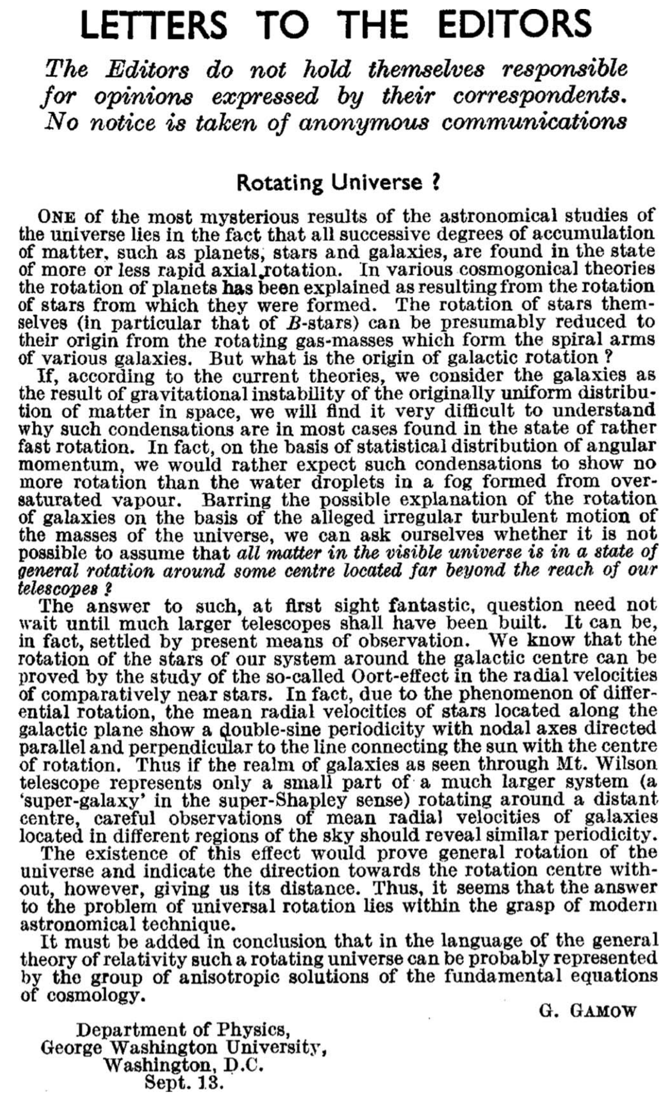
The situation becomes even worse with the irregular spacetimes of General Relativity that correspond to real-life irregular matter distributions. Only in the idealized homogeneous-isotropic universes introduced by Friedman in 1922 and 1924, of which the 1917 static Einstein universe was a special case, do we find an absolutely (geometrically) determined worldwide time. These universes (except the Einstein universe) expand with a single expansion function, and their intrinsically determined time slices correspond to constant values of their steadily diminishing density. Thus the objective (or absolute) view of time got a reprieve from Friedmanian cosmology—which Gödel dismissed as accidental. His purported aim in Ref. 1 was to show that in more general cosmologies no such objective time need exist.

Today we can learn a lot about the details of how Gödel arrived at his universe, from the three different versions in which he presented it, as well as from his correspondence with the Festschrift editor and with his mother. There is, first of all, his brief (six page) essay in the Einstein Festschrift.[1] This contains not a single equation and concentrates mainly on the physics and philosophy of time. The manuscript was completed by the end of March 1949. Almost simultaneously there appeared, in a special Einstein issue of *Reviews of Modern Physics*,[3] Gödel's technical paper which describes, with similar brevity, the quantitative properties of the model as well as an outline of its construction. But, most illuminating of all, there is a lecture on Rotating Universes given by Gödel on 7 May 1949 at the Institute for Advanced Study. It was published only posthumously.[2] This lecture begins as follows:

"A few years ago, in a note in *Nature*, Gamow[4] (1946) suggested that the whole universe might be in a state of uniform rotation and that this rotation might explain the observed rotation of the galactic systems…." Gamow's idea was that if the cells of primordial matter, which eventually collapsed under their own gravity to form galaxies, had no initial angular momentum, then all their matter would simply fall into the center and there form a compact mass. (But, in fact, the initial rotation needed to form a galaxy is so small that it can be explained simply by natural random turbulence.) Importantly, however, Gamow ends his brief note (see Fig. 2) with the conjecture that rotating universes can probably be constructed within General Relativity.

In his lecture Gödel immediately proceeds to exhibit a beautifully simple Newtonian version of a rotating universe. And only from there does he gradually build up his relativistic analog. It is therefore tempting to contemplate the following route whereby Gödel might actually have arrived at his universe:

It is May 1946. Einstein's 70th birthday is a little less than 3 years away. Paul Arthur Schilpp, a philosophy professor at Northwestern University and editor of *The Library of Living Philosophers*, is already planning an Einstein Festschrift for the occasion. He visits Princeton and seeks out Gödel. Gödel promises a contribution. Soon thereafter, in correspondence,[5,6] he offers to write about three pages under the title "Some remarks about the relation between the theory of relativity and Kant." And although Schilpp presses him for a much longer paper, Gödel keeps insisting that three to five pages is all he needs for what he has to say. Evidently he is not yet thinking of inventing a new universe.



LETTERS TO THE EDITORS

The Editors do not hold themselves responsible for opinions expressed by their correspondents. No notice is taken of anonymous communications

Rotating Universe ?

ONE of the most mysterious results of the astronomical studies of the universe lies in the fact that all successive degrees of accumulation of matter, such as planets, stars and galaxies, are found in the state of more or less rapid axial rotation. In various cosmogonical theories the rotation of planets has been explained as resulting from the rotation of stars from which they were formed. The rotation of stars themselves (in particular that of *B*-stars) can be presumably reduced to their origin from the rotating gas-masses which form the spiral arms of various galaxies. But what is the origin of galactic rotation ?

If, according to the current theories, we consider the galaxies as the result of gravitational instability of the originally uniform distribution of matter in space, we will find it very difficult to understand why such condensations are in most cases found in the state of rather fast rotation. In fact, on the basis of statistical distribution of angular momentum, we would rather expect such condensations to show no more rotation than the water droplets in a fog formed from over-saturated vapour. Barring the possible explanation of the rotation of galaxies on the basis of the alleged irregular turbulent motion of the masses of the universe, we can ask ourselves whether it is not possible to assume that *all matter in the visible universe is in a state of general rotation around some centre located far beyond the reach of our telescopes ?*

The answer to such, at first sight fantastic, question need not wait until much larger telescopes shall have been built. It can be, in fact, settled by present means of observation. We know that the rotation of the stars of our system around the galactic centre can be proved by the study of the so-called Oort-effect in the radial velocities of comparatively near stars. In fact, due to the phenomenon of differential rotation, the mean radial velocities of stars located along the galactic plane show a double-sine periodicity with nodal axes directed parallel and perpendicular to the line connecting the sun with the centre of rotation. Thus if the realm of galaxies as seen through Mt. Wilson telescope represents only a small part of a much larger system (a 'super-galaxy' in the super-Shapley sense) rotating around a distant centre, careful observations of mean radial velocities of galaxies located in different regions of the sky should reveal similar periodicity.

The existence of this effect would prove general rotation of the universe and indicate the direction towards the rotation centre without, however, giving us its distance. Thus, it seems that the answer to the problem of universal rotation lies within the grasp of modern astronomical technique.

It must be added in conclusion that in the language of the general theory of relativity such a rotating universe can be probably represented by the group of anisotropic solutions of the fundamental equations of cosmology.

G. GAMOW

Department of Physics,
George Washington University,
Washington, D.C.
Sept. 13.

Fig. 2. The "trigger:" Gamow's letter to *Nature*. [Reprinted with permission from Macmillan Publishers Ltd: G. Gamow, Nature (London) **158**, 549 (1946). Copyright 1946.]

And then, serendipitously, he comes across Gamow's note in *Nature* with its challenge to find a rotating universe consistent with General Relativity. Suddenly he has a problem worthy of his genius and the perfect gift for Einstein! But that is not all. Gödel seems to have recognized quite early that in a rotating universe there would be no absolute time, so that his Kantian ambition of superseding Friedman would also come true. (Rotation implies twisting worldlines of the galaxies and hence the nonexistence of preferred time slices.) Thus fortune played into his hand not only a significant relativistic problem, but one which even fell within his original Kantian program, and which, when it was all done, turned out to be far more beautiful than could possibly have been foreseen.

No wonder Gödel soon immerses himself happily into this work. On 15 July 1947 he writes to Schilpp[6] regretting the delay, but saying that there is still an important point to settle, depending "on the solution of a mathematical problem, at which I am working now." By September 1947 he seems to have the outline of his model. But more problems keep cropping up. Only on 10 May 1948 can he write to his mother in Vienna[5] that he had intended to write long before, but that for several weeks he had been beset by a problem which had driven everything else out of his mind, and that at last he had "settled the matter enough to be able to sleep well again." What he had just found was that in his universe one could travel into the past! And yet Schilpp had to wait another 10 months for the final manuscript.

IV. GÖDEL'S NEWTONIAN ROTATING UNIVERSE

In his May 1949 lecture Gödel, after quoting Gamow, recalls that "Newtonian physics gives a surprisingly good approximation for the expanding [nonrotating Friedman-] universes…." And so he now proceeds ingeniously to construct a Newtonian rotating (but nonexpanding) universe. We shall outline his arguments here, partly because this Newtonian model is intrinsically interesting and surprising, and partly because it still seems to be the best approach to the full Gödel model. [Incidentally, this Newtonian model was foreshadowed in Ref. 17, where already Eq. (5) below appears.]

In Newton's theory, we start with absolute space. In this absolute space we pick one fixed axis about which the universe is to rotate uniformly and rigidly. Its density ρ must then be constant in time, and for the sake of the cosmological principle we take it to be constant in space also. As in the Friedman case, we use Newton's law of gravitation only in its differential form:

$$\nabla^2\Phi = 4\pi G\rho, \tag{1}$$

where G is Newton's constant of gravity. The solution we want is

$$\Phi = \pi G\rho(x^2 + y^2) = \pi G\rho r^2, \tag{2}$$

if the field is radially away from the axis, as symmetry demands. The gravitational force towards the axis,

$$\vec{f}_{\text{grav}} = -\nabla\Phi = -2\pi G\rho\vec{r}, \tag{3}$$

is then precisely balanced by the centrifugal force

$$\vec{f}_{\text{cent}} = \Omega^2\vec{r}, \tag{4}$$

provided the angular velocity Ω satisfies

$$\Omega^2 = 2\pi G\rho. \tag{5}$$

At first sight this may seem to be an unlikely model universe, since, in defiance of the cosmological principle, it has a center, or at least a central axis. But, on closer inspection, it turns out that there is complete *empirical* symmetry among the galaxies. Each moves freely, that is, each sits still without constraint on the rigidly moving substratum. There is no observable "gravitational" force relative to the substratum anywhere. At each point of the substratum, however, there is the same Coriolis force

$$\vec{f}_{\text{Cor}} = 2\vec{v} \times \vec{\Omega} \tag{6}$$

acting on any particle that moves relative to the substratum with velocity $\vec{v}$. Empirically, therefore, each galaxy can consider itself to be at rest on the axis of a universe that rotates rigidly at angular velocity $\vec{\Omega}$ around it!

Because of the close analogy with the later relativistic model, it is worth noting that all free orbits in the Newtonian model are circles, if started in a "horizontal" plane z=const, and circular helices otherwise, with respect to the substratum. This follows from the uniformity of the Coriolis force (6). Suppose, first, that a particle is projected with some velocity $\vec{v}$ relative to the substratum in a plane z=const. Since there is no force on it in the direction of $\vec{\Omega}$ it stays in its original plane. And since it experiences only a sideways force, its speed remains constant. Consequently the magnitude of the sideways Coriolis force also remains constant and the particle traces out a circle of radius

$$r = v/2\Omega. \tag{7}$$

It is easily seen that this circle is described in the sense opposite to that of $\vec{\Omega}$. If, on the other hand, the particle's initial velocity also has a component in the direction of $\vec{\Omega}$, that component stays constant, while only the "horizontal" velocity component determines the radius of the resulting helix according to Eq. (7).

We may note from Eq. (7) that the magnitude ω of the angular velocity of the orbiting particle is given by

$$\omega = 2\Omega. \tag{8}$$

And this is to be expected: if the mass of some central "vertical" cylinder of the substratum (itself rotating at angular velocity Ω) allows a free substratum particle on its surface to orbit its axis at angular velocity $\vec{\Omega}$, it must also allow a particle to orbit its axis at angular velocity $-\vec{\Omega}$, and that is $-2\vec{\Omega}$ relative to the substratum.

Gödel's Newtonian universe as discussed here and in his lecture is not quite the correct Newtonian analog of his relativistic universe. The reason is that his relativistic universe is predicated on a negative cosmological constant Λ, for which allowance can be made in Newtonian theory, but such allowance was not made by Gödel. The Newtonian field equation to which Einstein's field equation with Λ reduces in first approximation[7,8] is

$$\nabla^2\Phi + \Lambda c^2 = 4\pi G\rho, \tag{9}$$

instead of Eq. (1), c being the speed of light and Φ the *joint* potential for gravity and the Λ force. Gödel's relativistic universe, it turns out, crucially needs

$$\Lambda = -4\pi G\rho/c^2. \tag{10}$$

If we use this value in Eq. (9), then instead of Eq. (2), which satisfies Eq. (1), we need

$$\Phi = 2\pi G\rho(x^2 + y^2), \tag{11}$$

which satisfies Eqs. (9) and (10). Consequently, instead of Eq. (5) we now find

$$\Omega^2 = 4\pi G\rho, \tag{12}$$

which is actually the precise relation that holds in Gödel's relativistic model. Gödel considered Eq. (5) as already a "surprisingly good approximation."[2]

It should be noted, however, that whereas in the case of Einstein's static universe of 1917 the Newtonian analog "explains" the need for a positive Λ (a repulsive force is needed to counteract gravity), the Newtonian rotating universe seems to throw no light on the need for a *negative* Λ in Gödel's universe. Somehow, via Einstein's field equation, it may contribute to the puzzling equilibrium in the z-dimension.[8]

V. STATIONARY METRICS IN GENERAL, AND THE FORM OF GÖDEL'S METRIC IN PARTICULAR

Our next step here, as it was in Gödel's lecture, is to construct the relativistic model in close analogy to the Newtonian model. But Gödel's original method is geometrically quite demanding, depending as it does on clever tricks with Clifford parallels in hyperspheres, quaternions, etc., all of which modern relativists are not so familiar with. Here, therefore, we shall give an elementary derivation that uses only standard results from the general theory of stationary spacetimes.[9]

Loosely speaking, a gravitational field is stationary if it does not change with time, and it is said to be static if, additionally, "there is no rotation." One can think of the fixed points in the stationary field as forming a rigid (and generally curved) three-dimensional lattice. Relative to this lattice there is generally a permanent gravitational field. And if the field is only stationary but not static, there is also a unique permanent rotation of the "compass of inertia" at each lattice point.

Four-dimensionally speaking, each point of the lattice has its worldline in spacetime, and static spacetimes are characterized by the property that the set of these "fundamental" worldlines is irrotational. This is equivalent to the existence of a unique set of identical (isometric) hypersurfaces (three-dimensional subspaces) cutting orthogonally across all the fundamental worldlines. It is these hypersurfaces that constitute the unique simultaneities in static spacetimes.

In merely stationary spacetimes the fundamental worldlines twist and are not "hypersurface-orthogonal." Infinitely many sets of parallel and isometric (and, of course, generally curved) hypersurfaces can then be drawn across the fundamental worldlines, forming different sets of equally admissible simultaneities. Gödel's universe is of this kind.

It can be shown that for the most general stationary spacetime one can always find an "adapted" global time t (nonuniquely) of which the metric coefficients are independent. It is then usual to write the metric in the following "canonical" form:

$$ds^2 = e^{2\Phi/c^2}(cdt - c^{-2}w_i dx^i)^2 - h_{ij}dx^i dx^j, \tag{13}$$

where Φ is called the scalar potential, w_i the vector potential $(i=1,2,3)$, and where x^i are the coordinates for the lattice, whose metric tensor is h_{ij}; Φ, w_i, h_{ij} are functions of the x^i only. [Einstein's summation convention applies in Eq. (13), and in all subsequent formulas.] One of the advantages of this canonical metric is that it allows us to "read off" the effective gravitational field and the rotation. The effective gravitational field $\vec{g}$ is given by

$$\vec{g} = -\,\text{grad}\,\Phi \tag{14}$$

and the proper (i.e., with respect to proper time at a given lattice point) rotation rate of the lattice relative to the local gyrocompass is given by

$$\vec{\Omega} = \frac{1}{2c}e^{\Phi/c^2}\,\text{curl}\,\vec{w}, \tag{15}$$

both grad and curl referring to the metric h_{ij}.[10]

In the case of Gödel's universe, where the lattice is free-floating, we need $\vec{g}=0$ and can therefore set $\Phi=0$. This still leaves the following "gauge" freedom, namely the freedom to transform the metric (13) into an equivalent one:

$$x^i \rightarrow x^{i'} = x^{i'}(x^i), \tag{16a}$$

$$t \rightarrow t' = t + f(x^i), \tag{16b}$$

$x^{i'}$ and f being any well behaved functions of the x^i. The effect of (16b) on Φ and w_i is as follows:

$$\Phi' = \Phi, \quad w_i \rightarrow w_i' = w_i + c^3 f_{,i}, \tag{17}$$

where ", i" denotes partial differentiation with respect to x^i. We shall return to this gauge freedom presently. Note that if we know the metric of the lattice, plus, at each point, the gravitational field and the rotation rate, then the metric (13) is uniquely determined up to gauge transformations. (This argument can eventually be used to verify that Gödel's universe is indeed homogeneous.)

To construct the lattice for Gödel's universe we need, first of all, a set of parallel lines (in the sense of constant orthogonal distance between neighboring members of the set) to serve as local rotation axes for the universe. The two-spaces orthogonal to these lines must be homogeneous, if we wish to make the whole lattice homogeneous; but, in accordance with General Relativity, they need not be flat. They can, in fact, be two-spaces of constant curvature. There are three types of such two-spaces, having positive, negative, or zero curvature, k/a^2 ($k=1,\ -1$, or 0). Their respective metrics are[11]

$$dl^2 = a^2(dr^2 + \Sigma^2 d\phi^2), \tag{18}$$

with

$$\Sigma = \sin r, \quad \Sigma = \sinh r, \quad \Sigma = r, \quad \text{for } k = 1, -1, 0, \text{respectively}. \tag{19}$$

Here ϕ is the angle around the origin, and ar measures radial ruler distance from the origin. (There are other forms for these metrics, but for our purposes these are the most convenient.) If z is distance along the rotation lines, the three-metric of the lattice becomes

$$dl^2 = a^2(dr^2 + \Sigma^2 d\phi^2) + dz^2. \tag{20}$$

By the rotational symmetry about the "central" z-line, all the coefficients of the full four-metric must be independent of ϕ; by homogeneity in the z- and t-dimensions, they must also be independent of z and t. We have already justified setting the scalar potential equal to zero: $\Phi=0$. The vector potential w_i, now reduced to dependence on r only, cannot have a z-component, since its curl is to point in the z-direction; a possible r-component can be transformed away by a gauge transformation (17). Consequently the full metric may be written in the form

$$ds^2 = [dt - w(r)d\phi]^2 - a^2(dr^2 + \Sigma^2 d\phi^2) - dz^2, \tag{21}$$

where here, and from now on unless otherwise stated, units are chosen so as to make $c=1$.

Following Gödel, we now redefine t, w, and z,

$$t \rightarrow at, \quad w \rightarrow aw, \quad z \rightarrow az, \tag{22}$$

so that the metric (21) takes on the "conformal" form

$$ds^2 = a^2\{[dt - w(r)d\phi]^2 - (dr^2 + \Sigma^2 d\phi^2 + dz^2)\}. \tag{23}$$

Let us first deal with the rotation rate of the lattice, as calculated from Eq. (15). For the canonical metric (13) (with

$c=1$), the magnitude Ω of the rotation vector is given explicitly[12] by

$$\Omega = \frac{1}{2\sqrt{2}} e^{\Phi}[h^{ik}h^{jl}(w_{i,j}-w_{j,i})(w_{k,l}-w_{l,k})]^{1/2}. \tag{24}$$

The metric (23) falls under the category of Eq. (13) with

$$a^2 = e^{2\phi} = h_{rr} = h_{zz} = h_{\phi\phi}/\Sigma^2. \tag{25}$$

If we now set

$$(r,\phi,z) = (x^1,x^2,x^3), \tag{26}$$

we see that $w' := dw/dr = w_{2,1}$ is the only nonvanishing derivative $w_{i,j}$ and, with that, formula (24) yields

$$\Omega = \frac{w'}{2a\Sigma}. \tag{27}$$

Since the coordinates (26) are right-handed if ϕ is measured in the anti-clockwise sense, it can be seen from Eqs. (15) and (27) that $\vec{\Omega}$ acts in the sense of increasing or decreasing ϕ according as w' is positive or negative.

For a homogeneous universe we clearly need Ω=const, and thus, by Eq. (27), $w' = \alpha\Sigma$ for some constant α. We could now integrate this and substitute the explicit function $w(r)$ into Eq. (23) before applying the field equations to that metric. In practice, however, the field equations look a little simpler if w is left generic. Because of the symmetry of all the other ingredients, the field equations themselves will imply $w' = \alpha\Sigma$.

At this point we still have enough gauge freedom left [see Eqs. (16) and (17)] to add a constant to $w(r)$, and simultaneously a suitable multiple of ϕ to t. We can use this freedom to associate a geometrically preferred—or canonical—global time with any galaxy X. The obvious choice is that time which makes the hypersurface t=const cut X's worldline orthogonally, so that t then coincides locally with X's inertial time. If X is the origin-galaxy of the metric (23), this simply requires

$$w(0) = 0, \tag{28}$$

a condition we shall now impose. Of course, the canonical time slices determined by the galaxy X differ from those of another galaxy Y, just as they do in Special Relativity from one inertial observer to another. But that is precisely what made this universe so attractive to Gödel. It stands in contrast to the situation in the Friedman universes, where the canonical time slices cut orthogonally across all the fundamental worldlines, and are thus shared by all the galaxies. In rotating universes the fundamental worldlines twist, making such unicity impossible.

VI. DO THE FIELD EQUATIONS PERMIT A GÖDEL UNIVERSE?

So far we have used a wish list of properties—mainly suggested by the Newtonian analogy—to arrive at the metric (23), and we even know $w(r)$ up to a constant [see after Eq. (27)]. We also know that in order to be on the same footing as the Friedman models, the source matter should be pressureless "dust." So there is almost no wiggle room left; only the sign of the curvature index k [cf. Eq. (19)] is still free. It is time to take our metric before the judge—will it pass?

The judge is Einstein, his field equations are the law. According to General Relativity, matter (in the form of the energy tensor) and geometry (in the form of the metric) must jointly satisfy this law. In units that make $c=1$, Einstein's field equations with cosmological constant Λ read as follows:

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{29}$$

Here $G_{\mu\nu}$ stands for the Einstein tensor $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$, which is built out of the metric and represents the geometry. We adopt for it the same sign convention as did Gödel (which also coincides with that of Ref. 7). Greek indices run from 1 to 4 and we augment Eq. (26) by

$$t = x^4. \tag{30}$$

On the rhs of Eq. (29), $T_{\mu\nu}$ represents the sources. It is usual to assume that the mechanical properties of the substratum—the smoothed-out universe—are equivalent to those of "dust," which is the technical term for a pressureless perfect fluid. Its energy tensor is then given by the alternative formulas[13]

$$T^{\mu\nu} = \rho U^\mu U^\nu \quad \text{or} \quad T_{\mu\nu} = \rho g_{\mu\alpha} g_{\nu\beta} U^\alpha U^\beta, \tag{31}$$

where ρ is the proper density (a constant in Gödel's universe) and $U^\mu = dx^\mu/d\tau$ is the four-velocity of the substratum (τ =proper time). The substratum satisfies x^i=const, whence, from Eq. (23), $d\tau = a dt$, so that

$$U^\mu = (0,0,0,a^{-1}). \tag{32}$$

Then, since for Eq. (23) we have

$$g_{\mu\nu} = a^2 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & w^2-\Sigma^2 & 0 & -w \\ 0 & 0 & -1 & 0 \\ 0 & -w & 0 & 1 \end{pmatrix}, \tag{33}$$

we find from Eq. (31) the following to be the only nonzero components of $T_{\mu\nu}$:

$$T_{22} = \rho a^2 w^2, \quad T_{24} = -\rho a^2 w, \quad T_{44} = \rho a^2. \tag{34}$$

To calculate the Einstein tensor $G_{\mu\nu}$ for the metric (33)—a formidable task if done by hand—it is nowadays best to use a computer program.[14] The following values are found:

$$G_{12} = G_{13} = G_{14} = G_{23} = G_{34} = 0, \tag{35}$$

$$G_{11} = -\frac{1}{4}\frac{w'^2}{S^2}, \tag{36}$$

$$G_{33} = \frac{1}{4}\frac{w'^2}{S^2} + k, \tag{37}$$

$$G_{44} = -\frac{3}{4}\frac{w'^2}{S^2} - k, \tag{38}$$

$$G_{24} = \frac{3}{4}\frac{ww'^2}{S^2} + \frac{1}{2}w'' - \frac{1}{2}\frac{C}{S}w' + kw, \tag{39}$$

$$G_{22} = -\frac{3}{4}\frac{w^2 w'^2}{S^2} - \frac{1}{4}w'^2 - ww'' + \frac{C}{S}ww' - kw^2. \qquad (40)$$

In these formulas, the meanings of S and C depend on the curvature index as follows:

$$\text{if } k = 1: \quad S = \sin r, \quad C = \cos r,$$

$$\text{if } k = -1: \quad S = \sinh r, \quad C = \cosh r, \qquad (41)$$

$$\text{if } k = 0: \quad S = 1, \quad C = 0.$$

We note that all the components $G_{\mu\nu}$ are independent of the constant scale factor a.

We are now ready to look at the Einstein field equations (29) explicitly, using the metric components from Eq. (33), the Einstein tensor components from Eq. (35)–(40), and the energy tensor components from Eq. (34). For $\mu\nu = 12, 13, 14, 23, 34$, these equations are trivially satisfied, each term vanishing separately. For $\mu\nu = 11$, the field equation reads

$$-\frac{1}{4}\frac{w'^2}{S^2} = a^2\Lambda, \qquad (42)$$

which shows that we need the cosmological constant Λ, and that it must be negative. Then for $\mu\nu = 33$ we find

$$\frac{1}{4}\frac{w'^2}{S^2} + k = a^2\Lambda, \qquad (43)$$

which, when added to Eq. (42), yields $k = 2a^2\Lambda$. So, because of the negativity of Λ, we must have

$$k = -1 \qquad (44)$$

and then

$$\Lambda = -\frac{1}{2a^2}. \qquad (45)$$

According to Eq. (44), we must choose the second line in Eq. (41). Then, with Eq. (45), Eq. (42) yields the expected relation [see after Eq. (27)]

$$w' = \pm\sqrt{2}\sinh r, \qquad (46)$$

and consequently, with Eq. (28),

$$w = \pm\sqrt{2}(\cosh r - 1). \qquad (47)$$

Using Eqs. (44)–(46), the field equation for $\mu\nu = 44$ now yields

$$\rho = \frac{1}{8\pi G a^2}. \qquad (48)$$

At this stage no more freedom is left, and the last two field equations (for $\mu\nu = 22$ and 24) either are or are not satisfied by the values for k, Λ, ρ, and w already found. Happily (and miraculously?) it turns out that they *are* satisfied. As Gödel pointed out, it is sufficient to check the field equations at a single point (most conveniently the origin), since the spacetime is homogeneous. And at the origin, from Eqs. (46) and (47), we have

$$w = w' = 0, \quad w'/S = w'' = \pm\sqrt{2}, \qquad (49)$$

which makes the checking trivial.

One more point needs clarification here. In Eqs. (46) and (47) we may pick the positive or the negative sign. The result will be a universe that rotates anti-clockwise, that is, in the same sense as ϕ ($w > 0$) or clockwise ($w < 0$): both obviously are equally possible. We prefer the choice $w > 0$, though Gödel chose $w < 0$, nevertheless asserting, apparently erroneously, that the rotation was in the sense of increasing ϕ.

So we have established the metric for Gödel's universe in the following form [cf. Eq. (23)]:

$$ds^2 = a^2\{[dt - \sqrt{2}(\cosh r - 1)d\phi]^2 - (dr^2 + \sinh^2 r\, d\phi^2 + dz^2)\}. \qquad (50)$$

The proper rotation rate of the lattice relative to the compass of inertia [cf. Eq. (27)] is now given in full units by

$$\Omega = \frac{c}{\sqrt{2}a}. \qquad (51)$$

Summarizing our findings, we thus have, again in full units,

$$-\Lambda = -\frac{1}{2}K = 4\pi G\rho/c^2 = \Omega^2/c^2 = 1/2a^2, \qquad (52)$$

where K is the curvature of the layers z=const of the lattice.

Note the required relation between Λ and ρ (which is actually the same, except for sign, as in Einstein's static universe of 1917). If Λ is indeed a constant of nature, Gödel's universe, just like Einstein's, would need a highly tuned creation—unless one regards Λ as *determined* by the ρ of the universe.

VII. LIGHT CONES, TIME TRAVEL, AND GEODESICS, IN GÖDEL'S UNIVERSE

The basic absolute structure in the (Minkowskian) spacetime of Special Relativity is the set of parallelly oriented hourglass-like light cones, one at each event. They are closely related to the invariance of the speed of light and to the well-known special-relativistic speed limit $v \le c$ for all particles and signals, which is needed to ensure nonparadoxical causality.

Let us consider this speed limit pictorially (see Fig. 3), under the usual suppression of one spatial dimension, say that of z. In an x, y, t spacetime diagram [see Fig. 3(i)], where t is always drawn vertically, consider a portion (dx, dy, dt) of a particle's worldline, starting at some event **P** [see Fig. 3(ii)]. Let the distance traveled be $dr = (dx^2 + dy^2)^{1/2}$. The inclination dr/dt of this worldline away from the t-axis measures the particle's speed. In ordinary units, the speed of light is very large, so that the worldline of a photon would make almost a right angle with the t-axis. For visual and dimensional convenience it is therefore preferable to choose "relativistic" units such that the speed of light is unity. Then photon worldlines are inclined at 45° to the vertical. All the 45° lines through an event **P** constitute the light cone at **P** [see Fig. 3(iii)], and ordinary particles passing through **P** must have their worldlines within the cone ($v < c$). An extended particle worldline must everywhere be inclined to the vertical at less than 45° and so must lie within the light cones all along its extent [see Fig. 3(iv)]. All events in the top half of the light cone at **P** can be influenced by signals from **P**, so this region is called the "absolute future" of **P** [see Fig. 3(v)]. Similarly all events in the lower half of

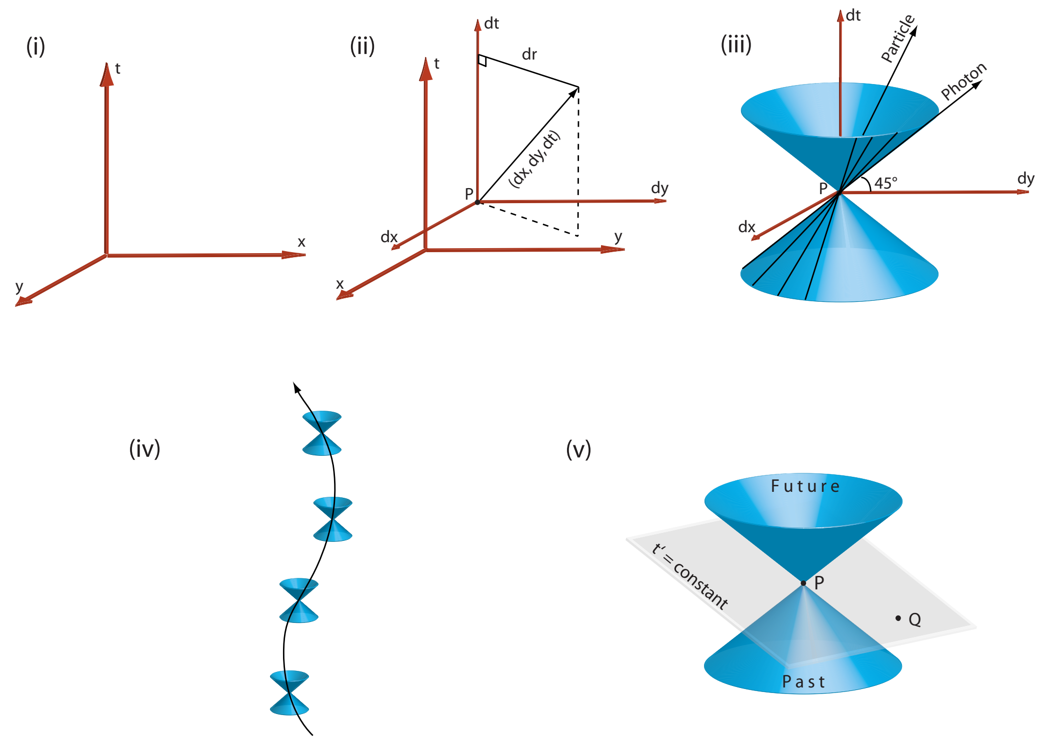


Fig. 3. The spacetime aspect of Special Relativity in a nutshell.

the cone can influence **P**, and so this region is called the "absolute past" of **P**. All events like **Q**, outside the cone, occur before **P** in some inertial frames, after **P** in other inertial frames, and are simultaneous with **P** in yet others, since any plane through **P** outside the cone represents a simultaneity $t' = \text{const}$ in *some* inertial frame (x', y', z', t').

In full Minkowski spacetime (x, y, z, t) the speed restriction $dr/dt < c$ on a portion (dx, dy, dz, dt) of a particle's worldline reads, in full units,

$$dx^2 + dy^2 + dz^2 < c^2 dt^2 \tag{53a}$$

or

$$ds^2 := c^2 dt^2 - dx^2 - dy^2 - dz^2 > 0, \tag{53b}$$

while the local equation of a light cone is

$$dx^2 + dy^2 + dz^2 = c^2 dt^2 \quad \text{or} \quad ds^2 = 0. \tag{54}$$

In Minkowski's language, ds^2 stands for the "squared displacement" between neighboring points (events) in the flat spacetime of Special Relativity, and it is the primary law of Special Relativity that every inertial observer obtains the same value for it. Evidently it is an analog of $dx^2 + dy^2 + dz^2$, which represents "squared distance" between neighboring points in Euclidean space—except that Minkowski's ds^2 can be negative as well as positive or zero for perfectly ordinary event pairs. [As we have seen, it is zero for neighboring events on a photon worldline, and negative for events like **P** and **Q** in Fig. 3(v) above.]

According to General Relativity, the spacetime in the presence of gravitating sources is curved, though locally Minkowskian. This is analogous to the situation in differential geometry, where, for example, the surface of a sphere is curved, but sufficiently small portions of it can be treated as flat and Euclidean to a high degree of accuracy. In fact, in every suitably small region of spacetime we can release an "Einstein cabin"—a freely falling nonrotating box, like a severed elevator cabin or an astronaut's capsule—inside of which gravity has disappeared, and Special Relativity holds. The ds^2 between neighboring events in the curved spacetime is the same as that measured in the local Einstein cabin. Hence $ds^2 = 0$ is still the equation of the local light cone, and $ds^2 \geq 0$ is still the local speed condition on particle and signal worldlines.

Now, whereas in the flat spacetime of Special Relativity the light cones are all parallelly aligned like soldiers on parade [see Fig. 4(i)], and no worldline can loop back on itself, general-relativistic spacetimes are curved, and the light cones can be all over the place [as in Fig. 4(ii)]. As early as 1914, a year before he had even completed his General Theory of Relativity, Einstein already worried[15] about the possible existence of curved spacetimes that allowed closed particle worldlines as in Fig. 4(ii), even though the light cones permit a consistent past/future labeling throughout. Einstein wrote: "This conflicts strongly with my physical intuition. But I am

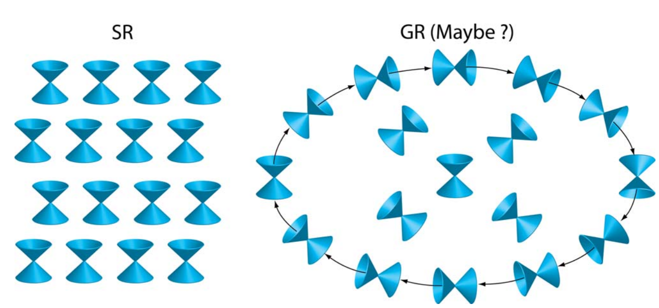


Fig. 4. The invariant orientation of the light-cones in Special Relativity and a permissible orientation of the light-cones in General Relativity.

unable to prove that the theory excludes the occurrence of such orbits." And he was right. Much earlier than 1949 spacetimes had indeed been discovered where such orbits exist (anti-de Sitter space,[16] van Stockum space,[17] etc.), but Gödel's universe was the cleanest example, certainly the one that caught the widest attention, and possibly the first where time loops were explicitly recognized.

On the scale of the universe, an observer is but a particle. Observers' worldlines must therefore also lie everywhere within the light cone. For an observer to travel back to an event already experienced, his worldline must be a closed loop, lying everywhere within the light cone. In other words, his worldline must satisfy $ds^2>0$ all along its length. Observe also, from Eq. (53b), that if you momentarily fly along with an orbiting particle in a free Einstein cabin, its worldline relative to the cabin will satisfy $dx=dy=dz=0$, and thus $ds^2=c^2dt^2$. So time elapsed at the particle is given by $\int ds/c$.

In Gödel's universe, the existence of closed particle worldlines is almost trivially easy to read off directly from the metric (50). Consider a circular path, r=const, in one of the layers z=const, and let it be traced out so that t=const! (This last condition is not nonsensical: time cannot stand still on an orbiting particle, nor on a given galaxy along its path, but the cosmic time coordinate t *can* be the same on successive galaxies passed by the particle.) If we accordingly set $dr=dz=dt=0$ in Eq. (50), we are left with

$$ds^2 = a^2[2(\cosh r - 1)^2 - \sinh^2 r]d\phi^2, \tag{55}$$

which is obviously positive for sufficiently large r. Specifically, Eq. (55) can be transformed into

$$ds^2 = a^2[(\cosh r - 2)^2 - 1]d\phi^2, \tag{56}$$

so that, provided

$$\cosh r > 3, \tag{57}$$

such circles are possible particle worldlines, and they clearly return to precisely the initial event after one complete revolution. With equality instead of inequality in Eq. (57), we would have $ds^2=0$, and the worldline would be that of a photon. Note, however, that none of these orbits are geodesics (free-fall paths), as we shall see below; the photon would have to be guided by mirrors, and the particle would have to be propelled by rocket motors.

As far as the requirement $ds^2>0$ is concerned, a circle satisfying the condition (57) could be described by a particle in the positive or the negative sense of ϕ. But in any properly *causal* spacetime—namely, one throughout which the two halves of the local light cones can be consistently labeled "past" and "future" (and Gödel's universe is of this kind, as we shall presently see)—all worldlines along their entire length must point into the future cones. And this will require the closed worldlines of the positively rotating universe to be described in the negative sense.

For the purpose of establishing this result, let us examine the light-cone structure of the metric (50) all over a surface t=const in spacetime. Such a surface is not a plane in the geometrical sense, but we are at liberty to draw a *map* in which it is a plane (see Fig. 5). Nor do the fundamental worldlines t=var (except that of the central galaxy) cut such a surface orthogonally, but we are at liberty to *map* them as vertical lines cutting t=const orthogonally. And we shall not restrict the generality of the argument by looking only at events satisfying z=const. In the plane t=const, let us now

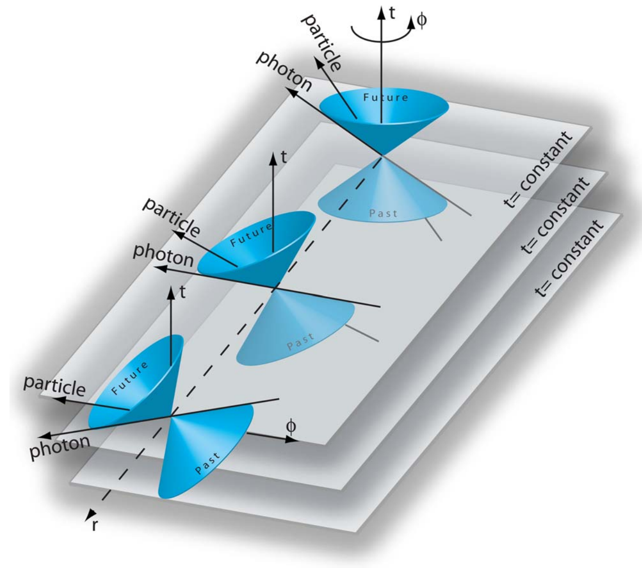


Fig. 5. A spacetime map of Gödel's universe; the map flattens the t=const sections into planes and progressively widens the intrinsically 90 deg angle of the light cones as we go away from the chosen center.

go out from the center $r=0$ along a radius ϕ=const and there look at directions satisfying r=const. For them to lie *on* the light cone they must satisfy $ds^2=0$ (with $dr=dz=0$), which, together with Eq. (50), implies

$$dt - \sqrt{2}(\cosh r - 1)d\phi = \pm (\sinh r)d\phi$$

or

$$dt/d\phi = \sqrt{2}(\cosh r - 1) \pm \sinh r. \tag{58}$$

Let us assume that we are sufficiently far from the origin to make the first term in the bracket in Eq. (55) greater than the second. Then both values for $dt/d\phi$ in Eq. (58) are positive. We know that the fundamental (galaxy-) worldline t=var at the event in question must point into the future cone. Hence the light cone there looks like the third one out in Fig. 5.

At the origin $r=0$, the metric (50), to first order in r, reduces to a Minkowskian form. Hence the central light cone has the usual special-relativistic appearance with a 45° semi-vertical angle. As we move away from the center, the cone tilts and widens, until at $\cosh r=3$ one of its edges touches the plane t=const, because one of the values of $dt/d\phi$ in Eq. (58) becomes zero. (The reason that its semi-angle is no longer 45° is that our map distorts angles.) Farther out still, the direction ϕ=var lies in its interior; and as the diagram shows, its future direction is that of decreasing ϕ. This is what we set out to establish.

Let us next inquire into the particle speeds needed to describe the orbits under discussion. Consider the ds^2 of a portion of such an orbit, satisfying r,z,t=const. It is given by Eq. (55). In terms of the *local* inertial coordinates associated with whatever galaxy the particle just passes, it is also given by $ds^2=d\tilde{t}^2-dl^2$, where $\tilde{t}$ is the local inertial time and, by Eq. (50), $dl=a(\sinh r)d\phi$ is the ruler distance traveled. (Note: whereas *at* coincides with the proper time along any galaxy-worldline, the hyperplane elements t=const are *inclined* to the hyperplane elements $\tilde{t}$=const everywhere except at the center.) The local speed of the particle is then given by

$$v = \frac{dl}{d\tilde{t}} = \frac{dl}{\sqrt{ds^2 + dl^2}} = \frac{\sinh r}{\sqrt{2}(\cosh r - 1)}. \tag{59}$$

For the critical circle we have $\cosh r = 3$ (and thus $\sinh r = 2\sqrt{2}$), and this gives $v = 1$, the speed of light. For larger circles the required v gradually decreases to $1/\sqrt{2}$ of the speed of light, which is therefore the minimum speed for such journeys into the past.

If, on a given circle r=const, the particle travels faster than with speed (59), which is required to keep t constant, it will actually return to its spatial starting point earlier than when it left. That can be seen particularly well from Fig. 5, where the v of Eq. (59) corresponds to a horizontal orbit to the left, and a greater v corresponds to an orbit closer to the lower edge of the cone, and thus going into earlier t values. Formally, from Eq. (59), a larger v makes the corresponding ds^2 smaller for a given dl and thus for a given $d\phi$. This leads to $dt < 0$ in Eq. (50), since $d\phi$ is negative. The corresponding orbit spirals downward below the t=const plane.

It is now evident that, as Gödel asserted in his Festschrift article, in his universe one can travel from any event to any other event, past or future. For suppose we wish to go from event **P** here and now to event **Q** occurring anywhere at time $t = t_0$. By going around circles of the right radius at the right speed, and, if necessary, repeatedly, we can travel to any past event on our original fundamental worldline. On the other hand, by simply sitting still we can go to any desired future event on this worldline. Once we have reached the event $t = t_0$ on our fundamental worldline, we can find any number of circular arcs in the "plane" $t = t_0$ to take us to **Q**.

As we mentioned earlier, the constant-time circular orbits discussed above are not geodesics in the Gödel universe, although all geodesics are circles in the lattice or circular helices, as is to be expected from the Newtonian analogy. The simplest method of determining these geodesics is the method of rotating coordinates.[18] According to this method, we take a uniformly rotating "vertical coordinate plane" $\phi = \omega t$ (ω=const) as the base plane $\tilde{\phi} = 0$ for a new angular coordinate $\tilde{\phi}$:

$$\tilde{\phi} = \phi - \omega t, \tag{60a}$$

$$d\phi = d\tilde{\phi} + \omega dt. \tag{60b}$$

The canonical stationary metric (50) is thereby transformed into another canonical stationary metric, one whose lattice rotates relative to the old lattice at coordinate angular velocity $d\phi/dt = \omega$. For our present purposes we need not calculate that entire metric; all we need is the coefficient of dt^2. Replacing $d\phi$ by $d\tilde{\phi}$ according to Eq. (60b) transforms Eq. (50) into

$$ds^2 = a^2\{[1 - \sqrt{2}(\cosh r - 1)\omega]^2 - (\sinh^2 r)\omega^2\}dt^2 + \cdots. \tag{61}$$

A fixed point P on this new lattice will move geodesically if the gravitational field, relative to this lattice, vanishes at P. According to Eq. (14) we thus need only set the r-derivative of the brace in Eq. (61) equal to zero. A simple calculation then yields a quadratic equation for ω which is of the form $A\omega + B\omega^2 = 0$, one of whose roots is zero (not surprisingly: particles sitting still in the original lattice also trace out geodesics). The other root is

$$\omega = \frac{-\sqrt{2}}{2 - \cosh r}. \tag{62}$$

Hence, within a coordinate plane z=const, a circle traced out at radius r with this coordinate angular velocity is a geodesic; the negative sign means that it must be traced out in the clockwise (negative) sense, if ϕ increases in the anticlockwise sense. Note that, as r increases from zero to $\cosh^{-1} 2$, ω numerically increases from $1/\sqrt{2}$ to infinity. But a particle (or photon-) orbit must be timelike (or null), and hence must satisfy $ds^2 \geq 0$. Since the dots in Eq. (61) stand for terms involving the new spatial differentials, we now set them equal to zero for the rotating lattice and substitute for ω from Eq. (62). After some manipulation this leads to

$$ds^2 = \frac{2 - \cosh^2 r}{(2 - \cosh r)^2} a^2 dt^2, \tag{63}$$

and hence to the following condition for $ds^2 \geq 0$:

$$\cosh r \leq \sqrt{2}. \tag{64}$$

Equality here corresponds to a photon geodesic, which is thus the largest circular geodesic. We may note that Eq. (64) is equivalent to $\cosh 2r \leq 3$; comparison with Eq. (57) then shows that the largest circular geodesic is half as big as the smallest time loop.

To determine the local velocity of a particle following a circular geodesic, we repeat the argument leading to Eq. (59), except that in the last step we now take ds^2 from Eq. (63) and in dl replace $d\phi$ by ωdt. This leads to the pleasant result

$$v = \sqrt{2} \tanh r, \tag{65}$$

in terms of which the condition (64) reads $v \leq 1$, just as one would expect.

The above analysis, in fact, determines *all* plane geodesics, namely those in coordinate "planes" z=const. For, given any initial particle velocity at any point of such a plane, there is a unique geodesic with that initial vector velocity, and it will be a circle: we need only to proceed orthogonally to the given velocity, along a geodesic of the spatial lattice, through a ruler distance ar, where r is given by Eq. (65), to find the center of the circular geodesic that fits the given initial data. Incidentally, it is well to bear in mind this relation of the coordinate r to radial ruler distance—implicit in the metric (50)—in all the above formulas.

We must also remember that not t but at in Eq. (50) corresponds to proper time at the galaxies. So if we wish to compare the ω of geodesics with the Ω of the entire Gödel universe, we need a factor a in the denominator of Eq. (62). And then, for small values of r, we find, from Eqs. (62) and (51), $\omega = 2\Omega$ numerically, just as in the Newtonian analog [cf. Eq. (8)].

To complete our discussion of the geodesics in Gödel's universe, we would expect from the Newtonian analogy that in the most general case they are helices in the lattice. And this is indeed so. For proof, we simply augment the rotation (60) by an additional uniform translation (up or down) in the z direction,

$$\tilde{z} = z - ut, \quad dz = d\tilde{z} + u dt \quad (u = \text{const}), \tag{66}$$

so that each lattice point of the $(r, \tilde{\phi}, \tilde{z})$ system now traces out a helix around the central axis $r = 0$. Algebraically, this merely brings an additional term $-u^2$ into the brace in Eq. (61), which leaves Eq. (62) unchanged. Consequently, any point fixed in the new lattice, now spiraling in the z-direction, still describes a geodesic when Eq. (62) is satisfied. But the condition for $ds^2 \geq 0$ is now obtained from the obvious modification of Eq. (63), which yields

$$\frac{2-\cosh^2 r}{(2-\cosh r)^2} \geq u^2, \tag{67}$$

with equality again corresponding to photons.

The local horizontal velocity component of the geodesic is still given by Eq. (65). The local vertical velocity component is *not* u, but must again be found by a suitable variation on Eq. (59) (putting $dl = u dz$). If we denote it by $\tilde{u}$, we find

$$\tilde{u} = \frac{u}{1+[2(\cosh r-1)/(2-\cosh r)]}. \tag{68}$$

With that, Eq. (67) is equivalent to $\tilde{u}^2 + v^2 \leq 1$, as one would expect.

The helicoidal geodesics here discussed are, in fact, the most general geodesics in the Gödel universe, in the sense that every possible geodesic is such a helix around *some* suitable axis. For, suppose at any point in the lattice we are given an initial local velocity having a "horizontal" component v and a vertical component $\tilde{u}$. Then v allows us, as before, to find r and ω and hence the axis of the helix; lastly $\tilde{u}$ together with r determines the u of the helix via Eq. (68).

In any universe, the propagation of light is of special importance for interpreting the observations. In Gödel's universe, as we have just seen, photons propagate along circles or, more generally, along helices with axes in the z-direction; each satisfies Eq. (65) and equality in Eq. (67). The larger the radius, the larger the horizontal velocity component (65), and therefore the smaller the vertical velocity component $\tilde{u}$ (since $v^2 + \tilde{u}^2 = 1$) and with it also the pitch $\tilde{u}/v$ of the helix. The largest possible radius, by Eq. (64), is $r = \cosh^{-1}\sqrt{2}$ when $v = 1$ and the helix reduces to a circle. Also, the larger the radius, the larger the angular velocity ω [by Eq. (62)], and hence the shorter the period. On the other hand, the pitch increases without bound as $r \to 0$ and the path ultimately becomes a straight line in the z-direction.

From the above remarks it is clear that galaxies on the same z-line can see each other directly along that straight line, traversable in either direction. All other light paths are traversable in one direction only, namely, clockwise. The farthest one can see from any galaxy "horizontally" is to a ruler distance $2a \cosh^{-1}\sqrt{2}$. A galaxy at that distance sends us light along a semicircle of half that radius, and, in turn, sees us along the complementary semicircle. That same full circle, of course, shows us "the back of our head." (And that galaxy, incidentally, lies on the smallest time-loop around ourselves.) Other galaxies see us along portions of suitable helices, and we see them along the complementary helices, the two being mirror images in the coordinate plane $\phi = \text{const}$ joining us to the galaxy in question. All galaxies that lie strictly within a cylinder of radius $2a \cosh^{-1}\sqrt{2}$ around us can receive light from us, and we from them. However, galaxies with large z-values relative to us may have to wait a while for the signal, since many turns of the required helix may be necessary. Also there will generally be multiple connecting paths, and thus multiple images, having left the source at different ages. This is most obvious for galaxy pairs on the same "vertical," which can be connected not only directly, but also via an infinite number of helicoidal light paths: the z-distance between successive loops of the helices need only be whole fractions of the z-difference between the galaxies. (Think of the cylinders on which the helices lie as touching the z-line joining the two galaxies.)

VIII. STABILITY

An important topic we have not yet touched upon is the stability or otherwise of Gödel's universe. Einstein's static model of 1917 is patently unstable (in hindsight!). The slightest contraction leads to total collapse, since it decreases the Λ repulsion (which is proportional to distance) and increases the gravity (which is inversely proportional to the square of the distance). Conversely, the slightest expansion becomes unstoppable.

An analogous Newtonian argument suggests that Gödel's universe, on the contrary, is stable to contraction or expansion. Consider a slight deformation $r_0 \mapsto r_0 + dr$ of a central cylinder $r \leq r_0$ in Gödel's model. The density ρ will decrease like $1/r^2$ (by mass conservation), as will the rotation rate Ω (by angular momentum conservation). Consequently we have

$$\rho = \rho_0 r_0^2/r^2, \quad \Omega = \Omega_0 r_0^2/r^2,$$

so that, with $k = 2\pi G\rho_0$, Eqs. (3), (4), and (12) yield

$$f_{\text{grav}} = -kr_0^2/r, \quad f_{\text{cent}} = 2kr_0^4/r^3.$$

Of course Λc^2 stays constant at its equilibrium value $-4\pi G\rho_0$, so that, by Eqs. (2), (3), and (11), we have, for the Λ force,

$$f_\Lambda = -kr.$$

At r_0 these three forces balance. Thereafter we have

$$\begin{aligned} df_{\text{grav}} + df_\Lambda + df_{\text{cent}} &= k(r_0^2/r^2)dr - kdr - 6k(r_0^4/r^4)dr \\ &= -6kdr, \end{aligned}$$

at $r = r_0$. The changes in f_{grav} and f_Λ just cancel each other, and there is now a restoring force provided by change in centrifugal force. This shows the radial equilibrium to be stable.

A very meticulous discussion of the stability problem can be found in Ref. 19.

IX. COMMENTS

This is as far as we shall here go in discussing the technicalities of Gödel's stationary model universe. In summary, this model provides a prime example of an anti-Machian distribution of matter, i.e., one where the rotation of the compass of inertia bears no relation to the mass distribution. This goes against our intuition, but need not be fatal to the model. Gödel's universe also neatly exhibits the two main "difficulties" with time in General Relativity. The first is the existence of closed time-loops, which play havoc with causality. A reasonable response to this is essentially Einstein's: one hopes that nature has some as yet undiscovered mechanism to prevent the formation of such universes, analogously perhaps to the speed limit in Special Relativity, which also serves to preserve causality. The second "difficulty" is the non-unicity of simultaneity in relativity, even in some relativistic cosmologies, though not in others. But while this dependence of the objectivity of time on the cosmic mass distribution may well be a problem for the philosophers, it presents neither physical nor logical problems for physicists, who have long learned to live and work with this state of affairs.

One last question might be asked. How original were the remarkable features of Gödel's model? There is little doubt that both Gödel and Einstein regarded Gödel's discovery of time loops as unprecedented. Yet solutions with time loops undoubtedly existed before 1949, such as anti-de Sitter space,[16] Lanczos space,[20] and the van Stockum spaces.[17] In anti-de Sitter space the time-loops can be removed by simple topological extension, but in the Lanczos and van Stockum spaces they are irremovable, just as in Gödel space. On the other hand, an admittedly incomplete literature search failed to uncover any evidence that the time loops in those earlier solutions had actually been recognized as such until Gödel's work made people look again. Rotating solutions, on the other hand, were definitely known and recognized as such much earlier,[17] though certainly not rotating homogeneous universes.

In our Epilogue (Sec. X below) we briefly report on the above-mentioned model by Lanczos, simply because of the historical irony that Lanczos came so uncannily close to discovering Gödel's universe exactly 25 years earlier.

Gödel himself continued his interest in rotating universes well beyond 1949. However, only one more publication resulted from this work, in 1952, being the text of a lecture delivered in September 1950[21] entitled "Rotating universes in general relativity theory." In this lecture he was "setting forth the main results (for the most part without proofs) to which my investigations on rotating universes have led me so far." The models he examines are spatially finite and homogeneous, and expanding. We shall here content ourselves with reporting only one very interesting result from that paper. Since in an expanding universe the density ρ must steadily decrease, we can consider the set of hypersurfaces ρ=const cutting across the fundamental worldlines as determining a kind of global time. However, they cannot cut across those worldlines orthogonally, for then we would have a Friedman universe and no rotation. Imagine therefore a locally parallel set of such constant-density surfaces, going from greater to lesser density, and the worldline of a fundamental observer (a substratum particle) cutting across them obliquely. The observer's proper surface of simultaneity is orthogonal to his worldline, and therefore cuts across the constant-density surfaces. At any given instant, therefore, the observer "sees" more galaxies in one half of the sky (where his simultaneity dips into the greater density) than in the other half. In fact, Gödel long persisted in searching for this effect, both through personal calculations from published data, and by urging the astronomers.[22] But no such indication of rotation has ever been found.

Schücking, who later talked with Gödel, has conjectured[23] that the absence of proofs from this paper may have been due to Gödel's dissatisfaction with the inelegance of his private calculations. But, as described in detail by Ellis,[23] this paper contains a number of truly seminal ideas and results that later became part and parcel of the various extensions of theoretical cosmology beyond the Friedman models. In fact, it is probably fair to say that Gödel's 1952 paper was one of the main impulses for many of these extensions.

X. EPILOGUE: LANCZOS'S MODEL UNIVERSE OF 1924

In 1924, 25 years before Gödel, Kornel (later Cornelius) Lanczos, one of the distinguished theoreticians of the 20 century, invented a model universe[20] possessing many of the striking features of Gödel's later model. Excusably, however, for that period, Lanczos failed to recognize the two most striking ones, rotation and time-loops. His universe was rediscovered in 1937 by van Stockum,[17] who at least recognized its rotation, if not yet its time loops. The great defect of Lanczos's model as a cosmology is that it is not homogeneous.

Lanczos, like Einstein, still takes it for granted that the universe is unchanging in time. Friedman's ground-breaking work of 1922 (Ref. 24 in the same journal!), where he introduced expanding universes, had not yet percolated into the scientific consciousness of the day, nor would it for many more years to come. (It took Hubble's forceful personality to finally convince Einstein in 1931 that the universe was really expanding.) So Lanczos only has Einstein's 1917 universe to look at, and he finds it unsatisfactory. It requires a specific relation, $\Lambda c^2 = 4\pi G\rho$ (numerically the same as Gödel's!), to hold between the density and the cosmological constant. Such a relation, Lanczos writes, "would be a mere coincidence and is therefore not a satisfactory assumption." He therefore sets out to construct a stationary universe without Λ. The field equations with dust sources do not permit such a universe to be spherically symmetric, so Lanczos opts for the next-best thing: rotational symmetry about an axis.

Guided partly by symmetry, partly by simplicity, he then arrives at the following ansatz:

$$ds^2 = dt^2 - G(dr^2 + dz^2) - 2Q\,d\phi\,dt - P\,d\phi^2, \tag{69}$$

where G, Q, and P are functions of r only. [For comparison purposes, we here reversed the sign of Lanczos's metric, and wrote r, ϕ, z for his x, ψ, y.] It would still not be too late to get Gödel's metric (50) out of this! But Lanczos proceeds down another road. Setting $\Lambda = 0$ in the field equations, he now necessarily finds

$$ds^2 = dt^2 - e^{-r^2}(dr^2 + dz^2) - 2r^2\,d\phi\,dt - (r^2 - r^4)\,d\phi^2, \tag{70}$$

apart from an overall dimension-giving constant factor, which he sets equal to unity. The density function corresponding to this metric is given by

$$\rho = \frac{e^{r^2}}{2\pi G}, \tag{71}$$

where we have written ρ for Lanczos's μ and used Gödel's units in which $c=1$ but $8\pi G \neq 1$. Note that $\rho \to \infty$ as $r \to \infty$. But Lanczos's r is only a conventional coordinate, related to ruler distance l by

$$l = \int_0^r e^{-r^2/2}\,dr, \tag{72}$$

so that $l \to \sqrt{\pi/2}$ as $r \to \infty$. Lanczos's universe is thus recognized as a cylinder, infinite in the z-dimension, but with a singular edge and infinite density at ruler radius $\sqrt{\pi/2}$.

For ease of comparison, we recast Gödel's metric into an often useful alternative form, which results when we set

$$t=2\tilde{t}, \quad r=2\tilde{r}, \quad z=2\tilde{z} \quad (\text{and } \tilde{S}=\sinh\tilde{r}), \tag{73}$$

namely,

$$ds^2=4a^2\{d\tilde{t}^2-(d\tilde{r}^2+d\tilde{z}^2)-2\sqrt{2}\tilde{S}^2 d\phi dt-(\tilde{S}^2-\tilde{S}^4)d\phi^2\}. \tag{74}$$

The analogy with Lanczos's metric is now quite striking.

As is easily verified nowadays [cf. formula (24) above], Lanczos's universe rotates locally (but not rigidly) relative to the compass of inertia with proper angular velocity

$$\Omega=e^{r^2/2}, \tag{75}$$

although he did not know it. Even so, he determined the quasi-helicoidal shape of the light paths in his model (which might well have suggested rotation). Together, Eqs. (75) and (71) imply $\Omega^2=2\pi G\rho$, as compared to Gödel's $\Omega^2=4\pi G\rho$.

Lanczos's universe also has closed time-loops. He actually goes so far as to remark that the simultaneity surfaces t =const eventually cut into the light cone (just as in our Fig. 5), in fact, when $r>1$. But he also remarks that this is nothing to worry about. Evidently he was only a hair away from realizing that all loops t, z, r=const with $r>1$ in Eq. (70) are possible closed particle worldlines (as follows immediately from inspection of the metric: $ds^2>0$). Alas.

Later in his paper Lanczos does include Λ. But unfortunately and inexplicably he takes it as an axiom that $\Lambda\geq 0$, so nothing significantly new emerges. Alas, once more.

ACKNOWLEDGMENTS

The author wishes to thank Mr. Chris Allison for his fine execution of the diagrams, Dr. Donato Bini and Dr. Andrea Geralico for help with computing the field equations, and Professor Mustapha Ishak for extricating him from a bad computer crash. Very special thanks go to Pamela Contractor, the developmental editor of *Kurt Gödel and the Foundations of Mathematics*, for smoothing the process all along the way.

[a)]Electronic mail: rindler@utdallas.edu

[1]Kurt Gödel, "A remark about the relation between relativity theory and idealistic philosophy," in *Albert Einstein, Philosopher-Scientist*, edited by P. A. Schilpp (Library of Living Philosophers, Vol. 7, Evanston, IL. 1949). Reprinted in Ref. 2, Vol. II (1990), pp. 202–207.

[2]Kurt Gödel, "Lecture on rotating universes," in *Collected Works*, edited by S. Feferman, J. W. Dawson, Jr., W. Goldfarb, C. Parsons, and R. N. Solovay (Oxford U. P., Oxford, 1995), Vol. III pp. 269–287.

[3]Kurt Gödel, "An example of a new type of cosmological solutions of Einstein's field equations of gravitation," Rev. Mod. Phys. **21**, 447–450 (1949). Reprinted in Ref. 2, Vol. II (1990), pp. 190–198.

[4]G. Gamow, "Rotating Universe?" Nature (London) **158**, 549 (1946).

[5]John W. Dawson, Jr., *Logical Dilemmas: The Life and Work of Kurt Gödel* (Peters, Wellesley, MA, 1997), pp. 177–183.

[6]See Ref. 2, Vol. V (2003), pp. 232–237.

[7]Wolfgang Rindler, *Relativity: Special, General, and Cosmological*, 2nd ed. (Oxford U. P., Oxford, 2006), Eq. (14.20).

[8]I. Ozsvath and E. L. Schücking, "Approaches to Gödel's rotating universe," Class. Quantum Grav. **18**, 2243–2252 (2001), Sec. 3(c).

[9]Reference 7, Chap. 9.

[10]Reference 7, p. 197.

[11]Reference 7, Eq. (16.19).

[12]Reference 7, Eq. (9.23).

[13]Reference 7, Eq. (7.86).

[14]Dr. Donato Bini and Dr. Andrea Geralico, of CNR, Roma, gave valuable help with this.

[15]A. Einstein, "Die formale Grundlage der allgemeinen Relativitätstheorie," Sitzungsber. K. Preuss. Akad. Wiss. **XLI**, 1030–1085 (1914) (see p. 1079.) I am indebted to J. Ehlers for drawing my attention to this reference.

[16]For discussion of anti-de Sitter space see J. L. Synge, *Relativity: The General Theory*, (North Holland—Interscience, New York, 1960) p. 264, or Ref. 7, Sec. 14.6. I was unable to determine when its time-loops were first noticed.

[17]W. J. van Stockum, "The gravitational field of a distribution of particles rotating about an axis of symmetry," Proc. R. Soc. Edinburgh **57**, 135 (1937).

[18]W. Rindler and V. Perlick, "Rotating Coordinates as Tools for Calculating Circular Geodesics and Gyroscopic Precession," Gen. Relativ. Gravit. **22**, 1067–1081 (1990).

[19]J. D. Barrow and C. G. Tsagas, "Dynamics and stability of the Gödel universe," Class. Quantum Grav. **21**, 1773–1789 (2004). This paper, with its numerous references, will be useful to anyone desiring a deeper immersion into the literature.

[20]K. Lanczos, "Über eine stationäre Kosmologie im Sinne der Einsteinschen Gravitationstheorie," Z. Phys. **21**, 73–110 (1924).

[21]Kurt Gödel, "Rotating universes in general relativity theory," Proc. Int. Congress Math., Cambridge, MA, 1950, I:175–181 (Am. Math. Soc. 1952). Reprinted in Ref. 2, Vol. II (1990) pp. 208–216.

[22]See Ref. 5, p. 182.

[23]G. F. R. Ellis, "Contributions of K. Gödel to Relativity and Cosmology," University of Capetown preprint September 1996, Institute of Theoretical Physics and Astrophysics. Also in *Gödel '96*, edited by P. Hayek (Springer Lecture Notes in Logic 6, Springer-Verlag, Heidelberg, 1996).

[24]A. Friedman, "Über die Krümmung des Raumes," Z. Phys. **10**, 377–386 (1922).