JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 42, 191–197 (1973)

Invariance and the n-Body Problem

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Submitted by Richard Bellman

1. INTRODUCTION

The classical theorem of E. Noether [1] on invariant variational problems states briefly that if the variational integral is invariant under an infinitesimal group of transformations, then a certain number of identities hold true. Under the additional assumption that the Euler equations for the system are satisfied, these identities reduce to expressions which are constant along the extremals. In different words, the Noether identities lead to conservation laws for the system. In this manner, E. Bessel-Hagen [2] in 1921 applied the Noether theorem using the ten parameter Galilean group to derive the ten conservation laws for the classical n-body problem involving gravitational forces.

In this paper we shall consider a type of converse problem, namely how the Noether theorem can be used to deduce the general form of a Lagrangian which has specified invariance properties. In particular, we shall characterize the Lagrangians which possess the same invariance properties under the Galilean group as the Lagrangian for the n-body problem. Interestingly enough, there is a large class of Lagrangians which possess these properties. We shall show in Section 3 that these Lagrangians can be written as the difference between the classical kinetic energy and a scalar potential which depends upon the magnitudes of the relative positions, the magnitudes of the relative velocities, and scalar products between the relative positions and velocities.

2. NOETHER'S THEOREM

In this section we state those facts which will be necessary in later discussions and at the same time establish the notation that will be used. We shall often make use of the summation convention of summing repeated indices that appear on different levels.

We consider the variational integral

$$J = \int_{t_0}^{t_1} L(t, q(t), q'(t))\, dt, \tag{2.1}$$

where $q(t)$ and $q'(t)$ are $3n$-dimensional vectors with $q'(t)$ being the derivative of $q(t)$. The total variation ΔJ of J with respect to the infinitesimal transformation

$$\bar{t} = t + \Delta t,$$
$$\bar{q}^k = q^k + \Delta q^k, \qquad k = 1, 2, \ldots, 3n,$$

is given by (see for example Gelfand and Fomin [3]),

$$\begin{aligned} \Delta J[\Delta t, \Delta q^k] &= \int_{t_0}^{t_1} \Phi_k(\Delta q^k - q'^k \Delta t)\, dt \\ &\quad + \int_{t_0}^{t_1} \frac{d}{dt}\left[\left(L - q'^k \frac{\partial L}{\partial q'^k}\right) \Delta t + \frac{\partial L}{\partial q'^k} \Delta q^k\right] dt, \end{aligned} \tag{2.2}$$

where

$$\Phi_k = \frac{\partial L}{\partial q^k} - \frac{d}{dt}\frac{\partial L}{\partial q'^k}, \qquad k = 1, \ldots, 3n,$$

are the variational derivatives. The following definition makes precise the notion of invariance of J under a μ-parameter infinitesimal group of transformations. Let ϵ^α, $\alpha = 1, 2, \ldots, \mu$ be independent parameters.

DEFINITION 2.1. J is divergence-invariant under the transformation

$$\begin{aligned} \bar{t} &= t + \epsilon^\alpha \tau_\alpha(t, q, q'), \\ \bar{q}^k &= q^k + \epsilon^\alpha \chi_\alpha{}^k(t, q, q'), \end{aligned} \tag{2.3}$$

$\alpha = 1, \ldots, \mu$, if there exist functions $\psi_\alpha(t, q, q')$ for which

$$\Delta J[\epsilon^\alpha \tau_\alpha, \epsilon^\alpha \chi_\alpha{}^k] = \int_{t_0}^{t_1} \frac{d}{dt}(\epsilon^\alpha \psi_\alpha)\, dt, \tag{2.4}$$

for every t_0 and t_1. If $\psi_\alpha = 0$ for all α, then we say J is absolutely invariant.

The following theorem then holds true. Its proof consists of equating (2.2) and (2.4) and then using the independence of the parameters and the arbitrariness of the limits of integration.

THEOREM 2.1 (Noether). *If J is divergence-invariant under the μ-parameter group of transformations* (2.3), *and if $\Phi_k = 0$, $k = 1,..., 3n$, then*

$$(L - q'^k(\partial L/\partial q'^k))\,\tau_\alpha + (\partial L/\partial q'^k)\,\chi_\alpha{}^k - \psi_\alpha = \text{constant}, \tag{2.5}$$

$\alpha = 1,..., \mu$.

The expressions given by (2.5) are constant along the extremals (solutions to $\Phi_k = 0$) and, consequently, are regarded as conservation laws for the system. The article by Hill [4] can be consulted for details.

The application of Theorem 2.1 to the n-body problem gives the ten conservation laws of the problem. To carry out a discussion of this application, we adhere to the following notation: Let

$$\begin{aligned} q^k &= x_k && \text{for } k = 1,..., n, \\ q^k &= y_{k-n} && \text{for } k = n+1,..., 2n, \\ q^k &= z_{k-2n} && \text{for } k = 2n+1,..., 3n. \end{aligned} \tag{2.6}$$

The Lagrangian L will have the meaning:

$$L = T - U, \tag{2.7}$$

where

$$T = \tfrac{1}{2} \sum_{i=1}^{n} m_i(x_i'^2 + y_i'^2 + z_i'^2) \tag{2.8}$$

and

$$U = -\sum G(m_i m_j/r_{ij}), \qquad 1 \leqslant i < j \leqslant n \tag{2.9}$$

are the kinetic and potential energies of the system, and m_i is the mass of the i-th body with position (x_i, y_i, z_i). G is the gravitational constant, and

$$r_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2}.$$

The ten-parameter Galilean group is given by:

(i) Time translation

$$\bar{t} = t + \epsilon^1,$$

$$\bar{x}_i = x_i, \qquad \bar{y}_i = y_i, \qquad \bar{z}_i = z_i.$$

(ii) Spatial translation

$$\bar{t} = t,$$

$$\bar{x}_i = x_i + \epsilon^2, \qquad \bar{y}_i = y_i + \epsilon^3, \qquad \bar{z}_i = z_i + \epsilon^4.$$

(iii) Galilean transformations

$$t = t,$$

$$\bar{x}_i = x_i + \epsilon^5 t, \qquad \bar{y}_i = y_i + \epsilon^6 t, \qquad \bar{z}_i = z_i + \epsilon^7 t.$$

(iv) Spatial rotations

$$\bar{t} = t,$$
$$\bar{x}_i = x_i + \epsilon^8 y_i + \epsilon^9 z_i ,$$
$$\bar{y}_i = y_i - \epsilon^8 x_i + \epsilon^{10} z_i ,$$
$$\bar{z}_i = z_i - \epsilon^9 x_i - \epsilon^{10} y_i .$$

Bessel-Hagen showed that $J = \int_{t_0}^{t_1} (T - U)\, dt$ is absolutely invariant with respect to time translations, spatial translations, and spatial rotations. This means

$$\psi_\alpha = 0 \qquad \text{for } \alpha = 1, 2, 3, 4, 8, 9, 10. \tag{2.10}$$

On the other hand, J is only divergence–invariant with respect to the Galilean transformations with the divergence terms given by

$$\psi_5 = \sum_{i=1}^{n} m_i x_i , \qquad \psi_6 = \sum_{i=1}^{n} m_i y_i , \qquad \psi_7 = \sum_{i=1}^{n} m_i z_i . \tag{2.11}$$

The conservation laws follow directly from Eq. (2.5). Invariance under time translations, spatial translations, spatial rotations, and Galilean transformations leads to conservation of energy, linear momentum, angular momentum, and uniform motion of the center of mass, respectively.

3. The Converse Problem

In this section we obtain the most general Lagrangian $\bar{L}(t, q, q')$ which exhibits the same invariance properties as the Lagrangian for gravitational interactions. The result will give a negative answer to the question: If $\int \bar{L}\, dt$ is divergence-invariant under the Galilean group with the ψ_α given by (2.10) and (2.11), then does it follow that $\bar{L} = L$, where L is given by (2.7)?

To begin the investigations, we show that the invariance of $\int \bar{L}(t, q, q')\, dt$ under a μ-parameter infinitesimal group of transformations leads to a system of μ first order partial differential equations which $\bar{L}$ must satisfy.

THEOREM 3.1. *If* $\int_{t_0}^{t_1} \bar{L}(t, q, q')\, dt$ *is divergence-invariant with respect to the transformation* (2.3), *then*

$$\frac{\partial \bar{L}}{\partial t} \tau_\alpha + \frac{\partial \bar{L}}{\partial q'^k} \chi_\alpha{}^k + \frac{\partial \bar{L}}{\partial q'^k} \chi_\alpha^{k'} = \frac{d\psi_\alpha}{dt}, \qquad \alpha = 1, 2, \ldots, \mu,$$

where (3.1)

$$\chi_\alpha^{k'} = (d/dt)\, \chi_\alpha{}^k.$$

Proof. Let $\Delta t = \epsilon^\alpha \tau_\alpha$, $\Delta q^k = \epsilon^\alpha \chi_\alpha{}^k$, and $\Delta q'^k = \epsilon^\alpha \chi_\alpha^{k'}$. Since

$$\Delta J = \int_{t_0}^{t_1} \Delta \bar{L}(t, q, q')\, dt,$$

with

$$\Delta \bar{L} = \frac{\partial \bar{L}}{\partial t} \Delta t + \frac{\partial \bar{L}}{\partial q^k} \Delta q^k + \frac{\partial \bar{L}}{\partial q'^k} \Delta q'^k,$$

it follows from (2.2) and the definition of divergence-invariance that

$$\int_{t_0}^{t_1} \Delta \bar{L}(t, q, q')\, dt = \int_{t_0}^{t_1} \frac{d}{dt} (\epsilon^\alpha \psi_\alpha)\, dt.$$

Since this equation holds for all t_0 and t_1, we conclude that

$$\frac{\partial \bar{L}}{\partial t} \Delta t + \frac{\partial \bar{L}}{\partial q^k} \Delta q^k + \frac{\partial \bar{L}}{\partial q'^k} \Delta q'^k = \frac{d}{dt} (\epsilon^\alpha \psi_\alpha).$$

Using the linear independence of the parameters $\epsilon^1, \ldots, \epsilon^\mu$, we obtain the system (3.1). This completes the proof.

We now write down the system (3.1) for $\bar{L}(t, q, q')$ when the group is the ten-parameter Galilean group, and the divergence terms ψ_α are given by (2.10) and (2.11). This means that we are requiring $\bar{L}$ to have the same invariance properties as the Lagrangian for the n-body problem. First, absolute invariance under time translations yields

$$\partial \bar{L} / \partial t = 0, \tag{3.2}$$

or $\bar{L}$ is independent of t. Absolute invariance under spatial translations gives

$$\sum \frac{\partial \bar{L}}{\partial x_i} = 0, \qquad \sum \frac{\partial \bar{L}}{\partial y_i} = 0, \qquad \sum \frac{\partial \bar{L}}{\partial z_i} = 0. \tag{3.3}$$

Divergence-invariance under Galilean transformations leads to

$$\sum \frac{\partial \bar{L}}{\partial x_i'} = \sum m_i x_i', \qquad \sum \frac{\partial \bar{L}}{\partial y_i'} = \sum m_i y_i', \qquad \sum \frac{\partial \bar{L}}{\partial z_i'} = \sum m_i z_i'. \tag{3.4}$$

Finally, absolute invariance under rotations implies

$$\begin{aligned} &\sum \left[\left(y_i \frac{\partial \bar{L}}{\partial z_i} - z_i \frac{\partial \bar{L}}{\partial y_i}\right) + \left(y_i' \frac{\partial \bar{L}}{\partial z_i'} - z_i' \frac{\partial \bar{L}}{\partial y_i'}\right)\right] = 0, \\ &\sum \left[\left(z_i \frac{\partial \bar{L}}{\partial x_i} - x_i \frac{\partial \bar{L}}{\partial z_i}\right) + \left(z_i' \frac{\partial \bar{L}}{\partial x_i'} - x_i' \frac{\partial \bar{L}}{\partial z_i'}\right)\right] = 0, \\ &\sum \left[\left(x_i \frac{\partial \bar{L}}{\partial y_i} - y_i \frac{\partial \bar{L}}{\partial x_i}\right) + \left(x_i' \frac{\partial \bar{L}}{\partial y_i'} - y_i' \frac{\partial \bar{L}}{\partial x_i'}\right)\right] = 0. \end{aligned} \tag{3.5}$$

All of the above sums range over $i = 1,..., n$. Equations (3.2)–(3.5) represent ten first order, quasilinear partial differential equations for $\bar{L}$. The method of characteristics provides a means of extracting information about $\bar{L}$ from these equations. Equation (3.3a) is equivalent to the system of equations

$$dx_1 = dx_2 = \cdots = dx_n,$$

which has the $n - 1$ constants,

$$x_i - x_1 = \text{constant}, \qquad i \neq 1. \tag{3.6}$$

Similar information from (3.3b) and (3.3c) gives

$$\begin{aligned} y_i = y_1 &= \text{constant}, \\ z_i - z_1 &= \text{constant}, \qquad i \neq 1. \end{aligned} \tag{3.7}$$

Equation (3.4a) is equivalent to the system of equations

$$dx_1' = dx_2' = \cdots = dx_n' = \frac{d\bar{L}}{\sum m_i x_i'},$$

which has n constants,

$$\begin{aligned} x_i' - x_1' &= \text{constant}, \qquad i \neq 1, \\ \bar{L} - \tfrac{1}{2} \sum m_i x_i'^2 &= \text{constant}. \end{aligned} \tag{3.8}$$

Hence, from (3.2), (3.6), (3.7) and (3.8) we conclude that $\bar{L}$ is given implicitly by

$$\begin{aligned} \bar{L} = \Lambda \Big(\bar{L} - \tfrac{1}{2} \sum m_i x_i'^2, x_i - x_1, y_i - y_1, z_i - z_1, \\ \times\, x_i' - x_1', y_i' - y_1', z_i' - z_1'\Big). \end{aligned}$$

Consequently,

$$\bar{L} = \tfrac{1}{2}\sum m_i x_i^2 + \Omega(x_i - x_1, y_i - y_1, z_i - z_1, x_i' - x_1', y_i' - y_1', z_i' - z_1').$$

Successive applications of Eqs. (3.4b) and (3.4c) yield

$$\bar{L} = T - Y(\mathbf{R}_i - \mathbf{R}_1, \mathbf{R}_i' - \mathbf{R}_1'), \tag{3.9}$$

where we have denoted $\mathbf{R}_i = (x_i, y_i, z_i)$, and T is given by (2.8). Therefore, we have shown so far that $\bar{L}$ is equal to the difference between the usual kinetic energy and a function which depends only on the relative positions and velocities. Fortunately, it is possible to obtain information about $\bar{L}$ concerning its invariance under rotations without finding the characteristics of Eq. (3.5). Since $\bar{L}$ is assumed to be absolutely invariant under spatial rotations, and since T is absolutely invariant under rotations (this is easily shown), it follows from (3.9) that Y must be invariant. Therefore, Y must be of the form

$$Y(|\mathbf{R}_i - \mathbf{R}_1|, |\mathbf{R}_i' - \mathbf{R}_1'|, (\mathbf{R}_i - \mathbf{R}_1) \cdot (\mathbf{R}_i' - \mathbf{R}_1')). \tag{3.10}$$

We record the following theorem.

THEOREM 3.2. *If* $\int \bar{L}\, dt$ *is divergence-invariant under the Galilean group with the divergence terms given by* (2.10) *and* (2.11), *then*

$$\bar{L} = T - Y,$$

where T *is given by* (2.8), *and* Y *given by* (3.10) *is a function depending upon the magnitudes of the relative positions, the magnitudes of the relative velocities, and scalar products of the relative positions and velocities.*

ACKNOWLEDGMENT

The author expresses his appreciation to Professor Stefan Drobot of the Ohio State University for many suggestions relating to this paper.

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