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des Lauriers, Guérard
Sur les systèmes différentiels du second ordre qui admettent un groupe continu fini de transformations. (French)
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The author studies systems of n differential equations of second order which admit a Lie group of transformations. He remarks that a particular case of this problem is the problem of finding all the Riemannian spaces, the geodesic lines of which admit such a group; this problem has already been solved by the reviewer [Mem. Accad. Sci. Torino (2) **43** (1903); Atti Accad. Naz. Lincei. Rend. **14** (1905)].

The first chapter is concerned with two kinds of systems of differential equations. Systems I define the second derivatives of unknown functions x_i $(i=1,2,\cdots,n)$ with respect to a parameter t as functions of the x_j and their first derivatives (t not appearing explicitly in the equations). Systems II give $d^2x_i/dx_n{}^2$ $(i=1,2,\cdots,n-1)$ as functions of the x_j and of the dx_j/dx_n. Given two systems I, the author derives conditions under which it is possible to transform one into the other by means of a change of the independent variable t, and similarly conditions that a system I be equivalent to a system II. He finds that a special rôle is played by the systems III of n equations:

$$\frac{d^2x_i}{dt^2}=\sum_{r,s} f^i_{rs}\frac{dx_r}{dt}\frac{dx_s}{dt}+\frac{dx_i}{dt}R,$$

where the f^i_{rs} are functions of the x_j only and R is independent of i and a function of the x_j and dx_j/dt. For such systems the author generalizes Riemann's symbols of the second kind and Ricci's calculus in Riemannian spaces.

Next conditions are stated under which a given system III admits an infinitesimal transformation

$$X=\sum_i \xi^i\frac{\partial}{\partial x_i},\quad \xi^i \text{functions of the } x_j,$$

and the corresponding partial equations for the ξ are found. Using the conditions of integrability for these equations, the author shows that the second derivatives of the ξ are linear functions of the ξ and their first partial derivatives with respect to the x_j. There exist special systems which, by a suitable choice of the x_j and of the independent variable, can be transformed into the system $x_i''=0$. This case excepted, the author finds the maximum number M of parameters on which a group transforming a system III into itself may depend. For example, if $n\geq 3$, then $M=(n-1)(n-2)+3$. In studying the infinitesimal transformations of a group Γ which transforms a system III into itself, he takes into account their order r; if Taylor's development of the ξ begins with terms of degree r, the transformation X is of order r. From the preceding theorems it follows that, in the most important cases, r is equal either to zero or to 1. The following investigation concerns the set of transformations of Γ of order zero, and the set γ of transformations of order 1 (which transforms the origin into itself), and the relations among such transformations. Moreover, a study is made of the group $\overline{\gamma}$ generated by the infinitesimal transformations of γ, where all the terms which are not of first order in the development of the ξ are omitted. Owing to the long calculations a detailed description of them is impossible. Next the author studies systems having a group Γ

depending on M or $M-1$ parameters; here the properties of the corresponding group $\overline{\gamma}$ are important. Despite their length the calculations in the paper are not complete. Finally the author determines all the systems of three equations which admit a group. Many results are summarized in synoptic tables. *G. Fubini*