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Henri Poincaré's Mathematical Contributions to Relativity and the Poincaré Stresses

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Henri Poincaré was the first to introduce four-vectors, the Lorentz group and its invariants (including the space–time metric), "Poincaré stresses," as well as making other valuable contributions to relativity theory. We owe to him the names: "Principle of Relativity," "Lorentz group," "Lorentz transformation," and "invariants of the Lorentz group." It will be shown that his main contributions to relativity were those of a mathematical nature. This has not been sufficiently recognized, although his physical contributions to relativity have been much discussed recently. Frequent misunderstanding of the work of Poincaré and Einstein has resulted in controversy tending to obscure the main achievements of Poincaré. The Poincaré stresses are discussed because of widespread ignorance regarding the theory of classical fundamental charges. The emphasis in this article is on those accomplishments of Poincaré by which he should be better known today. In addition, some misconceptions about the history of relativity and classical electron theory will be corrected.

INTRODUCTION

The study of the history of relativity is not only interesting, challenging, and instructive, but it also aids in removing misconceptions about the physics of relativity and the present state of the theory, and can give valuable insight into the future course of relativity. For this purpose, one should not exaggerate the importance of determining the exact historical sequence of the events in relativity. However, knowledge of this sequence should correct misconceptions and give credit to the original workers whom we frequently slight, as we tend to take results for granted.

In contrast to general relativity, the special theory resulted from the work of many men. Of special importance was the work of Lorentz, Poincaré, and Einstein, each bringing the theory closer to completion. In studying their work, one must carefully distinguish between their mathematical formalism and their physical ideas because of deceptive similarities both in the equations and the words used. Nevertheless, each of the three great men had different viewpoints and relativities.[1] One must be most careful in analyzing Poincaré's contributions because of his somewhat inconsistent position[2] of transition between classical and relativistic physics. A well-known misinterpretation of the historical facts is found in the work of Whittaker,[3] a brilliant writer and mathematician. He failed to appreciate the physical approach by Einstein and gave Poincaré most credit for supplying the principle of relativity. Moreover, he misinterpreted[4] Poincaré's achievements in relativity as being mainly physical, and those of Lorentz as being mainly mathematical. Generally, insufficient[5] credit in literature has been given to Poincaré for his mathematical contributions to relativity. This is regrettable because this part of his work has kept its unchanged importance, while some of his physical contributions must be revised after the advances of his followers. Otherwise, Poincaré is known to physicists usually in reference to Poincaré stresses and the Poincaré group. This does not adequately represent his work.

Frequently one may also tend to misinterpret the work of Einstein, in ascribing to him many results which others obtained. For example, Semat[6] credits Einstein with the mass-increase

[1] H. Dingle, Brit. J. Phil. Sci. **16**, 242–246 (1966).

[2] A. Sesmat, *Systemes de Reference et Mouvements* (Hermann & Cie., Paris, 1937), pp. 601–7, 674–682.

[3] E. Whittaker, *History of the Theories of Aether and Electricity* (Harper Torchbooks, New York, 1960), Vol. 2, pp. 27–40; criticized, e.g., by G. Holton, Am. J. Phys. **28**, 627–636 (1960).

[4] Ref. 3, p. 36.

[5] As pointed out by H. Schwartz, Am. J. Phys. **33**, 170 (1965).

[6] (a) H. Semat, *Fundamentals of Physics* (Holt, Rinehart and Winston, Inc., New York, 1966), 4th ed., pp. 614, 618. (b) H. Semat, *Introduction to Atomic and Nuclear Physics* (Holt, Rinehart and Winston, Inc., New York, 1962), 4th ed., p. 36.

equation $m=m_0(1-v^2/c^2)^{-1/2}$, and he uses the name "Einstein–Lorentz" transformations. The following list of a few interesting events in special relativity will serve to point out some misconceptions and show for instance that Minkowski, Planck, and others contributed important results. For example, Lorentz[7] in 1904 was the first one to present the equation $m=m_0(1-v^2/c^2)^{-1/2}$ for electrons and other particles. It was Planck[8] in 1906 and 1908, and Minkowski[9] in 1908, who obtained the equations of dynamics of a particle in special relativity. Einstein[10] in 1905 considered $[m_0c^2(1-v^2/c^2)^{-1/2}-m_0c^2]$ as the kinetic energy of an electron and showed that $m=E/c^2$ was approximately valid for a charged particle radiating electromagnetic energy. He *assumed* it to be true in general, but Lorentz[11] in 1911 *showed* that in general all types of energies must be included in E. Minkowski[9] in 1908, and not Einstein, completed the relativistic treatment of the electrodynamics of matter in general (including magnetization). Surprisingly, "Lorentz" transformations were already used by Larmor[12] in 1900 (for x, y, z, t only). One should also know exactly which marvellous simplifications and contributions Einstein's genius made in special relativity. Again, Poincaré did not seem to consider the Poincaré group, but only the Lorentz group. It is also important to note that the classical theory of the elementary charged particle has been recently brought close to solution by Rohrlich,[13] thus modifying Feynman's[14] claim that the classical theory breaks down. The name "relativity" can be misleading. It emphasizes the reality of the relative (velocity times mass, etc.), and implies, perhaps, that "all views are relative." However, the basis and applications of the theory of relativity are characterized by phenomena that do not change according to relative observers. The two Einstein postulates state that the laws of physics and speed of light do not change for relatively moving reference systems (being covariant and invariant, respectively). To clarify many such facts and prepare for an appreciation of the role of Poincaré, a brief outline of the *theoretical* development of special relativity is given first.

I. HISTORICAL OUTLINE OF SPECIAL RELATIVITY

The theory of relativity developed from attempts to develop a satisfactory electrodynamics of moving bodies. Newtonian mechanics was known to be Galilean invariant, while the laws of electrodynamics were not. Since Maxwell's unification of optics and electrodynamics, the electromagnetic ether assumed further importance as the medium of propagation of light waves. Its rest frame where light moves with speed c and Maxwell's equations are valid was thought to be an absolute reference frame. Experiments failed to detect this absolute frame and did not show that modifications of the laws of electrodynamics were required for "moving" systems. Hence, attempts were made at a theoretical explanation by a study of invariance of the equations of electrodynamics, by a search for principles to explain the experiments, and by attempts to revise the existing laws.

Maxwell's equations were developed in classical physics. Therefore, it is surprising that they are not Galilean[15] invariant. The Lorentz transformations could have been derived by Maxwell, but it required several more decades. The efforts of many physicists, notably Lorentz, Poincaré, Einstein, Planck, and Minkowski solved the problems. Newtonian mechanics was revised, while electrodynamics was left unchanged and shown to be Lorentz invariant. Other major results were the rejection of absolute time and the ether, constancy of the speed of light (its being also the maximum signal velocity attain-

[7] H. Lorentz, Amst. Proc. **6,** 809 (1904); reprinted in A. Einstein *et al.*, *Principle of Relativity* (Dover Publications, Inc. New York, 1923), pp. 11–34.

[8] M. Planck, Verhandl. Deut. Phys. Ges. **8,** 136–141 (1906); **10,** 728–732 (1908).

[9] H. Minkowski, Nachr. Königl. Ges. Wiss. Göttingen, 53–111 (1908).

[10] (a) A. Einstein, Ann. Phys. **17,** 891–921 (1905); (b) transl. in Ref. 7, p. 37; (my modified translation in quoting); (c) Ann. Phys. **18,** 639–641 (1905); transl. in Ref. 7, p. 69.

[11] H. Lorentz, Amst. Versl. **20,** 87–98 (1911).

[12] J. Larmor, *Aether and Matter* (Cambridge University Press, Cambridge, England, 1900), Chap. 11.

[13] F. Rohrlich, Phys. Rev. Letters **12,** 375–7 (1964); *Classical Charged Particles* (Addison–Wesley Publ. Co., Inc., New York, 1965).

[14] R. Feynman, *The Feynman Lectures on Physics* (Addison–Wesley Publ. Co., Inc., 1964), Vol. 2, Chap. 28.

[15] The name "Galilean transformation" was introduced by P. Frank, Ann. Phys. **34,** 825 (1911).

able), and the connection between mass and energy.

A. Lorentz, Larmor, and Planck

An early forerunner of Lorentz transformations is found in the work of Voigt[16] in 1887. This work has recently received more attention.[17] Voigt studied the Doppler principle for waves propagating with speed w in an elastic medium. He obtained the invariance of the wave equation with respect to coordinate and time transformations, equal to the modern Lorentz transformation (with the speed of light c replaced by w) multiplied by $(1-v^2/w^2)^{1/2}$, where v is the uniform speed of a new reference frame. One can even construct an electrodynamics invariant with respect to the Voigt transformations, but it gives some wrong results.[17]

Not knowing of Voigt, Lorentz[18] in 1892–1895 started to develop his electron theory of matter. In contrast to Maxwell's phenomenological theory, the electron theory explained many macroscopic phenomena of electrodynamics by the microscopic behavior of electrons (and atoms) and their interaction with the field. Lorentz tried to show the invariance of form of the equations of electrodynamics in uniformly moving reference frames, for this would have meant agreement with experiments showing the equivalence of all such frames. By transforming fields, coordinates, and time he had the Lorentz transformations to first order in v/c, and this accounted for experiments not involving more than the first power of v/c. This meant a change in the origin of time, without a contraction of the time scale, but the physical implications were not recognized, for the change was regarded as just a mathematical convenience. To account for the Michelson–Morley experiment involving v^2/c^2, Lorentz assumed a contraction of moving bodies in their direction of motion by a factor $(1-2v^2/c^2)$ in 1892[19] or better $(1-v^2/c^2)^{-1/2}$ in 1895. In 1899[20] he even considered a change in time scale in a transformation.

[16] W. Voigt, Nachr. Königl. Ges. Wiss. Göttingen, 41 (1887); reprinted in Physik. Z. **16**, 381 (1915).

[17] A. Gluckman, Am. J. Phys. **36**, 226–231 (1968).

[18] H. Lorentz, Arch. Neerl. **25**, 363–551 (1892); *Versuch einer Theorie der elektrischen und optischen Erscheinungen in betwegten Körpern* (E. Brill, Leiden, 1895).

[19] H. Lorentz, Amst. Versl. **1**, 74–79 (1892).

[20] H. Lorentz, Amst. Proc. **1**, 427–442 (1899).

In 1900 Larmor[12] in his electron-theory of matter already used the "Lorentz" transformation equations

$$x' = \epsilon^{1/2}(x - vt),$$
$$y' = y,$$
$$z' = z,$$
$$t' = \epsilon^{-1/2}[t - v\epsilon(x - vt)/c^2],$$

where $\epsilon = \gamma^2$. The time part reduces to our familiar $t' = \epsilon^{1/2}(t - vx/c^2)$. As his electromagnetic fields were not properly transformed he obtained invariance only to order v^2/c^2. From the transformation equations he was the first one to derive the Lorentz contraction.

Further developing his electron theory, Lorentz[7] in 1904 proved the invariance of Maxwell's equations in *free space* under the transformation

$$x' = l\gamma(x - vt),$$
$$y' = ly,$$
$$z' = lz,$$
$$t' = lt/\gamma - l\gamma v(x - vt)/c^2,$$

or

$$t' = l\gamma(t - vx/c^2),$$

where l is a function of velocity which Lorentz showed to be equal to one. The invariance of Maxwell's equations including charges and currents was only approximate in Lorentz's treatment because he incorrectly transformed velocity, current, and charge density. Neither Lorentz nor Poincaré used this kinematics to derive the Lorentz contraction. The contraction of moving bodies was for them a physical effect. It remained for Einstein to show it is not physical but a kinematical appearance, and that it is mutual with respect to two frames depending only on relative motion. Lorentz explained the contraction by the assumption that molecular forces change in motion as electrostatic forces do, for the shape of a body depends on the molecular forces. Assuming also that electrons become flattened ellipsoids in motion, he derived the variation of mass with velocity. These assumptions were not satisfactory for they just covered up the difficulties but did not remove them. He did not use a principle of relativity, although he indicated the advantage of using some general and fundamental postulate.

Thus starting from Maxwell's equations, Lorentz established their invariance of form and, therefore, the invariance of the laws of electrodynamics with respect to moving reference frames, e.g., the motion of the earth.

The dependence of the electron mass on velocity was considered in theory and experiment before Lorentz, but in 1904 he obtained the famous mass formula for a particle. He made assumptions on the structure (shape, charge distribution, etc.) of the electron. Using the fundamental equations he obtained $F'=l^2(1, \gamma, \gamma)F$ and $a'=l(\gamma^3, \gamma^2, \gamma^2)a$, for the force and acceleration transformed to the instantaneous (primed) rest frame of the electron. From this follows $m'=l(1/\gamma^3, 1/\gamma, 1/\gamma)m$, where $m'=m_0$ is the rest mass, and m is the mass in the laboratory frame. Then, $m_\perp=\gamma m_0$, $m_{||}=\gamma^3 m_0$ for the masses, transverse and longitudinal with respect to the velocity which means a tensor mass as the force and acceleration are in general in different directions. Although others considered a tensor mass before Lorentz, their results were incorrect. In 1905 Einstein[10] obtained the unusual appearing relations $m_\perp=\gamma^2 m_0$, which are not incorrect but are a result of his inconvenient definition of transverse force as $Q\gamma(\mathbf{E}+\mathbf{v}\times\mathbf{B}/c)_{\perp v}$ instead of $Q(\mathbf{E}+\mathbf{v}\times\mathbf{B}/c)_{\perp v}$. But Einstein made no assumptions on the structure of the electron, and thus he freed the theory from assumptions about the theory of matter. Finally, Planck[8] in 1906 obtained the usual formula for the mass without assumptions about electron structure. He also completed dynamics in special relativity by generalizing Newton's second law and obtaining the Hamiltonian and Lagrangian equations for a particle. In 1908 Planck[8] generally defined relativistic momentum and mass–energy. Minkowski[9] in 1908 continued Planck's study of relativistic dynamics. Introducing the proper time he obtained the relativistically invariant form of Planck's new equations of dynamics.

Lorentz had also generalized[7] his mass–velocity relation for the electron to all particles: "... the proper relation between the forces and accelerations will exist ... if we suppose that the masses of all particles are influenced by a translation in the same degree as the electromagnetic[21] mass of the electrons." But, Einstein[10] in 1905 justified it by a better argument: "We remark that these results as to the mass are also valid for ponderable material points because a ponderable material point can be made into an electron by the addition of an electric charge, as small as desired."

[21] Sec. III below.

B. Poincaré

Poincaré's main contribution to the theory of relativity was his mathematical[22] work of 1905 briefly outlined in an article of 5 June 1905, and presented fully in January 1906 (dated July 1905). The 1905 article was completed just before Einstein's work[10] dated 30 June, was published in September. In this work Poincaré analyzed the transformation properties of many physical quantities, gave the complete treatment of the covariance of Maxwell's equations, introduced four-vectors and Poincaré stresses, and proved the group character of the "Lorentz transformation," introducing this name and the name "Lorentz group." Lorentz[23] had summarized and discussed Poincaré's important work of 1905 but a more detailed and modern account is given below (in Secs. II and III).

Poincaré's physical contributions to relativity included his work on the two postulates of relativity, his treatment of time, electromagnetic momentum, mass–energy, and gravitation. In 1902[24] Poincaré recognized the importance of relativity of displacement, i.e., homogeneity of space. He started to put forward the principle of relativity[25] for uniformly moving systems in 1895,[26] named it first in 1902,[24] and postulated it in 1904[27]: "... according to which the laws of physical phenomena must be the same for a fixed observer as for an observer who has a uniform motion of translation; so that we have not, and cannot have any means of discerning whether we are, or are not, involved in such a movement."

[22] (a) H. Poincaré, Compt. Rend. **140,** 1504 (1905); reprinted in *Oeuvres de H. Poincaré* (Gauthier-Villars, Paris, 1954), Vol. 9, p. 489; henceforth called *Oeuvres*; (b) H. Poincaré, Palermo Rend. **21,** 129 (1906); reprinted in *Oeuvres*, Vol. 9, p. 494.

[23] H. Lorentz, Acta Math. **38,** 293 (1921); reprinted in *Oeuvres*, Vol. 9, p. 683.

[24] H. Poincaré, *Science and Hypothesis* (Dover Publications, Inc., New York, 1902 reprint), p. 78.

[25] C. Scribner, Am. J. Phys. **32,** 672–678 (1964).

[26] H. Poincaré, L'Eclairage Electrique **5,** 5–14 (1895); reprinted in *Oeuvres*, Vol. 9, p. 412.

[27] H. Poincaré, Bull. Sci. Math. **28,** 306 (1904); transl. in Bull. Am. Math. Soc. **12,** 240ff (1906).

One must be careful in one's analysis because Poincaré's views changed[1] every few years. Although he sometimes questioned[28] the existence of the ether, he retained it.[2] His principle had not the same sense as Einstein's (although the wording could be the same as Einstein's) despite such statements as, in 1905: "It seems that this impossibility of experimentally demonstrating the absolute motion of the earth is a general law of nature; we are led to admit this law which we shall call the *postulate of relativity*, and admit it without restriction." It seems that Poincaré meant[2] by his principle that the laws in uniformly moving frames have the same form, and that one cannot detect the absolute motion because of the compensatory effects of the motion but that an absolute frame is conceptually distinguishable. This is inconsistent, for how is one to assign an absolute system? Not only was his work not fully relativistic, but a more complicated and less general formalism resulted in contrast to Einstein's (two universal postulates). Poincaré started to recognize the principle of constancy of the speed of light in 1898.[29] He did not mean this in Einstein's sense, for his speed of light did not appear as a universal constant but as a result of compensating effects[2] of motion on length and time.

Poincaré also preceded Einstein in his further treatment[29,30] of the concept of time in which he utilized the constancy of the speed of light for synchronization of clocks by observers communicating by light signals. In 1900[30] Poincaré defined the local time of Lorentz ($t'=t-vx/c^2$) as the time measured by clocks synchronized in such a way. In 1904[31] he amplified his ideas to include the motion of observers and concluded: "The clocks synchronized in this way do not indicate the true time, but they indicate what one can call *the local time*, so that one lags behind the other. This hardly matters, for we have no means of perceiving it. All the phenomena in A, for example, will be retarded, but they will be so equally, and the observer will not perceive it for his clock lags too; thus, as the principle of relativity requires, he will not have any means of knowing whether he is at rest or in absolute motion." Again Poincaré was inconsistent for he stated[32] that there is no absolute time and simultaneity of two events, yet elsewhere he distinguished the absolute system at rest with respect to the ether as evidenced by his words *true* and *local* time, instead of using quotation marks as Einstein did in 1905. The inconsistency in Poincaré's work resulted from his different standpoints[33] as philosopher, physicist and mathematician, and from his personal approach.[2] His difficulties also originated from his problems with accelerated[34] reference systems in special relativity.

Despite all inconsistencies, Poincaré must be credited with great insight[35]: "From all these results, if they will be confirmed, will start a completely new mechanics characterized by this fact that no speed could exceed that of light (for the bodies would oppose a growing inertia to the causes tending to accelerate their movement, and this inertia would become infinite when one approached the speed of light), moreover that no temperature can fall below the absolute zero. For an observer, carried along in translation, of which he is aware, no apparent speed could exceed that of light; that would be a contradiction, if one did not remember that the observer would not be using the same watches as a fixed observer, but those indicating 'local time.' " The conclusion to Poincaré's 1904 article is[36]: "Perhaps also we must construct a new mechanics, which we can only dimly see now, where the inertia growing with speed, the speed of light would become an impassable limit. The common mechanics, as more simple, will stay as a first approximation as it would be valid for speeds not too great, so that one would find again the old mechanics under the new one."

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[28] (a) H. Poincaré, Rev. Gen. Sci. **11**, 1171–3 (1900); transl. in The Monist **12**, 516 (1902); (b) Ref. 24, p. 205–222.

[29] H. Poincaré, Rev. de Metaphys. **6**, 11 (1898); reprinted in his *La Valeur de la Science* (Flammarion, Paris, 1917), pp. 54–55.

[30] H. Poincaré, Arch. Neerl. **5**, 252 (1900); reprinted in *Oeuvres*, Vol. 9, p. 483.

[31] Ref. 27, p. 311.

[32] Ref. 24, p. 90.

[33] S. Goldberg, Am. J. Phys. **35**, 934–944 (1967).

[34] Ref. 24, p. 113–114.

[35] Ref. 27, p. 316–317.

[36] Ref. 27, p. 324.

In 1900[37] Poincaré introduced[38] the momentum of the electromagnetic field, in his units as

$$\mathbf{G}=\int \mathbf{E}\times\mathbf{B}\,dV.$$

Then he generalized Newton's third law for electromagnetic phenomena by obtaining the law of conservation of momentum

$$\sum m\mathbf{v}+\int \mathbf{E}\times\mathbf{B}\,dV=\text{const.}$$

Poincaré stated[39] without proof, that the angular momentum of the field obeys a conservation equation analogous to those of energy and momentum. Thus Poincaré restored agreement of the Lorentz theory with Newton's third law, which the theory had appeared to violate. Again he was inconsistent,[33] for in his later work[40] he rejected his results on Newton's third law. It is interesting to note that his further[41] interpretation of the results lead to a connection between electromagnetic[21] mass and energy but without proof. He regarded the electromagnetic energy as a fictitious (not indestructible) fluid of mass density J/c^2 (with $J=B^2/8\pi+2\pi c^2E^2$ is his energy density of the field) which is equivalent to $E=mc^2$ for the electromagnetic case. None of the general implications were recognized by Poincaré. One of his examples[42] was: "The electromagnetic energy behaving . . . as a fluid endowed with inertia, one must conclude that if any device after having produced electromagnetic energy, sends it by radiation into a certain direction, this device will recoil as a cannon recoils when it launches a projectile."

Poincaré was the first one to study[43] Newton's law of gravitation in his special theory of relativity. In 1905[44] he used even the term "gravitational waves" for he considered the propagation of gravitational effects with the speed c. His relativity required the Lorentz invariance of the laws of gravitation and prohibited a propagation speed of gravity greater than that of light. He modified Newton's law by constructing functions of invariants of the Lorentz group and requiring agreement with the old law for $v\ll c$ (where v is the speed of the attracted body, at the time of arrival of the gravitational wave). The deviation from Newton's law contains terms of the order of v^2/c^2 only. This is barely measurable but it gives some modification[45] for the motion of the planet Mercury although not enough to account for the anomalous motion. Others later also considered the law of force instead of using the differential equations of the field and a new geometry; only Einstein finally succeeded in solving the problem.

C. Einstein

Almost simultaneously with Poincaré's main work, Einstein[10] in 1905 with superior clarity and simplicity obtained all the main previous results and some new ones by starting from two general postulates (of relativity and constancy of the speed of light). His postulates were completely general so that the concepts of ether and its "true time" disappeared. It makes no sense to speak of a medium of propagation of light in vacuum, contrary to the case of sound. No experiments could detect the ether, so why keep it? For according to E. Mach, one should omit from experimental science those concepts which cannot be verified experimentally.

". . . the special[46] theory of relativity does not depart from classical mechanics through the postulate of relativity, but through the postulate of the constancy of the speed of light in *vacuo*, from which, in combination with the special principle of relativity, there follow in the well-known way, the relativity of simultaneity, the Lorentzian transformation, and the related laws for the behavior of moving bodies and clocks." Einstein derived the Lorentz transformations

[37] Ref. 30, p. 465–7.

[38] J. J. Thomson in 1893 had already considered ". . . momentum in the field . . . ." as **E × B** per unit volume in his *Notes on Recent Researches in Electricity and Magnetism* (Oxford University Press, Oxford, England, 1893), p. 9.

[39] Ref. 30, p. 471; Abraham derived it: Physik. Z. **4**, 57–63 (1902).

[40] H. Poincaré, *Science and Method* (Dover Publications, Inc., New York, 1908), p. 225.

[41] Ref. 30, p. 467–468.

[42] Ref. 30, p. 471.

[43] Ref. 22(b), Sec. 9.

[44] Ref. 22(a), p. 492.

[45] W. de Sitter, M. N. Roy. Astron. Soc. **71**, 388 (1911).

[46] A. Einstein, Ann. Phys. **49**, 769 (1916); transl. in Ref. 7, pp. 111–164.

from the postulate of constancy of the speed of light alone, along with considerations of symmetry, linearity, and homogeneity. He proved for a spherical wave of light the covariance of $\mathbf{x}^2=c^2t^2$ in all systems, and noted that the Lorentz transformations could have been derived more simply from this fact. This invariance shows that the two fundamental principles are compatible. One should clarify the difference between an ordinary material sphere and a spherical wave of light considered from a "rest" frame S and a "moving" frame S'. The wave of light[47] is still considered spherical in S'; while the other sphere becomes an ellipsoid. Einstein "considered"[48] both cases and did not mean "seeing," for the latter act occurs at an instant of time and gives a different interpretation.[49]

Then he applied the transformations to moving rods obtaining the Lorentz contraction. He first indicated[50] that the Lorentz contraction is a reciprocal property. It is evident that to him the Lorentz contraction was an apparent kinematical consequence and not a complicated actual physical contraction as for Lorentz. Einstein's work on the contraction has this interesting view[51]: "For $v=c$ all moving bodies viewed from the 'stationary' system shrink into plain figures. For velocities greater than that of light our considerations become meaningless; we shall, however, find in the following considerations, that the velocity of light in our theory plays the role physically of an infinitely great velocity." It was seen that Poincaré realized the upper limit on speeds in 1904. Einstein also applied the Lorentz transformation to moving clocks and obtained the dilatation of time. He even considered[50] a "twin problem."

Einstein demonstrated the invariance of Maxwell's equations including distributions of electrons, as Poincaré also did, but he did not consider macroscopic matter described by the four electromagnetic field vectors in general. Therefore, his title *On the Electrodynamics of Moving Bodies* is inconsistent. It remained for Minkowski[9] in 1908 to complete the job; he transformed to a moving system the equations for stationary matter that Lorentz had obtained from Maxwell's equations by averaging. From the transformations of the electromagnetic fields Einstein derived the first transformations[52] of the wave number vector $\mathbf{k}$ and wave frequency ω. This yielded the formulas for aberration and the relativistic Doppler principle, including the important transverse shift.

He obtained important results on mass and energy: The kinetic energy of an electron or any ponderable mass point was given as

$$m_0c^2[(1-v^2/c^2)^{-1/2}-1],$$

and in his second[10] 1905 article Einstein proved that the relation $\Delta m=\Delta E/c^2$ holds approximately for an electron in a process involving a decrease in electromagnetic energy by radiation. Then he *assumed* the latter to hold true in general: "The mass of a body is a measure of its energy content . . . ." As we saw before, Poincaré's statement of $m=E/c^2$ was without proof. Further progress[53] in this direction was made by Planck[8] in 1906 and 1908; and Lorentz[11] in 1911, who *showed* that all types of energies must be included in E.

Einstein's work of 1905 was characterized by great and ingenious *simplicity*, both physically and mathematically. He started with basic tasks by first building tools for the job. He claimed[54] only to have obtained a kinematics of translation and the kinetic energy of a particle and to have considered clocks, rigid bodies, and light signals. Starting from a simple set of two assumptions he achieved a simple formalism and derived all his results, whereas Lorentz and Poincaré started by assuming electrons, ether, Maxwell's equations, and some fragmentary hypotheses. Einstein derived the Lorentz transformations, while Lorentz and Poincaré assumed them in order to get the invariance of the electromagnetic equations. Although Larmor and Lorentz obtained the transformation equations, Einstein showed their basic significance. His main new mathematical results were the formulas for the relativistic transformation of velocities, aberration and Doppler shift, and the relativistic kinetic energy of a particle. His overlooking of difficulties with accelerated

[47] J. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1962), pp. 354–355.
[48] Ref. 10(a), pp. 46, 48, and 57.
[49] Ref. 6(b), p. 40.
[50] Ref. 10(a), p. 49.
[51] Ref. 10(a), p. 48.
[52] Ref. 10(a), p. 56.
[53] Ref. 3, pp. 51–54.
[54] A. Einstein, Ann. Phys. **23**, 206–208 (1907).

frames of reference was also a simplification without which he could not have achieved a simple special relativity theory, and that was later removed in the general theory. In his first simple theory Einstein did not consider a theory of electrons, for the task was too complicated. Indeed it required several more decades to approach a solution classically. One must further note the *universality* of Einstein's special relativity. Lorentz and Poincaré based their theory on electrodynamics but Einstein's theory was based on general assumptions, which was more appropriate as he felt that Maxwell's equations were only approximate and were to be later replaced by some (quantum) theory. Although the principle of relativity is subject to a possible experimental disproof in the future, the importance of the postulational approach is that it freed relativity from electrodynamics as a basis and made special relativity more universal. There is no experimental decision between Lorentz's and Einstein's theories, but Einstein's is preferable for it is far simpler, more universal, and predicted many new results. Einstein's *audacity* was also impressive. His great passion for the physical explanation of the laws of nature resulted in his abandoning ether and absolute time, thus radically modifying long-established Newtonian space–time. Thus, he was the first true relativist.

One must neglect neither of his predecessors, who prepared his way, nor his followers. Poincaré undoubtedly approached Einstein closer than Lorentz or Larmor. There is evidence[55] of Einstein's using ideas of Poincaré, for example, the name of the principle of relativity and the method of synchronization of clocks by light signals. But, each man had different theories and relativities.[1] Lorentz had many of the mathematical results of Einstein, but his physical principles were not relativistic. Poincaré perfected Lorentz's mathematical results and extended the mathematics (and physics) of relativity.

II. POINCARE'S MATHEMATICAL ACHIEVEMENTS IN 1905

Poincaré continued Lorentz's study of Lorentz transformations and named them[56]: "Lorentz's idea can be summarized as follows: without modifying any of the apparent phenomena, if one imparts to the whole system a common translation, then the electromagnetic equations are not altered under certain transformations, which we call *Lorentz transformations*; two systems, one immobile, the other translating, become the exact image of each other." The transformations as used by Poincaré had a similar form to Lorentz's, but his units of length and time made the speed of light equal to unity so that his transformation defining parameter $\epsilon=-\beta=-v/c$ corresponded to $(-v)$ of Lorentz and his l was a function of β; l was later shown by group theory to be equal to unity. Modern notation (β, γ) is used throughout, and it is used for the field vectors **E** and **B**.

The first one to consider the group properties[57] of the Lorentz transformation was Poincaré. He showed that the Lorentz transformation forms a group named by him ("Lorentz group"). First, two successive transformations produce a new Lorentz transformation.[58] Second, the inverse transformation (written by changing $\beta\rightarrow-\beta$, $l\rightarrow 1/l$ in the basic transformation) belongs to the group if $l=1$, for it must be equivalent to the transformation obtained by changing the signs of x, x', z, z' by a rotation of 180° around y (or changing the sign of β in the basic transformation). This proves the group property (for the identity transformation $v=0$ is trivial). Lorentz demonstrated $l=1$ in another way in 1904. Before requiring $l=1$, Poincaré considered a more general Lorentz group for any l. First, characterized it by the following infinitesimal transformations[58]: (1) T_0, permutable with all others; (2) T_1, T_2, and T_3; and (3) the rotations $[T_1, T_2]$, $[T_2, T_3]$, $[T_3, T_1]$. These operators were given in terms of their infinitesimal generating transformations and written also in operator form according to Lie. Second, he also characterized his group as decomposed into $x'=lx$, $y'=ly$, $z'=lz$, $t'=lt$ plus a linear transformation leaving $x^2+y^2+z^2-t^2$ invariant. Third, he had $x'=\gamma l(x-\beta t)$, $y'=ly$, $z'=lz$, and $t'=\gamma l(t-\beta x)$ preceded and followed by a rotation. Thus, one sees that Poincaré considered the homogeneous Lorentz group plus the three-dimensional rotations as his Lorentz group, which

[55] G. Keswani, Brit. J. Phil. Sci. **15**, 286 (1965); **16**, **19**, 273 (1965).

[56] Ref. 22(b), p. 495.

[57] Ref. 22(b), Sec. 4; Einstein briefly considered it in Ref. 10(a), p. 51.

[58] Appendix.

is a six-parameter group (three velocity components plus three rotations). This does not include translation of the origins of space and time (four more parameters) so that Poincaré did not consider the inhomogeneous Lorentz group or "Poincaré group" with 10 parameters.

Poincaré obtained the correct transformations[22] for charge and current density, for Lorentz's one's were wrong. From $d\mathbf{x}/dt$, $d\mathbf{x}'/dt'$ Poincaré proved the velocity transformations.[58] Similarly, he considered the transformation of an element of volume of space.[58] It was obtained from $d\tau'\, dt' = l^4\, d\tau\, dt$ and $dt' = l\gamma(1-\beta v_x)\, dt$, using the Jacobian determinant of x', t' with respect to x, t:

$$\partial(x_\mu')/\partial(x_\nu) = l^4,$$

in our notation. From demanding invariance of charge for the electron: $\rho' d\tau' = \rho\, d\tau$, follow the charge density and current transformations.[53] Thereby, the Maxwell–Lorentz equations became rigorously invariant under Lorentz transformations for the first time. In addition, Poincaré had the correct transformations[58] for the electromagnetic scalar and vector potentials, force per unit volume $\mathbf{f} = \rho\mathbf{E} + \rho\mathbf{v}\times\mathbf{B}$, force (per unit charge) $\mathbf{F} = \mathbf{f}/\rho$. The transformation equations for the acceleration[58] are also Poincaré's. These formulas are difficult to find in the literature even up to the present. Poincaré also used the symbol

$$\Box = \Delta - \partial^2/\partial t^2$$

in relativity (which symbol was introduced by Cauchy) and showed $\Box' = \Box/l^2$.

Another important contribution was Poincaré's introduction[59] of four-vectors, anticipating Minkowski[9]:

"Let us regard $x, y, z, it; \delta x, \delta y, \delta z, i\delta t; \cdots$; as coordinates of . . . points . . . in a space of four dimensions. We see that the transformation of Lorentz is just a rotation of this space about the origin, regarded as fixed. We shall have as invariants only the . . . distances between the . . . points among themselves and the origin, or, if one prefers,

$$x^2+y^2+z^2-t^2, \qquad x\delta x+y\delta y+z\delta z-t\delta t,$$

$$\ldots \text{and} \qquad \delta x^2+\delta y^2+\delta z^2-\delta t^2 \ldots."$$

A little below he wrote: ". . . Let $T = \mathbf{f}\cdot\mathbf{v}$; we see that [the transformation equations of the force per unit volume] can be written . . . so that f_x, f_y, f_z, T have the same transformations as x, y, z, t." ". . . the Lorentz transformation acts on F_x, F_y, F_z, T_1 in the same manner as on f_x, f_y, f_z, T, with the difference that the expressions are in addition multiplied by $\rho/\rho' = \gamma_0/\gamma_0'$." "In the same way it acts on $v_x, v_y, v_z, 1$ . . . ." "Let us consider F_x, F_y, F_z, iT_1 as the coordinates of a . . . point . . . ." In a later passage one reads: ". . . the following systems of quantities:

$$\begin{array}{cccc} x & y & z & t \\ \gamma_0 F_x & \gamma_0 F_y & \gamma_0 F_z & \gamma_0 T_1 \\ \gamma_0 v_x & \gamma_0 v_y & \gamma_0 v_z & \gamma_0 \end{array}$$

experience the same linear relations when one applies the transformations of the Lorentz group."

Poincaré used the invariants of the Lorentz group as a powerful and convenient tool. Among his first invariants (taking $l=1$ for the Lorentz group) was that of electric charge $\rho' dV' = \rho dV$ and of the four-element of volume $d\tau' dt' = l^4 d\tau dt$. Further, he proved[60] the invariance of the basic functions of the field $\mathbf{E}'^2 - \mathbf{B}'^2 = (\mathbf{E}^2 - \mathbf{B}^2)/l^4$ and $\mathbf{B}'\cdot\mathbf{E}' = \mathbf{B}\cdot\mathbf{E}/l^4$. As $\mathbf{E}'^2 + \mathbf{B}'^2 = \mathbf{E}^2 + \mathbf{B}^2 - 4\boldsymbol{\beta}\cdot\mathbf{E}\times\mathbf{B}$, the field energy density is not an invariant.[61] Then followed the invariance[62] of the action integral

$$J = \int dt dV(\mathbf{E}^2 - \mathbf{B}^2)/2.$$

This is important, for from the principle of least action $\delta J = 0$ he derived all the equations of electrodynamics. Moreover, as this derivation is carried out in the same way in all Lorentz frames, the invariance of J shows the invariance of electrodynamics. In his work on gravitation[43] in special relativity, Poincaré used invariants extensively, i.e., products of four vectors x_μ, v_μ, f_μ, F_μ. He also considered the invariant space–time metric $dx_\mu{}^2 = d\mathbf{x}^2 - dt^2$. Using functions of these invariants Poincaré built his equations; a similar procedure is frequently used in modern physics.

In his calculation[63] of the field of an accelerated electron, Poincaré was the first one to make use of simplification by Lorentz transformation to the rest frame of a particle. The "velocity field" of the electron was obtained by Poincaré by the inverse Lorentz transformation of the simple

59 Ref. 22(b), pp. 542, 547.

60 Ref. 22(b), p. 520.

61 Ref. 22(b), p. 513.

62 Ref. 22(b), p. 510–511.

63 Ref. 22(b), Sec. 5.

electrostatic field in the rest frame. Choosing $\mathbf{v}=(v_x, 0, 0)$, $\mathbf{v}'=0$, and $\beta=v_x$. Then in the rest frame of the electron $\mathbf{B}'=0$, $\mathbf{E}'=Q(\mathbf{x}'-\mathbf{x}_1')/4\pi r'^3$, where $\mathbf{x}'$ is the observation point, $\mathbf{x}_1'$ is the position of the electron, and $r'=|\mathbf{x}'-\mathbf{x}_1'|$. By Lorentz transformation one replaces $(\mathbf{x}'-\mathbf{x}_1')$ by $[\mathbf{x}-\mathbf{x}_1-\mathbf{v}(t-t_1)]$ to get $\mathbf{E}=Q\gamma l^3\{\mathbf{x}-[\mathbf{x}_1+\mathbf{v}(t-t_1)]\}/4\pi r'^3$ and $\mathbf{B}=\mathbf{v}\times\mathbf{E}$ (where t is the time at the observation point, and t_1 is the time at the electron position). The magnetic field is perpendicular to the velocity and to the electric field, and the electric field is directed from $\mathbf{x}_1+\mathbf{v}(t-t_1)$, the present position of the electron, which it would reach at t, if it kept moving at the uniform velocity $\mathbf{v}$, which it had at t_1. One might expect the electric field to be directed from the position the electron had at t_1, but relativity thus shows otherwise.

The acceleration field was also reduced to the rest-frame case. Poincaré used the result that this is a radiation field at a distant point, consisting of transverse electromagnetic waves with the fields perpendicular to each other and to the radius vector from the electron. In particular, Poincaré applied this to Hertz's results for an electron executing oscillations of small displacement and velocity, but finite acceleration. To generalize the results to finite velocity, Poincaré proved the Lorentz invariance of the typical perpendicularity properties of a transverse electromagnetic wave:

$$\mathbf{E}^2-\mathbf{B}^2=0,\quad \mathbf{E}\cdot\mathbf{B}=0,\quad \mathbf{E}\cdot\mathbf{x}=0,\quad \mathbf{B}\cdot\mathbf{x}=0,$$

where $\mathbf{x}$ is the direction of propagation of the ray.

III. POINCARÉ STRESSES

In 1954 de Broglie[64] wrote: "Let us note the capital point of Poincaré's article: the discovery that an electron, as conceived by Lorentz, is not stable under the sole action of electromagnetic forces, that its stability requires the influence of another force, of unknown nature, ... This 'Poincaré stress' which can be interpreted as indicating the incomplete character of our usual conception of the electromagnetic field, has to the present hour, kept all of its importance and often there is mention of it in the most recent works on the nature of the electron." This statement may be exaggerated, but it shows first that Poincaré is perhaps best known for these stresses, which is one of his main achievements, and second, that the stresses are frequently mentioned in modern works. However, Whittaker strangely does not even mention the stresses.

In fact, these stresses and the classical electron theory are a somewhat misunderstood or ignored topic today. The classical theory of the electron has only recently approached a solution by Rohrlich.[13] Thus one must qualify Feynman's statement[14]: ". . . this tremendous edifice (the theory of electromagnetism), which is such a beautiful success in explaining so many phenomena, ultimately falls on its face . . . . You can appreciate that there is a failure of all classical physics because of quantum mechanical effects . . . . It is interesting, though, that the classical theory of electromagnetism is an unsatisfactory theory all by itself . . . . The concepts of simple charged particles and the electromagnetic field are in some way inconsistent." Pauli[65] also thought that Maxwell–Lorentz electrodynamics is not compatible with charged particles, but Rohrlich showed that a classical approach is possible. Even Einstein, in 1905, realized that a particle theory was premature at the time; therefore, he concentrated on building a special theory of relativity first.

The microscopic particle (e.g., the electron) was not considered in Maxwell's macroscopic theory for he only considered macroscopic charge distributions. Lorentz constructed a microscopic electrodynamics. He and others then tried to achieve a theory of the electron itself. Abraham in 1903 used a rigid sphere as the model of the electron. But this is really a small macroscopic body with electrostatic repulsive forces between its parts rendering it unstable, tending to "blow it apart" because missing are the intermolecular forces that hold a macroscopic sphere of matter stable. Stability is of course a quantum-mechanical property; according to classical electromagnetic theory, a collection of positive and negative charges is unstable. Abraham[66] and others calculated the mass of the electron. Incorrectly[67] using the concept of electromagnetic momentum, Abraham obtained results violating relativity for they gave the wrong relationship between the

[64] L. de Broglie, "Preface," *Oeuvres*, Vol. 9.

[65] W. Pauli, *Theory of Relativity* (Pergamon Press, Inc., New York, 1958), pp. 184–186.

[66] M. Abraham, Ann. Phys. **10**, 105–179 (1903); Physik. Z. **5**, 576–579 (1904); Ref. 3, p. 51 ff.

[67] Ref. 13, pp. 13, 133.

mass and momentum: $\mathbf{p}=4m\mathbf{v}/3$, where $m=U/c^2$ and U is the electromagnetic self-energy. The factor of $\frac{4}{3}$ was eliminated by Fermi[68] in 1922. In 1904 it became evident that the electron should contract in motion to form an ellipsoid. Further there was instability and the incorrect $\frac{4}{3}$ factor so that the model had to be changed. Hence, Abraham[66] in 1903–4 qualitatively realized the necessity of a nonelectromagnetic contribution to the structure and mass, an internal potential energy of the electron. Up to then the mass or inertia of electrons was thought to be of electromagnetic origin, i.e., the field of the electron would oppose an accelerating force because of induction. Lorentz[7] held: ". . . that there is no other, no 'true' or 'material' mass." What were the nonelectromagnetic cohesive forces required for stability? The gravitational force is too weak; For instance between two isolated electrons, it is far smaller than the electrostatic force. Langevin,[69] in a lecture in 1904, suggested unknown forces holding the electron in equilibrium. Poincaré in 1905[22] expanded this idea by postulating unknown stresses. This stabilizes the electron and can compensate for the $\frac{4}{3}$ factor.[70]

Poincaré's first[71] illustration of the reason for the Poincaré stresses was expressed in the following words: "If the inertia of the electrons is exclusively of an electromagnetic origin and if in addition they are subjected only to forces of electromagnetic origin, the condition for equilibrium requires in the interior of the electrons $\mathbf{f}=\rho(\mathbf{E}+\mathbf{v}\times\mathbf{B})=0$.[65] Now, (by the transformation equations) that is equivalent to $\mathbf{f}'=0$. The conditions for equilibrium of electrons are not changed by the transformation. Unfortunately, such a simple hypothesis is inadmissible. If, indeed, one supposes $\mathbf{v}=0$, the conditions $\mathbf{f}=0$ require $\mathbf{E}=0$, and consequently $\nabla\cdot\mathbf{E}=0$, or $\rho=0$. In the general case one arrives at analogous results. In addition to the electromagnetic forces, one must then admit other forces and 'binding' ('liaison'). One must then seek conditions to be satisfied by these forces or the 'binding', so that the equilibrium of the electrons is not troubled by the transformation . . . ."

Poincaré's discovery of the stresses originated from his examination[22] of the problem of the deformation of the electron in motion. At rest the particle was supposed to be a sphere, but in motion, a flattened ellipsoid. This contraction as considered by Lorentz and Langevin differed in some details. Poincaré showed by complicated calculations the agreement of the Lorentz model with the principle of relativity. To accomplish this he had to introduce a supplementary potential proportional to the volume of the electron[72]: "Which are these forces engendering the potential . . . ? They can be considered as a pressure acting in the interior of the electron . . . a constant internal pressure (independent of the volume); the work of such a pressure is evidently proportional to the variations in volume." At rest, the equilibrium of the electromagnetic and "Poincaré forces" resulted in a spherical electron, but in motion they required a contraction. Thus his pressure explained[73] the contraction of the electron. It is inconsistent that he sometimes thought of the pressure as internal[72] yet elsewhere[74] as external. Despite the nonelectromagnetic stresses, he assumed a purely electromagnetic nature for the electron.[72] He even strangely assumed[75] all mass of matter to be purely electromagnetic. The stresses remain unknown. His only hint in reference to their nature was[76]: ". . . the pressure creating our supplementary potential is proportional to the fourth power of the experimental mass of the electron. As the Newtonian attraction is proportional to this experimental mass, one is tempted to conclude that there is some relationship between the cause creating gravitation and that creating the supplementary potential."

The classical theory of fundamental charged particles for a *finite* electron obtained stability by means of the nonelectromagnetic Poincaré stresses and mathematical agreement with special relativity. It seemed that the problem of the finite electron was solved[77] classically. However, this theory was unsatisfactory because of the unknown nature of the Poincaré stresses and because the particle is not purely electromagnetic

[68] E. Fermi, Physik. Z. **23**, 340 (1922).

[69] P. Langevin, Rev. Gen. Sci. **16**, 257–276 (1905).

[70] Ref. 65, Sec. 63; Ref. 13, pp. 16–18; Ref. 47, pp. 591–2.

[71] Ref.22(b), p. 503.

[72] Ref.22(b), p. 537.

[73] Ref. 24(a), p. 491; Ref. 24(b), p. 524.

[74] Ref. 24(a), p. 491; Ref. 24(b), p. 496.

[75] Ref. 24(b), p. 496; Ref. 41, p. 207.

[76] Ref. 24(b), p. 538.

[77] Ref. 47, p. 593.

after the introduction of a nonelectromagnetic part. The *point* electron, as shown by Rohrlich is stable because it has no structure and as such there are no repulsive self-interactions between its parts, and there is no electrostatic self-energy. Then no nonelectromagnetic cohesive forces are required. Although a few difficulties[78] remain, Rohrlich achieved a meaningful theory. The necessity of the point electron is quantum-mechanical for the problem is outside of classical physics and one must not consider a spatial structure of the electron. Rohrlich finally showed that Poincaré stresses can be interpreted as strong interactions and are relevant to the corresponding neutral or charged particles.

IV. CONCLUSION

Many decades have passed since Poincaré's work. After the work of Einstein and after the confusing discussions of Poincare's work in recent years, for what contributions to relativity should Poincaré be remembered? Whether he did or did not anticipate Einstein, it is clear that he came closest to him. But many of his results are very relevant today for they are still valid. The relevance of Poincaré's work in relativity to contemporary physics is evident in our frequently using some of his words or ideas in relativity. His mathematical contributions, especially four-vectors, the Lorentz group and its invariants, and the transformations of many quantities are in common use. The Poincaré stresses can be important for some fundamental particles.

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APPENDIX

(A) The new parameters $l''=ll'$, $\beta''=(\beta+\beta')/(1+\beta\beta')=(1-\beta''^2)^{-1/2}$ for the transformation $(x, t)\to(x'', t'')$ are obtained from $(x, t)\to(x', t')$ with parameters l, β, γ; and $(x', t')\to(x'', t'')$ with l', β', γ'.

(B) The infinitesimal generating transformation of T_1 produces a change $\delta x=-\beta t$, $\delta y=\delta z=0$, $\delta t=-\beta x$ (obtained by letting $l=1$ and β infinitesimal). T_2 and T_3 are obtained by interchanging axes $x\to y$ or $x\to z$. T_0 produces $\delta x_\mu=x_\mu\delta l$ (obtained from $\beta=0$ and $l=1+\delta l$). Poincaré wrote the groups in operator form, e.g., $T_1=t\partial/\partial x+x\partial/\partial t$, $T_0=x_\mu\partial/\partial x_\mu$, and $[T_1, T_2]=x\partial/\partial y-y\partial/\partial x$ (using modern four-vector notation).

(C) Transformations (with primed system moving with speed β in x direction):

(1) Velocity $(\gamma_0\mathbf{v}, \gamma_0)$: $v_x'=(v_x-\beta)/(1-\beta v_x)=\gamma\gamma_0(v_x-\beta)/\gamma_0'$, $v_y'=v_y/(1-\beta v_x)\gamma=\gamma_0 v_y/\gamma_0'$, $v_z'=v_z/(1-\beta v_x)\gamma=\gamma_0 v_z/\gamma_0'$ and $\gamma_0'\gamma\gamma_0(1-\beta v_x)$, where $\gamma_0=(1-v^2)^{-1/2}$, $\gamma_0'=(1-v'^2)^{-1/2}$, $\gamma=(1-\beta^2)^{-1/2}$, and $\mathbf{v}$, $\mathbf{v}'$ are the velocities in the two frames.

(2) Volume element: $d\tau'/d\tau=l^3/(1-\beta v_x)\gamma=l^3\gamma_0/\gamma_0'$.

(3) Charge density and current $(\rho\mathbf{v}, \rho)$: $\rho'=\gamma(\rho-\beta\rho v_x)/l^3$, $\rho' v_x'=\gamma(\rho v_x-\beta\rho)/l^3$, $\rho' v_y'=\rho v_y/l^3$, and $\rho' v_z'=\rho v_z/l^3$.

(4) Potentials $(\mathbf{A}, \psi)$: $\psi'=\gamma(\psi-\beta A_x)/l$, $A_x'=\gamma(A_x-\beta\psi)/l$, $A_y'=A_y/l$, and $A_z'=A_z/l$.

(5) Force per unit volume $(\mathbf{f}, \mathbf{f}\cdot\mathbf{v})$: $f_x'=\gamma(f_x-\beta T)/l^5$, $f_y'=f_y/l^5$, $f_z'=f_z/l^5$, $T'=\gamma(T-\beta f_x)$, where $T=\mathbf{f}\cdot\mathbf{v}$.

(6) Force $(\gamma_0\mathbf{F}, \gamma_0\mathbf{F}\cdot\mathbf{v})$: $F_x'=\gamma\rho(F_x-\beta T_1)/l^5\rho'$, $F_y'=\rho F_y/l^5\rho'$, $F_z'=\rho F_z/l^5\rho'$, $T_1'=\gamma\rho(T_1-\beta F_x)/\rho'$; where $T_1=\mathbf{F}\cdot\mathbf{v}$, and $\rho/\rho'=l^3\gamma_0/\gamma_0'$ may be used.

(7) Acceleration $(\gamma_0^2\mathbf{a}+\gamma_0^4\mathbf{v}\cdot\mathbf{a}\mathbf{v}, \gamma_0^4\mathbf{v}\cdot\mathbf{a})$: $a_x'=\gamma_0^3 a_x/\gamma_0'^3$, $a_y'=\gamma_0^2 a_y/\gamma_0'^2+\gamma\gamma_0^3\beta v_y a_x/\gamma_0'^3$, $a_z'=\gamma_0^2 a_z/\gamma_0'^2+\gamma\gamma_0^3\beta v_z a_x/\gamma_0'^3$, and $P'=\gamma_0^3 P/\gamma_0'^3-\gamma\gamma_0^6\beta a_x/\gamma_0'^4$, where $P=\mathbf{v}\cdot\mathbf{a}=\mathbf{F}\cdot\mathbf{v}/\gamma_0^3$.

(D) Invariants: $x_\mu x_\mu=\mathbf{x}^2-t^2$, $x_\mu v_\mu=\gamma_0(t-\mathbf{x}\cdot\mathbf{v})$, $v_\mu^1 v_\mu^2=\gamma_1\gamma_2(1-\mathbf{v}_1\cdot\mathbf{v}_2)$, $f_\mu f_\mu=\mathbf{f}^2-T^2$, $x_\mu f_\mu=\mathbf{x}\cdot\mathbf{f}-Tt$, $F_\mu F_\mu=\gamma_0^2(\mathbf{F}^2-T_1^2)$, $x_\mu F_\mu=\gamma_0(\mathbf{x}\cdot\mathbf{F}-T_1 t)$, and $v_\mu F_\mu=\gamma_0^2(\mathbf{F}\cdot\mathbf{v}-T_1)$, where $\gamma_1=(1-v_1^2)^{-1/2}$ and $\gamma_2=(1-v_2^2)^{-1/2}$.

[78] Ref. 13, p. 260.