Painlevé's Conjecture

Florin N. Diacu

A-t-on tout à fait le droit d'établir une séparation entre les deux grands aspects de la vie de Painlevé, son côté scientifique et son côté humain? Ce n'est point certain et, devant nous, récemment, l'homme d'État qui a peut-être été le plus près de sa pensée et de son action, faisait ressortir l'unité secrète par laquelle toutes les manifestations de cette admirable nature sont solidaires les unes des autres.

Jacques Hadamard: L'oeuvre scientifique de Paul Painlevé
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This is a story about celestial mechanics and mathematics and about a question older than Bieberbach's conjecture; a question that died close to its 100th birthday but which—like any good question—left behind it many other unanswered questions as well as a universe of intellectual achievements.

The *n*-Body Problem

The roots of the n-body problem get lost somewhere in the early history of humankind, but we can easily recognize its modern birth certificate signed by Isaac Newton in his fundamental *Philosophiae Naturalis Principia Mathematica,* published for the first time in 1687. The clear formulation of the problem in terms of differential equations is based on the inverse-square law of mutual attraction between particles and can be stated in the following way: Consider n particles in the ambient space whose positions are given by the vectors $\mathbf{q}_i$, $i = 1, \cdots, n$ (with respect to a fixed frame), and let $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$ be the *configuration* of the system. Determine the motion of the n particles by finding the general solution $(\mathbf{q}, \dot{\mathbf{q}})$ of the second-order system

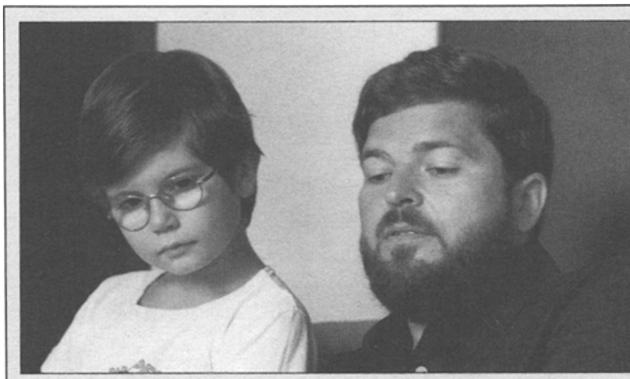
$$\ddot{\mathbf{q}} = M^{-1}\nabla U(\mathbf{q}),$$

where $U: \mathbf{R}^{3n}\setminus\boldsymbol{\Delta} \to \mathbf{R}_+$, $U(\mathbf{q}) = \Sigma m_i m_j |\mathbf{q}_i - \mathbf{q}_j|^{-1}$ is called the *potential function* (or *force function*) of the system of particles, $\boldsymbol{\Delta} = \cup\{\mathbf{q}|\mathbf{q}_i = \mathbf{q}_j\}$ is the *collision set,* and $M = \operatorname{diag}(m_1, m_1, m_1, \ldots, m_n, m_n, m_n)$ is a $3n$-dimensional diagonal matrix, $m_1, m_2, \ldots, m_n$ being the masses of the n particles. The usual formulation is that of an initial-value problem for a system of $6n$ differential equations: Solve

$$\begin{aligned} \dot{\mathbf{q}} &= M^{-1}\mathbf{p}, \\ \dot{\mathbf{p}} &= \nabla U(\mathbf{q}) \end{aligned} \tag{1}$$

subject to the initial conditions $(\mathbf{q}, \mathbf{p})(0) \in (\mathbf{R}^{3n}\setminus\boldsymbol{\Delta}) \times \mathbf{R}^{3n}$, where $\mathbf{p} = M\dot{\mathbf{q}}$ denotes the *momentum* of the system.

For $n = 2$, the problem is not difficult, and its solution can be found in any celestial mechanics or astronomy textbook under the name of the *two-body problem* or the *Kepler problem* (in honour of the famous German astronomer Johannes Kepler who actually provided Newton the inspiration for the inverse-square attraction law). Depending on the initial conditions, the motion of one particle with respect to the other can be an ellipse (including possibly a circle), a parabola, a



Florin N. Diacu (with son Răzvan)

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branch of a hyperbola, or a line. This last case, of *rectilinear* motion, is the only one when collisions between the two particles can take place. It is interesting that the complete solution as described above was not given by Newton as one would expect, but by Johann Bernoulli, and only in 1710 (see [24]).

For $n \geqslant 3$, the problem is still open even after three centuries of intense efforts to find its solution. Almost all important mathematicians up to the first quarter of this century attacked some aspect of the n-body problem, bringing important contributions to the understanding of the subject. In spite of this, the global image we have today is still far from complete.

There are several ways to approach the problem. A modern method for tackling systems of differential equations in 19th-century mathematics was to find *first integrals* and, consequently, to reduce the dimension of the system. More precisely, a function

$$F: (\mathbf{R}^{3n} \setminus \boldsymbol{\Delta}) \times \mathbf{R}^{3n} \rightarrow \mathbf{R}$$

is said to be a *first integral* for Equations (1) if $F(\mathbf{q}, \mathbf{p}) = c$ (constant), along a solution $(\mathbf{q}, \mathbf{p})$ of it. A relation like this between the components of a solution reduces the dimension of the system by 1. It is known that systems of k equations have (locally) k linearly independent first integrals, and it was an important goal to find as many integrals as possible. For Equations (1), 10 of them were easy to obtain: three integrals of the momentum, three integrals of the center of mass, three of the angular momentum, and one energy integral, namely,

$$\Sigma \mathbf{p}_i = \mathbf{a}, \qquad \Sigma m_i \mathbf{q}_i - \mathbf{a}t = \mathbf{b},$$
$$\Sigma \mathbf{q}_i \times \mathbf{p}_i = \mathbf{c}, \; T(\mathbf{p}) - U(\mathbf{q}) = h,$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are constant vectors and h is a real constant with T denoting the *kinetic energy*.

Any further attempt to find new ones was unsuccessful, and people started to look for other methods. The decisive result which stopped completely the search for first integrals was published in 1887 by Bruns. In a long paper [2] he proved the following negative statement:

THEOREM 1. *The only linearly independent integrals of Equations* (1), *algebraic with respect to* $\mathbf{q}$, $\mathbf{p}$, *and* t, *are the* 10 *described above.*

This was an important moment in the history of mathematics, which changed the way of thinking prevalent since Galilei. After a long period of quantitative methods, mathematicians understood that the class of problems solvable in this way is very small, and a large window on qualitative methods was opened. The new era was signaled by Liapunov stability criteria, obtained approximately at the same time, and also motivated by celestial mechanics.

Approximately one hundred years ago, interest in the problem reached a high level. Advised by Gustav Mittag-Leffler (at that time Editor-in-Chief of *Acta Mathematica*), King Oscar II of Sweden and Norway, a protector and supporter of science and especially of mathematics, established in 1887 an important prize for solving the 3-body problem. The formulation was very precise: *one must obtain, for any choice of the initial data, a solution expressing the coordinates as a power series, convergent for all real values of the time variable.* The idea of attacking the problem in this way is attributed to Dirichlet (see [19]). Bruns's result was at that time still too fresh to change the belief in quantitative methods. Unexpectedly, nobody could provide the desired solution. In spite of this, the prize was awarded to Henri Poincaré in 1889 for his memoir *Le problème des trois corps et les équations de la dynamique,* published in *Acta Mathematica* one year later [13]. This was in recognition of this paper's stimulating value for further research in mathematics and mechanics, and indeed this choice was a good one. Poincaré's interest was aroused by this success and he continued investigation into the mysterious n-body problem for many years. He also wrote the famous *Les nouvelles méthodes de la mécanique céleste,* in three volumes [14], where the idea of *chaos* appears for the first time.

Not only were many mathematical theories born from the study of the n-body problem but also the strength of new theories is checked today by trying to find applications of them to this old problem. It has been studied by classical analysis, differential equations, and sometimes function theory, but nowadays also by new fields like dynamical systems, differential topology, differential geometry, Morse theory, algebraic geometry, algebraic topology, symplectic manifolds, Lie groups and algebras, ergodic theory, numerical analysis and computers, operator theory, and C^*-algebras.

The Conjecture of Painlevé

In 1895, at 32 years of age, Paul Painlevé was already one of the most famous mathematicians of his time, and King Oscar II invited him to give a series of lectures at the University of Stockholm in September–November of that year. The event was considered of paramount importance and even the King attended the introductory lecture. The notes were published in 1897 in handwritten form under the title *Leçons sur la théorie analytique des équations différentielles* [11] and can be found today also in Painlevé's Complete Works [12]. The last pages contain an application of the results to the 3-body problem and an opinion of the author concerning the n-body case, formulated as a statement which was known afterwards as the *Conjecture of Painlevé.* First, let us try to understand its natural birth.

Standard results of differential equations ensure, for any $(\mathbf{q}, \mathbf{p})(0) \in (\mathbf{R}^{3n} \setminus \boldsymbol{\Delta}) \times \mathbf{R}^{3n}$, the existence and uniqueness of an analytic solution of Equations (1) defined locally on (let's say) (t^-,t^+), with 0 contained in this interval. Due to the symmetry of mechanical laws with respect to the past and future, one can study the problem on $(t^-,0]$ or on $[0,t^+)$, without loss of generality. Because many scientists have a natural desire to predict future phenomena, let us choose the second interval. We can extend the solution analytically to a maximum interval $[0,t^*)$, with $0 < t^+ \leqslant t^* \leqslant \infty$. In case $t^* = \infty$, the motion is called regular, whereas if t^* is finite, we say that the solution experiences a *singularity*. What is the physical meaning of such a singularity and is it important? One obvious possible way for a solution to encounter a singularity is for a collision to occur. Indeed, the configuration vector $\mathbf{q}$ will then so tend to the set $\mathbf{\Delta}$ that at least two position vectors have the same value, consequently ∇U tends to infinity and the equations of motion (1) become meaningless. The creation of the prize made the importance of such a study very clear. Because a series expansion of the coordinates convergent for every real value of t was asked, solutions leading to singularities were expected to be extended somehow beyond the singularity.

Although very young in 1887, Painlevé was working on his doctoral thesis and knew about the famous problem. He tried, therefore, to understand whether in the 3-body problem the only possible singularities are collisions. His worry about the occurrence of other singularities was motivated by the possible appearance of large oscillations (suspected already by Poincaré). For example, one particle could oscillate between the other two without colliding but coming closer and closer to a collision at each close encounter. Under such circumstances, one can find a subsequence t_n of times converging to a finite t^* such that $\nabla U(\mathbf{q}(t_n)) \to \infty$. This again makes Equations (1) meaningless, and such t^* is also a singularity. In modern terminology,

DEFINITION. Let $(\mathbf{q}, \mathbf{p})$ be a solution of Equations (1) defined on $[0,t^*)$ with t^* a singularity. Then t^* will be called a *collision singularity* if $\mathbf{q}(t)$ tends to a definite limit when $t \to t^*$, $t < t^*$. If the limit does not exist, then the singularity will be called a *pseudocollision* or *noncollision singularity*.

It is clear that these singularities (especially the noncollision ones) are an important obstacle to accomplishing King Oscar's goal. Indeed, one might try to extend a collision as an elastic bounce and possibly obtain a globally convergent power series, but how to do that with pseudocollisions? Painlevé doubted that pseudocollisions can actually appear and he proved for the 3-body case

THEOREM 3. *For $n = 3$, any solution of the Equations* (1) *defined on $[0,t^*)$ with t^* a finite singularity, experiences a collision when $t \to t^*$.*

Attempts to extend this result to the n-body problem ($n > 3$) failed, and the intuition of Painlevé was that pseudocollisions may, indeed, arise for more than 4 bodies. Thus, his Stockholm lectures end with the following:

CONJECTURE. *For $n \geqslant 4$, Equations* (1) *admit solutions with noncollision singularities.*

Painlevé understood that this is a very hard problem; his subsequent mathematical work contains some papers dealing with singularities, none, however, attempting to prove the conjecture. After 1905, Painlevé's scientific activity becomes less intense because of his deep involvement in politics. Paul Painlevé was elected several times as deputy, holding the War, then Finance, and finally Air portfolios, and serving as President of the Chamber of Deputies of France. In 1918, he became Président de l'Académie des Sciences, and in 1927 the University of Cambridge offered him the title of Doctor Honoris Causa. Indeed a remarkable and successful life! His famous conjecture remained open, however, for more than half a century after his death.

It is interesting to note that collision orbits are very improbable. Donald Saari proved that in the n-body problem they are of *Lebesgue measure zero* and of the *first Baire category*. Moreover, this is true for all singularities in the 4-body problem (see [15,17]). Some of these results were generalized and are expressed in terms of lower-dimensional manifolds [18]. It is also expected that, for any n, singularities are improbable. However, these results did not diminish the interest in the study of singularities.

Singularity Criteria

Many of Painlevé's contemporaries tried to find examples of solutions with pseudocollisions but no one succeeded. Their attention was, therefore, directed towards understanding theoretical aspects and especially towards criteria for obtaining noncollision singularities.

A way of finding singularities had already been found, but it is quite hard to discern when and by whom:

THEOREM 4. *Consider a solution $(\mathbf{q}, \mathbf{p})$ of Equations* (1). *Then t^* is a singularity of this solution iff*

$$\liminf_{t \to t^*} \min_{i<j} q_{ij}(t) = 0, \tag{2}$$

where $q_{ij} = |\mathbf{q}_i - \mathbf{q}_j|$.

Painlevé himself improved this result [11] in proving Theorem 3. He showed that condition (2) can actually be replaced by

$$\lim_{t \to t^*} \min_{i<j} q_{ij}(t) = 0.$$

The first important condition for the occurrence of noncollision singularities was found and published only in 1908 by a Swedish mathematician of German origin, Hugo von Zeipel [26]. His result has not only a nice formulation but also an unusual history and played a fundamental role in the story of Painlevé's conjecture.

THEOREM 5. *If t^* is a collision singularity for a solution $(\mathbf{q}, \mathbf{p})$ of Equations (1), then $J(\mathbf{q}(t))$ tends to a definite limit when $t \to t^*$, where $J(\mathbf{q}) = \Sigma m_i|\mathbf{q}_i|^2$ is the moment of inertia.*

This implies, of course, that a necessary condition for having a noncollision singularity is that the motion become unbounded in finite time, because the moment of inertia is a measure of the distribution of particles in space.

What is obvious is that at a singularity the whole $|(\mathbf{q}, \mathbf{p})|$ has to become unbounded. This always happens at a collision instant because the velocities are infinite. It is not clear what would happen in the configuration space (i.e., for the vector $\mathbf{q}$), and here lies von Zeipel's contribution. His paper appeared in a less famous journal (see [26]) and was, therefore, not well known. Personally I have tried to find it in several good university libraries in Eastern and Western Europe as well as in North America, but without success. An article of Dick McGehee [10], who spent a period in Stockholm and was interested in this subject, makes it less necessary to read the original.

The French astronomer Jean Chazy had announced Theorem 5 without making any reference to von Zeipel's paper [3]. Aurel Wintner wrote in 1941 that the proof of the Swedish mathematician has some gaps and there is no complete argument for the theorem [24]. Thirty years later, Hans Sperling gave a detailed proof [20], apparently ending the dispute. However, McGehee's paper cited above provides a translation in modern mathematical language of von Zeipel's proof, showing that it was actually correct from the beginning.

Today we know a beautiful generalization of this result which is due to Donald Saari from Northwestern University [16]. He proves that if $J \circ \mathbf{q}$ is a *slowly varying* function as $t \to t^*$ for a solution $(\mathbf{q}, \mathbf{p})$ of Equations (1), then the singularity t^* is necessarily a collision.

Theorem 5 is a fundamental contribution to the subject of singularities in the n-body problem, and the elucidation of Painlevé's conjecture would have been hard to imagine without it.

The Computer and the Idea

As has happened many times, the idea that was to solve Painlevé's conjecture came by looking for something else, and depended on electronic computers.

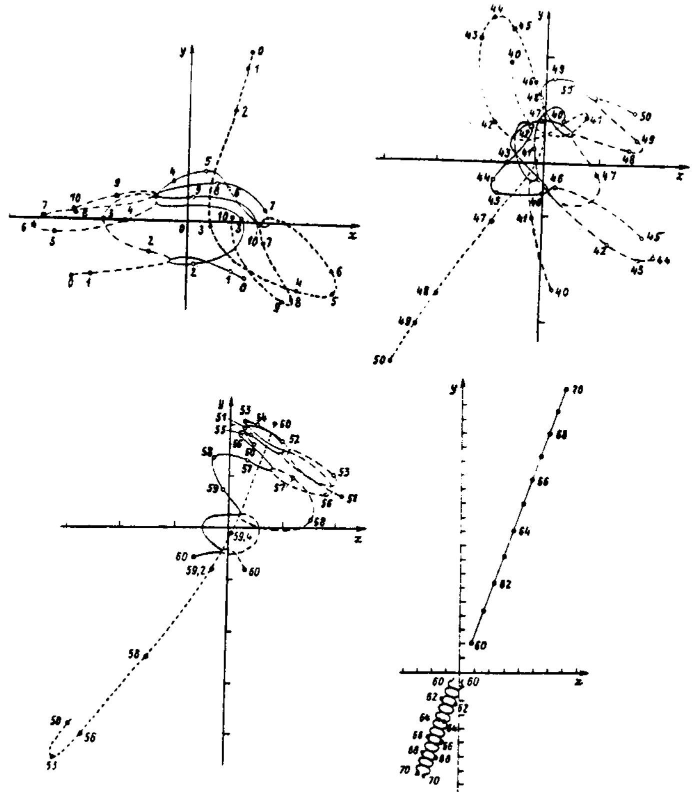

Figure 1: Numerical results in the Pythagorean problem.

In 1893, Meissel proposed the investigation of a so-called Pythagorean problem, in which three gravitationally attracting particles of masses 3, 4, and 5 are initially located at the vertices of a triangle with sides 3, 4, and 5 such that the corresponding point masses and sides are opposite. Releasing the particles with zero initial velocities from their positions, how will they move in the future? Burrau investigated the problem numerically in 1913 but without reaching important conclusions. Several computer investigations in 1966 and 1967 [21] helped to go much further by showing a surprising qualitative behavior: After passing close to a triple collision, two particles will form a binary while the other one is expelled with high velocity in the opposite direction, as in Figure 1 (see also [1]). The formation of the binary was an interesting point for astronomers, whereas the high-speed escape of the third particle attracted the attention of mathematicians. It provided the idea that it might be possible to construct an example of a noncollision singularity solution. The main reason for this qualitative feature is the triple approach of the particles, as was recognized in [8], [9], [22], [23].

We sketch crucial ideas from Dick McGehee's 1974 paper. He considered the case of the rectilinear 3-body problem, i.e., when the masses m_1, m_2, m_3 move all the time on a fixed line. He was interested in understanding the flow in a neighborhood of a triple collision solution. This was, indeed, a hard problem because previous numerical investigations suggested chaotic behavior near a *total collapse* (i.e., a simultaneous collision of all bodies). Only qualitatively speaking, the

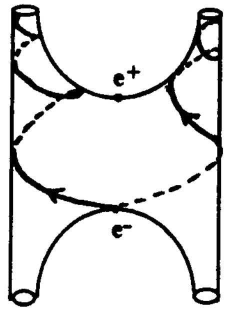

Figure 2: The flow on the collision manifold.

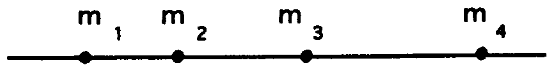

Figure 3: The example of Mather and McGehee.

particles behave by forming a binary and an escape, numerical investigations showing a highly sensitive dependence with respect to initial data. For example, for some initial conditions the particles m_1 and m_2 form a binary and m_3 escapes, whereas, perturbing the data a little bit, it may be that m_1 and m_3 form a binary and m_2 escapes. Two such solutions look very different in a phase-space picture, in spite of being close to one another at some initial moment of time.

McGehee's idea was to restrict the equations of motion to an arbitrary energy level, then to blow up the singularity (by using certain transformations which today bear his name), and finally to introduce a so-called *collision manifold* which is proved to be independent of the chosen energy level. In the rectilinear 3-body problem, the collision manifold happens to be a sphere missing four points, as in Figure 2.

Roughly speaking, the McGehee coordinates are polar coordinates for the configuration vector and a decomposition of the velocity into a radial and a tangential component, rescaled by a suitable transformation of time, which makes the collision manifold be approached asymptotically by the real flow when the new (fictitious) time variable goes to infinity.

The equations of motion restricted to the collision manifold do not describe a real physical situation. However, due to the continuity property of the solutions with respect to initial data, a study of the flow on this manifold provides valuable information on the behavior of solutions passing close to a triple collision. Many interesting theoretical results were proved in McGehee's paper using these powerful techniques, including a theorem on the occurrence of solutions with high-velocity escapes. Studies on collision singularities are hard to imagine today without McGehee's transformations.

The Example of Mather and McGehee

One of the results McGehee announced (without proof) in his 1974 paper is the construction of a solution with noncollision singularities in the rectilinear 5-body problem, using the idea of a high-speed escape. The trouble is not that collisions always appear in a rectilinear problem but that they always arise before an impending pseudocollision, as was shown by Saari [16]. I may now have confused the reader, for I said before that the solution was defined on a maximal interval $[0,t^*)$, t^* (finite) representing either a collision or a noncollision singularity. There is no inconsistency. Binary collision solutions can be analytically extended by a mathematical procedure called *regularization*. There is a vast literature on this subject (see, e.g., [4]). Physically, this means that an elastic bounce, without loss or gain of energy, takes place. I hope the sense of Saari's result is now clear.

Mather and McGehee [7] were later able to prove completely that a noncollision singularity can occur in the rectilinear 4-body problem, but only after an infinity of (regularized) binary collisions. Here is their scenario.

Four bodies of suitably chosen masses m_1, m_2, m_3, m_4 lie on a straight line at some initial moment (see Fig. 3). The initial data (positions and velocities) are such that the particles m_1 and m_2 stay close together, so we say that they form a binary system. The particle m_3 oscillates between the binary system and the particle m_4. The motion is regularized beyond the binary collisions which take place at the instants $t_1, t_2, \ldots, t_k, \ldots$. This sequence converges as k goes to infinity. Meanwhile, the binary m_1, m_2 goes to $-\infty$, m_4 goes to $+\infty$, and m_3 bounces back and forth, with increased velocity after every close passage to a triple collision. This is possible because the distance between m_1 and m_2 tends to zero, the loss of potential energy of the binary being transferred into kinetic energy for the particle m_3. The proof of Mather and McGehee is not at all easy.

Whatever its mathematical beauty and interest for dynamical systems theory, the above example is not accepted as a proof of Painlevé's conjecture because the pseudocollision appears only after (infinitely many) collisions.

Gerver's First Example

In 1984, Joe Gerver from Rutgers University proposed a solution of a planar 5-body problem in which the particles escape to infinity in finite time [5]. Although he does not give a complete proof, he provides a lot of support for the existence of such a solution. We reproduce his scenario.

Consider the planar motion of five particles $m_1, \ldots, m_5$, with $m_3 = m_4$, m_2 somewhat greater but of the same order of magnitude as m_3, m_1 much smaller than m_2, and m_5 much smaller than m_1 (see Fig. 4). Initially, m_1 is in a roughly circular orbit around m_2, whereas m_3 and m_4 are much further away. The bodies m_2, m_3, m_4 are approximately at the vertices of an obtuse triangle. Initially the triangle is slowly expanding while maintaining its shape. Meanwhile, m_5 moves rapidly around the triangle, coming close to each of the other

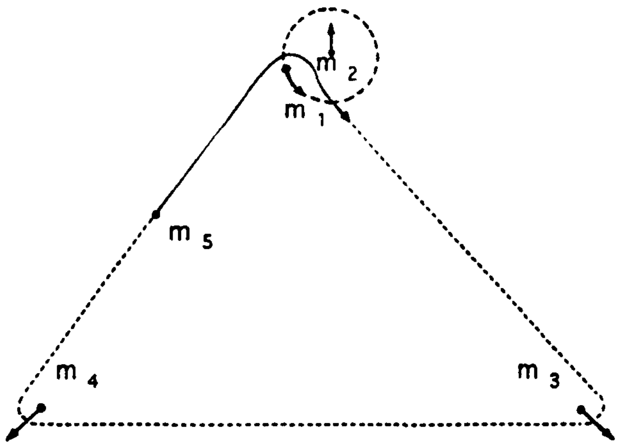

Figure 4: Gerver's heuristic example.

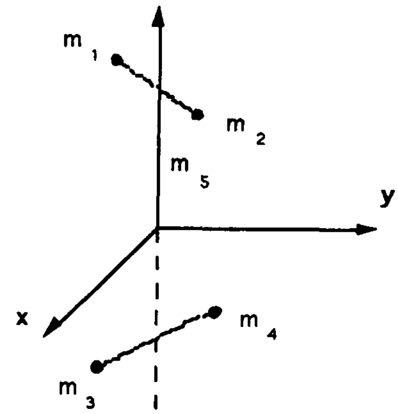

Figure 5: The example of Xia.

four bodies, the velocity of m_5 being much greater than that of m_1. Each time m_5 passes close to m_1, it picks up a small amount of kinetic energy. This causes m_1 to fall into a lower orbit around m_2 such that the mean kinetic energy of m_1 in its orbit actually increases by about the same factor as for m_5. A small fraction of the kinetic energy of m_5 is transferred to m_2, m_3, and m_4, causing faster expansion of the triangle. The time required for one trip of m_5 around the triangle decreases each time (in spite of the expansion) by a factor slightly less than 1. After a finite time, the geometric progression of the time instants $t_1, t_2, \ldots, t_k, \ldots$ measuring a round trip will converge, and m_5 will have travelled an infinite number of times around the triangle. In the meantime, the triangle has become infinitely large.

Xia's Example

In his doctoral thesis written under the supervision of Donald Saari at Northwestern University, Jeff Xia proved in 1988 that a certain type of solution in the spatial 5-body problem leads to a noncollision singularity without involving an infinite number of binary collisions, as was the case in the example of Mather and McGehee. Painlevé's conjecture was finally proved.

The author considers two pairs of bodies, the particles in the same pair having equal masses, plus a fifth particle of small mass. The bodies in a pair move in highly eccentric orbits parallel with the (x,y)-plane (see Fig. 5). The binaries are on opposite sides with respect to the (x,y)-plane and have an opposite rotation. The motion of the small particle is restricted to the z-axis, so that the total angular momentum is zero. The small particle will oscillate between the two binaries, determining an unbounded motion in finite time. More precisely, suppose the particle m_5 intersects the line connecting m_3 with m_4 from above, at a moment when these particles come near to their closest approach, the motion of m_3, m_4, and m_5 thus being close to a triple collision. The body m_5 goes a little under the line m_3m_4, whereas the particles m_3 and m_4 are at their closest approach. Thus, m_5 is strongly attracted backwards. It intersects the line m_3m_4 again when these point masses start to separate. This separation reduces the retaining force on the small particle which consequently moves very fast towards the other binary system. The action–reaction effect forces the binary m_3, m_4 to move further away from the plane (x,y). The same situation described above is now repeated (in mirror image) for the binary m_1, m_2. Iterating this procedure with higher and higher accelerations for m_5, the two binaries will be forced to tend to infinity in finite time. Simple though this scenario sounds, it is very hard to prove it is possible. For example, because the motion becomes unbounded in finite time, the acceleration effects on the small particle have to become infinitely large. The point masses in each binary must come closer and closer together, making it hard to guarantee nonoccurrence of collisions.

There were mistakes in the first attempt of Xia but he was able to correct them. The paper appeared in *Annals of Mathematics*.

His example can be extended to similar symmetric problems for any $N > 5$.

In spite of his youth (not even 30 years old in 1992), today associate professor at Georgia Tech, Xia has already brought a tremendous contribution to the field. He recently proved a new magnificent result, namely, that the very rare (and hard to detect) phenomenon called *Arnold diffusion* (a kind of chaos) takes place in a very natural system, the elliptical restricted 3-body problem. Arnold himself constructed in the 1960s a very sophisticated and artificial system to show for the first time that such a phenomenon exists. It is expected that Xia will make many other important contributions in years to come.

Gerver's Second Example

The idea of using radial symmetry, combined with the experience obtained by trying to prove his previous heuristic example, led Joe Gerver to the following solution for the planar case. Consider $3n$ bodies (n sufficiently large) in a plane as in Figure 5. $2n$ of the particles are arranged in n nearly circular orbiting pairs and all have the same mass. The center of mass of each binary lies at one of the vertices of a regular polygon.

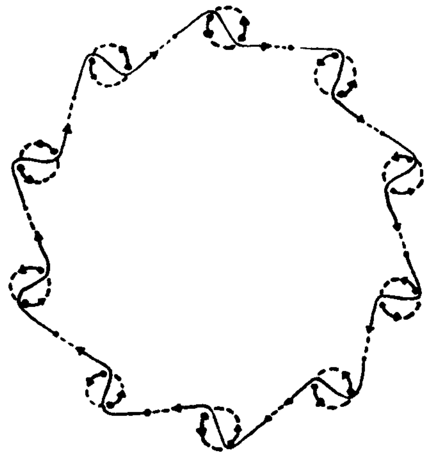

Figure 6: Gerver's planar example.

The other n bodies have small equal masses and move rapidly from one pair to the other as in Figure 6. When a small particle comes close to the binary it takes some kinetic energy from the pair and transfers some momentum to it, forcing the binary to move into a tighter orbit and concomitantly to increase its distance from the center of the polygon. Iterating this process for a suitably chosen n, suitable values of the masses, and of the initial velocities, the size of the configuration will increase by each close encounter of a small particle with a binary. The sequence of times from one encounter to the next will converge to a finite value, whereas the system becomes unbounded in finite time. The complete proof contains very many computations and is, therefore, quite hard to follow (see [6]).

Gerver found out about Painlevé's conjecture 19 years before he gave the solution. Xia succeeded in proving his example about six months before Gerver. However, Gerver's is the first confirmation of the conjecture for the case of planar solutions and is also very elementary, using mainly 19th-century mathematics. Seeing the proof, one sees that the conjecture would have been possible for Painlevé's contemporaries to prove, but nobody did it.

It was not the first time Gerver attacked a famous problem. As a graduate student at Columbia University in 1969, he proved a conjecture of Riemann on the nowhere differentiability of the function $\Sigma_{n=1}^{\infty}\sin(n^2x)/n^2$. But this was long before his work on Painlevé's conjecture started.

A comparison between the two solutions is hard to make. Each is interesting and valuable in its own way. Xia opened a new direction of work bringing fresh air into the field, whereas Gerver used the old methods showing that they can be successful too. Surely both achieved a most remarkable feat in an old and hard field where good new results are not at all easy to obtain.

Acknowledgments

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