$m = 0$ such a basis is given by the vectors $P_4^k$, $C_1^k$, and $C_2^k$.

## § 25. The Galilean group.

### 25.1. Motion of a particle.

The relationships among the conservation laws in classical mechanics are determined by the structure of the Galiliean group. We consider first the simplest example of the free motion of a material point, described by the equation

$$m\ddot{x} = 0 \ , \qquad (25.1)$$

where $m$ is the mass of the particle, $x = (x^1, x^2, x^3)$, and the dots denote differentiation with respect to $t$. The operators

$$X_\mu = \frac{\partial}{\partial x^\mu} \ , \quad X_{\mu\nu} = x^\nu \frac{\partial}{\partial x^\mu} - x^\mu \frac{\partial}{\partial x^\nu} \ ,$$
$$X_4 = \frac{\partial}{\partial t} \ , \quad Y_\mu = t \frac{\partial}{\partial x^\mu} \ , \quad \mu, \nu = 1,2,3, \qquad (25.2)$$

constitute a basis of the Lie algebra $L$ of the Galilean group admitted by Eq. (25.1).

When one shifts from the Lorentz to the Galilean group (by formally letting $c \to \infty$), the equalities (24.3) and (24.4) become

$$(L^A)^\ell (X_i) = \{X_\mu\}, \ (L^A)^\ell (Y_1) = \{X_\mu, Y_\mu\} \ , \qquad (25.3)$$

and

$$(L^A)^\ell (X_{12}) \quad \{X_\mu, X_{\mu\nu}, Y_\mu\} \qquad \ell \geq 2 \ . \qquad (25.4)$$

Therefore, algebra $L$ is generated by two elements: $X_4$ and another element, for example, $X_{12}$. Consequently, the fundamental integrals of motion of a particle in classical mechanics are the energy

$$E = \frac{m}{2}|\dot{x}|^2, \tag{25.5}$$

and the angular momentum M, and it actually suffices to take only one component,

$$M^3 = m\left(x^1\dot{x}^2 - x^2\dot{x}^1\right). \tag{25.6}$$

From (25.4) it follows that, by acting repeatedly on $M^3$ with the adjoint algebra, we can produce the momentum

$$P = m\dot{x} , \tag{25.7}$$

the angular momentum

$$M = x \times P \tag{25.8}$$

and the integral

$$Q = m(x - t\dot{x}) , \tag{25.9}$$

corresponding to the operators $Y_\mu$: since the vector M can be obtained from $M^3$ through obvious rotations, it suffices to consider P and Q. Note that by (25.3), the momentum is also related to the energy, and is obtained from it through the action of ad $Y_\mu$:

$$\text{ad } Y_\mu(E) = \left(t\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial \dot{x}^\mu}\right)\left(\frac{m}{2}|\dot{x}|^2\right) = m\dot{x}^\mu = P^\mu .$$

The same conclusion may be reached by using the Galilean transformation with generator $Y_\mu$:

$$x' = x + ta , \quad \text{where} \quad a = (a^1, a^2, a^3). \tag{25.10}$$

In fact, this transformation takes every solution of Eq. (25.1) again into a solution, and so the energy is taken into the integral

$$E' = \frac{m}{2}|\dot{x}'|^2 = \frac{m}{2}|\dot{x} + a|^2 = E + P\cdot a + \frac{m}{2}|a|^2.$$

Since the group parameter a is arbitrary, this shows that

$P$ is an integral of motion. One can similarly derive the vectors $P$ and $Q$ from the angular momentum. Indeed, the translations $x' = x + a$ take $M$ into the integral $M' = M - P \times a$, and hence, as above, $P$ is an integral of motion. To obtain the vector $Q$ from $M$, we use transformation (25.10):

$$M' = mx' \times \dot{x}' = m(x + ta) \times (\dot{x} + a) = M + Q \times a .$$

This simple example reveals the following general properties of the mechanical systems invariant under the Galilean group. In contrast to relativistic mechanics, where the angular momentum provides a basis for the conservation laws, in classical mechanics both the energy and the angular momentum are fundamental.*) The above computations also show that both Lemma 22.4 and the transformations of the conservation laws under the group admitted by the given differential equation lead to the same results. However, for the more complex systems that are considered below, application of the adjoint algebra is easier and, in addition, enables us to exhibit a basis of conservation laws.

The construction of the Lie-Bäcklund algebra for the Lagrangian equations of motion of a particle reduces (§ 17.1) to the solution of the defining equation (17.3) for the operator $X = \eta^i(t,x,v)\frac{\partial}{\partial x^i} + \ldots$, where $v = \dot{x} = D_t(x)$. Consider, for example, the two-body problem, i.e., the motion of a particle in Newton's gravitational field $U = \frac{\alpha}{r}$, $r = |x|$, $\alpha = \text{const}$. Here the defining equation has the form

$$m\left(\frac{d}{dt}\right)^2\eta^i = \frac{\alpha}{r^3}\left(\eta^i - 3\,\frac{x^i}{r^2}\sum_{k=1}^{3} x^k\eta^k\right),$$

where

*) It is of interest to note that by the time the fundamental principles of mechanics were established, Daniel Bernoulli (in a letter addressed to Euler in February 1744; published in the book of Fuss [1]) already advanced the idea that the linear and angular momenta might not be independent; he assumed that the conservation of the angular momentum could be derived from the conservation of momentum. It seems that Euler regarded these two conservation laws as independent (see Truesdell [1], p. 256).

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} + \frac{\alpha}{mr^3} x^i \frac{\partial}{\partial v^i} .$$

To the infinitesimal operators of the invariance group of point transformations there corresponds a solution of this equation that is linear in $x$ and $v$:

$$\eta^i = (3at + b)v^i + \left(c^i_k - 2a\delta^i_k\right)x^k ,$$

where $a, b, c^i_k = \text{const.}$, and $c^i_k + c^k_i = 0$. Substitution of functions $\eta^i$ of the more general form

$$\eta^i = a^i_k(t,x)v^k + b^i(t,x)$$

in the defining equation yields three additional Lie-Bäcklund operators:

$$X_k = \left(2x^k v^i - x^i v^k - (x\cdot v)\delta^i_k\right) \frac{\partial}{\partial x^i} + \ldots, \quad k = 1,2,3,$$

which leave invariant the equation of motion,

$$m\ddot{x} = \alpha \frac{x}{r^3} .$$

For these operators and the Lagrangian $L = \frac{m}{2}|v|^2 - \frac{\alpha}{r}$ we have

$$X_k(L) = m\left[2(v\cdot\dot{v})x^k - (x\cdot\dot{v})v^k - (x\cdot v)\dot{v}^k\right] - D_t\left(\alpha \frac{x^k}{r}\right).$$

On the particle's trajectories these equalities become $X_k(L) = D_t(-2\alpha \frac{x^k}{r})$, and hence Theorem 22.3 and Remark 22.2 yield the Laplace integrals *)

$$A_k = m\left(|v|^2 x^k - (x\cdot v)v^k\right) + \alpha \frac{x^k}{r} , \quad k = 1, 2, 3.$$

These are the components of the vector

$$A = v \times M + \alpha \frac{x}{r} ,$$

---

*) Laplace [1], Livre II, Ch. III n° 18, formula (P). In Quantum mechanics, the vector A is also known as the Runge-Lenz vector.

the conservation of which yields Kepler's First Law after we substitute the expression of $A$ in the scalar product $A \cdot x$.

As Theorem 22.4 shows, $A$ can be alternatively produced by letting the operators $X_k$ act on the angular momentum $M$.

In Hamiltonian mechanics, associated with every first integral of motion, $F(x,p)$, there is a one-parameter group of canonical transformations with the infinitesimal operator

$$Y = \frac{\partial F}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial F}{\partial x^i}\frac{\partial}{\partial p_i},$$

which preserves the Hamiltonian (see Goldstein [1], § 8.6). The operator corresponding to the integral $A_k$ through this formula is

$$Y_k = \left(2x^k v^i - x^i v^k - (x \cdot v)\delta^i_k\right)\frac{\partial}{\partial x^i} + \left(v^i v^k + \frac{\alpha}{mr^3} x^i x^k - (|v|^2 + \frac{\alpha}{mr})\delta^i_k\right)\frac{\partial}{\partial v^i},$$

$(v = \frac{1}{m} p)$. The Lie equation for the one-parameter group of canonical transformations with operator $Y_k$ (that is, the Hamiltonian system with Hamiltonian $A_k$) has three pairwise-commuting integrals, $A_k$, $E = \frac{1}{2m}|p|^2 + \frac{\alpha}{r}$, and $M_{ij} = x^i p_j - x^j p_i$ $(i,j \neq k)$, and thus, according to Liouville's theorem, can be integrated by quadratures. With respect to Lagrangian mechanics, $Y_k$ is not a Lie-Bäcklund operator, because it does not preserve the equality $v = D_t(x)$. However, due to the fact that $Y_k = X_k$ on the particle's trajectories, the canonical transformations generated by $Y_k$ are well defined on these trajectories, and form a group of Bäcklund transformations which leave invariant the Langrangian equation of motion.

## 25.2. Perfect gas.

Consider the equations describing the motion of a perfect polytropic gas

$$v_t + (v \cdot \nabla)v + \frac{1}{\rho}\nabla p = 0,$$

$$\rho_t + v \cdot \nabla\rho + \rho \operatorname{div} v = 0, \qquad (25.11)$$

$$p_t + v \cdot \nabla p + \gamma p \operatorname{div} v = 0, \quad \gamma = \text{const.},$$

where $t$ and $x = (x^1, \ldots, x^n)$ are the independent variables, while the differential variables are the components of velocity $v = (v^1, \ldots, v^n)$, the pressure $p$, and the density $\rho$; $n$ assumes the values 1,2, or 3 for a one-dimensional, planar, or spatial flow, respectively. System (25.11) is invariant under a vector representation of the Galilean group and a 3-parameter dilatation group. A basis of the corresponding Lie algebra is supplied by the operators

$$X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j} + v^j \frac{\partial}{\partial v^i} - v^i \frac{\partial}{\partial v^j},$$

$$X_{n+1} = \frac{\partial}{\partial t}, \quad Y_i = t \frac{\partial}{\partial x^i} + \frac{\partial}{\partial v^i}, \quad X_{n+2} = t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i},$$

$$Z_1 = 2t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} - v^i \frac{\partial}{\partial v^i} + 2\rho \frac{\partial}{\partial \rho}, \quad i,j = 1, \ldots, n, \tag{25.12}$$

and

$$X_{n+3} = \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p}. \tag{25.13}$$

This group is maximal in the case of an arbitrary adiabatic exponent $\gamma$, whereas for

$$\gamma = \frac{n+2}{n} \tag{25.14}$$

it can be enriched: to (25.12) and (25.13) one adds the operator

$$Z_2' = t^2 \frac{\partial}{\partial t} + tx^i \frac{\partial}{\partial x^i} + \left(x^i - tv^i\right) \frac{\partial}{\partial v^i} - nt\rho \frac{\partial}{\partial \rho} - (n+2)tp \frac{\partial}{\partial p} \tag{25.15}$$

(Ovsyannikov [2]). The operator $X_{n+3}$ commutes with all the others, and has no role in the construction of conservation laws; we therefore omit it from further consideration. The structure of our algebra is defined by equalities (25.3) and (25.4), and the following commutation relations (which are the only ones we shall use):

$$\mathrm{ad}\ Z_2\left(X_{n+1}\right) = - Z_1\left(\mathrm{mod}\ X_{n+3}\right),\ \mathrm{ad}\ Z_2\left(Z_1\right) = - 2Z_2. \tag{25.16}$$

In hydrodynamics, it is customary to write conservation laws (22.6) in the form

$$D_t(\tau) + \mathrm{div}\ (\tau v + \xi) = 0. \tag{25.17}$$

Let $\Omega(t)$ be an arbitrary n-dimensional domain representing a volume of fluid in motion, and denote by $S(t)$ and $\nu$ the boundary of $\Omega(t)$ and the unit outer normal to $S(t)$, respectively. By the standard procedure (i.e., integrating over the (n+1)-dimensional cylinder $\Omega \times [t_1, t_2]$ and using the Gauss-Ostrogradskii formula) we can rewrite the differential conservation law (25.17) in the integral form

$$\frac{d}{dt}\int_{\Omega(t)} \tau dx = -\int_{S(t)} \xi\cdot\nu dS \tag{25.18}$$

which has a suitable physical interpretation. The classical conservation laws of the mass, energy, momentum, and angular momentum for system (25.11) have the form

$$\frac{d}{dt}\int_{\Omega(t)} \rho dx = 0 ,$$

$$\frac{d}{dt}\int_{\Omega(t)} \left(\frac{1}{2}\rho|v|^2 + \frac{p}{\gamma-1}\right) dx = - \int_{S(t)} pv\cdot\nu dS ,$$

$$\frac{d}{dt}\int_{\Omega(t)} \rho v dx = - \int_{S(t)} p\nu dS , \tag{25.19}$$

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x \times v) dx = - \int_{S(t)} p(x \times \nu) dS.$$

In addition, there is the conservation law

$$\frac{d}{dt}\int_{\Omega(t)} \rho(tv - x) dx = - \int_{S(t)} tp\nu ds , \tag{25.20}$$

which can be written, using the conservation of mass, as the center-of-mass theorem:

$$\frac{dR}{dt} = V ,$$

where

$$R = \frac{1}{M}\int_{\Omega(t)} \rho x dx, \quad V = \frac{1}{M}\int_{\Omega(t)} \rho v dx, \quad M = \int_{\Omega(t)} \rho dx.$$

By analogy with the mechanics of material points, it is natural to expect that, in view of (25.3) and (25.4), the energy and the angular momentum constitute a basis of conservation laws, despite the fact that the conditions of Theorem 22.4 are not fulfilled in our case (Eqs. (25.11) do not have a Lagrangian). We verify this by a straightforward computation, observing first that it suffices to examine the densities of our conservation laws. We choose, for example, an operator $Y_i$, and write it in the canonical form

$$\overline{Y}_i = t\frac{\partial}{\partial x^i} + \frac{\partial}{\partial v^i} - tD_i .$$

using the formula (16.17). We have

$$\overline{Y}_i\left(\frac{1}{2}\rho|v|^2 + \frac{p}{\gamma-1}\right) = \rho v^i - D_i\left[t\left(\frac{1}{2}\rho|v|^2 + \frac{p}{\gamma-1}\right)\right] \approx \rho v^i .$$

Hence, under the action of ad $Y_i$, the energy becomes the momentum. The other transformations are effected in a similar manner. In particular,

$$\overline{Y}_i\left(\rho v^j\right) = \rho\delta_i^j - D_i\left(t\rho v^j\right) \approx \rho\delta_i^j , \tag{25.21}$$

i.e., ad $Y_i$ takes the momentum into the mass, although $[Y_i, X_j] = 0$; we remove this formal discrepancy with diagram (22.21) by considering potential gas flows, which admit a variational formulation.

Under the assumption that the adiabatic exponent $\gamma$ and the dimension $n$ are related as in (25.14), there are two additional conservation laws. They were originally discovered by applying Noether's theorem to the 2-parameter group with operators $Z_1$ and $Z_2$, admissible for our problem (Ibragimov [12]). These conservation laws can be easily derived from the energy by using Lemma 22.4, together with relation (25.16). Rewriting $Z_2$ in the canonical form:

$$\overline{Z}_2 = \left(t^2 v^i_t + t x\cdot\nabla v^i + t v^i - x^i\right) \frac{\partial}{\partial v^i} +$$

$$+ t\left(t\rho_t + x\cdot\nabla\rho + n\rho\right) \frac{\partial}{\partial\rho} + t\left(tp_t + x\cdot\nabla p + (n+2)p\right) \frac{\partial}{\partial p} ,$$

we find that

$$\overline{Z}_2\left(\frac{1}{2}\rho|v|^2 + \frac{n}{2} p\right) = \frac{1}{2}|v|^2[t^2\rho_t + \mathrm{div}(t\rho x)] +$$

$$+ \frac{n}{2}[t^2 p_t + 2tp + \mathrm{div}(tpx)] + \rho v\cdot[t^2 v_t + t(x\cdot\nabla)v + tv - x] =$$

$$= t\left(\rho|v|^2 + np\right) - \rho x\cdot v + \mathrm{div}\left[t\left(\tfrac{1}{2}\rho|v|^2 + \frac{n}{2} p\right)(x - tv) - t^2 pv\right].$$

This shows that the conservation of energy is taken into a new conservation law (25.18), with density

$$\tau_1 = t\left(\rho|v|^2 + np\right) - \rho x\cdot v . \qquad (25.22)$$

The vector $\xi$ is found from the equality

$$\tau_1 v + \xi = \overline{Z}_2\left[\left(\tfrac{1}{2}\rho|v|^2 + \left(\tfrac{n}{2} + 1\right)p\right)v\right] + D_t\left[\tfrac{t}{2}\left(\rho|v|^2 + np\right)(x - tv) - t^2 pv\right]$$

and can be shown to be equal to

$$\xi_1 = p(2tv - x).$$

Applying $\overline{Z}_2$ to the functions $\tau_1$ and $\xi_1$, we obtain another conservation law, with density

$$\tau_2 = t^2\left(\rho|v|^2 + np\right) - \rho x\cdot(2tv - x) \qquad (25.23)$$

and vector

$$\xi_2 = 2tp(tv - x).$$

Therefore, for $\gamma = (n+2)/n$, one can add the following conservation laws to (25.19) and (25.20):

$$\frac{d}{dt}\int_{\Omega(t)}\left[t\left(\rho|v|^2 + np\right) - \rho x\cdot v\right]dx = -\int_{S(t)} p(2tv - x)\cdot\nu dS, \quad (25.24)$$

and

$$\frac{d}{dt}\int_{\Omega(t)}\left[t^2\left(\rho|v|^2 + np\right) - \rho x\cdot\left(2tv - x\right)\right]dx = -\int_{S(t)} 2tp(tv-x)\cdot\nu dS. \quad (25.25)$$

To explain the conservation laws in gas dynamics in terms of Noether's theorem, we may consider a potential isentropic flow. Let $v = \nabla\Phi$ and entropy $S = \text{const}$. The equation of state $p = e^S\rho^\gamma$ becomes

$$p = c\rho^\gamma, \quad c = \text{const.}, \quad (25.26)$$

we can take $c = 1$, for example. Then the Lagrange-Cauchy integral is written in the form

$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + \frac{\gamma}{\gamma-1}\rho^{\gamma-1} = 0, \quad (25.27)$$

and system (25.11) is replaced by the following second-order equation for the potential $\Phi(t,x)$:

$$\Phi_{tt} + 2\nabla\Phi\cdot\nabla\Phi_t + \nabla\Phi\cdot\left(\nabla\Phi\cdot\nabla\right)\nabla\Phi + \left(\gamma-1\right)\left(\Phi_t + \frac{1}{2}|\nabla\Phi|^2\right)\Delta\Phi = 0. \quad (25.28)$$

This equation is given by the Lagrangian

$$L = \left(\Phi_t + \frac{1}{2}|\nabla\Phi|^2\right)^{\gamma/(\gamma-1)}$$

and inherits the group properties of the original system (25.11). For arbitrary $\gamma$, a basis of the admissible algebra is supplied by the operators

$$X_o = \frac{\partial}{\partial\Phi}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{n+1} = \frac{\partial}{\partial t}, \quad X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j},$$

$$Y_i = t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial\Phi}, \quad X_{n+2} = t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} + \Phi \frac{\partial}{\partial\Phi},$$

$$Z_1 = x^i \frac{\partial}{\partial x^i} + \frac{2\gamma+n(\gamma-1)}{\gamma+1} t \frac{\partial}{\partial t} + \frac{2-n(\gamma-1)}{\gamma+1} \Phi \frac{\partial}{\partial\Phi}; \qquad (25.29)$$

for $\gamma = (n+2)/n$, we add to these the operator

$$Z_2 = t^2 \frac{\partial}{\partial t} + tx^i \frac{\partial}{\partial x^i} + \frac{1}{2}|x|^2 \frac{\partial}{\partial\Phi}. \qquad (25.30)$$

From here, by extension to the first derivatives of $\Phi$, we can obtain the operators (25.12) and (25.15) using the Lagrange-Cauchy integral. All the operators (25.29) (except $X_{n+2}$) and (25.30) satisfy the conditions of Noether's theorem, and therefore Theorem 22.4 applies. In particular, the conservation of mass corresponds to the operator $X_o$, while the commutation relations $[X_i Y_j] = \delta_i^j X_o$ explain (25.21). To derive the conservation laws listed above from Noether's theorem, we may construct them first for Eq. (25.28), and subsequently eliminate $\Phi$ by using the equalities $\nabla\Phi = v$, (25.27), and (25.26). We can do this for all conservation laws, except for the one corresponding to $Z_1$ and having the form

$$\frac{d}{dt}\int_{\Omega(t)} \left[\frac{2\gamma+n(\gamma-1)}{\gamma+1} t\left(\frac{1}{2}\rho|v|^2 + \frac{p}{\gamma-1}\right) - \rho x\cdot v - \frac{n(\gamma-1)-2}{\gamma+1}\rho\Phi\right]dx =$$

$$= -\int_{S(t)} p\left(\frac{2\gamma+n(\gamma-1)}{\gamma+1} tv - x\right)\cdot\nu dS. \qquad (25.31)$$

From here, we can eliminate the potential only if condition (25.14) is fulfilled, and then (25.31) becomes (25.24). This means that $Z_1$ leads to a new conservation law, either for arbitrary gas flows with adiabatic exponent $\gamma = (n+2)/n$, or for potential flows of an arbitrary polytropic gas.

In connection with the conservation laws (25.24) and (25.25) the question arises of whether there is a field in which the motion of a material point has analogous integrals of motion. Since, for $n = 3$, formula (25.14) gives $\gamma = 5/3$ for the adiabatic exponent, i.e., characterizes a monoatomic gas, we actually deal with a field describing a monoatomic gas. Therefore, we first consider a Coulomb field with potential

$$U = \frac{\alpha}{|x|}, \quad \alpha = \text{const.} \tag{25.32}$$

But, in this field, the integrals of motion are the energy $E = \frac{m}{2}|\dot{x}|^2 + \frac{\alpha}{|x|}$, the angular momentum $M = m(x \times \dot{x})$, and the Laplace vector $A = \dot{x} \times M + \alpha \frac{x}{|x|}$ which is peculiar to the field (25.32) ($m$ is the mass of the particle). Comparing these integrals with formulas (25.22) and (25.23) we see that the Coulomb field is not appropriate. Thus, let us take a more general, central field with an arbitrary power potential

$$U = \alpha|x|^k . \tag{25.33}$$

One can show that a nontrivial integral of motion corresponds to the dilatation group (with some operator of the form (5.1)) only for $k = -2$, i.e., only for the potential

$$U = \frac{\alpha}{|x|^2}, \quad \alpha = \text{const.} \tag{25.34}$$

For the equation

$$m\ddot{x} = 2\alpha \frac{x}{|x|^4}, \tag{25.35}$$

describing the motion of a particle in this field, the Galilean group is represented by the operators $X_4$ and $X_{\mu\nu}$

from (25.2); the absence of $X_\mu$ and $Y_\mu$ is explained by our choice of the fixed center - the origin of coordinates. In addition, Eq. (25.35) is invariant under the 2-parameter group consisting of dilatations with operator

$$Z_1 = 2t\frac{\partial}{\partial t} + x^\mu \frac{\partial}{\partial x^\mu}, \tag{25.36}$$

and of projective transformations with operator

$$Z_2 = t^2\frac{\partial}{\partial t} + tx^\mu \frac{\partial}{\partial x^\mu}. \tag{25.37}$$

The prolongations of these operators to $v = \dot{x}$ coincide with $Z_1$ and $Z_2$ from (25.12) and (25.15), if one eliminates the variables $\rho$ and $p$. According to (25.16), to construct the integrals of motion corresponding to $Z_1$ and $Z_2$, it suffices to transform the energy

$$E = \frac{m}{2}|\dot{x}|^2 + \frac{\alpha}{|x|^2} \tag{25.38}$$

by using ad $Z_2$. We thus get

$$\overline{Z}_2 = \left(tx^\mu - t^2\dot{x}^\mu\right)\frac{\partial}{\partial x^\mu} + \left(x^\mu - t\dot{x}^\mu - t^2\ddot{x}^\mu\right)\frac{\partial}{\partial \dot{x}^\mu}, \tag{25.37'}$$

and hence

$$\overline{Z}_2(E) = m\dot{x}\cdot x - t\left(m|\dot{x}|^2 + 2\frac{\alpha}{|x|^2}\right) - t^2\dot{x}\cdot\left(m\ddot{x} - 2\alpha\frac{x}{|x|^4}\right).$$

This says that the integral

$$I_1 = 2tE - mv\cdot x. \tag{25.39}$$

corresponds to the operator $Z_1$. Letting operator (25.37') act on $I_1$, we obtain the second integral

$$I_2 = 2t^2E - mx\cdot(2tv - x), \tag{25.40}$$

which corresponds to $Z_2$. Therefore, Eq. (25.35) possesses six first integrals, given by (25.38)-(25.40) and the angular momentum $M = m(x \times v)$. The integrals (25.39) and (25.40) coincide with the densities (25.22) and (25.23), so that the field with potential (25.34) has the required properties. This potential belongs to the class of the so-called integrable power potentials. Namely, the equation of the orbit of a particle moving in a central field with potential (25.33) can be integrated by means of elementary (trigonometric) functions only in one of the following three cases (see, for example, Goldstein [1], § 3.5):

$$k = 2, -1, -2,$$

i.e., for the harmonic oscillator, for the Coulomb field (or Newton's gravitational field), and for the potential (25.34). In the last case, the solutions of the equation of motion (25.35) are found by using the six first integrals given above, with no additional quadratures.

## 25.3. Incompressible fluid.

For flows of an incompressible fluid, the Galilean relativity principle is replaced by the more general invariance under the shift to an arbitrary coordinate system in translational motion. For the equations

$$v_t + (v\cdot\nabla)v + \nabla p = 0, \quad \operatorname{div} v = 0, \tag{25.41}$$

describing a perfect, imcompressible fluid flow (here the density is set equal to one), this generalized relativity principle is characterized by the operator

$$X_f = f^i \frac{\partial}{\partial x^i} + f'^i \frac{\partial}{\partial v^i} - (x\cdot f'') \frac{\partial}{\partial p}, \tag{25.42}$$

which generalizes the $X_i$ and $Y_i$ given by (25.12); here $f = \left(f^1(t),\ldots,f^n(t)\right)$ is an arbitrary vector-function of