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SEVEN MINIATURES ON GROUP ANALYSIS

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Experience gained from solving concrete problems and reading lectures on various aspects of mathematical physics has convinced me that quite a few phenomena (diffusion, wave, etc.) can be modeled directly in group-theoretical terms and that immediate consequences such as differential equations, conservation laws, solutions to boundary value problems, and so forth can be obtained from these models. It is this idea that has inspired the major part of the present paper. Various results on group analysis are presented here in the form of relatively independent, self-contained short stories, so that the exposition is concise and the reader does not have to read the entire paper.

1. THE GALILEI GROUP AND DIFFUSION

The invariance principle with respect to the Galilei group can be used to describe heat propagation in the linear approximation and to replace the Fourier law (or the Nernst law, which describes diffusion in solutions etc.) As is shown in [1] (see also [2]), the fundamental solution can immediately be derived from the invariance principle and the heat equation itself plays only an auxiliary role. The reasoning is as follows.

First, consider the Galilei group for the case in which space is one-dimensional. This is a three-parameter group formed by the translations with respect to time t and the space variable x and by the Galilei transformations, which describe the passage to steadily moving coordinate systems. We add a dependent variable u to the independent variables t and x. It will denote the temperature, and, by definition, will be transformed in passing to a moving (with constant velocity $2a$) coordinate system according to the law $\overline{u} = ue^{-(ax+a^2t)}$. This extension of the Galilei group yields the three-parameter family of transformations, generated by the operators

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial x}, \quad Y = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}.$$

This family is not a group since the linear span of these operators is not closed with respect to the commutator. The closure of this vector space yields a four-dimensional Lie algebra with the basis

$$X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial x}, \quad T_1 = u\frac{\partial}{\partial u}, \quad Y = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}. \tag{1.1}$$

Thus, in the one-dimensional theory of heat conductivity (diffusion) the Galilei group is realized as the four-parameter group formed by the translations with respect to t and x, the Galilei transformations, and dilatations of u. In [1] this realization was named the *heat representation of the Galilei group*.

Lemma 1. *The Lie algebra with the basis (1.1) can be extended to a five-dimensional algebra by adding operators of the dilatation group (scale transformations of t and x). This extension is unique and is determined by adding the operator*

$$T_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \tag{1.2}$$

to the basis (1.1).

Proof. Let us add a dilatation operator of general form $T = \alpha t\frac{\partial}{\partial t} + \beta x\frac{\partial}{\partial x}$ to the basis (1.1) and write down the conditions that the resultant space is closed with respect to the commutator. We obtain $\alpha = 2$ and $\beta = 1$.

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Theorem 1. *Any linear second-order equation that admits the Lie algebra with basis (1.1) has the form*

$$u_t = u_{xx} + cu, \quad c = \text{const}, \tag{1.3}$$

and the only second order equation that admits the extended algebra with basis (1.1), (1.2) is the heat equation

$$u_t = u_{xx}. \tag{1.4}$$

Proof. The general linear equation that admits the operators X_0, X_1, and T_1 in (1.1) is the equation

$$Au_{xx} + Bu_{xt} + Cu_{tt} + au_x + bu_t + cu = 0$$

with constant coefficients. It remains to write out the criterion for this equation to be invariant with respect to Y, and then with respect to T_2.

Remark. Equation (1.4) also admits the group of projective transformations with the operator

$$Z = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - \frac{1}{4}\left(2t + x^2\right)u\frac{\partial}{\partial u}. \tag{1.5}$$

The generalization to the multi-dimensional case is obvious. Thus, the heat equation

$$u_t = \Delta u \tag{1.6}$$

with n space variables $x = (x^1, \ldots, x^n)$ admits the Lie algebra with the basis

$$\begin{gathered} X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, \\ T_1 = u\frac{\partial}{\partial u}, \quad T_2 = 2t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}, \quad Y_i = 2t\frac{\partial}{\partial x^i} - x^i u\frac{\partial}{\partial u}, \\ Z = t^2\frac{\partial}{\partial t} + tx^i\frac{\partial}{\partial x^i} - \frac{1}{4}\left(2nt + r^2\right)u\frac{\partial}{\partial u}, \quad i,j = 1,\ldots,n. \end{gathered} \tag{1.7}$$

In conjunction with $X = \varphi(t,x)\frac{\partial}{\partial u}$ [where $\varphi(t,x)$ is an arbitrary solution to (1.6)] this algebra forms the maximum symmetry algebra of Eq. (1.6).

Theorem 2 (see [1, 2]). *The equation*

$$u_t - \Delta u = \delta(t,x) \tag{1.8}$$

with the Dirac δ-function on the right-hand side admits the algebra with basis X_{ij}, Y_i, $T_2 - nT_1$, Z. The fundamental solution to the heat equation is an invariant solution with respect to the group generated by that algebra.

2. ON THE NEWTON-COTES POTENTIAL

Consider the motion of a particle of mass m in a central potential field. The Lagrange function is

$$L = \frac{m}{2}v^2 + U(r) \quad \left(\mathbf{v} = \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt}\right), \tag{2.1}$$

and the equation of motion is

$$m\dot{\mathbf{v}} = \frac{U'(r)}{r}\mathbf{x}. \tag{2.2}$$

Since scale transformations are important in mechanics, let us find out for what potentials $U(r)$ the group of dilatations with the operator

$$T = kt\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}, \quad k = \text{const}, \tag{2.3}$$

satisfies the Noether theorem on conservation laws [i.e., when (2.3) is a Noether symmetry].

It is convenient to represent (2.3) as the canonical Lie-Bäcklund operator [3]

$$X = \Big(x^i - ktv^i\Big)\frac{\partial}{\partial x^i} \tag{2.4}$$

and to consider the infinitesimal coordinate transformation $\tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$, where

$$\delta\mathbf{x} = (\mathbf{x} - kt\mathbf{v})a. \tag{2.5}$$

The differentiation of (2.5) with respect to t yields the velocity increment

$$\delta\mathbf{v} = \Big((1-k)\mathbf{v} - kt\dot{\mathbf{v}}\Big). \tag{2.6}$$

For small coordinate and velocity variations $\delta\mathbf{x}$ and $\delta\mathbf{v}$ the main part of increment of the Lagrange function (2.1) is equal to

$$\delta L = m\mathbf{v}\cdot\delta\mathbf{v} + \frac{U'(r)}{r}\mathbf{x}\cdot\delta\mathbf{x}. \tag{2.7}$$

As applied to this case, the familiar Noether theorem asserts that *if the increment of the Lagrange function is the total derivative*

$$\delta L = \frac{dF}{dt}, \tag{2.8}$$

then equation (2.2) has the integral

$$J = m\mathbf{v}\cdot\delta\mathbf{x} - F. \tag{2.9}$$

Theorem 3. *Condition (2.8) is valid for the Lagrange function (2.1) and for the infinitesimal transformations (2.5) and (2.6) of the dilatation group if and only if*

$$U = -\frac{\alpha}{r^2}, \quad \alpha = \text{const}. \tag{2.10}$$

Moreover, we have $k = 2$, and consequently, the equation

$$m\ddot{\mathbf{x}} = 2\alpha\frac{\mathbf{x}}{r^4} \tag{2.11}$$

admits the dilatation group with the operator

$$T = 2t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i}. \tag{2.12}$$

Equation (2.11) also admits the group of projective transformations with the operator

$$Z = t^2\frac{\partial}{\partial t} + tx^i\frac{\partial}{\partial x^i}. \tag{2.13}$$

Both operators satisfy condition (2.8), and formula (2.9) provides the corresponding integrals

$$J_1 = 2tE - m\mathbf{x}\cdot\mathbf{v}, \quad J_2 = 2t^2E - m\mathbf{x}\cdot(2t\mathbf{v} - \mathbf{x}), \tag{2.14}$$

where $E = \frac{m}{2}v^2 + \frac{\alpha}{r^2}$ is the energy.

Proof. With (2.5), (2.6), and (2.2) taken into account, formula (2.7) can be rewritten as

$$\delta L = \frac{d}{dt}\Big((1-k)m\mathbf{x}\cdot\mathbf{v} - 2ktU\Big)a + ka(rU' + 2U).$$

Thus, we should find out under which conditions the expression $rU' + 2U$ is a total derivative. This is possible if and only if its variational derivative vanishes,

$$\frac{\delta}{\delta x^i}\Big(rU' + 2U\Big) = \frac{\partial}{\partial x^i}\Big(rU' + 2U\Big) = \Big(rU' + 2U\Big)'\,\frac{x^i}{r} = 0.$$

Hence we have the second-order equation $(rU' + 2U)' = 0$ for the unknown function $U(r)$. Its solution, to within an additive constant, is the potential (2.10). The other statements of the theorem can be verified by standard computations (also see [3, Sec. 25.2]).

The central potential (2.10) possesses a number of specific properties (we ascribe this to its projective invariance), and that is why it occurs in various problems in Newtonian mechanics. Newton [4] considered the motion of a body under the action of a central force proportional to the inverse cube of the distance. Later, such motion was investigated in more detail by Cotes (Roger Cotes, 1682–1716, English astronomer and mathematician, who prepared the second edition of "Principia") in his "Harmonia Mensurarum."

There is also a remarkable connection between the Newton-Cotes potential (2.10) and a monoatomic gas. This connection is realized by the conservation laws (2.14) (see [3, Sec. 25.2]), which are also inherited by the Boltzmann equations [5].

3. THE LIE-BÄCKLUND GROUP INSTEAD OF NEWTON'S APPLE

Had not Newton lain in the garden under an apple tree and had not an apple suddenly fallen on his head, we might be still unaware of the motion of celestial bodies and about a great number phenomena, which depend upon it.

L. Euler

As far as we know, the "law of inverse squares" of Newton's gravitation theory has not yet been derived from symmetry principles in literature. In this section (which is, surely, only of methodical significance) such an attempt is made. The main point is here to state Kepler's first empirical law in group-theoretical terms.

It was shown as early as by Laplace [6] that Kepler's first law (the planets move along ellipses with the Sun in one of the focuses) is a direct corollary of the conservation law for the vector*

$$A = \mathbf{v} \times \mathbf{M} + \alpha \frac{\mathbf{x}}{r} \quad (\mathbf{M} = m\mathbf{x} \times \mathbf{v}) \tag{3.1}$$

for the equation of motion

$$m\dot{\mathbf{v}} = \alpha \frac{\mathbf{x}}{r^3} \tag{3.2}$$

in the central field with potential

$$U = -\frac{\alpha}{r}, \quad \alpha = \text{const.} \tag{3.3}$$

In addition, Eq. (3.2) admits the infinitesimal Lie-Bäcklund transformation $\tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$ with the vector-parameter $\mathbf{a} = (a^1, a^2, a^3)$, where

$$\delta\mathbf{x} = 2(\mathbf{a}\cdot\mathbf{x})\mathbf{v} - (\mathbf{a}\cdot\mathbf{v})\mathbf{x} - (\mathbf{x}\cdot\mathbf{v})\mathbf{a}. \tag{3.4}$$

The *Hermann-Bernoulli-Laplace vector* (3.1) can be obtained from the Noether theorem according to formula (2.9) (see [3, Sec. 25.2]).

The cited Lie-Bäcklund symmetry is a natural generalization of the rotational symmetry of the Kepler problem. To visualize this fact, it suffices to note that an infinitesimal transformation from the rotational group is determined by the increment $\delta\mathbf{x} = \mathbf{x} \times \mathbf{a}$ and that the expression (3.4) can be represented as the symmetrized double vector product of the vectors $\mathbf{x}$, $\mathbf{v}$, and $\mathbf{a}$,

$$\delta\mathbf{x} = (\mathbf{x} \times \mathbf{v}) \times \mathbf{a} + \mathbf{x} \times (\mathbf{v} \times \mathbf{a}). \tag{3.5}$$

The symmetry with respect to the Lie-Bäcklund group with infinitesimal increment (3.5) is just Kepler's first law expressed in group-theoretical terms.

Theorem 4. *Condition (2.8) for the applicability of the Noether theorem is valid for a central field $U(r)$ and the increment (3.5) if and only if the potential has the form (3.3).*

* According to [7], this vector appeared in literature as the integration constant for the orbit equation in 1710 in the publications by Jacob Hermann, a student of brothers Bernoulli, and by Johann Bernoulli.

Proof. The reasoning is the same as in the preceding section. In this case we have

$$\delta L = 2\Big[\frac{d}{dt}\Big((\mathbf{x}\cdot\mathbf{a})U\Big) - (\mathbf{v}\cdot\mathbf{a})(rU' + U)\Big].$$

Hence condition (2.8) takes the form

$$\frac{\delta\Phi}{\delta\mathbf{x}} \equiv \frac{\partial\Phi}{\partial\mathbf{x}} - \frac{d}{dt}\frac{\partial\Phi}{\partial\mathbf{v}} = 0,$$

where $\Phi = (\mathbf{v}\cdot\mathbf{a})(rU' + U)$, and so

$$\frac{\delta\Phi}{\delta\mathbf{x}} = (rU' + U)'(\mathbf{v}\cdot\mathbf{a})\frac{\mathbf{x}}{r} - \frac{d}{dt}\Big(rU' + U\Big)\mathbf{a} = \frac{1}{r}\Big(rU' + U\Big)'\mathbf{v}\times(\mathbf{x}\times\mathbf{a}) = 0.$$

It follows that $(rU' + U)' = 0$, which implies potential (3.3) to within an unessential additive constant.

4. IS THE PARALLAX OF MERCURY'S PERIHELION CONSISTENT WITH THE HUYGENS PRINCIPLE?

In the concluding "Common scholium" of his "Principia," Newton wrote: "The gravity to the Sun is the gravity to its isolated particles and it is reduced with the distance to the Sun. This reduction is proportional to the square of the distance even up to the Saturn orbit, which follows from the fact that planets' aphelions are at rest, and even up to the farthest comets' aphelions provided that these aphelions are at rest. But so far I have not been able to derive the reason for these gravity properties from the phenomena, and I do not devise any hypotheses... It is enough to know that gravity really exists, acts according to the laws stated, and is sufficient to explain all motions of celestial bodies and of the sea."

About 150 years later it was however found that the planets' aphelions (or perihelions) are not at rest, but are slowly moving. For example, the observed parallax of Mercury is the only about 43″ per century. However, small effects are sometimes of fundamental significance for the theory, all the more so if we deal with description of real phenomena. Specifically, the anomaly in the motion of Mercury has not been given a satisfactory explanation on the basis of Newton's gravitation law, despite the efforts of greatest scientists. The explanation given by Einstein in 1915 was the first experimental justification of the general relativity theory. All these facts are well known, and a wonderful critical survey can be found in [8, Chap. 8, Sec. 6]. Here we consider a point that has not apparently been taken into account so far.

Einstein's explanation of the parallax of Mercury's perihelion is based on the assumption that the space near the Sun is not plane but has the Schwartzschild metric (1916), evaluated by Einstein to within the second approximation. But in passing from the plane Minkowski space-time to the Schwartzschild metric the Huygens principle is violated. This principle implies the existence of back front for sonic, light, and other waves that carry localized perturbations. This is because the Minkowski space belongs to the family of Riemann spaces with nontrivial conformal group, whereas the Schwartzschild space has a trivial conformal group and therefore does not satisfy the Huygens principle (for details see [9, Chap. 4] or [3, Chap. 2]).

Thus, in connection with the theoretical explanation of the observed anomaly in the planets' motion a new problem arises. It can be stated as the following alternative.

1. The explanation by passing to the Schwartzshild metric is adequate to the phenomenon in the approximation required. Then the Huygens principle is not valid, and hence sonic, light, and other signals undergo distortions. And we should estimate the distortion level from the view point of possible observation.

2. The Huygens principle holds in the real world. Then we should explain the parallax of Mercury's perihelion without any contradiction with this principle. This task requires thorough physical analysis of the equations of motion for a particle in a distorted space-time with nontrivial conformal group. The problem is facilitated by the fact that the family of such spaces can be described completely. Namely, any space-time with nontrivial conformal group can be defined in an appropriate coordinate system by the metric [3, Sec. 8.5]

$$ds^2 = e^{\mu(x)}[(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - 2f(x^1 - x^0)dx^2\,dx^3 - g(x^1 - x^0)(dx^3)^2], \tag{4.1}$$

where f and g are functions only of $x^1 - x^0$, such that $g - f^2 > 0$, and the function μ can depend on all variables x.

5. INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS THAT CONTAIN A SMALL PARAMETER AND ADMIT AN APPROXIMATE GROUP

The theory of approximate groups [10] can be used to classify and integrate differential equations with a small parameter. Let us consider a model example that illustrates the method of integrating ordinary differential equations in the framework of the regular perturbation theory for symmetry groups.

The second-order equation

$$y'' - x - \varepsilon y^2 = 0 \tag{5.1}$$

with small parameter ε does not admit a group of point transformations in the usual sense and hence cannot be integrated by the Lie method. However, it admits the following approximate symmetry operators [here we do not care whether there are any other approximate symmetry operators for equation (5.1)]:

$$\begin{aligned} X_1 &= \frac{2}{3}\varepsilon x^3 \frac{\partial}{\partial x} + \left[1 + \varepsilon\left(yx^2 + \frac{11}{60}x^5\right)\right]\frac{\partial}{\partial y}, \\ X_2 &= \frac{\varepsilon}{6}x^4 \frac{\partial}{\partial x} + \left[x + \varepsilon\left(\frac{1}{3}yx^3 + \frac{7}{180}x^6\right)\right]\frac{\partial}{\partial y}. \end{aligned} \tag{5.2}$$

These operators form a two-dimensional Abelian Lie algebra, and a two-parameter approximate group corresponds to them [10].

The method of successive order reduction (see [2, Chap. 2]) provides the following technique to integrate Eq. (5.1) approximately by means of the approximate symmetry operators (5.2). In what follows all equations should be interpreted as approximate equations to within $o(\varepsilon)$.

The change of variables

$$t = y - \varepsilon\left(\frac{1}{2}x^2y^2 + \frac{11}{60}yx^5\right), \quad u = x - \frac{2}{3}\varepsilon yx^3 \tag{5.3}$$

transforms X_1 into the translation operator $X_1 = \frac{\partial}{\partial t}$. The transformed equation (5.1) for the function $u(t)$ reads

$$u'' + u(u')^3 + \varepsilon\left[3u^2u' + \frac{1}{6}(u^2u')^2 - \frac{11}{60}(u^2u')^3\right] = 0$$

and does not depend upon t. Therefore, it can be integrated by the standard substitution $u' = p(u)$, which yields

$$p' + up^2 + \varepsilon\left(3u^2 + \frac{1}{6}u^4p - \frac{11}{60}u^6p^2\right) = 0. \tag{5.4}$$

Let us now integrate Eq. (5.4) by using the second operator in (5.2). To this end we rewrite X_2 in terms of the variables (5.3), extend the resultant operator to the variable $p = u'$, and consider its action in the space of the variables (u, p). As a result, we obtain the following approximate symmetry operator for equation (5.4):

$$\widetilde{X}_2 = \frac{\varepsilon}{2}u^4\frac{\partial}{\partial u} + \left[p^2 + \varepsilon\left(2u^3p - \frac{13}{15}u^5p^2\right)\right]\frac{\partial}{\partial p}. \tag{5.5}$$

The change of variables

$$z = u + \varepsilon\frac{u^4}{2p}, \quad q = -\frac{1}{p} + \varepsilon\left(\frac{u^3}{p^2} - \frac{13}{15}\frac{u^5}{p}\right) \tag{5.6}$$

takes the operator (5.5) into the translation operator $\widetilde{X}_2 = \frac{\partial}{\partial q}$ and Eq. (5.4) into the explicitly integrable form

$$q'(z) + z + \frac{11}{60}\varepsilon z^6 = 0.$$

Hence we have

$$q = -\frac{1}{2}z^2 - \frac{11}{420}\varepsilon z^7 + C_1, \quad C_1 = \text{const.} \tag{5.7}$$

If we substitute the expressions (5.6) for z and q into (5.7), and then resolve the obtained equation as $p = f(u)$, a single quadrature yields the solution $t = \int f(u)du$. Then the solution of the original equation (5.1) can be found via the change (5.3).

6. SPECIFIC FEATURES OF GROUP MODELING IN THE DE SITTER WORLD

Two astronomers who live in the de Sitter world and have different de Sitter clocks might have an interesting conversation concerning the real or imaginary nature of some world events.

F. Klein [11]

Here we sketch a future more detailed work on some new approaches and effects that are possible if the curvature of our universe is small but nonzero. We discuss only the following two features of the transition from the Minkowski geometry to the de Sitter world (space-time of constant curvature).

The first feature is the possibility to interpret the de Sitter group in terms of the theory of approximate groups and to consider it as the perturbation of the Poincaré group by introducing a small constant curvature. Indeed, according to the cosmological data, the curvature of our universe is so small (about $10^{-54}\mathrm{cm}^{-2}$) that it suffices to calculate only the first-order perturbations. The resultant Lie equations can easily be solved and provide an approximate representation of the de Sitter group [12]. This permits us to simplify the formulas of the exact theory dramatically.

The second feature is connected with the use of a special complex transformation, which, in the case of the Minkowski space, is a very simple conformal transformation, admitted by the Dirac equation with zero mass and is not significant there. But if the curvature is not zero, then the transformation becomes a nontrivial equivalence transformation on the collection of spaces with constant curvature. The addition of this transformation to the de Sitter group results in an interesting combination of the three possible types of spaces of constant curvature: elliptic, hyperbolic (Lobachevskii spaces), and parabolic (Minkowski spaces considered as the limit case in which the curvature is zero).

The metric of the de Sitter space is

$$d\tau^2 = \left(1 + \frac{K}{4}\varrho^2\right)^{-2} (c^2dt^2 - dx^2 - dy^2 - dz^2), \tag{6.1}$$

where

$$\varrho^2 = r^2 - c^2t^2, \quad r^2 = x^2 + y^2 + z^2, \quad K = \text{const.} \tag{6.2}$$

With the standard notation $(x^1, x^2, x^3, x^4) = (x, y, z, ict)$, $ds = id\tau$ we have

$$ds^2 = \left(1 + \frac{K}{4}\sigma^2\right)^{-2} \sum_{\mu=1}^{4} (dx^\mu)^2, \quad \sigma^2 = \sum_{\mu=1}^{4} (x^\mu)^2. \tag{6.3}$$

The motions of the metric (6.3) form the de Sitter group, which differs from the Poincaré group in that simple translations of the coordinates x^μ are substituted by more complicated transformations known as "generalized translations". For example, generalized translation with respect to the coordinate x^1 is generated by the operator

$$X_1 = \left(1 + \frac{K}{4}[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]\right)\frac{\partial}{\partial x^1} + \frac{K}{2}x^1\left(x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} + x^4\frac{\partial}{\partial x^4}\right), \tag{6.4}$$

and the translation itself has the form

$$\begin{aligned} \tilde{x}^1 &= 2\frac{x^1\cos(a\sqrt{K}\,) + (1 - \frac{K}{4}\sigma^2)\frac{1}{\sqrt{K}}\sin(a\sqrt{K}\,)}{1 + \frac{K}{4}\sigma^2 + (1 - \frac{K}{4}\sigma^2)\cos(a\sqrt{K}\,) - x^1\sqrt{K}\sin(a\sqrt{K}\,)}, \\ \tilde{x}^l &= 2\frac{x^l}{1 + \frac{K}{4}\sigma^2 + (1 - \frac{K}{4}\sigma^2)\cos(a\sqrt{K}\,) - x^1\sqrt{K}\sin(a\sqrt{K}\,)}, \quad l = 2, 3, 4, \end{aligned} \tag{6.5}$$

where a is the group parameter.

If the constant curvature K is small, one can use the theory of approximate groups [10]. The approximate Lie equation for the operator (6.4) can be solved easily. As a result, we obtain the following simple approximate representation of the generalized translation (6.5) (see the detailed calculation in [12]):

$$\tilde{x}^1 = x^1 + a + \frac{K}{4}\left\{[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]a + x^1a^2 + \frac{1}{3}a^3\right\} + o(K),$$
$$\tilde{x}^l = x^l + \frac{K}{4}x^l(2ax^1 + a^2) + o(K), \quad l = 2, 3, 4. \tag{6.6}$$

The free motion of a particle in the de Sitter world is described by the Lagrange function

$$L = -mc^2\theta^{-1}\sqrt{1-\beta^2}, \tag{6.7}$$

where $\theta = \left[1 + \frac{K}{4}(r^2 - c^2t^2)\right]$, $\beta^2 = \frac{v^2}{c^2}$; m and $\mathbf{v}$ are the mass and the velocity of the particle; we write $\mathbf{x} = (x^1, x^2, x^3)$, so that $\mathbf{v} = \frac{d\mathbf{x}}{dt}$. Starting from this known formula, we shall find the Lagrange statement of Kepler's problem in the de Sitter world. The invariance of the classical Kepler problem with respect to the rotations and time translations the known limit value of the potential (3.3) for $K = 0$ and $\beta^2 \to 0$, and formula (6.7) will serve as heuristics in our consideration. Hence we seek the Lagrange function in the form

$$L = -mc^2\theta^{-1}\sqrt{1-\beta^2} + \frac{\alpha}{r}\theta^m(1-\beta^2)^n; \quad m, n = \text{const}. \tag{6.8}$$

The action integral $\int L dt$ is invariant with respect to rotations. Therefore, the invariance with respect to the generalized time translations with the operator

$$X_4 = \frac{1}{c^2}\left[1 - \frac{K}{4}(c^2t^2 + r^2)\right]\frac{\partial}{\partial t} - \frac{K}{2}t\sum_{l=1}^{3} x^l \frac{\partial}{\partial x^l} \tag{6.9}$$

[which is obtained from (6.4) by replacing x^1 with x^4] will be the only additional condition.

Theorem 5. *The action integral $\int L dt$ with the Lagrange function (6.8) is invariant with respect to the group with the operator (6.9) if and only if $m = 0$ and $n = \frac{1}{2}$, i.e., when*

$$L = -mc^2\theta^{-1}\sqrt{1-\beta^2} + \frac{\alpha}{r}\sqrt{1-\beta^2}. \tag{6.10}$$

Proof. The required invariance condition is [3]

$$[\widetilde{X}_4 + D_t(\xi)]L = 0, \tag{6.11}$$

where $\widetilde{X}_4$ is the extension of operator (6.9) to $\mathbf{v}$, and ξ is the coordinate of this operator at $\frac{\partial}{\partial t}$. The calculation gives

$$[\widetilde{X}_4 + D_t(\xi)]L = \frac{\alpha K}{2r}\theta^m(1-\beta^2)^n\left[(1-2n)\frac{\mathbf{x}\cdot\mathbf{v}}{c^2} + mt\right]. \tag{6.12}$$

Therefore, the statement of the theorem follows from (6.11).

Remark. A similar theorem on the uniqueness of an invariant Lagrangian is not valid in the Minkowski space. Indeed, as follows from (6.12), for $K = 0$ condition (6.11) identically holds for the Lagrangian (6.8) with any n. Thus, the theorem proved is an effect characteristic of nonzero curvature.

The spinor analysis in a curvilinear space has been developed from various points of view by many authors. A good exposition of its techniques and a general review of literature on the topic are given in [13]. According [13, Sec. 2], the Dirac equation in the metric (6.3) can be rewritten in the form

$$\left(1 + \frac{K}{4}\sigma^2\right)\gamma^\mu\frac{\partial\psi}{\partial x^\mu} - \frac{3}{4}K(x\cdot\gamma)\psi + m\psi = 0, \quad m = \text{const}, \tag{6.13}$$

where γ^μ $(\mu = 1, \ldots, 4)$ are the usual four-row Dirac matrices in the Minkowski space and $(x\cdot\gamma)$ denotes the four-dimensional inner product $(x\cdot\gamma) = \sum_{\mu=1}^{4} x^\mu\gamma^\mu$. Here we are interested only in the equation for neutrino $(m = 0)$ in the linear approximation with respect to K:

$$\gamma^\mu\frac{\partial\psi}{\partial x^\mu} - \frac{3}{4}K(x\cdot\gamma)\psi = 0. \tag{6.14}$$

Equation (6.13) admits the de Sitter group, whose action on the wave function ψ is defined as follows. Let us write out the infinitesimal transformation of the de Sitter group as $\delta x = a\xi$. Then $\delta\psi = aS\psi$, where

$$S = \frac{1}{8}\sum_{\mu,\nu=1}^{4} \frac{\partial \xi^\mu}{\partial x^\nu}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu - 3\delta_{\mu\nu}) + \frac{3}{4}K\left(1 + \frac{K}{4}\sigma^2\right)^{-1}(x\cdot\xi). \tag{6.15}$$

Let us now return to Eq. (6.14). It can be reduced (in the linear approximation with respect to K) to the common Dirac equation

$$\gamma^\mu \frac{\partial \varphi}{\partial x^\mu} = 0 \tag{6.16}$$

by the change

$$\varphi = \psi(x)\exp\left[\frac{1}{2}\left(1 + \frac{K}{4}\sigma^2\right)^{-3}\right]. \tag{6.17}$$

Equation (6.16) is invariant with respect to the transform

$$\bar{x}^\mu = ix^\mu, \quad i = \sqrt{-1}. \tag{6.18}$$

Therefore, instead of (6.17), we choose the representation of φ as

$$\varphi = \chi(\bar{x})\exp\left[\frac{1}{2}\left(1 + \frac{K}{4}\sigma^2\right)^{-3}\right]. \tag{6.17'}$$

From (6.16) we obtain the equation (the bar over $\bar{x}$ is omitted)

$$\gamma^\mu \frac{\partial \chi}{\partial x^\mu} + \frac{3}{4}K(x\cdot\gamma)\chi = 0, \tag{6.19}$$

which coincides with Eq. (6.14) in the de Sitter space with curvature of the opposite sign.

In the Minkowski space the transformation (6.18) takes timelike intervals into spacelike ones and *vice versa*. The same is true of the de Sitter space, with the simultaneous change of sign of the space curvature. Actually, assigning the subscript K to the interval (6.3) and using (6.18), we have

$$ds^2_{(K)} = -d\bar{s}^2_{(-K)}. \tag{6.20}$$

Formulas (6.17) and (6.17′) can be interpreted as the "splitting" of a neutrino into two neutrinos, that are described by equations (6.14) and (6.19) and differ only if $K \neq 0$. System of equations (6.14), (6.19) admits the approximate transformation group whose infinitesimal transformations are defined by the increments

$$\delta x = a\xi, \quad \delta\psi = aS\psi, \quad \delta\chi = aT\chi, \tag{6.21}$$

where the vector $\xi = (\xi^1, \ldots, \xi^4)$ belongs to the 15-dimensional Lie algebra of the group of conformal transformations of the Minkowski space (e.g., see [3]) and the matrices S and T are given by the formulas [see (6.15)]

$$\begin{aligned} S &= \frac{1}{8}\sum_{\mu,\nu=1}^{4} \frac{\partial \xi^\mu}{\partial x^\nu}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu - 3\delta_{\mu\nu}) + \frac{3}{4}K(x\cdot\xi), \\ T &= \frac{1}{8}\sum_{\mu,\nu=1}^{4} \frac{\partial \xi^\mu}{\partial x^\nu}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu - 3\delta_{\mu\nu}) - \frac{3}{4}K(x\cdot\xi). \end{aligned} \tag{6.22}$$

7. THE TWO-DIMENSIONAL ZABOLOTSKAYA-KHOKHLOV EQUATION COINCIDES WITH THE LIN-REISSNER-TSIEN EQUATION

The group admitted by the equation

$$\varphi_x\varphi_{xx} + 2\varphi_{tx} - \varphi_{yy} = 0, \tag{7.1}$$

which describes a nonsteady potential flow of a gas with transonic velocities [15], has been calculated in [14]. The Lie algebra of this group is infinite-dimensional and contains five arbitrary functions of the variable t.

On the other hand, in [16] the infinite-dimensional symmetry algebra of the equation

$$\frac{\partial^2}{\partial q_1^2}\left(\frac{u^2}{2}\right) - \frac{\partial^2 u}{\partial q_1 \partial q_2} + \frac{\partial^2 u}{\partial q_3^2} = 0, \tag{7.2}$$

which is the two-dimensional version of the Zabolotskaya-Khokhlov equation [17] known in nonlinear acoustics, has been found in [16]. This algebra contains three arbitrary functions of the variable q_2. The comparison of the two algebras suggests the possibility of a nonpoint (since the dimensions of the symmetry algebras are different) correspondence between equations (7.1) and (7.2). In fact, these equations can easily be identified by introducing a potential. Namely, if we set $x = -q_1$, $t = -2q_2$, and $y = q_3$ in Eq. (7.1), differentiate it with respect to q_1, and denote

$$u = \frac{\partial \varphi}{\partial q_1}, \tag{7.3}$$

then we obtain Eq. (7.2). The passage from the symmetry algebra of Eq. (7.1) to the corresponding algebra for Eq. (7.2) can easily be accomplished by applying the formulas from [3, Sec. 19.4] to the differential substitution (7.3).

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