Sophus Lie and Harmony in Mathematical Physics, on the 150th Anniversary of His Birth

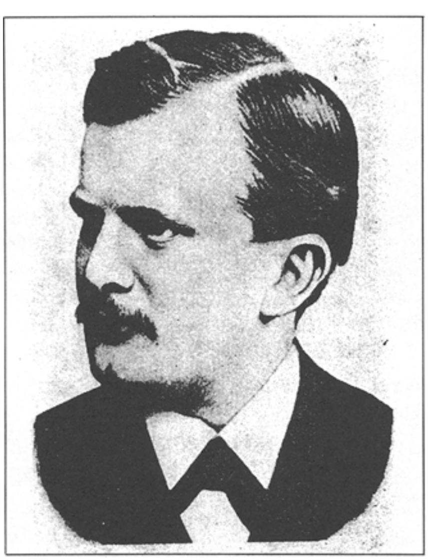
Nail H. Ibragimov

"The extraordinary significance of Lie's work for the general development of geometry can not be overstated; I am convinced that in years to come it will grow still greater" — so wrote Felix Klein [13] in his nomination of the results of Sophus Lie on the group-theoretic foundations of geometry to receive the N. I. Lobachevskii prize. This prize was established by the Physical-Mathematical Society of the Imperial University of Kazan in 1895 and was to recognize works on geometry, especially non-Euclidean geometry, chosen by leading specialists. The first three prizes awarded were to the following:

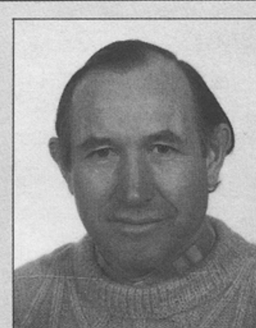
1897: S. Lie	(Nominator: F. Klein)
1900: W. Killing	(Nominator: F. Engel)
1904: D. Hilbert	(Nominator: H. Poincaré).

There can be no doubt that the work of Lie in differential equations merits equally high evaluation. One of Lie's striking achievements in this domain was the discovery that the majority of the known methods of integration, which until then had seemed artificial and not intrinsically related to one another, could be introduced all together by means of group theory. Further, Lie gave a classification of ordinary differential equations of arbitrary order in terms of the admitted group, thereby identifying the full set of equations which could be integrated or reduced to lower-order equations by group-theoretic considerations. But these and a rich store of other results of his did not lend themselves to popular expositions and remained for a long time the special preserve of a few. Today we find that this is the case with methods of solution of the problems of mathematical physics: Many of them have a group-theoretic nature yet are taught as though they were the result of a lucky guess.

It was my good fortune to get interested in application of groups to differential equations at the very beginning of my university work, and to write my first paper under the direction of Professor L. V. Ovsiannikov, who has done so much to awaken interest in this discipline and establish it as a contemporary scientific field. In my later



Friedrich Engel



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work I saw over and over how effective a tool Lie theory is for solving complicated problems. It significantly widens and sharpens the intuitive notion of symmetry, supplies concrete methods to apply it, guides one to the proper formulation of problems, and often discloses possible approaches to solving them.

This article presents my view of the role of Lie group theory in mathematical physics, drawing on parts of some of my lectures over the years at Moscow University and Moscow Institute of Physics and Technology.

His Life Story

Marius Sophus Lie was born 17 December 1842 in the town of Nordfjordeid, Norway, the sixth and youngest child of the Lutheran pastor Johann Herman Lie. He studied in Christiania (now Oslo) from 1857, first in gymnasium and then (1859–1865) at the University. Among the events of Lie's life which set his creative course, these stand out: his independent study in 1868 of the geometric works of Chasles, Poncelet, and Plücker; his travels in Germany and France in 1869–1870; his contacts there with Felix Klein, Chasles, Jordan, and Darboux; and his close friendship with Klein, leading to a long collaboration. Lie worked at the University of Christiania from 1872 to 1886, then from 1886 to 1898 at Leipzig. He died 18 February 1899 in Christiania.

The life and intellectual development and works of the greatest Norwegian mathematician are described in reminiscences of his colleagues and later biographies (see, for example, [7, 22, 27, 29], and references therein). I call special attention to the painstaking introduction of F. Engel to Lie's Collected Works [21]. These give detailed insight into the essence of Lie's ideas and a picture of him as a person.

Symmetry of Differential Equations

The notion of differential equations really has two components. For an ordinary first-order differential equation, for example, it is necessary

1. to specify a surface $F(x, y, y') = 0$ in the space of the three variables x, y, y'; we will call this surface the *skeleton* of the differential equation;
2. to define the class of solutions; for example, a smooth solution is a continuously differentiable function $\varphi(x)$ such that the curve

   $$y = \varphi(x), \qquad y' = \frac{\partial \varphi(x)}{\partial x}$$

   lies on the surface, i.e.,

   $$F\left(x, \varphi(x), \frac{\partial \varphi(x)}{\partial x}\right) = 0$$

   identically in x; going over to discontinuous or generalized solutions (keeping the same skeleton) changes the situation altogether.

A decisive move in integrating differential equations is simplifying the skeleton by means of a suitable change of variables. For this purpose, one uses the *symmetry group* of the differential equation (or its *admissible group*),

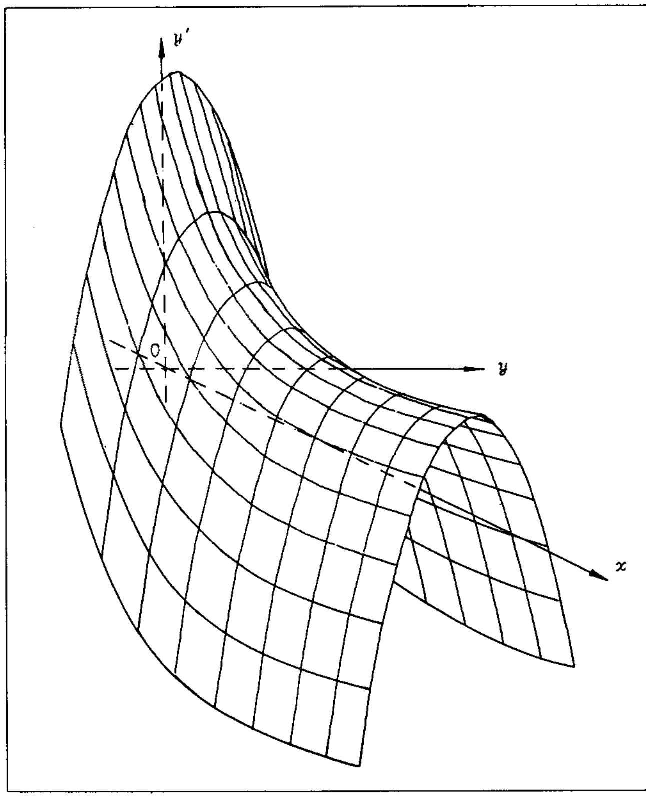


Figure 1. The skeleton of the Riccati equation $y' + y^2 - 2/x^2 = 0$ is a surface invariant under the group of inhomogeneous deformations $\bar{x} = xe^a, \bar{y} = ye^{-a}, \bar{y}' = y'e^{-2a}$.

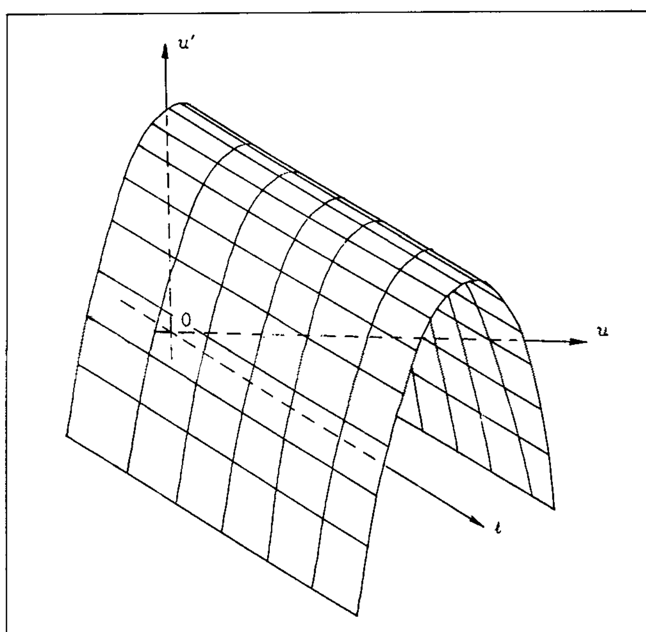


Figure 2. The skeleton of the equation $u' + u^2 - u - 2 = 0$, obtained from the Riccati equation $y' + y^2 - 2/x^2 = 0$ by the change of variables $t = \ln x, u = xy$.

Table 1. Lie's group classification of second-order equations.

Group	Basis of the Lie Algebra	Equation
G_1	$X_1 = \frac{\partial}{\partial x}$	$y'' = f(y, y')$
G_2	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}$	$y'' = f(y')$
	$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$y'' = \frac{1}{x} f(y')$
G_3	$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial x}, \quad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$	$y'' + 2\left(\frac{y' + Cy'^{3/2} + y'^2}{x-y}\right) = 0$
	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$	$y'' = Cy^{-3}$
	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}$	$y'' = Ce^{-y'}$
	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}$	$y'' = Cy'^{(k-2)(k-1)}, \quad k \neq 0, \frac{1}{2}, 1, 2$
G_8	$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = x\frac{\partial}{\partial y}, \quad X_4 = x\frac{\partial}{\partial x}, \quad X_5 = y\frac{\partial}{\partial x}$, $X_6 = y\frac{\partial}{\partial y}, \quad X_7 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$	$y'' = 0$

defined as the group of transformations of the (x, y)-plane whose extensions to the derivatives $y', \ldots$ leave the equation's skeleton invariant.

EXAMPLE. The Riccati equation $y' + y^2 - 2/x^2 = 0$ admits the group of transformations $\overline{x} = xe^a, \overline{y} = ye^{-a}$, for the equation's skeleton (Fig. 1) is invariant under the inhomogeneous stretching $\overline{x} = xe^a, \overline{y} = ye^{-a}, \overline{y}' = y'e^{-2a}$ which is obtained by extending the transformations of the group to the first derivative y'. The substitution $t = \ln x$, $u = xy$ leads to the differential equation $u' + u^2 - u - 2 = 0$. Thus, it straightens out the skeleton of the original Riccati equation, taking it to a parabolic cylinder (Fig. 2); concomitantly, the stretchings are replaced by a group of translations $\overline{t} = t + a$, $\overline{u} = u$, and $\overline{u}' = u'$.

Group Classification

> In a short communication to the Scientific Society of Göttingen (3 December 1874), I gave, among other things, a listing of all continuous transformation groups in two variables x, y, and specially emphasized that this might be made the basis of a classification and rational integration theory of all differential equations $f(x, y, y', \ldots, y^{(m)}) = 0$ admitting a continuous group of transformations. The great program sketched there I have subsequently carried out in detail. (S. Lie [16], p. 187)

This and the next two sections give some of the main results of implementing the program, as it applies to ordinary second-order differential equations. The restriction to second order is motivated not by anything essential about the method but by a desire to concentrate on concrete cases and give brief but definitive statements.

For second-order equations the group classification [16] looks especially simple. The second-order classification result is stated briefly and explicitly in [18], §3, and appears here as Table 1. Remember that Lie carried out his classification in the complex domain, using complex substitutions and complex bases of algebras as needed. For example, the equation

$$y'' = C(1 + y'^2)^{3/2} e^{q \arctan y'}, \quad C, q = \text{const},$$

admits a 3-dimensional Lie algebra with basis

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = (qx+y)\frac{\partial}{\partial x} + (qy-x)\frac{\partial}{\partial y}.$$

No real substitution takes this to any of the equations of Table 1. But it is transformed to the equation

$$\overline{y}'' = C\overline{y} \cdot \frac{k-2}{k-1}, \qquad k = \frac{q+1}{q-1}$$

by the complex substitution $\overline{x} = \frac{1}{2}(y - ix), \overline{y} = \frac{1}{2}(y + ix)$.

Algorithm of Integration

> I noticed that the majority of ordinary differential equations which were integrable by the old methods were left invariant under certain transformations, and that these integration methods consisted in using that property. Once I had thus represented many old integration methods from a common viewpoint, I set myself the natural problem: to develop a general theory of integration for all ordinary differential equations admitting finite or infinitesimal transformations. (S. Lie [17], p. iv)

If a second-order equation admits a Lie algebra of dimensionality $r \geq 2$, then it can be integrated by a group-theoretic quadrature method. This can be done in various ways, one of which is given in Table 3. It is based on the simple fact that in the complex case any Lie algebra of dimensionality $r > 2$ has a distinguished 2-dimensional subalgebra. But the structure of a 2-dimensional Lie algebra with basis

$$X_\alpha = \xi_\alpha(x, y)\frac{\partial}{\partial x} + \eta_\alpha(x, y)\frac{\partial}{\partial y}, \quad \alpha = 1, 2,$$

Table 2. Canonical form of 2-dimensional Lie algebras and invariant second-order equations.

Type	L_2 structure		Basis of L_2 in Canonical Variables		Equation
I	$[X_1, X_2] = 0,$	$X_1 \vee X_2 \neq 0$	$X_1 = \frac{\partial}{\partial x},$	$X_2 = \frac{\partial}{\partial y}$	$y'' = f(y')$
II	$[X_1, X_2] = 0,$	$X_1 \vee X_2 = 0$	$X_1 = \frac{\partial}{\partial y},$	$X_2 = x\,\frac{\partial}{\partial y}$	$y'' = f(x)$
III	$[X_1, X_2] = X_1,$	$X_1 \vee X_2 \neq 0$	$X_1 = \frac{\partial}{\partial y},$	$X_2 = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$	$y'' = \frac{1}{x} f(y')$
IV	$[X_1, X_2] = X_1,$	$X_1 \vee X_2 = 0$	$X_1 = \frac{\partial}{\partial y},$	$X_2 = y\,\frac{\partial}{\partial y}$	$y'' = f(x)y'$

can be described simply in terms of the commutator

$$[X_1, X_2] = X_1X_2 - X_2X_1$$

and the pseudo-scalar product

$$X_1 \vee X_2 = \xi_1\eta_2 - \eta_1\xi_2.$$

This description is given in Table 2; for details, see [4, 6, 11, 12, 19, 23, 32].

EXAMPLE. Let us apply the group algorithm to the equation

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}.$$

First step: finding the admissible algebra. This is done by use of the so-called *determining equation*. It turns out, as a consequence of standard and straightforward computations, that our equation admits the Lie algebra L_2 with basis

$$X_1 = x^2\,\frac{\partial}{\partial x} + xy\,\frac{\partial}{\partial y}, \qquad X_2 = -x\,\frac{\partial}{\partial x} - \frac{y}{2}\,\frac{\partial}{\partial y}.$$

From Table 3 we see that we can pass at once to the third step.

Third step: finding the type of the algebra L_2. We have

$$[X_1, X_2] = X_1, \qquad X_1 \vee X_2 = \tfrac{1}{2}x^2y \neq 0.$$

Consequently, the algebra L_2 belongs to type III of Table 2.

Fourth step: finding the integrating change of variables. From the equations

$$X_1(t) = 0, \qquad X_1(u) = 1,$$

we find the substitution

$$t = \frac{y}{x}, \qquad u = -\frac{1}{x},$$

taking the 1-parameter group generated by the operator X_1 (group of projective transformations) to the group of translations in u. After this substitution, the basis of L_2 takes the form

$$X_1 = \frac{\partial}{\partial u}, \qquad X_2 = \frac{t}{2}\,\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial u}$$

and coincides (up to the inessential coefficient $\frac{1}{2}$ in X_2) with the canonical basis for type III in Table 2. Here we exclude solutions of the form $y = Cx$. This substitution has put the equation into the integrable form

$$\frac{u''}{u'^2} + \frac{1}{t^2} = 0.$$

Solving it gives

$$u = -\frac{t^2}{2} + C \qquad \text{and} \qquad u = \frac{t}{C_1} + \frac{1}{C_1^2}\ln|C_1t - 1| + C_2.$$

Table 3. Algorithm for integrating a second-order equation using a 2-dimensional Lie algebra.

Step	Operation	Result
1	Compute the admitted Lie algebra L_r.	A basis of L_r: $X_1, \ldots, X_r$
2	If $r = 2$, go to the next step; if $r > 2$, then distinguish any 2-dimensional subalgebra L_2 of L_r. (If $r = 1$, the order of the equation may be lowered; if $r = 0$, the group method is not applicable.)	A basis of L_2: X_1, X_2
3	Determine the type of the algebra L_2 obtained by Table 2. For this one computes the commutator $[X_1, X_2]$ of X_1 and X_2 and their pseudo-scalar product $X_1 \vee X_2$; if $[X_1, X_2]$ is neither 0 nor X_1, then choose a new basis X_1', X_2', such that $[X_1', X_2'] = X_1'$.	Reduction of structure to a canonical form from Table 2
4	Bring the basis of L_2 into agreement with Table 2 by going over to canonical variables x, y. Rewrite the equation in canonical variables and integrate it.	Finding the integrating change of variables
5	Rewrite the solution in terms of the original variables.	Solution of the equation

Fifth step: finding the solution in the given variables. Now replace t, u in the foregoing formulas by their expressions, and recall the excluded special solutions $y = Cx$. We obtain the general solution of the given second-order differential equation in the form

$$y = Cx, \qquad y = \pm\sqrt{2x + Cx^2},$$

$$C_1 y + C_2 x + x \ln\left|C_1 \frac{y}{x} - 1\right| + C_1^2 = 0.$$

Linearization

In the study of ordinary differential equations it is useful to have simple tests for linearizability. Summing up Lie's results on this question, we can state the following theorem ([16], Part III, §1; see also [11, 12]).

THEOREM 1. *The following are equivalent:*

(i) *the second-order ordinary differential equation*

$$y'' = f(x, y, y') \tag{1}$$

can be linearized by change of variables;

(ii) *Equation* (1) *has the form* (2):

$$y'' + F_3(x,y)y'^3 + F_2(x,y)y'^2 + F_1(x,y)y' + F(x,y) = 0$$

with coefficients F_3, F_2, F_1, F satisfying the compatibility conditions of the auxiliary system

$$\begin{aligned} \frac{\partial z}{\partial x} &= z^2 - Fw - F_1 z + \frac{\partial F}{\partial y} + FF_2, \\ \frac{\partial z}{\partial y} &= -zw + FF_3 - \frac{1}{3}\frac{\partial F_1}{\partial x} + \frac{2}{3}\frac{\partial F_1}{\partial y}, \\ \frac{\partial w}{\partial x} &= zw - FF_3 - \frac{1}{3}\frac{\partial F_1}{\partial y} + \frac{2}{3}\frac{\partial F_1}{\partial x}, \\ \frac{\partial w}{\partial y} &= -w^2 + F_2 w + F_3 z + \frac{\partial F_3}{\partial x} - F_1 F_3; \end{aligned} \tag{3}$$

(iii) *Equation* (1) *admits an 8-dimensional Lie algebra;*

(iv) *Equation* (1) *admits a 2-dimensional Lie algebra with basis X_1, X_2 such that*

$$X_1 \vee X_2 = 0. \tag{4}$$

EXAMPLE 1. The equation $y'' = e^{-y'}$ from Table 1 is not linearizable, for it does not have the form (2) in (ii).

EXAMPLE 2. Suppose in Eq. (2) that $F_1 = F_2 = F_3 = 0$. Then Equations (3) take the form

$$\begin{aligned} z_x &= z^2 - Fw + F_y, \qquad & z_y &= -zw \\ w_x &= zw, & w_y &= -w^2, \end{aligned}$$

and the compatibility condition $z_{xy} = z_{yx}$ gives $F_{yy} = 0$. Consequently, the equation $y'' + F(x,y) = 0$ having an $F(x,y)$ not already linear in y cannot be linearized.

EXAMPLE 3. Let us see when the equation $y'' = f(y')$ in Table 1 can be linearized. According to (ii) of Theorem 1, it is required for linearizability that $f(y')$ be a polynomial of at most third degree, i.e., that the equation be of the form

$$y'' + A_3 y'^3 + A_2 y'^2 + A_1 y' + A_0 = 0 \tag{5}$$

with constant coefficients A_i. When one writes out the auxiliary system (3) for Eq. (5), one easily sees it is compatible. Consequently, Eq. (5) is linearizable for arbitrary coefficients A_i.

EXAMPLE 4. Now consider this equation from Table 1:

$$y'' = \frac{1}{x} f(y').$$

Linearizability requires it to be of the form (2), i.e.,

$$y'' + \frac{1}{x}(A_3 y'^3 + A_2 y'^2 + A_1 y' + A_0) = 0 \tag{6}$$

with constant coefficients A_i. The compatibility conditions of its auxiliary system (3) come out to

$$A_2(2 - A_1) + 9A_0A_3 = 0, \qquad 3A_3(1 + A_1) - A_2^2 = 0.$$

Setting $A_3 = -a$, $A_2 = -b$, this gives

$$A_1 = -\left(1 + \frac{b^2}{3a}\right), \qquad A_0 = -\left(\frac{b}{3a} + \frac{b^3}{27a^2}\right).$$

Hence, Eq. (6) can be linearized if and only if it is of the form

$$y'' = \frac{1}{x}\left[ay'^3 + by'^2 + \left(1 + \frac{b^2}{3a}\right)y' + \frac{b}{3a} + \frac{b^3}{27a^2}\right]. \tag{7}$$

It is convenient to find the linearizing substitution from assertion (iv). Let us do this for Eq. (7) in the case $a = 1$, $b = 0$:

$$y'' = \frac{1}{x}(y' + y'^3). \tag{8}$$

This equation admits the algebra L_2 with basis

$$X_1 = \frac{1}{x}\frac{\partial}{\partial x}, \qquad X_2 = \frac{y}{x}\frac{\partial}{\partial x}, \tag{9}$$

satisfying condition (4). This algebra L_2 belongs to type II of Table 2, and the linearizing substitution is obtained by going over to the canonical variables $\overline{x} = y$, $\overline{y} = x^2/2$, relative to which Eq. (9) take the form $X_1 = \partial/\partial\overline{y}$, $X_2 = \overline{x}(\partial/\partial\overline{y})$. Aside from the particular solutions $y = \text{const}$, this transforms Eq. (8) into $\overline{y}'' + 1 = 0$.

Invariant Solutions

Special types of exact solutions, now widely known as *invariant solutions,* have long been used to advantage on concrete problems. They have grown familiar in mathematics, mechanics, and physics even before there was any group theory, acquiring the status of folklore. Lie [20] elucidated their group-theoretical meaning and studied the possibility of integrating partial differential equations when the group is sufficiently rich (see [20], Chaps. III and IV).

Subsequently, group theory made it possible to clarify, sharpen, and extend many intuitive ideas, and incorporate the method of invariant solutions as an essential component of modern group analysis. It was exactly by the notion of invariant solution that group theory was able to transfer its area of application from ordinary differential equations to the problems of mathematical physics, especially thanks to the works [1, 5, 24, 26, 31].

EXAMPLE. Consider the equation

$$y'' = y^{-3}$$

from Table 1, admitting a 3-parameter group. Its solution,

$$y = \sqrt{1+x^2},$$

is invariant under a 1-parameter group, whose generator is

$$X_1 + X_3 = (1+x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.$$

Subjecting this invariant solution to the transformations of the 3-parameter admissible group gives

$$y = [C_1x^2 + 2\sqrt{C_1C_2 - 1}\,x + C_2]^{1/2},$$

which is the general solution. This means every solution of this equation is invariant under some 1-parameter subgroup of the admissible 3-parameter group (details in [12]).

The Invariance Principle in the Problems of Mathematical Physics

When we pass from ordinary to partial differential equations, it becomes impossible (with rare exceptions) and anyway not particularly useful to write out general solutions. But mathematical physics in any case seeks only those solutions that satisfy given side conditions—initial conditions, boundary conditions, etc. In solving many problems of mathematical physics it is advantageous to use the following semiempirical rule, which is rigorously based only in certain cases (see [5, §89; 25, §29; 28]).

THE INVARIANCE PRINCIPLE. *If a boundary-value problem is invariant under a group, then we should seek a solution among functions invariant under this group.*

Invariance of a boundary-value problem means invariance of the differential equation, also of the manifold where the data are given, and also of the data themselves.

When invariance of the boundary conditions is lost (as often happens), the principle stated can be put to use in other ways. This is what happens, for example, in the method of the majorant in the proof of the Cauchy–Kovalevskaya theorem (on the method of the invariant majorant, see [9]). Another example is the Riemann method, which reduces the Cauchy problem with arbitrary (hence not invariant) data to the special Goursat problem, which is invariant and can be solved by the invariance principle. Here, we only indicate briefly the essence of this approach, referring for details to [12].

The Group Approach to Riemann's Method

This section is an attempt at synthesis, at combining Riemann's method [30] of integrating linear hyperbolic second-order equations with Lie's group classification [16] of such equations. Also important here is the invariant formulation (in terms of Laplace invariants) of Lie's results, as given by Ovsiannikov [25].

Riemann's method reduces the problem of integrating the equation

$$\begin{aligned} L[u] &\equiv u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u \\ &= f(x,y) \end{aligned} \tag{10}$$

to the construction of an auxiliary function v satisfying the adjoint equation with given conditions on the characteristics:

$$L^*[v] = 0, \qquad v|_{x=x_0} = \exp\int_{y_0}^{y} a(x_0,\eta)\,d\eta,$$
$$v|_{y=y_0} = \exp\int_{x_0}^{x} b(\xi,y_0)\,d\xi. \tag{11}$$

Once v is found, the solution of the Cauchy problem for Eq. (10) with data on an arbitrary noncharacteristic curve is obtained by the known integral formula. The function v is called the *Riemann function*, and the boundary-value problem (11) which determines it is called the *characteristic Cauchy problem* or *Goursat problem.*

The quantities

$$h = a_x + ab - c, \qquad k = b_y + ab - c$$

are called the *Laplace invariants* for Eq. (10). They remain unaltered by any linear transformation of u with variable coefficients, with x, y not being transformed. In contrast, the quantities

$$p = \frac{k}{h}, \qquad q = \frac{1}{h}(\ln h)_{xy} \tag{12}$$

are invariant under the general equivalence transformation $\bar{x} = \alpha(x)$, $\bar{y} = \beta(y)$, $\bar{u} = \lambda(x,y)u$ of the homogeneous equation (10). These invariants are useful for the classification of Eq. (10) according to the dimensionality of the admissible group. Namely, *the homogeneous equation* (10) ($f = 0$) *admits a 4-dimensional Lie algebra* [more precisely, the quotient algebra with respect to the ideal generated by $X = \phi(x, y)$ with $\phi(x, y)$ an arbitrary solution of Eq. (10)] *if the quantities* (12) *are constant; whereas if at least one of them fails to be constant, then Eq.* (10) *can admit at most a 2-dimensional algebra.* For proof, see [25], §9.6. Using this result, one proves the following theorem (see [11] or [12]):

THEOREM 2. *Assume that Eq.* (10) *has constant invariants* (12). *Then the Goursat problem* (11) *admits a 1-parameter group. Therefore, the invariance principle is applicable, and the Riemann function can be found from a second-order ordinary differential equation.*

EXAMPLE 1. For the telegrapher's equation $u_{xy} + u = 0$ we have $p = 1, q = 0$. Hence, Theorem 2 applies. The Goursat problem (11), namely,

$$v_{xy} + v = 0, \qquad v|_{x=x_0} = 1, \qquad v|_{y=y_0} = 1 \tag{13}$$

must by Theorem 2 admit a 1-parameter group with generator

$$X = (x - x_0)\frac{\partial}{\partial x} - (y - y_0)\frac{\partial}{\partial y}.$$

Functionally independent invariants of this group are v and $z = (x - x_0)(y - y_0)$. Therefore, the invariant solution has the form $v = V(z)$, and after substitution in Eqs. (13), we get a form of Bessel's equation: $zV'' + V' + V = 0$ with condition $V(0) = 1$. Consequently, for the telegrapher's equation, a Riemann function is the Bessel function J_0.

EXAMPLE 2. Riemann ([30], §9) applied the technique he introduced to the equation

$$u_{xy} + \frac{\ell}{(x+y)^2}\, u = 0, \qquad \ell = \text{const} \neq 0. \tag{14}$$

In the corresponding problem (11), the condition on the characteristics has the form

$$v|_{x=x_0} = 1, \qquad v|_{y=y_0} = 1. \tag{14'}$$

Riemann reduced the problem (14), (14′) to an ordinary differential equation (leading to the special hypergeometric function of Gauss), considering v as a function of the variable

$$z = \frac{(x - x_0)(y - y_0)}{(x_0 + y_0)(x + y)}. \tag{15}$$

Here is how this looks from the group point of view. The invariants (12) of Eq. (14) are $p = 1, q = 2/\ell$. Hence, Theorem 2 applies here. Solving the determining equation, we find the operator

$$X = (x - x_0)(x + y_0)\frac{\partial}{\partial x} - (y - y_0)(y + x_0)\frac{\partial}{\partial y},$$

admissible for the Goursat problem (14), (14′). Invariants for this operator are v and the quantity z given by Eq. (15). Therefore, the invariant solution has the form $v = V(z)$. This is just the invariant solution found by Riemann!

EXAMPLE 3. Next take the equation

$$u_{xy} + \frac{\ell}{x+y}\, u = 0, \quad \ell = \text{const} \neq 0,$$

an "intermediate case" between the telegrapher's equation (13) and Eq. (14). The invariants (12) are $p = 1, q = 1/\ell(x + y)$. But q not being constant, Theorem 2 is not applicable.

A full catalog of equations to which Theorem 2 applies is in [11].

Fundamental Solutions

Keeping the same orientation as in the preceding section—the application of the invariance principle to boundary-value problems with arbitrary data by reduction to an invariant problem of a special form—let us see what Lie group theory can offer for the construction of fundamental solutions in the case of the three fundamental equations of mathematical physics. This natural line of development of group analysis, passing to the space of distributions, was sketched in [10], giving heuristic considerations and statement of the problems. Yurii Berest [3], my student, recently got remarkable results applying this method to wave equations in Riemannian manifolds with nontrivial conformal group. Some details of infinitesimal group techniques applicable to distributions may be found in [12].

The Laplace Equation. Let us consider the equation

$$\Delta u = \delta(x), \quad x \in \mathbb{R}^n, \tag{16}$$

for a fundamental solution as a boundary-value problem, where at a fixed point, the origin, a δ-function singularity is given. This boundary problem is invariant under the group of rotations and dilations, generated by the operators

$$\begin{aligned} X_{ij} &= x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, \quad i, j = 1, \ldots, n, \\ Z &= x^i\frac{\partial}{\partial x^i} + (2 - n)u\frac{\partial}{\partial u}. \end{aligned}$$

A basis of the invariants of this group consists of the single function $J = u|x|^{n-2}$. According to the invariance principle, the fundamental solution is to be sought as an invariant solution, determined by the equation $J =$ const. Thus,

$$u = C|x|^{2-n}. \tag{17}$$

Substituting Eq. (17) into Eq. (16), we find the value of the constant, $C = 1/(2 - n)\,\Omega_n$, where Ω_n is the measure of the surface of the unit sphere in n-space. Thus, *the fundamental solution was determined from the condition of invariance up to a constant multiple, and the differential equation served only for the normalization.*

The Heat Equation. The equation

$$u_t - \Delta u = \delta(t, x) \qquad (18)$$

with n-dimensional Laplace operator in the space of variables x^i is invariant under the group of rotations, Galilei transformations, and dilations, which is generated by

$$X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}, \qquad Y_i = 2t \frac{\partial}{\partial t} - x^i u \frac{\partial}{\partial u},$$
$$Z = 2t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} - nu \frac{\partial}{\partial u}.$$

This group has the invariant

$$J = ut^{n/2} e^{|x|^2/4t}.$$

Therefore, an invariant solution has the form

$$u = Ct^{-n/2} e^{-|x|^2/4t}. \qquad (19)$$

Equation (18) serves as a normalizing condition: Substitution of Eq. (19) into Eq. (18) yields the value of the constant, $C = (2\sqrt{\pi})^{-n}$.

The Wave Equation. For the equation

$$u_{tt} - \Delta u = \delta(t, x), \qquad (20)$$

the group of symmetries is generated by

$$X_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \qquad Y_i = t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t},$$
$$Z = t \frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} + (1-n)u \frac{\partial}{\partial u}, \quad i, j = 1, \ldots, n.$$

The operators X_{ij} and Y_i generate the group of rotations and Lorentz transformations and have two invariants: u and $\tau = t^2 - |x|^2$. Therefore, an invariant solution is to be sought of the form $u = f(\tau)$. The condition of invariance under the group of dilations with generator Z takes the form

$$2\tau f'(\tau) + (n-1) f(\tau) = 0. \qquad (21)$$

Let us look only at odd n, for in the case of even n we would have to use the method of balayage of Hadamard. Then, setting $n = 2m + 1, m = 0, 1, \ldots$, we write Eq. (21) as $\tau f'(\tau) + mf(\tau) = 0$, which is known to have general solution

$$f(\tau) = \begin{cases} C_1 \theta(\tau) + C_2 & \text{for } m = 0 \\ C_1 \delta^{(m-1)}(\tau) + C_2 \tau^{-m} & \text{for } m \neq 0. \end{cases}$$

Substitution of these formulas in Eq. (20) gives $C_1 = \frac{1}{2}\pi^{-m}, C_2 = 0$. In this way, the invariance principle yields the fundamental solution

$$u = \begin{cases} \frac{1}{2}\theta(t^2 - |x|^2), & n = 1 \\ \frac{1}{2}\pi^{(1-n)/2} \delta^{(n-3)/2}(t^2 - |x|^2), & n \geq 3, \end{cases}$$

where θ is the Heaviside function and $\delta^{(n-3)/2}$ is the derivative of order $(n-3)/2$ of the δ-function.

Kepler's Laws

The motion of a material point under a central force with potential $V = \alpha/|x|$ satisfies the obvious law of conservation of angular momentum $M = m(x \times v)$, where m is the mass of the particle and x and v are its coordinate and velocity, respectively. This conservation law, known in celestial mechanics as Kepler's Second Law, is a corollary of the invariance of the Lagrange equations of motion under the group of rotations and follows from Noether's theorem. Write the infinitesimal transformation of the rotation group with vector parameter $a = (a^1, a^2, a^3)$ in the form

$$\overline{x} = x + \delta x, \qquad \delta x = x \times a. \qquad (22)$$

Then from the group point of view, *Kepler's Second Law expresses the invariance of the problem under infinitesimal rotations* (22).

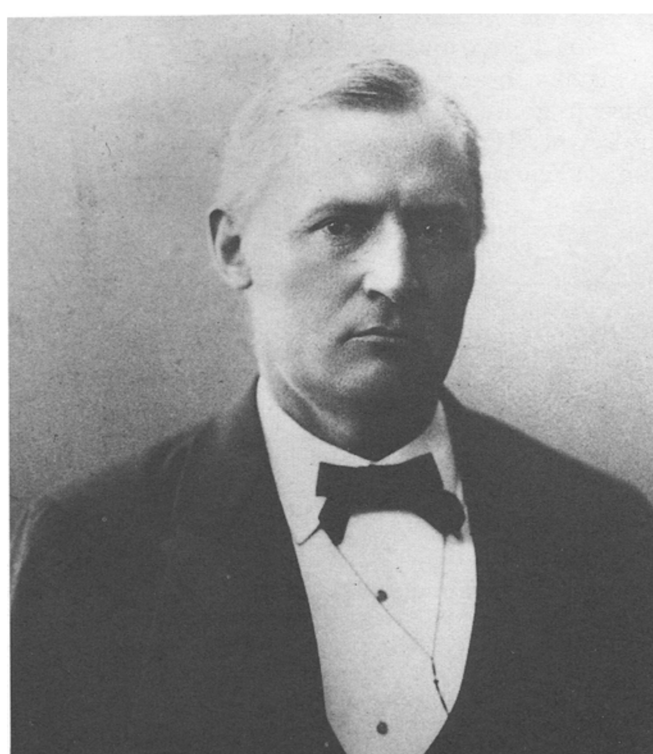
The Kepler problem is also invariant under the inhomogeneous dilation generated by the operator

$$X = 3t \frac{\partial}{\partial t} + 2x^i \frac{\partial}{\partial x^i}.$$

An invariant both of the rotation group and of this dilation is the quantity $J = t^2/r^3$. *The existence of this invariant is called in celestial mechanics Kepler's Third Law.*

Finally, the Kepler problem has a special group of symmetries, which in the notation of Eqs. (22), can be written

$$\overline{x} = x + \delta x, \qquad \delta x = (x \times v) \times a + x \times (v \times a). \qquad (23)$$



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This 3-parameter group differs from ordinary Lie groups of point and contact transformations, being more general; it is called the *group of Lie–Bäcklund transformations* [2]. The computation of the symmetry group (23) is carried out in [9], p. 346. From Noether's theorem one gets a vector integral of the motion

$$A = v \times M + \alpha \frac{x}{|x|},$$

found first by Laplace [14]. Taking the scalar product of Laplace's vector A with the radius vector x, one readily infers that the orbit of Keplerian motion is an ellipse. This is Kepler's First Law. So from the group point of view, *Kepler's First Law expresses the invariance of the problem under the 3-parameter Lie–Bäcklund group with infinitesimal transformation* (23).

Thus, all three of Kepler's Laws of celestial mechanics have a group-theoretic nature.

Concluding Remarks

I could continue in this spirit, for there are many entertaining applications of Lie theory, and nowadays newly developed methods of group analysis are awaiting application. But I hope what I have said is enough to convince you that acquaintance with the classical foundations and modern group-theoretic methods has become an important part of the mathematical culture of anyone constructing and investigating mathematical models of natural problems. For this, one can go to the beautiful books of Lie, Bianchi, and so on, and more recent works (see the References).

In conclusion, I would like to carry over to Lie theory in mathematical physics Einar Hille's remark [8]: "I hail a [semi-]group when I see one and I seem to see them everywhere! Friends have observed, however, that there are mathematical objects which are not [semi-]groups."

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